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Error Analysis for the One-Dimensional Convection-Diffusion Equation

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Our aim is to derive uniform error estimates for a class of Petrov-Galerkin mixed finite element methods for the one-dimensional convection-diffusion equation. In this we are only partially successful. We obtain such estimates for the flux of the solution, but not for the solution itself. However, we can obtain uniform error estimates for the difference between our discrete approximation and a problem dependent projection of the solution. In fact we get an estimate for the average of the absolute value of that difference in a cell. As the projection is close to the normal $L^2(\Omega)$-projection for all mesh cells where convection and diffusion are of the same order of magnitude, this shows that local singular perturbation does not have a global effect. To aid in the above analysis we prove several theorems on the regularity of the continuous problem.
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1 Introduction.

In this paper we give a general technique to obtain a discretisation scheme for the one-dimensional convection-diffusion equation starting from Raviart-Thomas[1] or Brezzi-Douglas-Marini[2] type elements. The technique can also be applied in two or more dimensions. The resulting schemes are equivalent to the schemes based on transformed variables (called Slotboom variables in semiconductor context) introduced by Brezzi, Marini and Pietra[3] but without the Lagrange multipliers used in the latter schemes. The purpose of this paper is to give an error analysis for such schemes that yields information on their local accuracy. For this purpose we adapt the technique used by Douglas and Roberts[4]. Our analysis differs in following two respects from the approach by O'Riordan and Stynes[5-10] or the approach by Reinhardt[11]. One: it deals with mixed finite elements as opposed to finite elements. And two: it attempts to deal with problems with localised singular perturbation. This last aspect is very important for semi-conductor problems, where we find such a situation in the continuity equations for the charge carriers. In that case the convection is given by the electric field. Singular perturbation may occur around junctions between differently doped materials, where very localised and very large electric fields can appear. We analyse the model equation,

$$-(au'-bu)'=f \text{ on } \Omega\,, \tag{1a}$$

on the domain $\Omega=(0,L)$ with homogeneous boundary conditions,

$$u(0) = u(L) = 0\,. \tag{1b}$$

Note the absence of a zero order term. In this respect our analysis is less general than that of the approaches of Stynes and O'Riordan and Reinhardt. Our analysis makes use of the regularity of the continuous problem and its adjoint. We take the adjoint problem to be

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$$-((av')'+bv')=F \quad \text{on } \Omega\,, \tag{2a}$$

with homogeneous boundary conditions,

$$v(0) = v(L) = 0\,. \tag{2b}$$

We proceed as follows. To derive error bounds for the discrete problem, we need to know the regularity of the solution of (1), upper bounds on the norm of the solution of (1) and upper bounds on the norm of the solution of the adjoint problem. In section 2, we discuss the regularity of problem (1) under the condition that b/a is strictly positive. Section 3 derives upper bounds for the norm of the solution of the adjoint problem. In section 4 we describe the discretisation. Section 5 derives special estimates for projections of the solution of the adjoint problem that are needed later. Section 6 uses the results from the sections 2 to 4 to derive a priori error estimates. In section 7 we give our conclusions.

2 Regularity of the problem.

We formulate a theorem on the regularity of problem (1), which gives general formulas for the solution u of (1) and its flux $\sigma=-(au'-bu)$. We postpone its proof to sections 2.2 and 2.3. In section 2.1 we recall some facts concerning differentiation and integration needed in the proof of this theorem.

Theorem 1.
We assume that,

$$\frac{1}{a} \in L^p(\Omega)\,, \quad \operatorname*{ess\,inf}_{x\in\Omega} \frac{1}{a} > 0\,, \tag{3a}$$

$$b \in L^q(\Omega)\,, \quad \operatorname*{ess\,inf}_{x\in\Omega} b > 0\,, \tag{3b}$$

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \text{ with } r \geqslant 1\,, \tag{3c}$$

$$f \in W^{k,1}(\Omega)\,, \tag{3d}$$

where

$$\operatorname*{ess\,inf}_{x\in\Omega} f = -\operatorname*{ess\,sup}_{x\in\Omega} -f = -\inf_{M\subset\Omega\,,\,\lambda(M)=0}\ \sup_{x\in\Omega-M} -f(x)\,,$$

with λ the Lebesgue measure on $\mathbb{R}$.

Note that (3b) implies $\frac{1}{b} \in L^\infty(\Omega)$.

Under the conditions (3a-d), equation (1) has a unique solution $u \in W^{1,1}(\Omega)$ and

$$\|u\|_{L^\infty(\Omega)} \leqslant \|1/b\|_{L^\infty(\Omega)} \|f\|_{L^1(\Omega)}\,, \tag{4a}$$

$$\|u\|_{W^{1,1}(\Omega)} \leqslant (L+\|b/a\|_{L^1(\Omega)})\|1/b\|_{L^\infty(\Omega)}\|f\|_{L^1(\Omega)} + \|1/a\|_{L^1(\Omega)}\|f\|_{L^1(\Omega)}\,, \tag{4b}$$

$$\|\sigma\|_{L^\infty(\Omega)} \leqslant \|f\|_{L^1(\Omega)}\,, \tag{4c}$$

$$\|\sigma\|_{W^{k+1,1}(\Omega)} \leqslant L\|f\|_{L^1(\Omega)} + \|f\|_{W^{k,1}(\Omega)}\,. \tag{4d}$$

Moreover, if we introduce

$$\psi(x) = \int_{t=0}^{x} \frac{b(t)}{a(t)}\,dt\,, \tag{5}$$

$$S(\xi,\eta) = \int_{t=\xi}^{\eta} \frac{\exp(-\psi(t))}{a(t)}\,dt\,, \tag{6}$$

then the functions ψ and S are well-defined and the solution and the flux have the following absolutely continuous representations,

$$u(x) = \frac{\exp(\psi(x))}{S(0,L)}\left[\int_{y=x}^{L} S(y,L)S(0,x)f(y)\,dy + \int_{y=0}^{x} S(0,y)S(x,L)f(y)\,dy\right], \tag{7}$$

$$-\sigma(x) = a(x)u'(x) - b(x)u(x) = \tag{8}$$

$$\frac{1}{S(0,L)}\int_{y=x}^{L} S(y,L)f(y)\,dy - \frac{1}{S(0,L)}\int_{y=0}^{x} S(0,y)f(y)\,dy .$$

The above results stay valid as long as a and b are of fixed sign and are bounded away from zero. Section 2.1 recalls some important facts concerning the integration and differentiation of Lebesgue integrable functions. In section 2.2 we use the Green's function for (1) to derive the formulas for the solution and the flux. In section 2.3 we prove the rest of the theorem.

2.1. Facts on integration and differentiation of Lebesgue integrable functions.

In preparation for our proof of theorem 1, we recall some facts concerning the integration and differentiation of Lebesgue integrable functions. We recall the definition of weak differentiability and the definition of the Sobolev space $W^{k,p}(\Omega)$. We assume that Ω is a bounded interval.

Definition 1.
Let the absolute value of u be integrable on compact subsets of Ω. A function v, whose absolute value is integrable on compact subsets of Ω, is called the k^{th} weak derivative of u if it satisfies,

$$\int_{\Omega} \phi v\, d\mu = (-1)^k \int_{\Omega} u \frac{d^k\phi}{dx^k} d\mu \quad \forall \;\; \phi \in C_0^{\infty}(\Omega) .$$

Cf. section 1, chapter 2 [12].

Definition 2.
The Sobolev space $W^{k,p}(\Omega)$ is the space of $L^p(\Omega)$ functions for which all weak derivatives up to order k are $L^p(\Omega)$ functions. We use the following norm on this space,

$$||f||_{W^{k,p}(\Omega)} = \left[||f||^p_{L^p(\Omega)} + \sum_{j=1}^{k} ||\frac{d^jf}{dx^j}||^p_{L^p(\Omega)} \right]^{1/p} \quad \forall \;\; f \in W^{k,p}(\Omega) .$$

Cf. section 1, chapter 2 [12].

Definition 3.
A real-valued function f defined on a closed bounded interval $\bar{\Omega}$ is said to be absolutely continuous on $\bar{\Omega}$ if, given $\epsilon > 0$, there is a $\delta > 0$ such that $\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \epsilon$, for every finite collection of non-overlapping sub-intervals $\{ (x_i,y_i) \}_{i=1}^{n}$ of Ω with $\sum_{i=1}^{n} |y_i - x_i| < \delta$. Cf. section 4, chapter 5 [13].

Theorem 2.
A function F is an indefinite integral if and only if it is absolutely continuous. Theorem 13, section 4, chapter 5 of [13].

Theorem 3.
Every absolutely continuous function F is the indefinite integral of its derivative F' and if f is an integrable function on $\bar{\Omega}$,

$$F(x) = F(0) + \int_{t=0}^{x} f(t)\,dt ,$$

then $F'(x) = f(x)$ for almost all x in Ω. Corollary 14, section 4, chapter 5 and Theorem 9, section 3, chapter 5 [13].

Lemma 1.
If f and g are absolutely continuous on Ω, then fg and $\exp(f)$ are absolutely continuous.
Proof.
Consider the condition

$$\sum_{i=1}^{n} |fg(y_i)-fg(x_i)| < \epsilon .$$

Continuous functions on a closed interval are bounded, so f and g are bounded. Take $M=\max(\|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)})$. Now there exists by definition a δ such that, for every finite collection of non-overlapping sub-intervals $\{ (x_i,y_i) \}_{i=1}^{n}$ of Ω with $\sum_{i=1}^{n} |y_i-x_i| < \delta$,

$$\sum_{i=1}^{n} |f(y_i)-f(x_i)| < \frac{\epsilon}{2M} \text{ and } \sum_{i=1}^{n} |g(y_i)-g(x_i)| < \frac{\epsilon}{2M} .$$

This implies that

$$\sum_{i=1}^{n} |fg(y_i)-fg(x_i)| \leqslant \sum_{i=1}^{n} |g(y_i)f(y_i)-g(x_i)f(y_i)+g(x_i)f(y_i)-g(x_i)f(x_i)| \leqslant$$

$$\sum_{i=1}^{n} M|g(y_i)-g(x_i)| + \sum_{i=1}^{n} M|f(y_i)-f(x_i)| \leqslant \epsilon .$$

Moreover, there is a δ such that, for every finite collection of non-overlapping sub-intervals $\{ (x_i,y_i) \}_{i=1}^{n}$ of Ω with $\sum_{i=1}^{n} |y_i-x_i| < \delta$,

$$\sum_{i=1}^{n} |f(y_i)-f(x_i)| < \epsilon \exp(-3M) .$$

In that case,

$$\sum_{i=1}^{n} |\exp(f(y_i))-\exp(f(x_i))| \leqslant$$

$$\sum_{i=1}^{n} \exp(f(x_i))\,|f(y_i)-f(x_i)|\,\exp(|f(y_i)-f(x_i)|) \leqslant \epsilon .$$

□

Theorem 4.
Let $\Omega=(\xi,\eta)$ be a bounded interval of $\mathbb{R}$. Let $C_0^\infty(\Omega)$ be the space of all $C^\infty(\Omega)$ functions with a compact support in Ω, Let $W_0^{1,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. All elements of $W_0^{1,p}(\Omega)$, where $1 \leqslant p \leqslant \infty$, are absolutely continuous. Cf. page 148 [14]
Proof.
We prove this to get an idea of the character of the space in question. For each $t \in W_0^{1,p}(\Omega)$ there is by definition a Cauchy sequence $\{ t_n \}_{n=1}^{\infty} \subset C_0^\infty(\Omega)$, that converges to t in the $W^{k,p}(\Omega)$-norm. We denote the first derivative of a function g by g'. We have, $t_n \to t$ in $L^p(\Omega)$ and $t_n' \to t'$ in $L^p(\Omega)$, so, if we define

$$T_n(x) = \int_{y=\xi}^{x} t_n'(y)\,dy \text{ and } T(x) = \int_{y=\xi}^{x} t'(y)\,dy \text{ for } x \in \Omega ,$$

then for all elements of the sequence $\{ t_n \}$, we have $t_n = T_n$ and theorem 3 implies that

$$\|t'-T'\|_{L^p(\Omega)} = 0 .$$

Moreover, for a given n,

$$||t-T||_{L^p(\Omega)} \leqslant ||t-t_n||_{L^p(\Omega)} + ||T_n-T||_{L^p(\Omega)} ,$$

so

$$||t-T||_{L^p(\Omega)} \leqslant ||t-t_n||_{L^p(\Omega)} + ||\int_{y=\xi}^{x} (t'-t_n')\, dy\,||_{L^p(\Omega)} \leqslant ||t-t_n||_{L^p(\Omega)} + (\eta-\xi)||t'-t_n'||_{L^p(\Omega)} \leqslant$$
$$(1+\eta-\xi)||t-t_n||_{W^{1,p}(\Omega)} .$$

This holds for all n, so $||t-T||_{L^p(\Omega)}=0$. This proves that t is the indefinite integral of t'. By theorem 2 this implies that t is absolutely continuous. □

2.2. The derivation of expressions for the solution and the flux.

We derive the expressions (7) and (8), we show that these functions satisfy (1), and we prove the statement about absolute continuity from theorem 1. We proceed as follows. In theorem 5 we construct the Green's function[15, 16] of (1) and use this to derive (7) and (8). We then substitute (7) in (1) and use theorem 3 to show that (7) and (8) satisfy (1). Absolute continuity of (7) and (8) is shown to follow from theorem 2. First we show that ψ is well-defined.

Lemma 2.
If (3a-d) hold, then the function ψ, defined by (5) is an absolutely continuous function on Ω and its derivative $\psi'(x)$ lies in $L^r(\Omega)$ and is equal to $b(x)/a(x)$.
Proof.
The Hölder inequality implies that

$$||fg||_{L^r(\Omega)} \leqslant ||f||_{L^p(\Omega)}||g||_{L^q(\Omega)} . \tag{9}$$

for all $p,q,r \in [1,\infty]$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. For superscripts of $L^p(\Omega)$ spaces only, we use the convention that $1/0=\infty$ and $1/\infty=0$. We assumed that $\frac{1}{a} \in L^p(\Omega)$, $b \in L^q(\Omega)$, so, according to (9) $b/a \in L^r(\Omega)$. According to theorem 2, ψ is absolutely continuous. Theorem 3 implies that $\psi' = b/a$ in almost all points of Ω. □

Theorem 5.
Assume (3a-d), take ψ as in (5) and S as in (6). If $f \in L^1(\Omega)$ then the function u defined below is a solution of the equation (1) with right hand side f and with homogeneous boundary conditions.

$$u(x) = \int_{y=0}^{1} G(x,y)f(y)\, dy , \tag{10}$$

with

$$G(x,y) := \frac{\exp(\psi(x))}{S(0,L)} \left[\theta(y-x)S(y,L)S(0,x) + \theta(x-y)S(0,y)S(x,L)\right] , \tag{11}$$

where θ is the Heaviside function,

$$\theta(z) = \begin{cases} 0 \text{ if } z < 0 , \\ ½ \text{ if } z=0 , \\ 1 \text{ if } z > 0 . \end{cases} \tag{12}$$

Proof.
We see immediately that $G \in C([0,L]\times[0,L])$. We use theorem 3 and the chain rule to derive (8) from (7). According to the chain rule and theorem 3,

$$u'(x) = \frac{b(x)}{a(x)}u(x) + \frac{1}{a(x)S(0,L)}\int_{y=x}^{L} S(y,L)f(y)\, dy - \frac{1}{a(x)S(0,L)}\int_{y=0}^{x} S(0,y)f(y)\, dy , \tag{13}$$

equation (8) follows immediately from (13) and the definition of the flux. Absolute continuity of the

solution constructed with the aid of the Green's function follows from theorem 2, lemma 1 and equations (7) and (8). We see immediately that u satisfies the homogeneous Dirichlet boundary conditions. When we apply theorem 3 to equation (8), we find, $\sigma'(x) = f(x)$. □
See also the books by Roach and Yosida[15, 16].

2.3. Upper bounds on the norms of the solution and the flux.

We complete the proof of theorem 1 by proving that (1) has a unique solution in $W^{1,1}(\Omega)$ and deriving the upper bounds on the norm of the solution and the flux from (7) and (8).

First we verify uniqueness of the solution as follows. Suppose (1) has two solutions $u_1, u_2 \in W_0^{1,p}(\Omega)$ for a given f. This implies that $w_0 = u_2 - u_1 \in W_0^{1,p}(\Omega)$ is a solution of (1) with $f = 0$. Now by definition,

$$((\exp(-\psi)w_0)', a\exp(\psi)\phi') = 0 \quad \forall \ \phi \in C_0^\infty(\Omega),$$

so $(\exp(-\psi)w_0)' = 0$. According to theorem 4, the function w_0 is absolutely continuous and according to lemma 1 the function $\exp(-\psi)$ is absolutely continuous, theorem 3 now implies that $\exp(-\psi)w_0$ is constant. The only $w_0 \in W_0^{1,p}(\Omega)$ that can give this result is $w_0 = 0$.

Before we can derive upper bounds on the norms of (7) and (8), we need to derive some bounds on $S(\xi,\eta)$.

Lemma 3.
Assume (3a-d) and take ψ as in (5). Let S be the function on $\Omega\times\Omega$ defined by (6). Then $S(0,x)$ and $S(x,L)$ are absolutely continuous functions. If $0 \leqslant \xi_0 \leqslant \xi < \eta \leqslant \eta_0 \leqslant L$ then

$$0 < S(\xi,\eta) \leqslant S(\xi_0,\eta_0), \tag{14}$$

and

$$S(\xi,\eta) \leqslant \|b^{-1}\|_{L^\infty(\Omega)} (\exp(-\psi(\xi)) - \exp(-\psi(\eta))). \tag{15}$$

Proof.
From (3a, b) and the positivity of the integrand (14) follows immediately. From (6) it follows that

$$S(\xi,\eta) = \int_{x=\xi}^{\eta} \frac{1}{b(x)} \frac{-d\exp(-\psi(x))}{dx}(x)\,dx .$$

We see, that $(-\exp(-\psi))' = \psi'\exp(-\psi) > 0$, so

$$S(\xi,\eta) \leqslant \|b^{-1}\|_{L^\infty(\Omega)} \int_{x=\xi}^{\eta} \frac{-d\exp(-\psi(x))}{dx}(x)\,dx .$$

As $\exp(-\psi)$ is absolutely continuous according to lemma 1, we find from theorem 3 that

$$\int_{t=\xi}^{\eta} (\exp(-\psi(t)))'(t)\,dt = \exp(-\psi(\eta)) - \exp(-\psi(\xi)) .$$

□

Next, we can prove the inequalities (4a-d). We assume that (3a-d) hold. Application of (14) to (7) yields the following upper bound on u,

$$|u(x)| \leqslant \frac{\exp(\psi(x))S(x,L)S(0,x)}{S(0,L)} \|f\|_{L^1(\Omega)} .$$

We use (14) and (15) to write this as,

$$|u(x)| \leqslant \|1/b\|_{L^\infty(\Omega)} \|f\|_{L^1(\Omega)} .$$

This proves (4a). Now (4b) follows immediately from (13). Next, we derive (4c). From (14) and (8) an estimate for σ follows immediately:

$$|a(x)u'(x)-b(x)u(x)| \leq \|f\|_{L^1(\Omega)} .$$

And (4d) follows from (4c) and the fact that (1) implies $\sigma' = f$.

3 The adjoint problem.

First, we derive a Green's function for (2). Then we give expressions for the solution and the flux of (2). Finally we derive upper bounds on the norms of the solution and the flux. The following theorem accomplishes our first two goals.

Theorem 6.
Assume (3a-d), take ψ as in (5) and S as in (6). If $F \in L^1(\Omega)$ then the function v defined below is a solution of the equation (2) with right hand side F and with homogeneous boundary conditions.

$$v(x) = \qquad (16)$$
$$\frac{1}{S(0,L)}\left[\int_{y=0}^{x} S(x,L)S(0,y)\exp(\psi(y))F(y)dy + \int_{y=x}^{L} S(0,x)S(y,L)\exp(\psi(y))F(y)dy\right] .$$

Proof.
The Green's function for the adjoint problem (2) is given by, $\bar{G}(x,y) = G(y,x)$. See also Roach or Yosida[15, 16]. According to theorem (2) v is absolutely continuous on $[0,L]$, so $v(0)=v(L)=0$. Moreover,

$$\tau(x) = -a(x)v'(x) = \qquad (17)$$
$$-\frac{\exp(-\psi(x))}{S(0,L)}\left[-\int_{y=0}^{x} S(0,y)\exp(\psi(y))F(y)dy + \int_{y=x}^{L} S(y,L)\exp(\psi(y))F(y)dy\right] .$$

And by differentiation of integrals, $\tau'(x) = -\frac{b(x)}{a(x)}\tau(x) + F(x)$. This in turn implies that v satisfies the adjoint problem. □
It now remains to give upper bounds on the norms of the solution and the flux.

Theorem 7.
Assume (3a-d), take ψ as in (5) and S as in (6). Assume $F \in W^{k,1}(\Omega)$. Now (2) has a unique solution $v \in W^{1,1}(\Omega)$. The solution v and the corresponding flux τ, defined by $\tau = -av'$, have the following properties:

$$\|v\|_{L^\infty(\Omega)} \leq \|1/b\|_{L^\infty(\Omega)} \|F\|_{L^1(\Omega)} , \qquad (18a)$$

$$\|v\|_{W^{1,1}(\Omega)} \leq \left[L\|1/b\|_{L^\infty(\Omega)} + \left[1+\frac{\|1/b\|_{L^\infty(\Omega)}}{S(0,L)}\right]\|1/a\|_{L^1(\Omega)}\right]\|F\|_{L^1(\Omega)} , \qquad (18b)$$

$$\|\tau\|_{L^\infty(\Omega)} \leq \left[1+\frac{\|1/b\|_{L^\infty(\Omega)}}{S(0,L)}\right]\|F\|_{L^1(\Omega)} . \qquad (18c)$$

Moreover, the solution and the flux are absolutely continuous.
Proof.
The solution is unique, because if it is not, then (2) with $F \equiv 0$ has a non-trivial solution in $W_0^{1,1}(\Omega)$. This in turn would imply that there is an absolutely continuous w_0 such that

$$(aw_0')'+bw_0' = 0 \text{ on } \Omega ,$$
$$w_0(0)=w_0(L)=0 .$$

According to theorem 1 there is a unique absolutely continuous $v \in W_0^{1,1}(\Omega)$ such that

$$(av'-bv)' = w_0 .$$

But this implies that

$$(w_0,w_0) = (w_0,(av'-bv)') = -(w_0',av'-bv) = -(aw_0',v')+(bw_0',v) .$$

We use the definition of weak differentiability to write this as,

$$(w_0,w_0) = ((aw_0')'+bw_0',v) = 0 .$$

This implies that $w_0=0$. Absolute continuity of the solution constructed with the aid of the Green's function follows from theorem 2, lemma 1 and equations (16) and (17). Uniqueness of the solution implies that we may derive upper bounds on the norm of the solution and the flux from the previously given expressions. We proceed as follows. Application of (14) and (15) to (16) yields the following estimate for v,

$$|v(x)| \leq \|1/b\|_{L^\infty(\Omega)} \|F\|_{L^1(\Omega)} .$$

This proves (18a). The inequality (18b) follows immediately from (17). Next, we derive the (18c). From (14) and (17) an estimate for τ follows immediately:

$$|a(x)v'(x)| \leq \left[1+\frac{\|b^{-1}\|_{L^\infty(\Omega)}}{S(0,L)}\right] \|F\|_{L^1(\Omega)} .$$

□

4 The discretisation.

We construct a Petrov-Galerkin mixed finite element discretisation. Our derivation uses trial spaces V_h and W_h that are defined as the ranges of the projections $\Pi_h: V \to V_h$ and $P_h: W \to W_h$, where we take $V = W^{1,1}(\Omega)$ and $W = L^1(\Omega)$. This approach was first used by Raviart and Thomas[1] and Fortin[17]. Our test spaces are derived from the trial spaces by multiplication with an exponential function. The final result will be equivalent to the standard mixed finite element discretisation for the symmetrised form of the equation but the special derivation allows us to obtain better a-priori error estimates. We proceed as follows. First we give conditions on the projections P_h and Π_h. We show that these conditions guarantee that $\frac{d}{dx}(V_h) = W_h$. Next we give an example of such projections. Finally we derive the discrete scheme and verify that the resulting discrete problem has a unique solution.

4.1. The projections onto the trial spaces for the solution and its flux.

As mentioned earlier, we derive our trial spaces from projections $P_h: W \to W$ and $\Pi_h: V \to V$. We assume these projections have finite dimensional ranges and satisfy the following conditions:

$$(s,P_h t) = (P_h s,t) \quad \forall \ s,t \in W , \tag{19}$$

and

$$P_h \frac{d}{dx} v = \frac{d}{dx} \Pi_h v \quad \forall \ v \in V , \tag{20a}$$

$$\Pi_h v(0) = v(0) \quad \forall \ v \in V . \tag{20b}$$

We define our approximation spaces as follows. We set $V_h = \mathcal{R}(\Pi_h)$ and $W_h = \mathcal{R}(P_h)$.

Theorem 8.

The map $\frac{d}{dx}: W^{1,1}(\Omega) \to L^1(\Omega)$ is continuous and surjective.

Proof.

Continuity follows immediately from the norms on these spaces. The map is surjective because, for all $f \in L^1(\Omega)$, theorem 2 shows that the function F, defined by

$$F(x) = \int_{y=0}^{x} f(y)\, dy \quad \text{for } x \in \Omega ,$$

is an element of $W^{1,1}(\Omega)$ with derivative f. □

Corollary 1.
The map $\frac{d}{dx}: V_h \rightarrow W_h$ is surjective.
Proof.
From (20) it follows that the image of V_h under $\frac{d}{dx}$ lies in W_h. From theorem 8 and (20) it follows that the image is in fact equal to W_h. □
The above use of projections can be found in [1, 17].

4.2. An example of a set of trial spaces.

An example of a set of spaces and projections that meet these criteria are the lowest order Raviart-Thomas spaces with the projections given in [1]. For the one dimensional case, this simply means that the image of a function under Π_h is obtained by linear interpolation between the values in mesh nodes and for P_h the image is obtained by taking cell-wise averages. Now V_h is the space of continuous functions that are linear on the mesh cells and W_h is the space of functions that are constant on mesh cells.

4.3. The discrete scheme.

We construct a Petrov-Galerkin mixed finite element method as follows. We take V_h as trial space for σ. As test space for σ we take $X_h = \exp(-\psi)V_h$. For u we take W_h as test space and $Y_h = \exp(\psi)W_h$ as trial space. Here ψ is defined as in (5). We define projections onto X_h and Y_h.

$$\hat{\Pi}_h\tau = \exp(-\psi)\Pi_h(\exp(\psi)\tau) , \tag{21}$$

$$\hat{P}_h t = \exp(\psi)P_h(\exp(-\psi)t) . \tag{22}$$

From (19) it follows that,

$$(s,\hat{P}_h t) = (\exp(-\psi)P_h(\exp(\psi)s),t) = (\exp(-\psi)P_h(\exp(\psi)s),\hat{P}_h t) = (\hat{P}_h^* s,\hat{P}_h t) , \tag{23}$$

where $\hat{P}_h^*$ is the adjoint operator of $\hat{P}_h$. By application of the defining formulas we find,

$$\left(\frac{d}{dx}\tau + \frac{b}{a}\tau, \hat{P}_h t\right) = \left(\left(\frac{d}{dx} + \frac{b}{a}\right)\hat{\Pi}_h\tau, \hat{P}_h t\right) . \tag{24}$$

The continuous solution of (1) satisfies

$$(\sigma,u) \in V \times W , \tag{25a}$$

$$\left(\sigma,\frac{\tau}{a}\right) - \left(\frac{d}{dx}\tau + \frac{b}{a}\tau, u\right) = 0 \quad \forall \ \tau \in H^1(\Omega) , \tag{25b}$$

$$\left(\frac{d}{dx}\sigma, t\right) = (f,t) \quad \forall \ t \in L^2(\Omega) . \tag{25c}$$

Our discrete scheme has the following form.

$$(\sigma_h,u_h) \in V_h \times Y_h , \tag{26a}$$

$$\left(\sigma_h,\frac{\tau_h}{a}\right) - \left(\frac{d}{dx}\tau_h + \frac{b}{a}\tau_h, u_h\right) = 0 \quad \forall \ \tau_h \in X_h , \tag{26b}$$

$$\left(\frac{d}{dx}\sigma_h, t_h\right) = (f,t_h) \quad \forall \ t_h \in W_h . \tag{26c}$$

We see that this scheme is equivalent to,

$$(\sigma_h, U_h) \in V_h \times W_h , \tag{27a}$$

$$\left[\sigma_h, \frac{\exp(-\psi)\tau_h}{a}\right] - \left[\frac{d}{dx}\tau_h, U_h\right] = 0 \quad \forall \ \tau_h \in V_h \ , \tag{27b}$$

$$\left[\frac{d}{dx}\sigma_h, t_h\right] = (f, t_h) \quad \forall \ t_h \in W_h \ . \tag{27c}$$

This last system has a unique solution. This can be demonstrated as follows. Suppose $f=0$. As σ_h is continuous, (20) and (27c) imply that σ_h is constant. Now take $\tau_h = 1$, from (27b) it follows that $\sigma_h \equiv 0$. Now corollary 1 implies $U_h = 0$. This completes the demonstration.

5 Properties of the projections.

In the section on a priori error estimates we shall need estimates of terms containing the difference between a function and its projection under one of the projections introduced in the previous section. In this section we give estimates for those terms. We start by considering $\tau - \hat{\Pi}_h\tau$. To do this we need the following auxiliary lemma.

Lemma 4.
If $f \in W^{1,1}(\Omega)$ then

$$f(x) - \Pi_h f(x) = \int_{y=0}^{x} (f' - P_h(f'))dy \ . \tag{28}$$

Proof.
The function f is continuous and differentiable, so

$$f(x) = f(0) + \int_{y=0}^{x} f'(y) \, dy \ .$$

Moreover (20) implies that,

$$\Pi_h f(x) = \Pi_h f(0) + \int_{y=0}^{x} P_h(f')(y) \, dy \ .$$

□

Lemma 5.
If $(v, \tau = -av')$ is the solution of the adjoint equation for the right hand side F, then

$$|(t, (\tau - \hat{\Pi}_h \tau)/a)| \leq \|b^{-1}\|_{L^\infty(\Omega)} \|F - \hat{P}_h F\|_{L^1(\Omega)} \|t\|_{L^\infty(\Omega)} \quad \forall \ t \in L^\infty(\Omega) \ .$$

Proof.
We know that

$$v(x) = \int_{y=0}^{L} \bar{G}(x,y) F(y) \, dy \ ,$$

where $\bar{G}(x,y)$ is Green's function for the adjoint problem. Now consider

$$av'(x) - \exp(-\psi(x)) \Pi_h(\exp(\psi) av') \ .$$

We can write this as,

$$\exp(-\psi(x)) \left[\exp(\psi) av'(x) - \Pi_h(\exp(\psi) av')\right] \ .$$

We wish to apply the previous lemma. To do this we need the first derivative of $\exp(\psi)av'$. Equation (17) implies that

$$(\exp(\psi) av')' = -\exp(\psi(x)) F(x) \ .$$

We use this to evaluate the expression $(t,(\tau-\hat{\Pi}_h\tau)/a)$,

$$(t,(\tau-\hat{\Pi}_h\tau)/a) = \int_{x=0}^{L} \frac{\exp(-\psi(x))}{a(x)} \int_{y=0}^{x} \exp(\psi)F - \mathrm{P}_h(\exp(\psi)F)\, dy\; t(x)dx =$$

$$\int_{x=0}^{L} \frac{\exp(-\psi(x))}{a(x)} \int_{y=0}^{x} \exp(\psi)(F-\hat{P}_hF)\, dy\; t(x)dx \;.$$

This implies,

$$(t,(\tau-\hat{\Pi}_h\tau)/a) \leqslant \left| \int_{y=0}^{L}\int_{x=y}^{L} \frac{\exp(\psi(y)-\psi(x))}{a(x)}(F-\hat{P}_hF)(y)\; t(x)dxdy \right| \leqslant$$

$$\int_{y=0}^{L} \|t\|_{L^\infty(\Omega)} \|b^{-1}\|_{L^\infty(\Omega)} \,|(F-\hat{P}_hF)(y)|\, dy \;.$$

□

Next, we consider $v-\mathrm{P}_h v$.

Lemma 6.
If $(v,\tau=-av')$ is the solution of the adjoint equation for the right hand side F, then

$$\|v-\mathrm{P}_hv\|_{L^2(\Omega)} \leqslant \sqrt{L}\,\|1/b\|_{L^\infty(\Omega)}\,\|F\|_{L^1(\Omega)} \;. \tag{29}$$

Proof.
This follows immediately from $\|v-\mathrm{P}_hv\|_{L^2(\Omega)} \leqslant \|v\|_{L^2(\Omega)}$ and (18a). □

6 A priori error estimates.

We derive estimates for $\|\sigma-\sigma_h\|_{L^\infty(\Omega)}$ and $\|\hat{P}_hu-u_h\|_{W^{k,1}(\Omega)'}$. We start by giving estimates for $\|\sigma-\sigma_h\|_{L^\infty(\Omega)}$ and $\|\sigma-\sigma_h\|_{W^{1,1}(\Omega)}$. We proceed as follows. First we show that there is a point $\xi\in\Omega$ where the function $\sigma-\sigma_h$ is zero, then we determine the first derivative of the function and use this to determine the desired estimates.

Lemma 7.
Given that (3a-d) are satisfied and σ satisfies (25c) and σ_h is a solution of (26c), there is at least one point ξ such that $(\sigma-\sigma_h)(\xi)=0$.
Proof.
We see immediately that $\exp(-\psi)\in X_h$. The solution of (1) satisfies (25b), so

$$\left(\sigma-\sigma_h, \frac{\exp(-\psi)}{a}\right) = 0 \,.$$

We know that $\exp(-\psi)$ and a are strictly positive and bounded from below, so there must be places where $\sigma-\sigma_h$ is negative. We know that $\sigma\in W^{1,1}(\Omega)$ from $-\sigma'=f$, and $\sigma_h\in V_h\subset W^{1,1}(\Omega)$ so $\sigma-\sigma_h$ is continuous. This implies that there is a ξ such that $(\sigma-\sigma_h)(\xi)=0$.

□

Theorem 9.
If (3a-d) hold and

$$C(f) = \left| \sup_{\xi,\eta\in\Omega} \int_{y=\xi}^{\eta} (f-\mathrm{P}_hf)(y)dy \right| , \tag{30}$$

then $C(f) \leqslant \|f-\mathrm{P}_hf\|_{L^1(\Omega)}$ and

$$\|\sigma-\sigma_h\|_{L^\infty(\Omega)} \leqslant C(f)\,, \tag{31}$$

$$\|\sigma-\sigma_h\|_{W^{1,1}(\Omega)} \leqslant L\,C(f) + \|f-P_hf\|_{L^1(\Omega)}\,. \tag{32}$$

Proof.
We take ξ to be a zero of $\sigma-\sigma_h$. We know that $\sigma-\sigma_h \in W^{1,1}(\Omega)$, so we may write,

$$(\sigma-\sigma_h)(x)-(\sigma-\sigma_h)(\xi) = \int_{y=\xi}^{x} (\sigma-\sigma_h)'(y)dy\,.$$

From (25) and (26) we see immediately that

$$(\sigma-\sigma_h)' = f-P_hf\,.$$

This implies,

$$(\sigma-\sigma_h)(x)-(\sigma-\sigma_h)(\xi) = \int_{y=\xi}^{x} (f-P_hf)(y)dy\,.$$

This implies that

$$\|\sigma-\sigma_h\|_{L^\infty(\Omega)} \leqslant \sup_{x\in\Omega}\int_{y=\xi}^{x} (f-P_hf)(y)dy\,.$$

□

We give an estimate for $\|\hat{P}_hu-u_h\|_{W^{k,1}(\Omega)^*}$. To derive this estimate we use the dual problem.

Theorem 10.
Under the conditions given in(3a-d),

$$\|\hat{P}_hu-u_h\|_{L^\infty(\Omega)} \leqslant 2\|b^{-1}\|_{L^\infty(\Omega)}(1 + C_k(\psi))\|f-P_hf\|_{L^1(\Omega)}\,,$$

where

$$C_k(\psi) := \sup_{F\in L^1(\Omega)} \frac{\|F-\hat{P}_hF\|_{L^1(\Omega)}}{\|F\|_{L^1(\Omega)}}\,,$$

and

$$\|\hat{P}_hu-u_h\|_{W^{k,1}(\Omega)^*} \leqslant 2\|b^{-1}\|_{L^\infty(\Omega)}(1 + D_k(\psi))\|f-P_hf\|_{L^1(\Omega)}\,,$$

where

$$D_k(\psi) := \sup_{F\in W^{k,1}(\Omega)} \frac{\|F-\hat{P}_hF\|_{L^1(\Omega)}}{\|F\|_{W^{k,1}(\Omega)}}\,.$$

Proof.
Regularity of the adjoint problem gives us a solution $(t,\tau=-at')$ of (2) for all $F\in L^1(\Omega)$. For this solution, we see that according to (26),

$$(\hat{P}_hu-u_h,F) = (\tau'+\psi'\tau,\hat{P}_hu-u_h) = ((\hat{\Pi}_h\tau)'+\psi'(\hat{\Pi}_h\tau),\hat{P}_hu-u_h) =$$
$$(\sigma-\sigma_h,\hat{\Pi}_h\tau/a) = (\sigma-\sigma_h,\tau/a) - (\sigma-\sigma_h,(\tau-\hat{\Pi}_h\tau)/a) =$$
$$((\sigma-\sigma_h)',t) - (\sigma-\sigma_h,(\tau-\hat{\Pi}_h\tau)/a) = (f-P_hf,t-P_ht) - (\sigma-\sigma_h,(\tau-\hat{\Pi}_h\tau)/a)\,.$$

We use lemma 5 and 6 and theorem 9 to derive from this that,

$$|(\hat{P}_hu-u_h,F)| \leqslant 2(\sqrt{L}\,\|F\|_{L^1(\Omega)} + \|F-\hat{P}_hF\|_{L^1(\Omega)})\|b^{-1}\|_{L^\infty(\Omega)}\|f-P_hf\|_{L^1(\Omega)}\,.$$

□

Corollary 2.
Assume that W_h contains the characteristic functions $\chi_{(x_{i-1},x_i)}$ of the cells of the partition $P = \{ 0=x_0 < x_1 < x_2 < \cdots < x_n=L \}$. As a direct consequence of theorem 10 and under the same conditions, we find

$$\frac{\| \hat{P}u-u_h \|_{L^1((x_{i-1},x_i))}}{x_i-x_{i-1}} \leqslant 2\| b^{-1} \|_{L^\infty(\Omega)} \|f-P_h f\|_{L^1(\Omega)} . \tag{33}$$

Proof.
We prove this as follows. For F in the proof of theorem 10, take $F = \exp(\psi)\chi_{(x_{i-1},x_i)}$. According to the Riesz representation theorem $L^1(\Omega)^* = L^\infty(\Omega)$. We find,

$$\| \exp(2\psi) \|_{L^1((x_{i-1},x_i))} \| \exp(-\psi)(\hat{P}u-u_h) \|_{L^\infty((x_{i-1},x_i))} \leqslant \tag{34}$$
$$2\| b^{-1} \|_{L^\infty(\Omega)} \| \exp(\psi) \|_{L^1((x_{i-1},x_i))} \|f-P_h f\|_{L^1(\Omega)} .$$

We see immediately that,

$$0 \leqslant (g-P_h g, g-P_h g) = \| g \|^2_{L^2(\Omega)} - \| P_h g \|^2_{L^2(\Omega)} .$$

This implies that,

$$\| \exp(2\psi) \|_{L^1((x_{i-1},x_i))} \geqslant \frac{\| \exp(\psi) \|^2_{L^1((x_{i-1},x_i))}}{x_i-x_{i-1}} .$$

We apply this to (34) and find,

$$\frac{\| \exp(\psi) \|_{L^1((x_{i-1},x_i))}}{x_i-x_{i-1}} \| \exp(-\psi)(\hat{P}u-u_h) \|_{L^\infty((x_{i-1},x_i))} \leqslant$$
$$2\| b^{-1} \|_{L^\infty(\Omega)} \| \exp(\psi) \|_{L^1((x_{i-1},x_i))} \|f-P_h f\|_{L^2(\Omega)} .$$

This implies,

$$\frac{\| \exp(\psi) \|_{L^1((x_{i-1},x_i))}}{x_i-x_{i-1}} \| \exp(-\psi)(\hat{P}u-u_h) \|_{L^\infty((x_{i-1},x_i))} \leqslant 2\| b^{-1} \|_{L^\infty(\Omega)} \|f-P_h f\|_{L^2(\Omega)} .$$

Which in turn implies that

$$\frac{\| \hat{P}u-u_h \|_{L^1((x_{i-1},x_i))}}{x_i-x_{i-1}} \leqslant 2\| b^{-1} \|_{L^\infty(\Omega)} \|f-P_h f\|_{L^2(\Omega)} .$$

□

7 Conclusions.

We see that the accuracy of the solution of the problem with homogeneous Dirichlet boundary conditions is entirely determined by two factors. One being the accuracy of the approximation of the right hand side f by $P_h f$ and the other being the quality of the approximation of $F \in W_1^k(\Omega)$ by $\hat{P}_h F$. As mentioned in the introduction, in the semi-conductor continuity equations the convection is given by the electric field. Singular perturbation may occur around junctions between differently doped materials, where locally very large electric fields can appear. From the uniform $L^\infty(\Omega)$ error estimate for the total error in the flux in theorem 9 it follows that local singular perturbation on the approximation has no influence on the error in the flux. In corollary 2 we get a uniform cell-wise estimate for the discretisation error with respect to a problem dependent projection that is close to $L^2(\Omega)$ projection on cells where the convection does not dominate the diffusion. The main problem that must be faced when extending this analysis to two or more dimensions, is the derivation of a useful estimate for $\| \sigma-\sigma_h \|$.

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