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Gray-scale Granulometries Compatible with Spatial Scalings

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Abstract

In the field of mathematical morphology granulometries form the major tool for the computation of size distributions for binary images. A granulometry can be defined as a one-parameter family of openings depending on a real parameter $\lambda > 0$ such that the opening becomes more and more active as λ increases. The granulometry is called a Euclidean granulometry if it is translation invariant and compatible with scalings. An important result due to Matheron states that one can build simple Euclidean granulometries by taking openings with convex compact structuring elements. In this paper we describe a general extension of binary Euclidean granulometries to gray-scale images using the notion of a spatial scaling (as opposed to umbral scaling). The main result of this paper is that one can build gray-scale Euclidean granulometries with one structuring function if and only if this function has a convex compact domain and is constant there (flat function).

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1. Introduction

Morphological granulometries were introduced by Matheron [20] to model the sieving of a random binary image according to the size and shape of grains within the image. Intuitively, as the mesh size of the sieve is increased, more of the image grains will fall through

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the sieve and the residual area of the filtered (sieved) image will decrease monotonically. These residual areas form a size distribution that is indicative of the image structure. Upon normalization, this size distribution becomes an increasing function from 0 to 1 and is a probability distribution function. Both it, and its derivative, which is a probability density, are called the *granulometric size distribution*, or, of more recent vintage, the *pattern spectrum* of the image. Moments of this size distribution serve as image features. Besides the original formulation of Matheron, granulometric size distributions are discussed by Serra [27] and Dougherty and Giardina [5,10]. Applications include those of Maragos [16, 17] for multiscale shape representation and symbolic image modeling, of Dougherty and Pelz [8] for the analysis of electrophotographic images, and of Dougherty et al [6,7,9] for segmentation and texture-based pixel classfication based on local granulometric features.

A particular class of binary granulometries merits special attention in Matheron's original theory, this class comprising the Euclidean granulometries. Besides translation invariance, these satisfy a certain property that makes them compatible (in a certain sense) with Euclidean scaling. Matheron [20] showed that the Euclidean granulometries possess a representation in terms of morphological openings (Thm. 3.7). As it stands, this representation requires a double union that makes it impractical for application. A key question concerns conditions under which this double union reduces to a tractable single union, for when it does we have a practical paradigm for the construction of size distributions. In one of his most profound results (Thm. 3.9), Matheron [20] gives necessary and sufficient conditions for this reduction. Together, Matheron's results, Theorems 3.7 and 3.9, characterize the design mechanism for application of morphological granulometries.

As noted by Serra [28, Chapter 5], the algebraic theory of granulometries extends at once to complete lattices, and therefore to gray-scale images. Direct extension of the Euclidean theory to the gray-scale is given by Dougherty [1,2,3]. We use the terminology "direct extension" in a specific manner. The theory of [1,2,3] employs a Euclidean scaling condition that is induced from the umbra formulation of gray-scale morphology and takes place relative to the image's graph. Consequently, whereas a binary Euclidean granulometry must be compatible with scaling, or magnification, of a d-dimensional binary image within Euclidean d-space, a gray-scale Euclidean granulometry must be compatible with scaling, or magnification, of the image graph within Euclidean $(d+1)$-dimensional space. From the geometrical perspective that is relevant to the present paper, there must be scaling compatibility in both the domain and range. As in the Matheron theory, there is a representation; however, it involves a double supremum. Application requires reduction of this double supremum to a single supremum and nonrestrictive sufficient conditions for this reduction are given in terms of umbra convexity. (Reduction will be subsequently discussed.) Application of this approach to the detection of osteoporosis in nuclear magnetic resonance images is given by Dougherty et al [4].

But there is another way to consider scaling compatibility: the granulometry must be compatible with scaling only in the domain of the image, that is, in the spatial direction. Fig. 1(a) illustrates signal scaling relative to both the domain and the range; Fig. 1(b) illustrates it for only the domain. Notice that the basic "shape" of the signal is unchanged under umbral scaling but is substantially changed under spatial scaling.

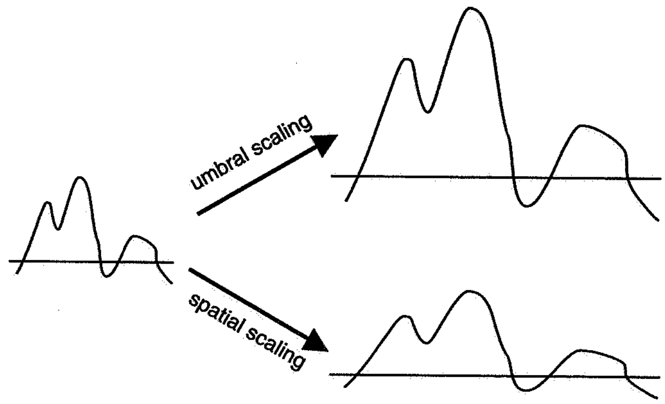


FIGURE 1. *Umbral scaling and spatial scaling.*

In line with the distinction between an umbral (overall) and a spatial (horizontal) scaling one can also distinguish among an umbral translation (both in spatial and gray-scale direction) and a spatial translation. This results in four different ways to generalize the concept of a Euclidean granulometry to gray-scale images. To distinguish among them we will use a double prefix "H-" or "T-", where the "H" stands for "horizontal" meaning that only translation or scaling in the horizontal direction is being considered, and where the "T" stands for "total" or "translation" meaning that translation or scaling is taken in all directions. For instance, "H-scaling" means scaling in the spatial direction only. Using the convention that the first prefix accounts for translation invariance and that the second prefix accounts for the scaling compatibility, we may distinguish between four types of granulometries for gray-scale images, namely (T,T)-, (T,H)-, (H,T)- and (H,H)-Euclidean granulometries. More precise definitions will be given in Sect. 5. If we speak of a T- or H-granulometry (and omit the second prefix), then we refer to the type of translation invariance being considered, and make no a priori assumption about scaling compatibility.

In this paper we are concerned with granulometries compatible with H-scalings (i.e., (T,H)- and (H,H)-Euclidean granulometries) whereas in [2] one deals exclusively with (T,T)-Euclidean granulometries; these are called Euclidean granulometries there. Once again we achieve a Matheron granulometric representation, and, as in the approach of [1,2,3], this is a double supremum. Nevertheless (and this is the main result of the present paper), we will derive a significant generalization of Matheron's theorem to provide necessary and sufficient conditions for reduction of the double supremum representation. Indeed, by employing the celebrated Krein-Milman theorem, we derive such necessary and sufficient conditions.

From a geometrical and applications perspective, the key theorem herein is a limiting theorem, in the sense that it shows that if we desire the granulometry to be compatible with spatial scalings, then we must pay a price: put simply and with details to follow, such granulometries have to be generated by flat structuring functions. This constraint greatly restricts the kind of granulometric information that can be extracted if we insist upon sizing compatibility with spatial scaling instead of umbral scaling.

In Sect. 2 we briefly recall some basic concepts from mathematical morphology. The emphasis in this section is on openings since these play an important role in Sect. 3 where we give an overview of the work of Matheron on binary (Euclidean) granulometries and convexity. In Sect. 4 we introduce the reader to the field of gray-scale morphology. Special attention is given to so-called T-openings and H-openings and to the construction of flat operators by thresholding. Granulometries for gray-scale images are introduced in Sect. 5. There it is argued that one can essentially distinguish among four types of Euclidean granulometries. In Sect. 6 we discuss (T,H)-Euclidean granulometries, in particular those which are generated by one structuring function. In Sect. 7 we present results very similar to those of Sect. 6, but now for (H,H)-granulometries. We end with some conclusions in Sect. 8.

2. A brief reminder on binary morphology

In this section we will briefly recall some of the basic concepts of mathematical morphology for binary images, modelled mathematically as sets. For a comprehensive exposition on mathematical morphology we refer to [20,27,28] and also [5, Chapter 3] and [10]. Since these are the most important in this paper the emphasis will be on openings.

For $X, A \subseteq \mathbb{R}^d$ the *Minkowski addition*, or *dilation*, is defined as

$$X \oplus A = \{x + a \mid x \in X \ , \ a \in A\} = \bigcup_{a \in A} X_a.$$

Here X_a is the translate of the set X along the vector a, that is, $X_a = \{x + a \mid x \in X\}$. The *Minkowski subtraction*, or *erosion*, is defined as

$$X \ominus A = \bigcap_{a \in A} X_{-a}.$$

Here A is called the *structuring element.* The composition of these two operations yields two other operations called the *opening* and the *closing* by A, respectively given by

$$X \bigcirc A = (X \ominus A) \oplus A$$
$$X \bullet A = (X \oplus A) \ominus A.$$

For the opening we have the alternative geometric expression

$$X \bigcirc A = \bigcup \{A_h \mid h \in \mathbb{R}^d \text{ and } A_h \subseteq X\}.$$

In words, the opening $X \bigcirc A$ consists of all translates A_h of the structuring element A which are completely contained within the original set X.

The dilation, erosion, opening and closing are instances of operators on $\mathcal{P}(\mathbb{R}^d)$ which are translation invariant. Recall that an operator $\psi : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$ is called *translation invariant* if

$$\psi(X_h) = [\psi(X)]_h,$$

for every $X \subseteq \mathbb{R}^d$ and $h \in \mathbb{R}^d$. If φ, ψ are operators on $\mathcal{P}(\mathbb{R}^d)$ then we write $\varphi \leq \psi$ if $\varphi(X) \subseteq \psi(X)$ for every $X \subseteq \mathbb{R}^d$.

An operator $\psi : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$ is called an *algebraic opening* [20] if

(i) ψ is increasing, i.e., $X \subseteq Y$ implies that $\psi(X) \subseteq \psi(Y)$,
(ii) ψ is anti-extensive, i.e, $\psi(X) \subseteq X$ for all X,
(iii) ψ is idempotent, that is, $\psi^2 = \psi$.

It can be shown that the opening $\psi(X) = X \bigcirc A$ is an algebraic opening which is translation invariant; this opening is sometimes called a *structural opening* because of the fact that it uses only one structuring element. The *invariance domain* of an operator ψ is defined as

$$\mathsf{Inv}(\psi) = \{X \subseteq \mathbb{R}^d \mid \psi(X) = X\}.$$

If ψ is translation invariant then $\mathsf{Inv}(\psi)$ is closed under translation. One can easily show that the invariance domain of an opening is closed under union.

The following representation theorem is due to Matheron [20].

2.1. Theorem. *Let $\psi : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$ be a translation invariant opening. Then ψ can be written as*

$$\psi(X) = \bigvee_{A \in \mathcal{A}} X \bigcirc A,$$

for some family of structuring elements $\mathcal{A}$.

In this theorem one may choose for $\mathcal{A}$ the invariance domain of ψ. More generally, one may choose for $\mathcal{A}$ any collection of structuring elements such that $\mathsf{Inv}(\psi)$ is the smallest subset of $\mathcal{P}(\mathbb{R}^d)$ which contains $\mathcal{A}$ and which is closed under union and translation.

For a proof of the following two results we refer to [24].

2.2. Proposition. *Let ψ, ψ' be openings on $\mathcal{P}(\mathbb{R}^d)$. The following assertions are equivalent:*

(i) $\psi \leq \psi'$
(ii) $\psi\psi' = \psi$
(iii) $\psi'\psi = \psi$
(iv) $\mathsf{Inv}(\psi) \subseteq \mathsf{Inv}(\psi')$.

We consider the special case that both ψ and ψ' are structural openings, that is, $\psi(X) = X \bigcirc A$ and $\psi'(X) = X \bigcirc B$. One can easily show that the assertions of Prop. 2.2 hold if and only if

$$A \bigcirc B = A. \tag{2.1}$$

In that case we say that A is B-open (see [27]).

The following property is elementary.

2.3. Proposition. *Let for every i in some index set I, ψ_i be an opening. Then $\bigvee_{i \in I} \psi_i$ is an opening as well.*

Many of the results concerning morphological operators on $\mathcal{P}(\mathbb{R}^d)$ have been generalized to the general algebraic framework of complete lattices; see [28] and [14,21,22,23,24]. In Sect. 4 we will give an extension to the space of gray-scale images, which also happens to be a complete lattice.

3. Granulometries for binary images

We briefly review salient points concerning Matheron's theory of binary granulometries. For more detailed discussions including explanation of the sieving model, see [20,27] and [5,10].

3.1. Definition. A granulometry on $\mathcal{P}(\mathbb{R}^d)$ is a one-parameter family $\{\psi_\lambda \mid \lambda > 0\}$ of operators on $\mathcal{P}(\mathbb{R}^d)$ such that

(G1) ψ_λ is increasing

(G2) ψ_λ is anti-extensive

(G3) $\psi_\lambda \psi_\mu = \psi_\mu \psi_\lambda = \psi_\mu, \quad \mu \geq \lambda.$

In the sequel we shall denote a granulometry by $\{\psi_\lambda\}$. Substituting $\mu = \lambda$ in (G3) we find that ψ_λ is idempotent and with (G1)–(G2) this means that ψ_λ is an opening. Furthermore, we derive from (G3) that $\psi_\mu = \psi_\mu \psi_\lambda \leq \psi_\lambda$ (cf. Prop. 2.2). Thus we arrive at the following alternative characterization.

3.2. Proposition. *A family of operators $\{\psi_\lambda\}$ on $\mathcal{P}(\mathbb{R}^d)$ is a granulometry if and only if*

(i) *every ψ_λ is an opening*

(ii) *$\psi_\mu \leq \psi_\lambda$ if $\mu \geq \lambda$.*

Instead of (ii) *we may also write*

(ii') $\mathsf{Inv}(\psi_\mu) \subseteq \mathsf{Inv}(\psi_\lambda), \quad \mu \geq \lambda.$

The last statement in this proposition is an immediate consequence of Prop. 2.2. If X is a finite collection of particles, one may think of $\psi_\lambda(X)$ as the subcollection of particles which cannot pass through the sieve with mesh width λ; see also the examples below.

3.3. Examples.

(a) Let α be an opening and define $\psi_\lambda = \alpha$ for every $\lambda > 0$. Then $\{\psi_\lambda\}$ defines a granulometry. More generally, let α_1, α_2 be openings and $\alpha_2 \leq \alpha_1$. Take $\lambda_1 > 0$ and define $\psi_\lambda = \alpha_1$ if $\lambda \in (0, \lambda_1]$ and $\psi_\lambda = \alpha_2$ if $\lambda > \lambda_1$. Then $\{\psi_\lambda\}$ defines a granulometry. It is easy to extend this example to the case where we have a finite collection of openings $\alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_1$.

(b) The example that we describe now is probably the most important one from a practical point of view. Besides that, it is also of great theoretical interest and plays a prominent role in the remainder of this paper. Let A be the unit square (below we will see that this example can be extended to arbitrary convex shapes) and define

$$\psi_\lambda(X) = X \bigcirc \lambda A.$$

Since μA is λA-open if $\mu \geq \lambda$ it follows immediately that $\psi_\mu \leq \psi_\lambda$ for $\mu \geq \lambda$.

3.4. Proposition. *Let I be an arbitrary index set and let $\{\psi_\lambda^i\}$ define a granulometry on $\mathcal{P}(\mathbb{R}^d)$ for every $i \in I$. Then $\{\bigvee_{i\in I} \psi_\lambda^i\}$ defines a granulometry as well.*

This result is an immediate consequence of Prop. 2.3 where it was stated that an arbitrary supremum of openings is again an opening. Since $\psi_\mu^i \leq \psi_\lambda^i$, $\mu \geq \lambda$ for every $i \in I$ the same is true for their supremum.

We mention two other ways to construct binary granulometries. First, assume that α_λ, $\lambda > 0$, is a one-parameter family of openings. Define $\psi_\lambda = \bigvee_{\mu \geq \lambda} \alpha_\mu$, then $\{\psi_\lambda\}$ defines a granulometry.

Second, for every $\lambda > 0$, let $\mathcal{B}_\lambda$ be a collection of subsets of $\mathbb{R}^d$ such that $\mathcal{B}_\mu \subseteq \mathcal{B}_\lambda$ for $\mu \geq \lambda$. Let ψ_λ be the opening generated by $\mathcal{B}_\lambda$. Then $\{\psi_\lambda\}$ defines a granulometry on $\mathcal{P}(\mathbb{R}^d)$

with invariance domain of ψ_λ the smallest family in $\mathcal{P}(\mathbb{R}^d)$ which contains $\mathcal{B}_\lambda$ and is closed under union.

We now turn to granulometries $\{\psi_\lambda\}$ that are translation invariant. This means that $\mathrm{Inv}(\psi_\lambda)$ has to be closed under translations for every $\lambda > 0$. Then ψ_λ is of the form

$$\psi_\lambda(X) = \bigcup_{B \in \mathcal{C}_\lambda} X \bigcirc B,$$

where $\mathcal{C}_\lambda = \mathrm{Inv}(\psi_\lambda)$. Note that $\mathcal{C}_\mu \subseteq \mathcal{C}_\lambda$ for $\mu \geq \lambda$ by Prop. 3.2.

Assume that ψ_λ is a granulometry given by $\psi_\lambda(X) = X \bigcirc B_\lambda$ for some $B_\lambda \subseteq \mathbb{R}^d$. The condition that $\psi_\mu \leq \psi_\lambda$ if $\mu \geq \lambda$ is equivalent to B_μ is B_λ-open for $\mu \geq \lambda$ (i.e., $B_\mu \bigcirc B_\lambda = B_\mu$, $\mu \geq \lambda$). The granulometry in Ex. 3.3(b) given by $\psi_\lambda(X) = X \bigcirc \lambda A$, where A is the unit square, satisfies this requirement. Introducing the notion of *Stieltjes-Minkowski integral* in the space of compact sets supplied with the Hausdorff metric, Matheron [20] was able to construct a large class of families of structuring elements $\{B_\lambda \mid \lambda > 0\}$ such that B_μ is B_λ-open for $\mu \geq \lambda$. As a special case we mention the following example. Let $A_1, A_2, \cdots, A_n \subseteq \mathbb{R}^d$ and $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{n-1}$. Define

$$B_\lambda = \begin{cases} A_1, & \text{for } 0 < \lambda \leq \lambda_1 \\ A_1 \oplus A_2, & \text{for } \lambda_1 < \lambda \leq \lambda_2 \\ \cdot & \\ \cdot & \\ \cdot & \\ A_1 \oplus A_2 \oplus \cdots \oplus A_{n-1}, & \text{for } \lambda_{n-2} < \lambda \leq \lambda_{n-1} \\ A_1 \oplus A_2 \oplus \cdots \oplus A_n, & \text{for } \lambda_{n-1} < \lambda. \end{cases}$$

Then B_μ is B_λ-open for $\mu \geq \lambda$.

Recalling the introduction, we next review the basic properties of binary Euclidean granulometries. We define the multiplication of a set X by a scalar λ in the usual way, namely $\lambda X = \{\lambda x \mid x \in X\}$.

3.5. Definition. A granulometry $\{\psi_\lambda\}$ on $\mathcal{P}(\mathbb{R}^d)$ is called a *Euclidean granulometry* if it satisfies the additional properties

(G4) every ψ_λ is translation invariant,

(G5) $\psi_\lambda(X) = \lambda\psi_1(\lambda^{-1}X)$, for every $\lambda > 0$ and $X \subseteq \mathbb{R}^d$.

We can restate these conditions in terms of the invariance domain of the granulometry.

3.6. Proposition. *Let $\{\psi_\lambda\}$ be a granulometry on $\mathcal{P}(\mathbb{R}^d)$.*

(a) *Condition* (G4) *holds if and only if* $\mathrm{Inv}(\psi_\lambda)$ *is closed under translations for every* $\lambda > 0$.

(b) *Condition* (G5) *holds if and only if* $\mathrm{Inv}(\psi_\lambda) = \lambda \mathrm{Inv}(\psi_1)$ *for every* $\lambda > 0$.

Assume that $\{\psi_\lambda\}$ is a Euclidean granulometry and let $\mathcal{C} = \mathrm{Inv}(\psi_1)$. Then $\mathcal{C}$ is closed under translation and $\lambda\mathcal{C} \subseteq \mathcal{C}$ for $\lambda \geq 1$. The latter inclusion follows from the fact that $\lambda\mathcal{C} = \mathrm{Inv}(\psi_\lambda) \subseteq \mathrm{Inv}(\psi_1) = \mathcal{C}$ for $\lambda \geq 1$. This suggests the following construction method for Euclidean granulometries. Let $\mathcal{B}$ be an arbitrary collection of subsets of $\mathbb{R}^d$ and define $\mathcal{C}$ to be the smallest subset of $\mathcal{P}(\mathbb{R}^d)$ which is closed under union, translation, and scalings by a factor ≥ 1. Let ψ_λ be the opening generated by $\lambda\mathcal{C}$. Then $\{\psi_\lambda\}$ defines a Euclidean granulometry. In fact, we obtain in this way all Euclidean granulometries as the following representation of Matheron [20] shows.

3.7. Theorem. *Let $\{\psi_\lambda\}$ be a Euclidean granulometry. Then there is a family $\mathcal{B} \subseteq \mathcal{P}(\mathbb{R}^d)$ such that*

$$\psi_\lambda(X) = \bigcup_{\mu \geq \lambda} \bigcup_{B \in \mathcal{B}} X \bigcirc \mu B. \tag{3.1}$$

Conversely, if $\mathcal{B} \subseteq \mathcal{P}(\mathbb{R}^d)$, then ψ_λ given by (3.1) defines a Euclidean granulometry.

In this case we say that *the granulometry $\{\psi_\lambda\}$ is generated by the family $\mathcal{B}$.*

Because the representation of Thm. 3.7 is necessary and sufficient, it provides a paradigm for constructing Euclidean granulometries. However, as it stands, even in the simple case where $\mathcal{B} = \{B\}$ is a singleton class, the Euclidean granulometry requires an infinite union over $\mu \geq \lambda$. If we can eliminate the outer union, then the representation reduces to a union of granulometries generated by the elements of $\mathcal{B}$, namely, $\psi_\lambda(X) = \bigvee_{B \in \mathcal{B}} X \bigcirc \lambda B$. In practice, this union is finite (and consists of very few elements).

3.8. Remark. If, in this paper, we speak of elimination of the outer union we mean elimination without enlargement of the family $\mathcal{B}$. It is easy to see that the outer union can also be eliminated by replacing $\mathcal{B}$ by the family $\bigcup_{\lambda \geq 1} \lambda \mathcal{B}$. Such a replacement, however, is only a rearrangement of terms and not a reduction.

The outer union is redundant if and only if $X \bigcirc \mu B \subseteq X \bigcirc \lambda B$ for $\mu \geq \lambda$ and $B \in \mathcal{B}$. But this is equivalent to

$$\lambda B \text{ is } B\text{-open for } \lambda \geq 1. \tag{3.2}$$

We have already met this condition in Ex. 3.3(b) where we considered openings with squares. For compact sets (and in practice these are the only relevant ones) a complete characterization of the shapes B for which λB is B-open when $\lambda \geq 1$ has been given by Matheron [20].

3.9. Theorem. *Let $B \subseteq \mathbb{R}^d$ be compact, then λB is B-open for every $\lambda \geq 1$ if and only if B is convex.*

The extension of this characterization to a particular class of gray-scale granulometries is the main result of the present paper. That compactness cannot be dropped is clear by means of the following example. Let $B \subseteq \mathbb{R}^2$ be the collection of points outside the open first quadrant, that is, $B = \{(x, y) \in \mathbb{R}^2 \mid x \leq 0 \ \vee \ y \leq 0\}$. Then $\lambda B = B$ for every $\lambda > 0$ and thus (3.2) is satisfied. However, B is *not* convex.

In analogy with the definition of a structural opening in Sect. 2, we call $\{\psi_\lambda\}$ a *structural granulometry* if every opening ψ_λ is a structural opening, that is

$$\psi_\lambda(X) = X \bigcirc B_\lambda,$$

for some structuring element B_λ. The results above show that if $\{\psi_\lambda\}$ is Euclidean and every B_λ is compact, then $B_\lambda = \lambda B$ with B convex. In that case the structuring elements satisfy the semigroup property

$$\lambda B \oplus \mu B = (\lambda + \mu) B. \tag{3.3}$$

A family B_λ, $\lambda > 0$, of compact nonvoid subsets of $\mathbb{R}^d$ is called a *continuous one-parameter semigroup* if the function $\lambda \to B_\lambda$ is continuous with respect to the Hausdorff metric and satisfies the semigroup property

$$B_\lambda \oplus B_\mu = B_{\lambda+\mu}. \tag{3.4}$$

It is easily seen that $\psi_\lambda(X) = X \bigcirc B_\lambda$ defines a structural granulometry if B_λ has the semigroup property (3.4). Matheron [20] showed that B_λ is a continuous one-parameter semigroup if and only if $B_\lambda = \lambda B$ with B compact and convex.

4. Gray-scale morphology

The theory of gray-scale morphology is now well-developed. The early work of Sternberg [29] (see also [27, Chapter XII]) dealing with the extension of binary morphology to gray-scale images, was of a geometrical nature. The key idea was to represent a function on $\mathbb{R}^d$ by means of an umbra, the points on and below the graph of the function (formal definition given below), which is a set in $(d+1)$-space. One can then apply the binary morphological operators to this set. However, the more recent studies by Serra [28], Matheron [Chapter 6 in 28], Heijmans and Ronse [14,24], Ronse [25], and Heijmans [11,12] have shown that the appropriate algebraic structure for studying mathematical morphology is the complete lattice. The work of Heijmans and Ronse [14,24] shows how most of the concepts of binary morphology, including translation invariance, can be extended to this general structure. Gray-scale functions, if defined in the right way, do fit perfectly well into this framework of complete lattices and most of the concepts of binary morphology carry over rather easily to gray-scale images.

In the present paper, we are concerned with four aspects of the domain space: (1) linear translation, (2) scaling, (3) topology, and (4) convexity. We are also concerned with scaling and topology in the range space. Throughout this paper we assume that the underlying domain space is $\mathbb{R}^d$ and that the set of gray-scales is $\overline{\mathbb{R}}$, the extended real line. However, many of the results carry over directly to the general case where the domain space is a locally convex topological vector space. Indeed, it is in the generality of a locally convex topological vector space that we apply the Krein-Milman theorem. More specifically, the main proposition of the present paper is Thm. 6.4, and the key part of that theorem holds when the domain space is a locally convex topological vector space; see Remark 6.5.

We shall now review the fundamentals of gray-scale mathematical morphology for functions. We denote by $\mathsf{Fun}(\mathbb{R}^d)$ the space of all functions mapping $\mathbb{R}^d$ into $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. Under the pointwise ordering this space becomes a complete lattice with supremum and infimum defined pointwise. All the basic concepts introduced in Sect. 2 can easily be extended to the function space $\mathsf{Fun}(\mathbb{R}^d)$. For example, the operator Ψ on $\mathsf{Fun}(\mathbb{R}^d)$ is called an opening if Ψ is increasing (i.e., $f \le g$ implies that $\Psi(f) \le \Psi(g)$), anti-extensive ($\Psi(f) \le f$), and idempotent ($\Psi^2 = \Psi$). However, in $\mathsf{Fun}(\mathbb{R}^d)$ we have to distinguish between two kind of translations, namely spatial translations $f \to f_h$, with $h \in \mathbb{R}^d$, given by

$$f_h(x) = f(x-h),$$

also called H-translations (where "H" stands for "horizontal"), and gray-scale (or vertical) translations $f \to f + v$, $v \in \mathbb{R}$, given by

$$(f+v)(x) = f(x) + v.$$

A translation $f \to f_h + v$ is called a T-translation.

4.1. Definition. An operator Ψ on $\mathsf{Fun}(\mathbb{R}^d)$ is called an H-*operator* if it is invariant under horizontal translations, i.e., $\Psi(f_h) = (\Psi(f))_h$ for $f \in \mathsf{Fun}(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$. If, in addition, Ψ is invariant under vertical translations, that is, $\Psi(f+v) = \Psi(f) + v$, for $f \in \mathsf{Fun}(\mathbb{R}^d)$ and $v \in \mathbb{R}$, then Ψ is called a T-*operator*.

An opening which is a T-operator will be called a T-opening, etc. Since openings will play a very prominent role in the sequel we devote some more words to them. First we point out that, with the necessary modifications, all results in Sect. 2 carry over. For instance, if Ψ_i are openings on $\mathsf{Fun}(\mathbb{R}^d)$ for all i in some index set I, then the supremum $\bigvee_{i \in I} \Psi_i$ is an opening as well; cf. Prop. 2.3. For the definition of a structural opening we have to distinguish between H- and T-openings, but apart from this, their definitions are very much alike. Let g be an element of $\mathsf{Fun}(\mathbb{R}^d)$ (called the *structuring function*). The *structural H-opening* of a function f by g denoted by $f \circledH g$ is defined as

$$f \circledH g = \bigvee \{ g_h \mid h \in \mathbb{R}^d \text{ and } g_h \leq f \}. \tag{4.1}$$

The *structural T-opening* of a function f by g denoted by $f \circledT g$ is defined as

$$f \circledT g = \bigvee \{ g_h + v \mid h \in \mathbb{R}^d,\ v \in \mathbb{R} \text{ and } g_h + v \leq f \}. \tag{4.2}$$

See Fig. 2 for an illustration of these two openings.

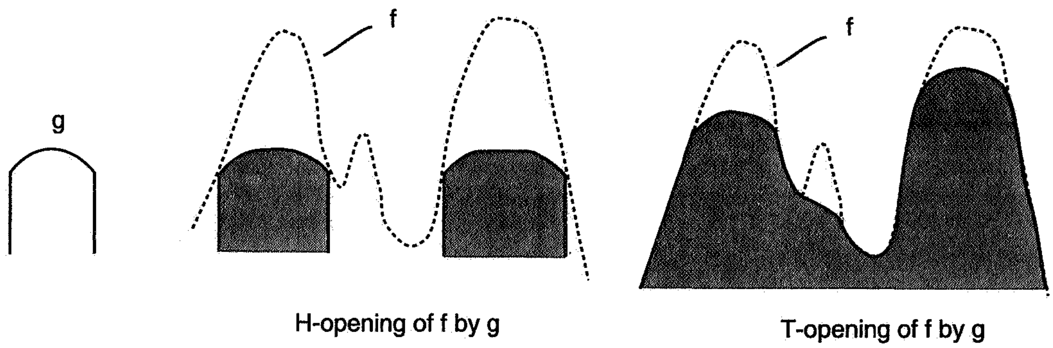


FIGURE 2. *H-opening (left) and T-opening (right) of a function.*

The invariance domain of an operator Ψ on $\mathsf{Fun}(\mathbb{R}^d)$ is defined as

$$\mathsf{Inv}(\Psi) = \{ f \in \mathsf{Fun}(\mathbb{R}^d) \mid \Psi(f) = f \}.$$

If Ψ is a T-operator then $\mathsf{Inv}(\Psi)$ is closed under T-translations. If Ψ is a H-operator then its invariance domain is closed under H-translations. Furthermore the invariance domain of an opening is closed under suprema.

Analogous to Prop. 2.1 the following results hold.

4.2. Proposition.

(a) *Let Ψ be an H-opening on $\mathsf{Fun}(\mathbb{R}^d)$ and let $\mathcal{G} \subseteq \mathsf{Fun}(\mathbb{R}^d)$ be a collection of structuring functions such that $\mathsf{Inv}(\Psi)$ is the smallest subset of $\mathsf{Fun}(\mathbb{R}^d)$ which contains $\mathcal{G}$ and is closed under suprema and horizontal translations, then*

$$\Psi(f) = \bigvee_{g \in \mathcal{G}} f \circledH g. \tag{4.3}$$

(b) *Let Ψ be a T-opening on $\mathsf{Fun}(\mathbb{R}^d)$ and let $\mathcal{G} \subseteq \mathsf{Fun}(\mathbb{R}^d)$ be a collection of structuring functions such that $\mathsf{Inv}(\Psi)$ is the smallest subset of $\mathsf{Fun}(\mathbb{R}^d)$ which contains $\mathcal{G}$ and is closed under suprema and horizontal and vertical translations, then*

$$\Psi(f) = \bigvee_{g \in \mathcal{G}} f \circledT g. \tag{4.4}$$

Also Prop. 2.2 carries over almost word for word to the gray-scale case.

4.3. Proposition. *Let $g, h \in \mathsf{Fun}(\mathbb{R}^d)$.*
(a) *$f \circledH h \leq f \circledH g$ for every $f \in \mathsf{Fun}(\mathbb{R}^d)$ if and only if $h \circledH g = h$.*
(b) *$f \circledT h \leq f \circledT g$ for every $f \in \mathsf{Fun}(\mathbb{R}^d)$ if and only if $h \circledT g = h$.*

If $h \circledH g = h$ then we say that h is g-open in the H-sense. Analogously, we say that h is g-open in the T-sense if $h \circledT g = h$. If it is clear from the context which of the two translation groups is meant then we only say that h is g-open.

An important and useful method to build function operators is by thresholding. This technique is extensively described in the literature and we refer to [11] for an overview. A formal but very general exposition can be found in [13]. For a function $f \in \mathsf{Fun}(\mathbb{R}^d)$ and gray-scale $t \in \mathbb{R}$ we define the *threshold set* $X_t(f)$ by

$$X_t(f) = \{x \in \mathbb{R}^d \mid f(x) \geq t\}.$$

We define the *domain* $D(f)$ of the function f by

$$D(f) = \{x \in \mathbb{R}^d \mid f(x) > -\infty\}.$$

Note that $D(f) = \bigcup_{t \in \mathbb{R}} X_t(f)$.

The $X_t(f)$ form a family of subsets of $\mathbb{R}^d$ which is non-increasing with respect to t and satisfy the continuity relation

$$X_t(f) = \bigcap_{s<t} X_s(f). \tag{4.5}$$

Conversely, if X_t, $t \in \mathbb{R}$, is a family in $\mathcal{P}(\mathbb{R}^d)$ which satisfies (4.5) then there is a unique element $f \in \mathsf{Fun}(\mathbb{R}^d)$ with $X_t(f) = X_t$. This function is given by

$$f(x) = \sup\{t \in \mathbb{R} \mid x \in X_t\}. \tag{4.6}$$

These observations form the basis for the extension of increasing set operators to increasing function operators. Namely, assume that ψ_t, $t \in \mathbb{R}$, is a non-increasing family ($\psi_t \leq \psi_s$ if $s \leq t$) of increasing set operators and let $f \in \mathsf{Fun}(\mathbb{R}^d)$. Then

$$Y_t = \bigcap_{s<t} \psi_s(X_s(f))$$

is a non-increasing family of subsets of $\mathbb{R}^d$ which satisfy the continuity requirement (4.5). Thus there is a unique function, which we call $\Psi(f)$, with

$$X_t(\Psi(f)) = Y_t.$$

It is easy to show that

$$\Psi(f)(x) = \sup\{t \in \mathbb{R} \mid x \in \psi_t(X_t(f))\}.$$

It is obvious that Ψ is increasing. We call Ψ a *semi-flat* operator and say that Ψ is generated by the family ψ_t. It has been shown [11] that Ψ inherits certain properties of the family ψ_t. Here we only list two of the properties which are relevant in the sequel.

4.4. Proposition. *Let ψ_t, $t \in \mathbb{R}$, be a non-increasing family of increasing set operators and let Ψ be the semi-flat function operator generated by this family.*

(a) *If every ψ_t is translation invariant, then Ψ is an H-operator.*

(b) *If every ψ_t is an opening then Ψ is an opening as well.*

If we take all ψ_t to be identical, say ψ, then our construction yields a so-called *flat* function operator Ψ given by the expression

$$\Psi(f)(x) = \sup\{t \in \mathbb{R} \mid x \in \psi(X_t(f))\}. \tag{4.7}$$

From the identity $X_t(f + v) = X_{t-v}(f)$, valid for all $t, v \in \mathbb{R}$, it follows immediately that a flat operator is invariant under vertical translations. In particular, the following result holds.

4.5. Proposition. *Let ψ be an increasing translation invariant operator on $\mathcal{P}(\mathbb{R}^d)$ and let Ψ be the function operator generated by ψ, then Ψ is an increasing T-operator.*

If $B \subseteq \mathbb{R}^d$ and $f \in \mathsf{Fun}(\mathbb{R}^d)$ then we denote by $f \bigcirc B$ the function extension of the set opening $X \to X \bigcirc B$. This opening is called a flat opening. See Fig. 3 for an example.

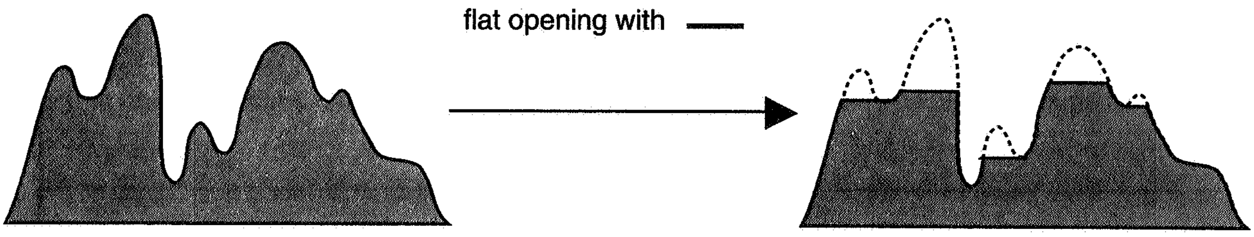


FIGURE 3. *Opening $f \bigcirc B$ of a function f by a flat structuring element B.*

From Prop. 4.4(b) it follows that Ψ is an opening if ψ is.

To get a geometric picture of the effect of morphological operators on gray-scale images one sometimes uses the *umbra* $U(f)$ of a function. This is defined as

$$U(f) = \{(x,t) \in \mathbb{R}^d \times \mathbb{R} \mid f(x) \geq t\}.$$

So the umbra is the region in the $(d+1)$-dimensional space which lies below the function. The umbra approach is used extensively in [10,27,29]. In [25] Ronse points out some drawbacks of this approach.

5. Granulometries for gray-scale images

As recognized by Serra [28], the definition of a binary granulometry applies at once for a family of mappings $\{\psi_\lambda\}$ on a complete lattice. Thus, letting $\leq$ denote function ordering, Def. 3.1 applies at once to gray-scale images. Since Prop. 3.2 also applies to lattices, we see that a gray- scale granulometry is a one-parameter family $\{\Psi_\lambda \mid \lambda > 0\}$ of openings on $\mathsf{Fun}(\mathbb{R}^d)$ such that $\Psi_\mu \leq \Psi_\lambda$ if $\mu \geq \lambda$. We speak of a *T-granulometry* (resp. *H-granulometry*) if every Ψ_λ is a T-operator (resp. H- operator).

5.1. Example. An easy way to build gray-scale granulometries is the following. Let $\{\psi_\lambda\}$ be a granulometry on $\mathcal{P}(\mathbb{R}^d)$. Take $f \in \mathsf{Fun}(\mathbb{R}^d)$ and define $\Psi_\lambda(f)$ to be the restriction of f to the opened domain $\psi_\lambda(D(f))$. Then $\{\Psi_\lambda\}$ defines a granulometry. If every ψ_λ is translation invariant, then $\{\Psi_\lambda\}$ is a T-granulometry. In this procedure we may replace $D(f)$ by any threshold set $X_t(f)$. However, vertical translation invariance is lost in this way and we may only conclude that $\{\Psi_\lambda\}$ is a H-granulometry if every ψ_λ is translation invariant.

An alternative way to construct gray-scale granulometries is to use the construction of (semi-) flat operators discussed in the previous section.

5.2. Proposition.
(a) *Let, for every $t \in \mathbb{R}$, $\{\psi_{t,\lambda}\}$ be a granulometry on $\mathcal{P}(\mathbb{R}^d)$ and let $\psi_{t,\lambda} \le \psi_{s,\lambda}$ if $s \le t$. Let Ψ_λ be the semi-flat operator generated by $\psi_{t,\lambda}$. Then $\{\Psi_\lambda\}$ is a granulometry on* $\mathsf{Fun}(\mathbb{R}^d)$*. If every $\psi_{s,\lambda}$ is translation invariant, then $\{\Psi_\lambda\}$ is an H-granulometry.*
(b) *Let $\{\psi_\lambda\}$ be a granulometry on $\mathcal{P}(\mathbb{R}^d)$, and let Ψ_λ be the flat operator generated by ψ_λ, then $\{\Psi_\lambda\}$ is a granulometry on* $\mathsf{Fun}(\mathbb{R}^d)$*. If every ψ_λ is translation invariant, then $\{\Psi_\lambda\}$ is a T-granulometry.*

The granulometry in (a) is called a *semi-flat granulometry*, whereas in case (b) we speak of a *flat granulometry.*

To generalize the concept of a Euclidean granulometry to gray-scale functions we have to give gray-scale analogues of the properties (G4) and (G5) introduced in Sect. 3. As to property (G4) concerning translation invariance there are, as we have seen in the previous section, two useful generalizations, namely T-invariance and H-invariance.

5.3. Definition. Let $\{\Psi_\lambda\}$ be a granulometry on $\mathsf{Fun}(\mathbb{R}^d)$. We say that $\{\Psi_\lambda\}$ is a *T-granulometry* (resp. *H-granulometry*) if every Ψ_λ is a T-operator (resp. H-operator).

But also with respect to property (G5), the scaling compatibility, there are two alternatives. Dougherty and co-workers [1,2,3] have defined multiplication as

$$(\lambda \times f)(x) = \lambda f(\frac{x}{\lambda}), \quad \lambda > 0. \tag{5.1}$$

This definition is based on scaling the umbra of f as a subset of $\mathbb{R}^{d+1}$. It is compatible with the view that gray-scale morphological operations can be viewed as operations on image umbrae. Specifically, $U[\lambda \times f] = \lambda U[f]$; see Fig. 1 for an illustration. Henceforth we will refer to this scaling as *T-scaling*, in accordance with our convention that the prefix "T-" means both in spatial and gray-scale direction. We are now ready to give a first generalization of the concept of a Euclidean granulometry for gray-scale functions.

5.4. Definition. A gray-scale granulometry $\{\Psi_\lambda\}$ is called a *(T,T)-Euclidean granulometry* if every Ψ_λ is a T-operator and $\{\Psi_\lambda\}$ is compatible with T-scalings, that is

$$\Psi_\lambda(f) = \lambda \times \Psi_1(\lambda^{-1} \times f), \quad \lambda > 0. \tag{5.2}$$

At the end of this section we will briefly summarize some of the results by [1,2,3] concerning (T,T)-Euclidean granulometries.

A second way to proceed is to consider compatibility with scaling in the spatial (horizontal) direction but not in the gray-scale direction. In mathematical terms

$$(\lambda * f)(x) = f(x/\lambda), \tag{5.3}$$

Henceforth we will refer to this scaling as the H-scaling. For an illustration we refer again to Fig. 1. For future reference we note that

$$X_t(\lambda * f) = \lambda X_t(f), \tag{5.4}$$

for every $f \in \mathsf{Fun}(\mathbb{R}^d)$ and $t \in \mathbb{R}$. The same relation holds for $D(f)$.

We can also use the H-scaling in combination with H- or T-translations to define gray-scale Euclidean granulometries.

5.5. Definition. A gray-scale granulometry $\{\Psi_\lambda\}$ is called a *(T,H)-Euclidean granulometry* if every Ψ_λ is a T-operator and $\{\Psi_\lambda\}$ is compatible with H-scalings, that is

$$\Psi_\lambda(f) = \lambda * \Psi_1(\lambda^{-1} * f), \quad \lambda > 0. \tag{5.5}$$

If Ψ_λ is only invariant under H-translations then $\{\Psi_\lambda\}$ is called a *(H,H)-Euclidean granulometry.*

Throughout the remainder of this section we briefly summarize the results by Dougherty [1,2,3] concerning (T,T)-Euclidean granulometries. So with "×" we denote the T-scaling defined in (5.1). It is obvious that the invariant classes of a (T,T)-Euclidean granulometry $\{\Psi_\lambda\}$ are determined by $\lambda \times \mathsf{Inv}(\Psi_1)$, just as in the binary setting. Moreover, it is shown that $\{\Psi_\lambda\}$ is a (T,T)-Euclidean granulometry if and only if there exists a class $\mathcal{G}$, such that

$$\Psi_\lambda(f) = \bigvee_{\mu \geq \lambda} \bigvee_{g \in \mathcal{G}} f \textcircled{T} \mu \times g, \tag{5.6}$$

where the openings in the representation are gray-scale openings. We say that the family $\mathcal{G}$ generates the granulometry $\{\Psi_\lambda\}$. We also point out that the invariance domain $\mathsf{Inv}(\Psi_1)$ is the closure of $\mathcal{G}$ under T-translations, suprema, and products $f \to \lambda \times f$, with $\lambda \geq 1$. As in the binary case, and as discussed in the introduction, elimination of the outer supremum is crucial for application. This is possible, leaving $\Psi_\lambda(f) = \bigvee_{g \in \mathcal{G}} f \bigcirc \lambda \times g$, if $\mu \times g \leq \lambda \times g$ for $\mu \geq \lambda$, and this occurs if $\mu \times g$ is $\lambda \times g$-open for $\mu \geq \lambda$. So long as the graphs of the generator elements are concave-down, which makes their umbrae convex in $\mathbb{R}^{d+1}$, this condition is indeed satisfied.

6. (T,H)-Euclidean granulometries

Throughout the remainder of this paper we let "$*$" denote H-scaling as defined in (5.3). As a first example of a (T,H)-granulometry we return to Ex. 5.1. Assume that $\{\psi_\lambda\}$ is a Euclidean granulometry on $\mathcal{P}(\mathbb{R}^d)$ and define $\Psi_\lambda(f)$ to be the restriction of f to $\psi_\lambda(D(f))$, where $D(f)$ is the domain of f. Since $D(\lambda * f) = \lambda D(f)$ it follows immediately that Ψ_λ is a (T,H)-Euclidean granulometry on $\mathsf{Fun}(\mathbb{R}^d)$. The next proposition shows that a flat function granulometry generated by a binary Euclidean granulometry is a (T,H)-Euclidean granulometry: see also Prop. 5.2(b).

6.1. Proposition. *Let $\{\psi_\lambda\}$ be a Euclidean granulometry on $\mathcal{P}(\mathbb{R}^d)$ and Ψ_λ the flat function operator generated by ψ_λ. Then $\{\Psi_\lambda\}$ defines a (T,H)-Euclidean granulometry on* $\mathsf{Fun}(\mathbb{R}^d)$.

PROOF. The only statement which needs to be proved is that $\Psi_\lambda(f) = \lambda{*}\Psi_1(\lambda^{-1}{*}f)$ if $\{\psi_\lambda\}$ is a Euclidean granulometry. We use (4.7) and (5.4):

$$\begin{aligned}\Psi_\lambda(f)(x) &= \sup\{t \in \mathbb{R} \mid x \in \psi_\lambda(X_t(f))\}\\ &= \sup\{t \in \mathbb{R} \mid x \in \lambda\psi_1(\frac{1}{\lambda}X_t(f))\}\\ &= \sup\{t \in \mathbb{R} \mid \frac{1}{\lambda}x \in \psi_1(X_t(\frac{1}{\lambda}{*}f))\}\\ &= \Psi_1(\frac{1}{\lambda}{*}f)(\frac{x}{\lambda})\\ &= \big(\lambda{*}\Psi_1(\frac{1}{\lambda}f)\big)(x).\end{aligned}$$

This concludes the proof. ∎

Since, by Prop. 4.2(b) every T-opening can be decomposed as a supremum of structural T-openings we know that the opening Ψ_1 corresponding with a (T,H)-Euclidean granulometry is of the form

$$\Psi_1(f) = \bigvee_{g \in \mathcal{G}} f \textcircled{\scriptsize T} g,$$

for some family $\mathcal{G} \subseteq \mathsf{Fun}(\mathbb{R}^d)$. Using that

$$\lambda{*}(\frac{1}{\lambda}f \textcircled{\scriptsize T} g) = f \textcircled{\scriptsize T} (\lambda{*}g),$$

we find that

$$\Psi_\lambda(f) = \bigvee_{g \in \mathcal{G}} f \textcircled{\scriptsize T} (\lambda{*}g). \tag{6.1}$$

If $\{\Psi_\lambda\}$ is a (T,H)-Euclidean granulometry then Ψ_λ can be written in the form (6.1). However, (6.1) defines a (T,H)-Euclidean granulometry if and only if $\mathcal{G}$ is closed under scaling with factor ≥ 1. The following holds (cf. Thm. 3.7).

6.2. Theorem. *Let $\{\Psi_\lambda\}$ be a (T,H)-Euclidean granulometry. Then there is a family $\mathcal{G} \subseteq \mathsf{Fun}(\mathbb{R}^d)$ such that*

$$\Psi_\lambda(f) = \bigvee_{\mu \geq \lambda} \bigvee_{g \in \mathcal{G}} f \textcircled{\scriptsize T} \mu{*}g. \tag{6.2}$$

Conversely, if $\mathcal{G} \subseteq \mathsf{Fun}(\mathbb{R}^d)$ then $\{\Psi_\lambda\}$ given by (6.2) defines a (T,H)-Euclidean granulometry.

Just as in the binary case practical application of gray-scale Euclidean representations requires elimination of the outer supremum in (6.2), which would leave $\Psi_\lambda(f) = \bigvee_{g \in \mathcal{G}} f \bigcirc \lambda{*}g$. Such an eliminiation requires that for each g in $\mathcal{G}$, we have $f \bigcirc \mu{*}g \leq f \bigcirc \lambda{*}g$ if $\mu \geq \lambda$, which is equivalent to

$$\lambda{*}g \text{ is } g\text{-open (in the T-sense) for } \lambda \geq 1. \tag{6.3}$$

Thm. 6.4 below, which is the main result in the present paper, gives a characterization of all such g under some extra assumptions which we formulate below.

6.3. Definition. The function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is said to be *upper semi-continuous* (*u.s.c.*) if for every $x \in \mathbb{R}^d$ and $t > f(x)$ there is a neighbourhood V of x such that $t > f(y)$ for $y \in V$ We say that the function f is *compact* if f is u.s.c. and its domain $D(f)$ is a compact set.

It is not difficult to show that a compact function attains it's maximum on its domain.

We are now ready to give a complete characterization of the functions g which satisfy (6.3) under the assumption that g is compact (cf. Thm. 3.9). In fact we shall prove the following result.

6.4. Theorem. *Let $g \in \mathsf{Fun}(\mathbb{R}^d)$ be compact. Then*

$$\lambda * g \text{ is } g\text{-open for } \lambda \geq 1$$

if and only if $D(g)$ is convex and g is constant on $D(g)$.

6.5. Remark. As stated here there are two conditions on g that are part of Thm. 6.4, and it is the second one, that g is constant on $D(g)$, that is of prime significance in the present paper (and in applications [4]). If we include in the hypothesis that $D(g)$ is convex, rather than attempting to conclude it, then the proof that g is constant on $D(g)$ carries over word for word to a locally convex topological vector space, and it is in proving g to be constant that we employ the Krein-Milman theorem. Because of this we state an alternative form of Thm. 6.4.

Let V be a locally convex topological vector space and let $g \in \mathsf{Fun}(V)$ be u.s.c. on a compact, convex domain. Then

$$\lambda * g \text{ is } g\text{-open for } \lambda \geq 1$$

if and only if g is constant on $D(g)$.

Whether or not Thm. 6.4 goes directly over to a locally convex topological vector space, with the convexity of $D(g)$ being part of the conclusion, is presently unknown. The sticking point here is that the convexity of $D(g)$ is proven in Lemma 6.8, and that lemma depends on the classic Matheron theorem (Thm. 3.9), which is not known for a locally convex topological vector space.

We show by means of two 1-dimensional examples that neither compactness of the domain nor upper semi-continuity of the function can be omitted.

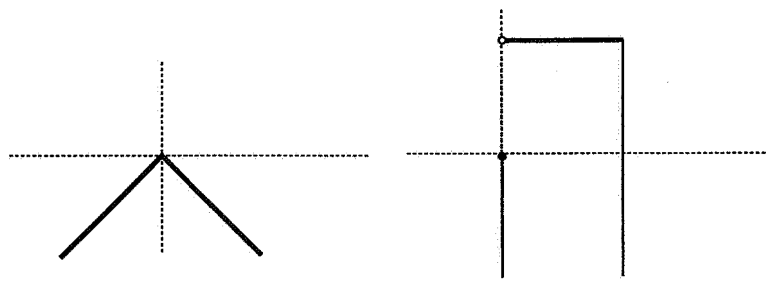
6.6. Examples.

(a) Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = -|x|$; see Fig. 4. Then $\lambda * g$ is g-open for $\lambda \geq 1$.

(b) Let

$$g(x) = \begin{cases} 0, & x = 0 \\ 1, & 0 < x \leq 1 \\ -\infty, & \text{elsewhere.} \end{cases}$$

Then $D(g)$ is compact but g is not u.s.c. at $x = 0$. It is easy to check that $\lambda * g$ is g-open for $\lambda \geq 1$.



(a) structuring function with infinite domain

(b) structuring function which is not u.s.c.

FIGURE 4. *If the domain of g is not compact (a) or if g is not u.s.c. (b) then the conclusion of Thm. 6.4 needs not hold; see also text above.*

Just as in Sect. 3 we can define a structural T-granulometry for gray-scale functions as a T-granulometry $\{\Psi_\lambda\}$ such that every opening Ψ_λ is structural, that is, $\Psi_\lambda(f) = f \circledT g_\lambda$ for some structuring function g_λ. For a structural (T,H)-Euclidean granulometries one gets immediately that $g_\lambda = \lambda * g$, where g is flat with convex domain (if g_λ is compact; see Thm.6.4). Such a granulometry is the flat extension of a structural Euclidean granulometry for binary images as described in Prop. 5.2(b).

6.7. Corollary. *Every structural (T,H)-Euclidean granulometry $\{\Psi_\lambda\}$ is the flat extension of a binary structural Euclidean granulometry, that is*

$$\Psi_\lambda(f) = f \bigcirc \lambda D,$$

where $D \subseteq \mathbb{R}^d$ is compact and convex.

We now turn to the proof of Thm.6.4. In fact, the "if"-statement is an immediate consequence of the binary theory, and the remainder of this section will be devoted to the demonstration of the "only if"-statement. First we will prove the following lemma.

6.8. Lemma. *If g is compact and $\lambda * g$ is g-open, then $D(g)$ is convex.*

PROOF. We notice that the operator $D : \mathsf{Fun}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$ which maps a function f to its domain $D(f)$ satisfies the property

$$D(\bigvee_{i \in I} f_i) = \bigcup_{i \in I} D(f_i),$$

for any collection f_i, $i \in I$, in $\mathsf{Fun}(\mathbb{R}^d)$. (In mathematical morphology such a mapping is called a *dilation*; see [14].) Assume that g is a compact function such that $\lambda * g$ is g-open. Then

$$\lambda * g = \bigvee \{g_h + v \mid h \in \mathbb{R}^d,\ v \in \mathbb{R} \text{ and } g_h + v \le \lambda * g\}.$$

Applying D at both sides and using that

$$D(\lambda * g) = \lambda D(g) \text{ and } D(g_h + v) = (D(g))_h,$$

we find that (putting $D = D(g)$)

$$\lambda D = \bigcup\{D_h \mid h \in \mathbb{R}^d,\ v \in \mathbb{R} \text{ and } g_h + v \leq \lambda * g\}.$$

Now we use that $g_h + v \leq \lambda * g$ is a stronger condition than $D(g_h + v) \subseteq D(\lambda * g)$, i.e., $D_h \subseteq \lambda D$. This yields that

$$\lambda D \subseteq \bigcup\{D_h \mid h \in \mathbb{R}^d \text{ and } D_h \subseteq \lambda D\} = \lambda D \bigcirc D.$$

But the other inclusion is trivial and we have derived that $\lambda D \bigcirc D = \lambda D$, for $\lambda \geq 1$. Furthermore, D is compact and we can apply Thm. 3.9 which yields that $D(g)$ is convex. ∎

We recall some definitions and results for convex sets. Let $B \subseteq \mathbb{R}^d$ be convex. A point $x \in B$ is called an *extreme point* of B if there do not exist two distinct points $b_1, b_2 \in B$ such that x lies on the open segment connecting b_1 and b_2. For example, the vertices of a convex polyhedron P in $\mathbb{R}^d$ are the extreme points of P. The extreme points of the unit ball in $\mathbb{R}^d$ are all the point on the unit sphere.

We denote the extreme points of D by $E(D)$. For a compact convex subset D of a locally convex topological vector space, such as $\mathbb{R}^d$, the extreme points play a crucial role. Such a set D contains at least one extreme point and $D = \overline{\text{co}}(E(D))$, where $\overline{\text{co}}(\cdot)$ denotes the closed convex hull. The latter result is the *Krein-Milman theorem* [15,26]. Often a property may be shown to hold for a compact convex set by showing that it holds for the extreme points. This is precisely the approach that will be used in establishing Thm. 6.4. We will show that a compact function g is maximized at its extreme points $E(D(g))$, and by a transfinite extension, that this holds for the closed convex hull of $E(D(g))$.

We adopt the following notation. If $x, y \in \mathbb{R}^d$ then we denote by $[x, y]$ the closed line segment connecting x and y, that is, $[x, y] = \{\lambda x + (1 - \lambda)y \mid 0 \leq \lambda \leq 1\}$. We use an open bracket "(" or ")" to denote that the left resp. right endpoint does not belong to the set.

Geometric intuition suggests that under magnification of a convex set D with a scalar $\lambda \geq 1$ there is only one way to translate D to fit within λD so that the fitting covers a given extreme point λe of λD. Namely, that translation is determined by the global translation T which moves e to λe, i.e., $T(x) = x + (\lambda - 1)e$. This observation is formalized in the following lemma, which gives an alternative characterization of an extreme point.

6.9. Lemma. *Let $D \subseteq \mathbb{R}^d$ be convex. Then $e \in E(D)$ if and only if for every $\lambda \geq 1$ the only $x \in D$ which satisfies*

$$D_{\lambda e - x} \subseteq \lambda D \tag{6.4}$$

is $x = e$.

PROOF. "only if": let $e \in E(D)$. Assume that $x \neq e$ and $D_{\lambda e - x} \subseteq \lambda D$. Then $y = \frac{1}{\lambda}(e + \lambda e - x) \in D$ and thus $e = \frac{1}{\lambda+1}x + \frac{\lambda}{\lambda+1}y$, yielding that $e \notin E(D)$, a contradiction.
"if": assume that $e \notin E(D)$. We show that there exists a $\lambda \geq 1$ and an $x \neq e$ such that (6.4) holds. There exist $u, v \in D$ such that $e = \beta u + (1 - \beta)v$ for some $\beta \in (0, 1)$. Take $\xi \in (0, \beta)$ and $x = \xi u + (1 - \xi)v$. We show that x solves (6.4) if $\lambda = (1 - \xi)/(1 - \beta) > 1$. First note that

$$e + \frac{1}{\lambda}(u - x) = u.$$

Take $y \in D$. Using the latter relation we find that

$$\begin{aligned}\frac{1}{\lambda}(y+\lambda e-x) &= e+\frac{1}{\lambda}(y-x)\\ &= e+\frac{1}{\lambda}(u-x)+\frac{1}{\lambda}(y-u)\\ &= u+\frac{1}{\lambda}(y-u)\\ &= \frac{1}{\lambda}y+(1-\frac{1}{\lambda})u \in D,\end{aligned}$$

since D is convex. This proves the result. ∎

The next lemma is the critical argument in the proof of our main Thm. 6.4. The lemma has the immediate consequence that if g has a maximum value at x, then g is maximized at every point on the line $[x,e]$ for any extreme point e.

6.10. Lemma. *Let g be a function such that*
(i) *its domain $D = D(g)$ is convex,*
(ii) *$\lambda * g$ is g-open for $\lambda \geq 1$ (in the T-sense).*
Let $x \in D$ with $g(x) \geq t$ and let $e \in E(D)$. Then $g(y) \geq t$ for every $y \in (e,x]$.

Proof. Let $e \in E(D)$ and $\lambda \geq 1$. For every $\varepsilon > 0$ there exists a $y \in \mathbb{R}^d$ and $v \in \mathbb{R}$ such that
(a) $g_y + v \leq \lambda * g$
(b) $g_y(\lambda e) + v > (\lambda * g)(\lambda e) - \varepsilon = g(e) - \varepsilon$.
From (a) we conclude that $D_y \subseteq \lambda D$ and from (b) that $\lambda e \in D_y$. So there exists an $x \in D$ such that $\lambda e = x + y$, i.e., $y = \lambda e - x$. This yields that

$$D_{\lambda e - x} \subseteq \lambda D,$$

and we conclude from Lemma 6.9 that $x = e$, and thus $y = (\lambda - 1)e$. Substitution in (b) yields $g(e) + v > g(e) - \varepsilon$, that is $v > -\varepsilon$. With (a) we obtain

$$g_{(\lambda-1)e} - \varepsilon \leq \lambda * g.$$

Since this inequality holds for every $\varepsilon > 0$ we conclude that

$$g_{(\lambda-1)e} \leq \lambda * g.$$

Now let $g(x) \geq t$. Substituting $y = x + (\lambda - 1)e$ in our last inequality yields

$$t \leq g(x) \leq g(\frac{1}{\lambda}x + (1-\frac{1}{\lambda})e).$$

This proves the assertion. ∎

Now we are ready to finish the proof of the "only if" statement of Thm. 6.4. In our proof we will use Zorn's lemma which we shall recall here for the convenience of the the reader.

6.11. Zorn's lemma. *If each chain in a partially ordered set $\mathcal{P}$ has an upper bound, then $\mathcal{P}$ possesses a maximal element.*

PROOF OF THEOREM 6.4

Assume that $\lambda * g$ is g-open for $\lambda \geq 1$. We have seen that the domain $D = D(g)$ is convex. Let $t = \sup_{x \in D} g(x)$. Since g is compact the value t is attained and therefore $X_t(g) \neq \emptyset$. Note that $g(x) = t$ for $x \in X_t(g)$. From Lemma 6.10 we conclude that $[x, e) \subseteq X_t(g)$ if $x \in X_t(g)$ and $e \in E(D)$. Since g is u.s.c. it follows that $g(e) = t$ and therefore $E(D) \subseteq X_t(g)$. Define $\mathcal{P}$ to be the partially ordered set consisting of all convex subsets of $X_t(g)$. If $\mathcal{C}$ is a chain in $\mathcal{P}$ then $\bigcup \mathcal{C}$ is an upper bound. Thus $\mathcal{P}$ has a maximal element which we denote by M. It is obvious that M is a closed set, otherwise $M \subseteq \overline{M} \subseteq X_t(g)$, and $\overline{M}$ is convex. This would contradict the maximality of M. Assume that $e \notin M$ for some $e \in E(D)$, then the convex hull M' of $\{e\} \cup M$ consists of all segments $[e, x]$ where $x \in M$, and with Lemma 6.10 we may conclude that $M' \subseteq X_t(g)$. But then M is not maximal, a contradiction. Thus we conclude that $E(D) \subseteq M$. Since M is closed and convex we find that $\overline{\text{co}}(E(D)) \subseteq M$. The Krein-Milman theorem yields that $D = \overline{\text{co}}(E(D))$ and therefore $M = D \subseteq X_t(g)$. This concludes the proof. ∎

If we take the definition of $\lambda * g$ (cf. (5.3)) that we have been employing in this section, the result is compatibility with H-scaling; however, there is a greater restriction on the use of structural granulometries. Specifically, according to Thm. 6.4 (if we assume compact g), $f \bigcirc \lambda * g$ is a (T,H)-Euclidean granulometry if and only if g is a flat structuring element defined on a convex set. Regarding singleton generators, this theorem says in effect that every structural (T,H)-Euclidean granulometry using a compact structuring function is generated by a structural Euclidean granulometry $X \bigcirc \lambda D$, where D is the domain of the generating structuring function. While compactness of the domain does not represent a further constraint on the kind of structuring functions we may employ, flatness certainly does. The upshot of the matter is that to obtain comaptibility with H-scaling, we have been forced to further restrict the kind of information we can obtain by granulometric analysis, at least insofar as practical application is concerned.

7. (H,H)-Euclidean granulometries

We start with an example which is closely related to the example discussed in the beginning of Sect. 6; see also Ex. 5.1. Let $\{\psi_\lambda\}$ be a granulometry on $\mathcal{P}(\mathbb{R}^d)$ and define for $f \in \mathsf{Fun}(\mathbb{R}^d)$, the opening $\Psi_\lambda(f)$ as the restriction of f to the set $\psi_\lambda(X_t(f))$ with $t \in \overline{\mathbb{R}}$ arbitrary but fixed. Then $\{\Psi_\lambda\}$ is a granulometry. Moreover, if $\{\psi_\lambda\}$ is a Euclidean granulometry, then $\{\Psi_\lambda\}$ is a (H,H)-Euclidean granulometry. To prove this one uses relation (5.4).

A rather general method to construct (H,H)-granulometries is to use semi-flat operator extensions as defined in Prop. 4.4 and Prop. 5.2(a).

7.1. Proposition. *Let Ψ_λ be the semi-flat function operator generated by $\psi_{t,\lambda}$. If, for every $t \in \mathbb{R}$, $\{\psi_{t,\lambda}\}$ is a Euclidean granulometry then $\{\Psi_\lambda\}$ is a (H,H)-Euclidean granulometry.*

The proof is almost similar to the proof of Prop. 6.1. Since

$$f \circledH \lambda * g = \lambda * [(\frac{1}{\lambda} * f) \circledH g],$$

we can also formulate the following analog of Thm. 6.2.

7.2. Theorem. *Let $\{\Psi_\lambda\}$ be a (H,H)-Euclidean granulometry. Then there is a family $\mathcal{G} \subseteq \mathsf{Fun}(\mathbb{R}^d)$ such that*

$$\Psi_\lambda(f) = \bigvee_{\mu \geq \lambda} \bigvee_{g \in \mathcal{G}} f \circledH \mu * g. \tag{7.1}$$

Conversely, if $\mathcal{G} \subseteq \mathsf{Fun}(\mathbb{R}^d)$ then $\{\Psi_\lambda\}$ given by (7.1) defines a (H,H)-Euclidean granulometry.

Again we are confronted with removing the outer supremum, and, as in the preceding cases, the key question concerns g-openness, i.e.,

$$\lambda * g \text{ is } g\text{-open (in the H-sense) for } \lambda \geq 1. \tag{7.2}$$

It turns out that this requirement gives rise to exactly the same class of structuring functions as in the T-invariant case.

7.3. Theorem. *Let $g \in \mathsf{Fun}(\mathbb{R}^d)$ be compact. Then*

$$\lambda * g \text{ is } g\text{-open in the H-sense for } \lambda \geq 1$$

if and only if $D(g)$ is convex and g is constant on $D(g)$.

Also the proof carries over almost word for word. Only the proof of Lemma 6.10 has to adapted slightly. For the sake of completeness we recall the lemma for the H-invariant case and supply the full proof.

7.4. Lemma. *Let g be a function such that*
(i) *the domain $D = D(g)$ is convex*
(ii) *$\lambda * g$ is g-open for $\lambda \geq 1$ (in the H-sense).*
Let $x \in D$ with $g(x) \geq t$ and let $e \in E(D)$. Then $g(y) \geq t$ for every $y \in (e, x]$.

Proof. Let $e \in E(D)$ and $\lambda \geq 1$. For every $\varepsilon > 0$ there exists a $y \in \mathbb{R}^d$ such that
(a) $\quad g_y \leq \lambda * g$
(b) $\quad g_y(\lambda e) > (\lambda * g)(\lambda e) - \varepsilon = g(e) - \varepsilon.$
From (a) we conclude that $D_y \subseteq \lambda D$ and from (b) that $\lambda e \in D_y$. So there exists an $x \in D$ such that $\lambda e = x + y$, i.e., $y = \lambda e - x$. This yields that

$$D_{\lambda e - x} \subseteq \lambda D,$$

and we conclude from Lemma 6.9 that $x = e$, and thus $y = (\lambda - 1)e$. Substitution in (a) yields

$$g_{(\lambda-1)e} \leq \lambda * g.$$

Now let $g(x) \geq t$. Substituting $y = x + (\lambda - 1)e$ in our last inequality yields

$$t \leq g(x) \leq g(\frac{1}{\lambda}x + (1 - \frac{1}{\lambda})e).$$

This proves the assertion. ∎

Let g be a structuring function as mentioned in Thm. 7.3, that is, $D = D(g)$ is compact and convex and g is constantly t_0 on D. Then the corresponding (H,H)-granulometry

$$\Psi_\lambda(f) = f \textcircled{H} \lambda * g$$

is semi-flat in the sense of Prop. 7.1. In fact $\{\Psi_\lambda\}$ is generated by the binary Euclidean granulometries $\{\psi_{t,\lambda}\}$ given by $\psi_{t,\lambda}(X) = X \bigcirc \lambda D$ if $t \leq t_0$ and $\psi_{t,\lambda} = \emptyset$ if $t > t_0$.

At first sight it may seem surprising that Thm. 6.4 and Thm. 7.3 yield the same class of structuring functions. This, however, does not imply that the corresponding granulometries are identical. In the case of a (T,H)-Euclidean granulometry the openings involved are T-openings, whereas for (H,H)-granulometries one deals with H-openings. As one clearly sees in Fig. 2 these operators act very differently on an image.

8. Conclusions

The present paper has considered granulometries as one-parameter families of mappings on $\mathsf{Fun}(\mathbb{R}^d)$ (or more generally, $\mathsf{Fun}(V)$ where V is a locally convex topological vector space) and in this context has introduced a new class of gray-scale granulometries that are compatible with H-scalings. It has been shown that the basic union/supremum representations of binary and Euclidean granulometries hold for spatial Euclidean granulometries. But more important, the Matheron theorem regarding convexity and binary scaling has been extended to gray-scale functions for two cases: (1) invariance under T-translations and compatibility under H-scalings, and (2) invariance under H-translations and compatibility under H-scalings. The main theorems concern a full characterization of those types of Euclidean granulometries that can be expressed as a supremum over a family of parameterized openings by H-scaled structuring elements. Specifically, such granulometries must exploit flat structuring functions.

So far we haven't said anything yet about (H,T)-Euclidean granulometries. The main reason for this neglect is the observation that from a conceptual point of view this combination of H-translations and T-scalings is rather strange. To a certain extent, however, the same objection against (T,H)-Euclidean granulometries can be raised. Yet there is no mathematical reason for ignoring (H,T)-Euclidean granulometries. We have a representation theorem very similar to Thm. 6.2 and the analog of condition (6.3) has to be satisfied to eliminate the outer supremum. In fact, such a condition gives rise to almost the same class of structuring functions. The only extra requirement these functions have to satisfy is that they are constantly zero (merely constant is not sufficient).

8.1. Theorem. *Let $g \in \mathsf{Fun}(\mathbb{R}^d)$ be compact. Then*

$$\lambda \times g \text{ is } g\text{-open for } \lambda \geq 1 \text{ in the H-sense} \tag{8.1}$$

if and only if $D(g)$ is convex and g is identically 0 on $D(g)$.

To prove this result we can use similar arguments as before. Assume that $\lambda \times g$ is g-open in the H-sense. The same argument as in Lemma 6.8 shows that $D(g)$ is convex. Let m be the maximum value attained by g, then λm is the maximum value attained by $\lambda \times g$. From (8.1) one gets immediately that $m = 0$. Now take $x \in D(g)$ with $g(x) = 0$, and let $e \in E(D)$. Using the same argument as in Lemma 7.4 we derive that $g(y) = 0$ for $y \in [e, x]$. We use the same argument as in the proof of Thm. 6.4 to show that g is identically 0 on its domain.

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