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Finite Interpolation of Random Fields

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Abstract
The best in the mean square sense linear interpolator for a stationary random field is constructed in terms of the orthogonal polynomials associated with the spectral distribution function of this field.

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1. Introduction

1.1. In this report useful formulas are presented for interpolating a random field, say $\{X_u\}_{u\in\mathbb{Z}^d}$, indexed by a d-dimensional integer-valued parameter $u = (u_1, ..., u_d)$. The observations on the random field in question are carried out within a certain set $\mathbb{U} \in \mathbb{Z}^d$ (cf. (2.4) below). However, only observations within a certain subset $\mathbb{U}\setminus\mathbb{J}$ are available, and the rest of the observations is missed. It is required to restore the missed values of $\{X_j\}_{j\in\mathbb{J}}$ by using the given values of $\{X_k\}_{k\in\mathbb{U}\setminus\mathbb{J}}$.

The traditional task consists of finding the best (in the least squares sense) linear interpolator for $\{X_j\}_{j\in\mathbb{J}}$, i.e. the linear approximation of each X_j with $j \in \mathbb{J}$ by the sums over $\mathbb{U}\setminus\mathbb{J}$ of type

$$\hat{X}_j = \sum_{k\in\mathbb{U}\setminus\mathbb{J}} c_{jk} X_k \tag{1.1}$$

where the coefficients c_{jk} are chosen so that the least squares criterion is satisfied: for any system of (complex) numbers $\{\xi_j\}_{j\in\mathbb{J}}$

$$\mathbb{E}\,|\sum_{j\in\mathbb{J}} \xi_j (X_j - \hat{X}_j)|^2 = \min. \tag{1.2}$$

The random variable $\hat{X}_j$ so defined is called the best (in the least squares sense) linear interpolator for X_j, $j \in \mathbb{J}$.

1.2. The solution to the above problem is sought under the assumption that the first and the second order characteristics of the random field in question are known. This allows

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us to assume, without loss of generality, that the field is centered by its expectation, and to express the least squares criterion (1.2) in terms of the known covariance function of the field (cf. section 2.1). It is well-known (see, e.g. Ripley (1988)) that the coefficients c_{jk} in (1.1), expressed in terms of the covariances, are then the roots of a system of linear equations which is typically large, as large as the number of points in $\mathbb{U}$. Therefore, even the numerical solution of this system is rather involved. As is shown below under the additional stationarity assumption, however, the difficulties are somewhat mitigated by making use of the Toeplitz character of the covariance matrix to be inverted. Extending, namely, the methods of Grenander and Szegö (1958) we will reduce the problem to minimizing certain Toeplitz forms associated with the spectral distribution function of the field in question; cf. also Rosenblatt (1985). The coefficients c_{jk} in (1.1) then will be expressed in terms of the coefficients of the orthogonal polynomials associated with this distribution function.

In section 2 we formulate the minimization problem of Toeplitz forms mentioned above, and indicate the relationship with the original interpolation problem. In section 3 we give an easy solution; see theorem 3.1 which can be viewed as an extension of theorem 2.2(a) in Grenander and Szegö (1958).

2. Minimization Problem

2.1. We assume, as was said in section 1.2, that the random field $\{X_u\}_{u\in\mathbb{Z}^d}$ in question is centered

$$\mathbb{E}\{X_u\} = 0 \text{ for all } u \in \mathbb{Z}^d$$

and wide sense stationary: the covariance function

$$\gamma_u = \mathbb{E}\{\overline{X}_t X_{t+u}\} \tag{2.1}$$

depends only on the lag u. Notice that complex-valued observations are allowed.

We will use throughout the following multi-index notation: for a d-dimensional variable $z = (z_1, ..., z_d)$ and a collection of integers $u = (u_1, ..., u_d)$

$$z^u \equiv z_1^{u_1} \dots z_d^{u_d}.$$

The relationship between the spectral distribution function of the random field in question, denoted by $F(\lambda)$ where $\lambda = (\lambda_1, ..., \lambda_d) \in \Lambda = [-\pi, \pi]^d$, and its Fourier coefficients γ_u (cf. (2.1)) can be written as follows:

$$\gamma_u = \int_\Lambda z^u \, dF(\lambda), \quad z = (e^{i\lambda_k}, k = 1, \dots d) \equiv e^{i\lambda}. \tag{2.2}$$

By applying first (2.1) and then (2.2) to the left-hand side of (1.2) we can express the least squares criterion (1.2) in spectral terms:

(2.3) $$\int_{\Lambda} |\sum_{j\in \mathbb{J}} \xi_j (z^j - \sum_{k\in \mathbb{U}\setminus\mathbb{J}} c_{jk} z^k)|^2 dF(\lambda) = \min, \quad z = e^{i\lambda}.$$

2.2. Let $N = (N_1, \dots, N_d) \in \mathbb{Z}_+^d$ with certain non-negative integers $N_1, \dots, N_d$. With each $N \in \mathbb{Z}_+^d$ we associate the set $\mathbb{N}$ of form

$$\mathbb{N} = [0, N] \equiv [0, N_1] \times \dots \times [0, N_d].$$

A polynomial in a complex d-dimensional variable $z = (z_1, \dots, z_d)$ of degree N is then defined by

$$p_N(z) = \sum_{n\in\mathbb{N}} a_n z^n.$$

Let $N, M \in \mathbb{Z}_+^d$ be such that if $N = (N_1, \dots, N_d)$ and $M = (M_1, \dots, M_d)$, then $N_i \geq M_i$ for all $i = 1, \dots, d$. For such N and M we write $M \leq N$, and define the set

$$[M, N] \equiv [M_1, N_1] \times \dots \times [M_d, N_d]$$

We assume that the sets $\mathbb{U}$ and $\mathbb{J}$ introduced in section 1.1 are of form

(2.4) $$\mathbb{U} = [0, U] \quad \text{and } \mathbb{J} = [m, M]$$

where $U \geq M \geq m$, and consider the following

Minimimization Problem. *Let $\{p_U^{(j)}(z)\}_{j\in\mathbb{J}}$ be a system of polynomials of degree* U, *with*

(2.5) $$p_U^{(j)}(z) = \sum_{u\in\mathbb{U}} a_{ju} z^u.$$

Look for a special system of polynomials which renders the (generalized) Toeplitz form

(2.6) $$\int_{\Lambda} |\sum_{j\in\mathbb{J}} \xi_j p_U^{(j)}(z)|^2 dF(\lambda), \quad z = e^{i\lambda}$$

as small as possible, for any system of (complex) numbers $\{\xi_j\}_{j\in\mathbb{J}}$, under the following constraints:

(2.7) $$a_{jk} = \delta_{jk}, \quad j, k \in \mathbb{J}.$$

Here δ is the usual Kronecker symbol: $\delta_{jk} = 1$ if $j = k$ and $= 0$ otherwise.

The relationship to the interpolation problem is obvious: if

(2.8) $$\underline{p}_U^{(j)}(z) = \sum_{u\in\mathbb{U}} \underline{a}_{ju} z^u, \quad j \in \mathbb{J}$$

are the minimizers of (2.6), subject to (2.7), then $c_{jk} = - \underline{a}_{jk}$ in (2.3). Hence the best linear interpolator for X_j, $j \in \mathbb{J}$ is

$$\hat{X}_j = -\sum_{k \in U \setminus J} \underline{a}_{jk} X_k.$$

3. Solution

3.1. With the spectral distribution function $F(\lambda)$, $\lambda \in \Lambda = [-\pi, \pi]^d$, we associate the system of polynomials

$$\{\phi_N(z)\}_{N \in Z_+^d}$$

of a d-dimensional complex variable z which are orthonormal on a unit d-dimensional circle $z = e^{i\lambda}$ with respect to the weight $dF(\lambda)$: they satisfy the following conditions.
(i) $\phi_N(z)$ is a polynomial of degree n with a real and positive coefficient of z^N.

(ii) $$\int_\Lambda \phi_N(z)\,\overline{\phi_M(z)}\,dF(\lambda) = \delta_{NM}, \quad z = e^{i\lambda}.$$

The system is uniquely defined by conditions (i) and (ii). The well-known condition on the density $f(\lambda) = dF(\lambda)\,/\,d\lambda$, sufficient for (i), is

$$\int_\Lambda \log f(\lambda)\,d\lambda > -\infty;$$

cf. Rosenblatt (1985).

Notice that $\phi_0(z) = \gamma_0^{-1/2}$. Further, for $1_k = (0...010...0)$ with the k^{th} coordinate equal 1, for instance, we have $\phi_{1_k}(z) = (\gamma_0 - |\gamma_{1_k}|^2 / \gamma_0)^{-1/2}(z_k - \gamma_{1_k} / \gamma_0)$.

The coefficient in $\phi_N(z)$ of z^n will be denoted by φ_{Nn}, i.e.

$$\phi_N(z) = \sum_{n \in [0,N]} \varphi_{Nn}\, z^n.$$

For $n \in [0, N]^c$ we set $\varphi_{Nn} = 0$.

3.2. We are now in position to formulate

Theorem 3.1. (i) *The system of polynomials* $\{\underline{p}_U^{(j)}(z)\}_{j \in J}$ *with*

(3.1) $$\underline{p}_U^{(j)}(z) = \sum_{n \in [m,U]} x_{jn}\, \phi_n(z)$$

solves the minimization problem in section 2.2, where the weights x_{jn} *are determined by solving the following system of linear equations*:

(3.2) $$\sum_{j \in J} \Phi_{kj}\, x_{jn} = \overline{\varphi}_{nk}, \quad k \in J, \ n \in [m, U]$$

with

(3.3) $$\Phi_{kj} = \sum_{n\in[m,U]} \overline{\varphi}_{nk}\, \varphi_{nj}.$$

(ii) *By writing the solution to (3.2) in the form*

(3.4) $$x_{jn} = \sum_{k\in \mathbb{J}} \Phi^{jk}\, \overline{\varphi}_{nk}, \quad j \in \mathbb{J}, \quad n \in [m, U],$$

we can express the error of approximation as follows:

(3.5) $$\int_{\Lambda} |\sum_{j\in \mathbb{J}} \xi_j\, p_U^{(j)}(z)|^2\, dF(\lambda) = \sum_{j,k\in \mathbb{J}} \Phi^{jk}\, \xi_j\, \overline{\xi}_k, \quad z = e^{i\lambda}.$$

Remark. By (3.2) - (3.4)

$$\sum_{n\in[m,U]} x_{jn}\, \overline{x}_{kn} = \Phi^{jk}, \quad j, k \in \mathbb{J}.$$

Hence (3.5) can be written alternatively:

(3.5') $$\int_{\Lambda} |\sum_{j\in \mathbb{J}} \xi_j\, p_U^{(j)}(z)|^2\, dF(\lambda) = \sum_{n\in[m,U]} |\sum_{j\in \mathbb{J}} \xi_j\, x_{jn}|^2, \quad z = e^{i\lambda}.$$

Proof of theorem 3.1. The system of polynomials defined by (3.1) satisfies (2.7). Indeed, with the notation (2.8) we have

$$a_{jk} = \sum_{n\in[m,U]} x_{jn}\, \varphi_{nk} = \sum_{l\in \mathbb{J}} \Phi^{jl}\, \Phi_{lk} = \delta_{jk}, \quad j, k \in \mathbb{J},$$

by (3.2) - (3.4).

We estimate the expression (2.6) by presenting first the polynomials (2.5) in the following form:

(3.6) $$p_U^{(j)}(z) = \sum_{u\in U} v_{ju}\, \phi_u(z).$$

We get ($z = e^{i\lambda}$)

(3.7) $$\int_{\Lambda} |\sum_{j\in \mathbb{J}} \xi_j\, p_U^{(j)}(z)|^2\, dF(\lambda) = \sum_{u\in U} |\sum_{j\in \mathbb{J}} \xi_j\, v_{ju}|^2 \geq \sum_{n\in[m,U]} |\sum_{j\in \mathbb{J}} \xi_j\, v_{jn}|^2.$$

The coefficients v_{ju} in (3.9) are restricted by condition (2.7) to

(3.8) $$\sum_{n\in[m,U]} v_{jn}\, \varphi_{nk} = \delta_{jk}, \quad j, k \in \mathbb{J}.$$

It will be shown now that under the restrictions (3.8) inequality (3.7) can be extended as follows:

(3.9) $$\sum_{n\in[m,U]} |\sum_{j\in \mathbb{J}} \xi_j\, v_{jn}|^2 \geq \sum_{n\in[m,U]} |\sum_{j\in \mathbb{J}} \xi_j\, x_{jn}|^2 = \sum_{j,k\in \mathbb{J}} \Phi^{jk}\, \xi_j\, \overline{\xi}_k,$$

which in view of (3.5) implies the desired assertions of the theorem.

To this end, consider the Hermitian kernel $\{K_{un}\}_{u,n\in[m,U]}$ with

$$K_{un} = \delta_{un} - \sum_{j,k\in \mathbb{J}} \Phi^{jk}\,\overline{\varphi}_{nk}\,\varphi_{uj}. \tag{3.10}$$

By (3.2) - (3.4) it satisfies

$$\sum_{n\in[m,U]} K_{un}\,\varphi_{nj} = 0,\ \ j\in\mathbb{J},\ \ u\in[m,U].$$

Therefore

$$\sum_{n\in[m,U]} K_{un}\,K_{nv} = K_{uv},\ \ u,v\in[m,U].$$

For any system of complex numbers $\{\eta_n\}_{n\in[m,U]}$ the last identity implies

$$\sum_{u,n\in[m,U]} K_{un}\,\overline{\eta}_u\,\eta_n = \sum_{u\in[m,U]} \Big|\sum_{n\in[m,U]} K_{un}\,\eta_n\Big|^2 \geq 0,$$

which by definition (3.10) means that

$$\sum_{n\in[m,U]} |\eta_n|^2 \geq \sum_{j,k\in\mathbb{J}} \Phi^{jk}\,\overline{y}_k\,y_j, \tag{3.11}$$

where

$$y_j = \sum_{n\in[m,U]} \overline{\eta}_n\,\varphi_{nj},\ \ j\in\mathbb{J}.$$

For the special choice of

$$\overline{\eta}_n = \sum_{j\in\mathbb{J}} \xi_j\,v_{jn},\ \ n\in[m,U]$$

(3.11) reduces to the desired inequality (3.9), since by (3.8) we have

$$y_k = \sum_{n\in[m,U]}\sum_{j\in\mathbb{J}} \xi_j\,v_{jn}\,\varphi_{nk} = \xi_k,\ \ k\in\mathbb{J}.$$

The proof is complete. <>

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