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Incidence and lattice calculus with applications to stochastic geometry and image analysis

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Incidence and Lattice Calculus with Applications to Stochastic Geometry and Image Analysis

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Abstract

Incidence between subsets is a basic concept of stochastic geometry and mathematical morphology. In this note we discuss a formal generalisation of incidence (and the dual notion of dominance) in the setting of complete lattices. We discuss applications to mathematical morphology, random set theory and combinatorial geometrical probability. We sketch other practical applications to transmission microscopy, digital image discretisation and robot motion planning.

The generalised incidence structure turns out to be equivalent to the established idea of a lattice adjunction. Using this, many problems in stochastic geometry (Buffon-Sylvester problem, local knowledge, overprojection effects) can be reformulated as lattice calculations.

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Introduction

Two subsets $X, Y \subset \mathbb{R}^n$ are 'incident' if $X \cap Y \neq \emptyset$. Incidence is a basic concept of stochastic geometry [12, 23]. Many classical problems [13, 19, 22] concern the probability that a random set will intersect a given fixed set. In the general theory of random sets [12, 15], a random subset X of an arbitrary space S is characterized *solely* by testing whether X intersects T for a sufficiently large class of test sets T.

Incidence information $X \cap T \neq \emptyset$ is of course equivalent to information about the partial order of set inclusion ($X \subseteq T$ iff $X \cap T^c = \emptyset$ where T^c denotes the set complement). In the related field of mathematical morphology [20] recent work [11, 16, 17, 18, 21] suggests that the partial order structure is more natural, and enables one to harness the theory of complete lattices [3, 8].

In this note we show that a generalisation of the incidence relation $X \sim Y \iff X \cap Y \neq \emptyset$ is trivially equivalent to the concept of an adjunction in lattice theory. Using this, many problems in stochastic geometry (including the theory of strong incidence functions [12], projection effects [6], and the Buffon-Sylvester problem [1, 2, 19]) can be reformulated as lattice calculations.

General theory is in Sections 1–3 and applications are discussed in §4–7. Section 1 recalls some lattice theory; Section 2 defines incidence and partial order structures and their (trivial) relationship to adjunctions. In section 3 we go through a calculation in lattice algebra that is found in different incarnations (local knowledge principle, conditional closure) in the applications. Our first application is in §4 which shows how classical mathematical morphology fits into the lattice setting. In section 5 we show that the non-probabilistic aspects of Kendall's theory of strong incidence functions in the general theory of random sets can be derived from the results in sections 1 and 2. In §6 we show use the conditional closure operation of 3 to handle the Buffon-Sylvester problem. More concrete applications (to projection effects in microscopy, image discretization, and robot motion planning) are described in section 7.

1 Complete lattices and adjunctions

This section recalls some basic lattice theory [3], in particular the concept of an adjunction [3, 8], and results about adjunctions from [11] that are important in the context of mathematical morphology.

A *complete lattice* is a partially ordered set $(\mathcal{L}, \leq)$ in which every subset $\mathcal{H} \subseteq \mathcal{L}$ has a supremum and infimum denoted by $\bigvee \mathcal{H}$ and $\bigwedge \mathcal{H}$ respectively. In particular there is a greatest element $\mathbf{1}_{\mathcal{L}}$ and a least element $\mathbf{0}_{\mathcal{L}}$. A complete lattice is *Boolean* if suprema distribute over infima and vice versa, and if every $X \in \mathcal{L}$ has a unique complement X^* such that $X \vee X^* = \mathbf{1}_{\mathcal{L}}$, $X \wedge X^* = \mathbf{0}_{\mathcal{L}}$.

For example the class $\mathcal{P}(E)$ of all subsets of an arbitrary set E, ordered by set inclusion $X \leq Y \Leftrightarrow X \subseteq Y$, is a complete Boolean lattice whose supremum, infimum and complement operations are equivalent to set union, intersection, and complement in E respectively. The closed subsets of a topological space form a complete lattice, where infimum is set intersection and supremum is the topological closure of the set union.

Definition 1 *Let $\mathcal{M}, \mathcal{L}$ be complete lattices and let $\epsilon : \mathcal{M} \to \mathcal{L}$ and $\delta : \mathcal{L} \to \mathcal{M}$. The pair (ϵ, δ) is called an* adjunction *between $\mathcal{M}$ and $\mathcal{L}$ if for every $X \in \mathcal{L}$ and $Y \in \mathcal{M}$,*

$$\delta(X) \leq Y \iff X \leq \epsilon(Y).$$

An example of a nontrivial adjunction is $\mathcal{L}$ = open sets, $\mathcal{M}$ = closed sets of a topological space, $\delta(X) = \mathrm{cl}(X)$ = topological closure of X, $\epsilon(Y) = \mathrm{int}(Y)$ = interior of Y.

Let $\mathcal{L}, \mathcal{M}$ be complete lattices. The identity mapping on $\mathcal{L}$ is denoted by $\mathbf{id}_{\mathcal{L}}$. A mapping $\psi : \mathcal{L} \to \mathcal{M}$ is called *increasing* if $X \leq X'$ implies that $\psi(X) \leq \psi(X')$. We say that ψ is a *dilation* if ψ distributes over suprema, that is for an arbitrary collection $\{X_i \mid i \in I\}$,

$$\psi(\bigvee_i X_i) = \bigvee_i \psi(X_i).$$

Taking I to be empty gives $\psi(\mathbf{0}) = \mathbf{0}$. Dually, ψ is called an *erosion* if it distributes over infima,

$$\psi(\bigwedge_i X_i) = \bigwedge_i \psi(X_i).$$

Taking I empty gives $\psi(\mathbf{1}) = \mathbf{1}$. It is obvious that dilations and erosions are increasing mappings.

Proposition 1 *Let $\mathcal{M}, \mathcal{L}$ be complete lattices.*

(a) *If (ϵ, δ) is an adjunction between $\mathcal{M}$ and $\mathcal{L}$, then δ is a dilation and ϵ an erosion.*

(b) *To every erosion $\epsilon : \mathcal{M} \to \mathcal{L}$ there corresponds a unique dilation $\delta : \mathcal{L} \to \mathcal{M}$ such that (ϵ, δ) is an adjunction.*

(c) *To every dilation $\delta : \mathcal{L} \to \mathcal{M}$ there corresponds a unique erosion $\epsilon : \mathcal{M} \to \mathcal{L}$ such that (ϵ, δ) is an adjunction.*

For proofs of (a)–(c) see [11], Proposition 2.5, Theorem 2.7 (ii) and (i) respectively.

If $\epsilon : \mathcal{M} \to \mathcal{L}$ is an erosion then, trivially, the adjoint dilation δ is given by

$$\delta(X) = \bigwedge\{Y \in \mathcal{M} \mid \delta(X) \leq Y\} = \bigwedge\{Y \in \mathcal{M} \mid X \leq \epsilon(Y)\}. \quad (1)$$

A similar expression holds for ϵ in terms of δ.

A mapping $\psi : \mathcal{L} \to \mathcal{L}$ is called a *closing* if it is increasing, extensive ($\psi \geq \mathbf{id}_{\mathcal{L}}$) and idempotent ($\psi^2 = \psi$). If ψ is increasing, anti-extensive ($\psi \leq \mathbf{id}_{\mathcal{L}}$) and idempotent, then it is called an *opening*.

Proposition 2 *Let (ϵ, δ) be an adjunction between $\mathcal{M}$ and $\mathcal{L}$. Then*

(a) *$\epsilon\delta \geq \mathbf{id}_{\mathcal{L}}$ and $\delta\epsilon \leq \mathbf{id}_{\mathcal{M}}$;*

(b) *$\delta\epsilon\delta = \delta$ and $\epsilon\delta\epsilon = \epsilon$;*

(c) *$\epsilon\delta$ is a closing on $\mathcal{L}$ and $\delta\epsilon$ is an opening on $\mathcal{M}$.*

This is trivial, but see [11], Prop. 2.6 and 2.8.

We say that $X \in \mathcal{L}$ is *closed with respect to* (ϵ, δ) if $\epsilon\delta(X) = X$. An element $Y \in \mathcal{M}$ is said to be *open with respect to* (ϵ, δ) if $\delta\epsilon(Y) = Y$.

Proposition 3 *Let (ϵ, δ) be an adjunction between $\mathcal{M}$ and $\mathcal{L}$.*

(a) *$X \in \mathcal{L}$ is closed if and only if $X = \epsilon(Y)$ for some $Y \in \mathcal{M}$.*

(b) *$Y \in \mathcal{M}$ is open if and only if $Y = \delta(X)$ for some $X \in \mathcal{L}$.*

(c) *Arbitrary infima of closed elements are closed; arbitrary suprema of open elements are open.*

(d) $\epsilon\delta(X)$ *is the smallest closed element* $\geq X$*;* $\delta\epsilon(Y)$ *is the largest open element* $\leq Y$.

Proof : If X is closed then $X = \epsilon\delta(X)$ and hence $X = \epsilon(Y)$ if one puts $Y = \delta(X)$. Conversely, if $X = \epsilon(Y)$, then $\epsilon\delta(X) = \epsilon\delta\epsilon(Y) = \epsilon(Y)$ by Prop 2(b). This proves (a). For (c), let $X_i \in \mathcal{L}$ be closed; then $X_i = \epsilon(Y_i)$ for some Y_i, hence $\bigwedge_i X_i = \bigwedge_i \epsilon(Y_i) = \epsilon(\bigwedge_i Y_i)$. Then (d) follows trivially. □

Proposition 4 *Let* $Y \in \mathcal{M}$ *be open. Then for* $X \in \mathcal{L}$

$$\delta(X) = Y \tag{2}$$

iff

$$\epsilon\delta(X) = \epsilon(Y). \tag{3}$$

One solution of (2) is $X = \epsilon(Y)$. *This is the largest solution and the unique closed solution.*

This follows from the proof of Proposition 3 and the fact that $\epsilon\delta$ is increasing and idempotent (Proposition 2(c)).

If both $\mathcal{L}$ and $\mathcal{M}$ are complete Boolean lattices, and if $\psi : \mathcal{L} \to \mathcal{M}$, then we can define the dual mapping $\psi^* : \mathcal{L} \to \mathcal{M}$ by

$$\psi^*(X) = [\psi(X^*)]^*.$$

Proposition 5 *Let* (ϵ, δ) *be an adjunction between the complete* **Boolean** *lattices* $\mathcal{M}$ *and* $\mathcal{L}$. *Then* (δ^*, ϵ^*) *is an adjunction between* $\mathcal{L}$ *and* $\mathcal{M}$. *An element* $X \in \mathcal{L}$ *is closed with respect to* (ϵ, δ) *iff* X^* *is open with respect to* (δ^*, ϵ^*).

Proof : For $X \in \mathcal{L}, Y \in \mathcal{M}$ we have $\epsilon^*(Y) \leq X$ iff $X^* \leq \epsilon(Y^*)$ iff $\delta(X^*) \leq Y^*$ iff $Y \leq \delta^*(X)$, so that (δ^*, ϵ^*) is an adjunction. Suppose that $X \in \mathcal{L}$ is closed with respect to (ϵ, δ); then

$$\epsilon^*\delta^*(X^*) = \epsilon^*([\delta(X)]^*) = [\epsilon\delta(X)]^* = X^*$$

so that X^* is open with respect to (δ, ϵ). The converse is proved dually. □

2 Incidence and dominance structures

This section defines generalisations of the concepts of incidence and dominance, and shows that they are broadly equivalent to lattice adjunctions.

Definition 2 *Let* $\mathcal{L}, \mathcal{M}$ *be complete lattices. A* dominance relation $\preceq$ *is a relation on* $\mathcal{L} \times \mathcal{M}$ *such that for arbitrarily large collections* $\{X_i : i \in I\} \subseteq \mathcal{L}$, $\{Y_j : j \in J\} \subseteq \mathcal{M}$,

$$\bigvee_i X_i \preceq \bigwedge_j Y_j \iff \forall i, j \ (X_i \preceq Y_j).$$

For example with $\mathcal{L} = \mathcal{M} = \mathcal{P}(S)$ for an arbitrary space S,

$$X \preceq Y \quad \text{iff} \quad X \subseteq Y$$

is a dominance relation. A dominance relation is never void since

$$\mathbf{0}_{\mathcal{L}} \preceq Y \ (Y \in \mathcal{M}) \quad \text{and} \quad X \preceq \mathbf{1}_{\mathcal{M}} \ (X \in \mathcal{L})$$

(taking I or J respectively to be empty) and it has the transitivity property that if $X \preceq Y$, $X' \leq X$ and $Y \leq Y'$ then $X' \preceq Y'$.

Theorem 1 *If (ϵ, δ) is an adjunction between $\mathcal{M}$ and $\mathcal{L}$, then $\preceq$ defined by*

$$X \preceq Y \iff \delta(X) \leq Y \iff X \leq \epsilon(Y) \tag{4}$$

is a dominance relation. Conversely if $\preceq$ is a dominance relation on $\mathcal{L} \times \mathcal{M}$ then there exists a unique adjunction (ϵ, δ) between $\mathcal{M}$ and $\mathcal{L}$ for which (4) holds, namely

$$\delta(X) = \bigwedge\{Y \mid X \preceq Y\}, \tag{5}$$

$$\epsilon(Y) = \bigvee\{X \mid X \preceq Y\}. \tag{6}$$

Proof : Let (ϵ, δ) be an adjunction and define $\preceq$ by (4). Suppose $\bigvee_i X_i \preceq \bigwedge_j Y_j$; then $\bigvee_i \delta(X_i) = \delta(\bigvee_i X_i) \leq \bigwedge_j Y_j$. By definition of $\wedge$ and $\vee$ it follows that $\delta(X_i) \leq Y_j$ for every i, j. Thus $X_i \preceq Y_j$ for all i, j. The converse follows a similar argument.

Let $\preceq$ be a dominance relation and δ, ϵ the maps constructed in (5)–(6). By definition of $\wedge$ if $X \preceq Y$ then $\delta(X) \leq Y$; conversely if $\delta(X) \leq Y$ then $X \preceq Y$ by the transitivity property stated above. Similarly for ϵ. That is, (ϵ, δ) is an adjunction. The adjunction is unique, since any adjunction satisfying (4) clearly must be of the form (5)–(6). □

Definition 3 *An* incidence relation $\sim$ *is a relation on $\mathcal{L} \times \mathcal{M}$ such that for arbitrarily large collections $\{X_i : i \in I\} \subseteq \mathcal{L}$, $\{Y_j : j \in J\} \subseteq \mathcal{M}$,*

$$\bigvee_i X_i \sim \bigvee_j Y_j \iff \exists i, j \ (X_i \sim Y_j).$$

Taking I or J to be empty gives

$$\mathbf{0}_{\mathcal{L}} \not\sim Y \ (Y \in \mathcal{M}) \quad \text{and} \quad X \not\sim \mathbf{0}_{\mathcal{M}} \ (X \in \mathcal{L}).$$

The standard example is $X \sim Y \iff X \cap Y \neq \emptyset$ for subsets X, Y of an arbitrary space. The attraction of incidence relations is that they are intuitively easier to define, and the symmetry of $\mathcal{L}, \mathcal{M}$ in the definition is a simple expression of projective duality, X hits Y iff Y hits X.

Theorem 2 *If $\mathcal{M}$ is Boolean, an incidence relation $\sim$ on $\mathcal{L} \times \mathcal{M}$ is equivalent to a dominance relation $\preceq$ on $\mathcal{L} \times \mathcal{M}$ through*

$$X \preceq Y \iff X \not\sim Y^*.$$

The proof is trivial. In this case the associated dilation and erosion can be expressed as

$$\begin{aligned} \delta(X) &= [\bigvee\{Y \in \mathcal{M} \mid X \not\sim Y\}]^* \\ \epsilon(Y) &= \bigvee\{X \in \mathcal{L} \mid X \not\sim Y^*\}. \end{aligned}$$

Remark: Even if $\mathcal{L}, \mathcal{M}$ are not Boolean, a dominance relation on $\mathcal{L} \times \mathcal{M}$ corresponds to an incidence relation on $\mathcal{L} \times \mathcal{M}'$ where $\mathcal{M}'$ is the dual lattice of $\mathcal{M}$ (i.e. with order reversed).

Suppose we have two Boolean complete lattices $\mathcal{L}, \mathcal{M}$ and an incidence relation $\sim$ on $\mathcal{L} \times \mathcal{M}$. This generates two dual adjunctions,

$$\begin{aligned} \delta(X) &= [\bigvee\{Y \in \mathcal{M} \mid X \not\sim Y\}]^* && (7) \\ \epsilon(Y) &= \bigvee\{X \in \mathcal{L} \mid X \not\sim Y^*\} && (8) \\ \delta^*(X) = \delta(X^*)^* &= \bigvee\{Y \in \mathcal{M} \mid X^* \not\sim Y\} && (9) \\ \epsilon^*(Y) = \epsilon(Y^*)^* &= [\bigvee\{X \in \mathcal{L} \mid X \not\sim Y\}]^* && (10) \end{aligned}$$

which in most cases are not identical. The analogues of (5)–(6) are

$$\begin{aligned} \delta^*(X) &= \bigvee\{Y \mid X^* \preceq Y^*\} && (11) \\ \epsilon^*(Y) &= \bigwedge\{X \mid X^* \preceq Y^*\} && (12) \end{aligned}$$

3 Local knowledge and conditional closing

Throughout this section we assume $\mathcal{L}, \mathcal{M}$ are complete Boolean lattices equipped with an incidence relation $\sim$.

Many calculations turn out to be expressible in the following context.

Definition 4 *Fix $N \in \mathcal{M}$ and let*

$$\mathcal{M}_N = \{M \in \mathcal{M} : M \leq N\}$$

with the inherited order relation. This is a complete Boolean lattice, with complement operation

$$c_N(Y) = Y^* \wedge N.$$

Lemma 1 *If we restrict the incidence relation $\sim$ to $\mathcal{L} \times \mathcal{M}_N$, then the associated adjunction is*

$$\begin{aligned} \delta_N(X) &= \delta(X) \wedge N && (13) \\ \epsilon_N(Y) &= \epsilon(Y \vee N^*). && (14) \end{aligned}$$

Proof : The associated dominance relation is clearly

$$X \preceq_N Y \Leftrightarrow X \not\perp (Y^* \wedge N).$$

By equations (7–8)

$$\begin{aligned}
\delta_N(X) &= c_N\left(\bigvee\{Y \in \mathcal{M}_N \mid X \not\perp Y\}\right) \\
&= c_N\left(\bigvee\{Y \in M \mid X \not\perp Y\} \wedge N\right) \\
&= (\delta(X)^* \wedge N)^* \wedge N \\
&= (\delta(X) \vee N^*) \wedge N \\
&= \delta(X) \wedge N \\
\epsilon_N(Y) &= \bigvee\{X \in \mathcal{L} \mid X \not\perp (Y^* \wedge N)\} \\
&= \bigvee\{X \in \mathcal{L} \mid X \not\perp (Y \vee N^*)^*\} \\
&= \epsilon(Y \vee N^*).
\end{aligned}$$

□

Since (ϵ_N, δ_N) is an adjunction between $\mathcal{M}_N$ and $\mathcal{L}$, the results of section 1 apply. For example the associated closing operator is

$$\begin{aligned}
\epsilon_N\delta_N(X) &= \epsilon\left((\delta(X) \wedge N) \vee N^*\right) \\
&= \epsilon(\delta(X) \vee N^*).
\end{aligned} \tag{15}$$

Notice that $\epsilon_N\delta_N(X)$ is the largest solution W in $\mathcal{L}$ of

$$\delta(W) \wedge N = \delta(X) \wedge N.$$

Since we are restricted to elements of $\mathcal{M}$ which are $\leq N$ we call $\epsilon_N\delta_N$ a **conditional closing**. We now specialise this to the case where N is of the form $N = \delta(Z)$ for some $Z \in \mathcal{L}$.

Proposition 6 *For fixed $X, Z \in \mathcal{L}$ the largest solution W of*

$$\delta(W) \wedge \delta(Z) = \delta(X) \wedge \delta(Z) \tag{16}$$

is

$$W = \epsilon(\delta(X) \vee \delta(Z)^*).$$

Proof : Setting $N = \delta(Z)$ in Lemma 1 we recognise (16) as the equation $\delta_N(W) = \delta_N(X)$. By Proposition 4 the largest solution is $W = \epsilon_N(\delta_N(X))$. But this is $W = \epsilon((\delta(X) \wedge N) \vee N^*) = \epsilon(\delta(X) \vee \delta(Z)^*)$. □

We give an example of the conditional closing in section 6.

Consider an element $Z \in \mathcal{L}$. Think of Z as a window which bounds the objects which we are able to perceive; that is, assume that for any element $X \in \mathcal{L}$ we only have information about the part $X \wedge Z$. From this local knowledge of X it is still possible to compute $\delta(X)$ inside a window $W \in \mathcal{M}$. The next result is dubbed the "*local knowledge principle*" after a result in mathematical morphology [20, pp. 11,49,62].

Proposition 7 *Fix $Z \in \mathcal{L}$. Then the largest element $W \in \mathcal{M}$ satisfying*

$$\delta(X) \wedge W = \delta(X \wedge Z) \wedge W \quad \text{for all} \quad X \in \mathcal{L} \tag{17}$$

is $W = \delta^*(Z)$.

Proof : Set $N = \delta^*(Z)$ in Lemma 1; the identity $\delta_N \epsilon_N \delta_N = \delta_N$ of Proposition 2(b) reads

$$\begin{aligned} \delta(X) \wedge \delta^*(Z) &= \delta(\epsilon[(\delta(X) \wedge \delta^*(Z)) \vee \delta(Z^*)]) \wedge \delta^*(Z) \\ &= \delta(\epsilon(\delta(X) \vee \delta(Z^*))) \wedge \delta^*(Z) \\ &= \delta\epsilon\delta(X \vee Z^*) \wedge \delta^*(Z) \\ &= \delta(X \vee Z^*) \wedge \delta^*(Z). \end{aligned}$$

Writing $X = (X \wedge Z) \vee (X \wedge Z^*)$ and applying distributivity of $\vee$ and δ gives the identity in (17) for fixed X, Z. Since $W = \delta^*(Z)$ does not depend on X the identity is true for all X.

By Proposition 4, $W = \delta^*(Z)$ is the largest element satisfying the identity in (17) for **fixed** X, Z. Again since W does not depend on X it is the largest element satisfying (17). □

Examples of the local knowledge principle will be given in the ensuing sections. There is also a dual identity

$$\epsilon^*(Y) \wedge \epsilon(Z) = \epsilon^*(Y \wedge Z) \wedge \epsilon(Z) \tag{18}$$

in which $\epsilon(Z)$ is the smallest W for which this identity holds.

4 Application to mathematical morphology

Mathematical morphology is a geometrical approach to quantitative image analysis based on set-theoretical operations such as Minkowski sum and difference. For a comprehensive treatment we refer to [15, 20]. Recently it has been shown [11, 16, 17, 18, 21] that mathematical morphology can be extended to arbitrary complete lattices. Adjunctions play a crucial role in this abstract formulation. In this section we show for illustration how the notions of incidence can be use to formalise classical Euclidean morphology.

4.1 Classical Euclidean morphology

Define the translation of a subset A by a vector $x \in \mathbb{R}^n$ to be

$$A_x = \{a + x : a \in A\}.$$

For subsets $X, A \subseteq \mathbb{R}^n$ define the *Minkowski sum*

$$X \oplus A = \bigcup_{a \in A} X_a = \{x + a : x \in X, a \in A\}$$

and *Minkowski difference*

$$X \ominus Y = \bigcap_{a \in A} X_{-a}$$

We can also write

$$\begin{aligned} X \oplus A &= \{z \in \mathbb{R}^n : \check{A}_z \cap X \neq \emptyset\} \\ X \ominus A &= \{z \in \mathbb{R}^n : A_z \subseteq X\} \end{aligned}$$

where $\check{A}$ is the reflection of A through 0,

$$\check{A} = \{-a : a \in A\}.$$

The *closing* and *opening* of X by A are defined as

$$\begin{aligned} X \bullet A &= (X \oplus A) \ominus A \\ X \circ A &= (X \ominus A) \oplus A \end{aligned}$$

respectively.

The closing $X \mapsto X \bullet A$ is

(i) increasing, i.e. $X \subseteq Y \Rightarrow X \bullet A \subseteq Y \bullet A$;

(ii) idempotent, i.e. $(X \bullet A) \bullet A = X \bullet A$;

(iii) extensive, i.e. $X \subseteq X \bullet A$.

See [20, pp. 52ff]. The opening operator also satisfies statements (i)–(ii) while (iii) is replaced by anti-extensivity, $X \circ A \subseteq X$.

4.2 Lattice reformulation

Let $\mathcal{L} = \mathcal{M} = \mathcal{P}(\mathbb{R}^n)$. Fix a subset $A \subseteq \mathbb{R}^n$ and define for $X, Y \subseteq \mathbb{R}^n$

$$X \sim Y \iff A_x \cap Y \neq \emptyset \text{ for some } x \in X;$$

this is clearly an incidence relation; note that it is not symmetric in X, Y but

$$X \sim Y \iff X \cap \check{A}_y \neq \emptyset \text{ for some } y \in Y.$$

The associated dominance relation is

$$X \preceq Y \iff A_x \subseteq Y \text{ for all } x \in X.$$

We have

$$\begin{aligned} \delta(X) &= \bigcap \{Y \mid X \preceq Y\} \\ &= \bigcap \{Y \mid \forall x \in X : A_x \subseteq Y\} \\ &= \bigcup_{x \in X} A_x \\ &= X \oplus A \end{aligned}$$

and

$$\begin{aligned}\epsilon(Y) &= \bigvee\{X : X \preceq Y\}\\ &= \{x : A_x \subseteq Y\}\\ &= Y \ominus A\end{aligned}$$

It follows that $\epsilon\delta(X) = X \bullet A$ and $\delta\epsilon(X) = X \circ A$.

In mathematical morphology [20, p. 53] a set $X \subseteq \mathbb{R}^n$ is called A-closed if $X \bullet A = X$. Note that this definition coincides with that given in Section 1. In particular we find that a set X is A-closed iff X is of the form $X = Y \oplus A$ for some $Y \subseteq \mathbb{R}^n$.

The dual adjunction is

$$\begin{aligned}\delta^*(X) &= \bigcup\{Y : X^c \not\preceq Y\}\\ &= Y \ominus \check{A}\\ \epsilon^*(Y) &= \left(\bigcup\{X : X \not\preceq Y\}\right)^c\\ &= Y \oplus \check{A}.\end{aligned}$$

The algebraic properties of the closing operator listed in (i)–(iii) of the previous subsection follow immediately from Proposition 2.

In this case the local knowledge principle stated in Proposition 7 reads as

$$(X \oplus A) \cap (Z \ominus \check{A}) = ((X \cap Z) \oplus A) \cap (Z \ominus \check{A})$$

for any $X, Z, A \subseteq \mathbb{R}^n$. This means that if a set X is only observed within a window Z, the dilation $X \oplus A$ can only be computed within the reduced window $Z \ominus \check{A}$. We refer to [20, pp. 11,49,62] for similar statements.

The dual identity (18) is a similar statement with A and $\check{A}$ exchanged.

5 Random set theory

5.1 Strong incidence functions

Kendall [12] introduced the following concepts. Let S be an arbitrary nonempty set, and $\mathcal{T} \subset \mathcal{P}(S)$ an arbitrary class of nonempty subsets of S (called 'traps') that cover S. The *incidence function* of X over $\mathcal{T}$ is

$$I_X(T) = \begin{cases} 1 & \text{if } X \cap T \neq \emptyset \\ 0 & \text{else} \end{cases}$$

The goal was to construct random sets as random 0,1-valued functions on $\mathcal{T}$; hence one needs to determine when an arbitrary function $f : \mathcal{T} \to \{0, 1\}$ is the incidence function of some subset X, and in that case, to find all solutions X. That is, to solve for X in

$$f = I_X.$$

Suppose X is a solution. Then $f(T) = 0$ implies $X \cap T = \emptyset$, or equivalently, $X \subseteq T^c$. This yields

$$\begin{aligned} X &\subseteq \bigcap \{T^c \mid T \in \mathcal{T} \text{ and } f(T) = 0\} \\ &= \left(\bigcap \{T \mid T \in \mathcal{T} \text{ and } f(T) = 0\}\right)^c. \end{aligned}$$

The following is a paraphrase of results in [12].

Definition 5 *The* $\mathcal{T}$**-support** *of an arbitrary function* $f : \mathcal{T} \to \{0, 1\}$ *is the set*

$$\mathbf{spt}\,(f, \mathcal{T}) = (\bigcup \{T \in \mathcal{T} : f(T) = 0\})^c.$$

The $\mathcal{T}$**-closure** *of an arbitrary set* $X \subseteq S$ *is*

$$\mathbf{clos}\,(X, \mathcal{T}) = \mathbf{spt}\,(I_X, \mathcal{T}) = (\bigcup \{T \in \mathcal{T} : T \cap X = \emptyset\})^c.$$

We say that X *is* $\mathcal{T}$**-closed** *if* $X = \mathbf{clos}\,(X, \mathcal{T})$.

Definition 6 *A function* $f : \mathcal{T} \to \{0, 1\}$ *is a strong incidence function (s.i.f.) if*

$$T \subseteq \bigcup_i T_i \ \text{ implies } \ f(T) \leq \max_i f(T_i)$$

for arbitrarily large collections $\{T_i : i \in I\}$.

Obviously every incidence function I_X is a strong incidence function. Conversely,

Theorem 3 (Kendall) *A function* $f : \mathcal{T} \to \{0, 1\}$ *can be written in the form* $f = I_X$ *iff it is a strong incidence function. In that case, one solution is* $X = \mathbf{spt}\,(f, \mathcal{T})$. *This is the largest solution, and the unique* $\mathcal{T}$*-closed solution. The other solutions* Y *are precisely those sets for which* $\mathbf{clos}\,(Y, \mathcal{T}) = X$.

The theorem is established in [12] by deducing a number of properties of the $\mathcal{T}$-support and the $\mathcal{T}$-closure.

5.2 Lattice reformulation

Let $S, \mathcal{T}$ be as in the previous subsection. We will show how the constructions described in Sections 1–2 'automatically' produce the operators of strong incidence function theory.

Let $\mathcal{L} = \mathcal{P}(S)$ with the partial order of set inclusion, and let $\mathcal{M}$ be the complete Boolean lattice of all functions $f : \mathcal{T} \to \{0, 1\}$ with pointwise order $f \leq f' \iff f(T) \leq f'(T)$ for all T. Of course $\mathcal{M}$ could be identified with $\mathcal{P}(\mathcal{T})$ via the correspondence $f \leftrightarrow \{T \in \mathcal{T} : f(T) = 1\}$.

Define an incidence relation by

$$X \sim f \iff \exists T \in \mathcal{T}\ (X \cap T \neq \emptyset \text{ and } f(T) = 1).$$

The associated dominance relation is

$$\begin{aligned} X \preceq f &\iff \forall T \in \mathcal{T}\ (f(T) = 0 \Rightarrow X \cap T = \emptyset) \\ &\iff \forall T \in \mathcal{T}\ (X \cap T \neq \emptyset \Rightarrow f(T) = 1). \end{aligned}$$

Applying Theorem 1, the associated adjunction has

$$\delta(X) \ = \ \bigwedge \{f : X \preceq f\},$$

i.e. the function on $\mathcal{T}$ which has the value 1 iff $X \cap T \neq \emptyset$. In other words, $\delta(X) = I_X$ is Kendall's incidence function of X. Further

$$\begin{aligned} \epsilon(f) &= \bigcup \{X : X \cap T \neq \emptyset \Rightarrow f(T) = 1\} \\ &= \bigcap \{T^c \mid f(T) = 0\} \\ &= \mathbf{spt}\,(f, \mathcal{T}). \end{aligned}$$

The associated closing is

$$\epsilon\delta(X) = \mathbf{spt}\,(I_X, \mathcal{T}) = \mathbf{clos}\,(X, \mathcal{T}),$$

i.e. $\epsilon\delta$ is Kendall's $\mathcal{T}$-closure operator, see Definition 5. On the other hand

$$\delta\epsilon(f) = I_{\mathbf{spt}(f,\mathcal{T})}$$

is the incidence function of the support of f; this is the largest strong incidence function below f. Hence $\delta\epsilon$ coincides with the operator $\mathbf{S}$ of Kendall [12, thm 7, p. 334].

A set X is closed under the adjunction, $\epsilon\delta(X) = X$, iff it is a $\mathcal{T}$-closed set in Kendall's sense (Definition 5). By Proposition 3(a) this is equivalent to $X = \epsilon(f) = \mathbf{spt}\,(f, \mathcal{T})$ for some f.

A function f is open under the adjunction, $\delta\epsilon(f) = f$, if and only if

$$f(T) = 1 \iff T \cap \mathbf{spt}\,(f, \mathcal{T}) \neq \emptyset$$

i.e.

$$f(T) = 0 \iff T \subseteq \bigcup_{f(T')=0} T'$$

i.e. f is a strong incidence function in the sense of Kendall. Theorem 3 then follows from Propositions 3(b) and 4.

In the dual adjunction, $\delta^*(X)$ is a 'containment function'

$$[\delta^*(X)](T) = \begin{cases} 1 & \text{if } T \subseteq X \\ 0 & \text{else} \end{cases}$$

while ϵ^* could be called the 'trace' operator

$$\epsilon^*(f) = \bigcup_{f(T)=1} T.$$

The dual opening is thus the "$\mathcal{T}$-interior"

$$\epsilon^*\delta^*(X) = \bigcup\{T \mid T \subseteq X\}.$$

The local knowledge principle (Proposition 7) states that for given $Z \subseteq S$ the largest set of traps $W \subseteq \mathcal{T}$ satisfying

$$I_X \equiv I_{X\cap Z} \quad \text{on } W$$

for all $X \subseteq S$, is $W = \delta^*(Z)$. In other words, if X is an unknown set but $X \cap Z$ is known, then the incidence function of X is known over the class of traps T satisfying $T \subseteq Z$ (and not over any larger class, in general).

The dual principle (18) states that for $Z \subset \mathcal{T}$ given, $\mathbf{spt}\,(Z, \mathcal{T})$ is the largest subset of S on which the trace of Y coincides with the trace of $Y \cap Z$ for all $Y \subset \mathcal{T}$. In other words if $f : \mathcal{T} \to \{0,1\}$ is an unknown function whose values are known only on $Z \subset \mathcal{T}$, then the trace of f can be reconstructed within $\mathbf{spt}\,(Z, \mathcal{T})$.

6 Buffon-Sylvester problem

6.1 Description

Let Λ be the set of all infinite straight lines in $\mathbb{R}^2$. For $A \subset \mathbb{R}^2$ define

$$[A] = \{\ell \in \Lambda : \ell \cap A \neq \emptyset\}.$$

and let $1_{[A]} : \Lambda \to \{0,1\}$ be its indicator function

$$1_{[A]}(\ell) = \begin{cases} 1 & \text{if } \ell \in [A] \\ 0 & \text{else} \end{cases}$$

The *Buffon-Sylvester problem* [1, 2], [13, pp. 60–61], [19, pp. 27–34], [22, pp. 65–70] concerns the probabilities of events $[A]$ and finite combinations $[A] \cap [B], [A] \cup [B]$, etc under a probability distribution P on Λ. The measure $P([A])$ is relatively straightforward to compute for convex compact A (see [2]). Sylvester [24] introduced the following arguments:

(1) if $X \subset \mathbb{R}^2$ is compact and path-connected, then $[X] = [\mathbf{co}\, X]$ where $\mathbf{co}\, X$ is the convex hull of X.

(2) If $A, B \subset \mathbb{R}^2$ are compact convex sets with $A \cap B \neq \emptyset$, then

$$1_{[A]\cap[B]} = 1_{[A]} + 1_{[B]} - 1_{[\mathbf{co}(A\cup B)]}$$

so that $P([A] \cap [B]) = P([A]) + P([B]) - P(\mathbf{co}(A \cup B))$ for any P; and

(3) If $A, B \subset \mathbb{R}^2$ are disjoint compact convex sets, then the sets A', B' indicated in Figure 1 have the property that $[A'] \cap [B'] = [A] \cap [B]$.

The construction in (3) allows us to apply argument (2) to A', B'. For further information see [2, 13, 22].

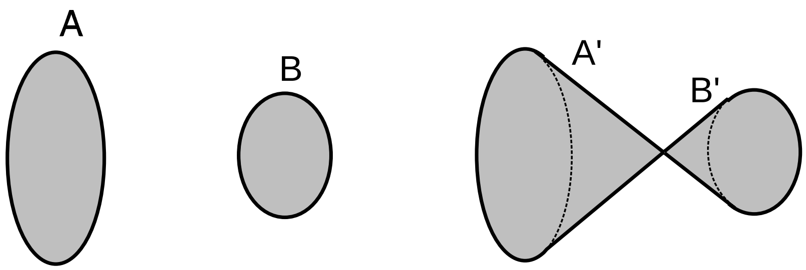

Figure 1: Two disjoint convex sets $A, B \subset \mathbb{R}^2$ (left) and the Sylvester construction (right).

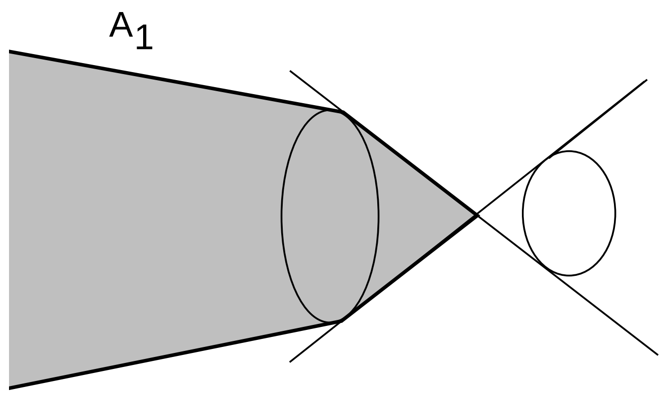

Figure 2: The conditional closing.

6.2 Lattice reformulation

Taking the lattices $\mathcal{L} = \mathcal{P}(\mathbb{R}^2)$ and $\mathcal{M} = \mathcal{P}(\Lambda)$ define

$$X \sim Y \iff \exists \ell \in Y : \ell \cap X \neq \emptyset.$$

Then analogously to §5.2 we find that

$$\begin{aligned} \delta(X) &= [X] \\ \epsilon(Y) &= \mathbf{spt}\,(Y; \Lambda) \\ &= \left(\bigcup\{\ell \mid \ell \notin Y\}\right)^c \\ \epsilon\delta(X) &= \left(\bigcup\{\ell \in \Lambda \mid \ell \cap X = \emptyset\}\right)^c \end{aligned}$$

Now argument (1) of the previous section follows from Propositions 2–3 and the fact that $\epsilon\delta(X) = \mathbf{co}\, X$ for path connected X. Argument (2) follows directly.

Consider argument (3). If we apply Proposition 6 with $X = A$ and $Z = B$ we find that the largest set A_1 satisfying

$$[A_1] \cap [B] = [A] \cap [B]$$

is

$$A_1 = \epsilon(\delta(A) \vee \delta(B)^*)$$

$$= \left(\bigcup\{\ell \in \Lambda \mid \ell \in [B], \quad \ell \notin [A]\}\right)^c$$

See Figure 2. A_1 can also be expressed as

$$\begin{aligned} A_1 &= \mathbf{spt}\,(I_A \vee (1 - I_B); \Lambda) \\ &= \mathbf{spt}(I_A; [B]) \\ &= \mathbf{clos}(A; [B]), \end{aligned}$$

for example with the first expression indicating that A_1 is the support of the logical implication "$[B]$ implies $[A]$".

A similar application of conditional closing to B yields a set B_1 such that

$$[A_1] \cap [B_1] = [A] \cap [B]$$

and A_1, B_1 are the largest such sets.

The usual sets A', B' of Figure 1 can be obtained with the same construction, by restricting $\mathcal{L}$ to subsets of $\mathbf{co}(A \cup B)$.

7 Practical applications

We now briefly indicate several other practical applications of the lattice formalism of sections 1 – 3.

7.1 Projection effects in microscopy

Let $\mathcal{L} = \mathcal{P}(\mathbb{R}^3)$, $\mathcal{M} = \mathcal{P}(\mathbb{R}^2)$, and let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be one of the standard coordinate projections. Fix a subset $C \subseteq \mathbb{R}^3$ and define

$$X \sim Y \iff \pi(X \cap C) \cap Y \neq \emptyset,$$

equivalently

$$X \preceq Y \iff \pi(X \cap C) \subseteq Y.$$

This is a model for the formation of images in simple optical transmission microscopes where sets in $\mathbb{R}^3$ are physical objects, C represents the microscope slide or physical sample of material, and π represents the projection of light onto the image plane $\mathbb{R}^2$.

Then we have the mappings of Figure 3:

$$\delta(X) = \bigcap\{Y : \pi(X \cap C) \subseteq Y\} = \pi(X \cap C)$$

and

$$\delta^*(X) = \bigcup\{Y : \pi(X^c \cap C) \subseteq Y^c\} = \pi(X^c \cap C)^c$$

which operations are known as *overprojection* and *underprojection* in microscopy [4, 5, 7, 6], [9, sec. 4, Fig 4.1]. Overprojection occurs when the object X is opaque and the surrounding medium X^c is transparent, so that $\delta(X)$ is the dark projected image of X against a light

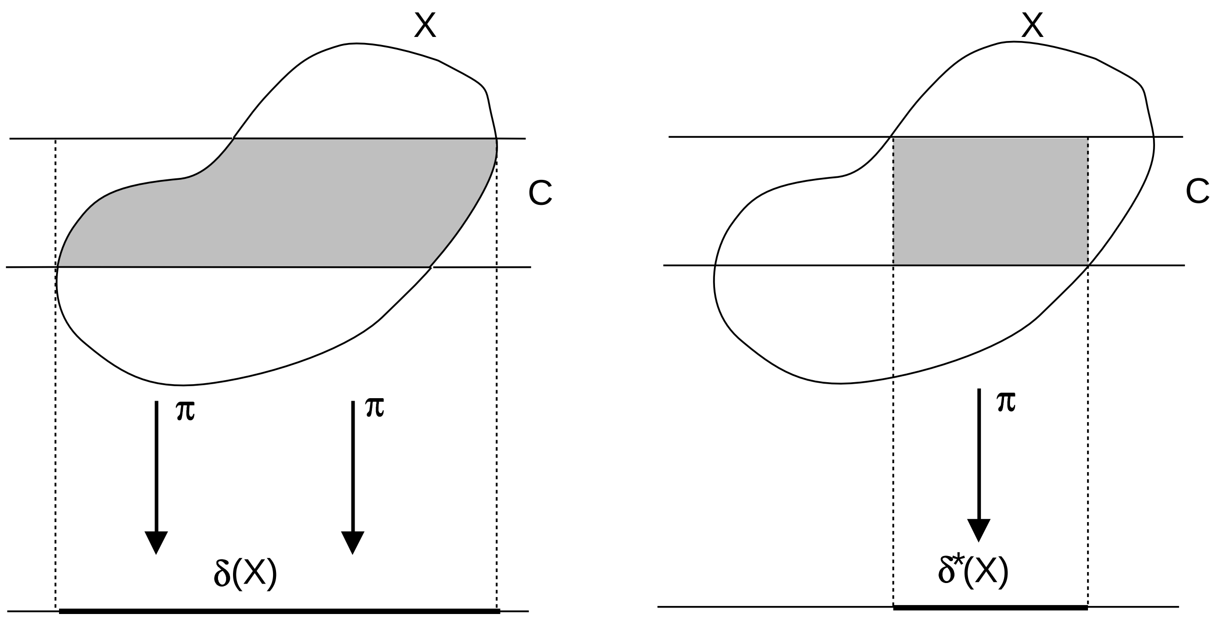

Figure 3: Overprojection and underprojection

background. Underprojection occurs when X is transparent and X^c is opaque, so that $\delta^*(X)$ is the light image of X against a dark background.

The operators ϵ and ϵ^* are two versions of the inverse projection. $\epsilon^*(Y)$ is the largest subset of C that will produce a given projected image $Y \subseteq \mathbb{R}^2$. One also has the equivalence

$$\pi(X \cap C) \cap Y \neq \emptyset \quad \text{iff} \quad X \cap \epsilon(Y) \neq \emptyset$$

which establishes a relationship between test sets Y in the two dimensional projection and test sets $\epsilon(Y)$ in three dimensions.

Proposition 7 states that if X is known within a region Z then the overprojection of X is known within the underprojection of Z.

7.2 Image discretization

A theory of image discretrization must contain the following two steps. First one has to describe a sampling procedure which replaces an image in continuous space by a discrete one. Since we are restricting consideration to subsets here, this amounts to an operator mapping $\mathcal{P}(\mathbb{R}^n)$ into $\mathcal{P}(\mathbb{Z}^n)$. Secondly, in order to compare the discretised image with the original one, we must represent any set $V \subseteq \mathbb{Z}^n$ as a subset of $\mathbb{R}^n$.

Let $\mathcal{L}$ be the complete lattice of all closed subsets of $\mathbb{R}^n$ and let $\mathcal{M}$ be the complete Boolean lattice $\mathcal{P}(\mathbb{Z}^n)$. Let $C \subset \mathbb{R}^n$ be an open neighbourhood of 0 so large that copies of C placed at integer positions $z \in \mathbb{Z}^n$ cover $\mathbb{R}^n$,

$$\bigcup\{C_z \mid z \in \mathbb{Z}^n\} = \mathbb{R}^n.$$

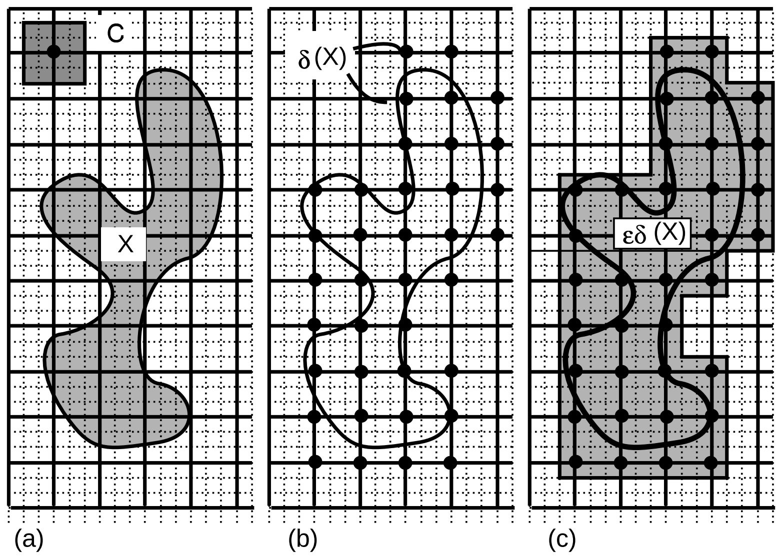

Figure 4: (a) Original set (b) sampled set $\delta(X)$ (c) reconstructed set $\epsilon\delta(X)$.

Define an incidence relation $\sim$ on $\mathcal{L} \times \mathcal{M}$ by

$$X \sim V \text{ iff } (X \oplus C) \cap V \neq \emptyset.$$

The associated adjunction (ϵ, δ) between $\mathcal{M}$ and $\mathcal{L}$ is given by

$$\begin{aligned} \delta(X) &= (X \oplus C) \cap \mathbb{Z}^n, \quad X \subseteq \mathbb{R}^n \text{ closed}, \\ \epsilon(V) &= \{x \in \mathbb{R}^n \mid C^*(x) \subseteq V\}, \quad V \subseteq \mathbb{Z}^n. \end{aligned}$$

Here $C^*(x) = \{z \in \mathbb{Z}^n \mid x \in C_z\}$. In this example δ has the interpretation of a sampling operator, and ϵ that of a representation operator. The closing $\epsilon\delta$ can be interpreted as a reconstruction or outer approximation operator. See Figure 4 for an illustration. The dual operators perform similar functions on the complement of X, so that $\delta^*\epsilon^*(X)$ is an inner approximation of X. For more details we refer to [10].

7.3 Robot motion planning

Following [14] we describe an abstract robot as a mapping $\mathcal{A} : \mathcal{C} \to \mathcal{P}(\mathbb{R}^n)$ where $\mathcal{C}$ is an arbitrary space representing all possible internal states of the robot, and $\mathcal{A}(c) \subset \mathbb{R}^n$ is the physical position (and shape) of the robot when it is in state $c \in \mathcal{C}$.

Let $\mathcal{L} = \mathcal{P}(\mathbb{R}^n)$ and $\mathcal{M} = \mathcal{P}(\mathcal{C})$. Members of $\mathcal{L}$ will be called *obstacles.* Say that the robot in state c *avoids* obstacle X if

$$\mathcal{A}(c) \cap X = \emptyset$$

and otherwise c *hits* X. Define an incidence relation between $\mathcal{L}$ and $\mathcal{M}$ by

$$\begin{aligned} X \sim Y &\iff \exists_{c \in Y} : c \text{ hits } X \\ X \preceq Y &\iff \forall_{c \notin Y} : c \text{ avoids } X. \end{aligned}$$

Then we have

$$\begin{aligned}
\delta(X) &= \{c \in \mathcal{C} : c \text{ hits } X\} \\
\epsilon(Y) &= \left(\bigcup_{c \notin Y} \mathcal{A}(c) \right)^c \\
\delta^*(X) &= \{c \in \mathcal{C} : \mathcal{A}(c) \subseteq X\} \\
\epsilon^*(Y) &= \bigcup_{c \in Y} \mathcal{A}(c)
\end{aligned}$$

The operators ϵ and ϵ^* deserve to be called *support* and *trace* respectively.

Latombe [14, pp. 10, 88] calls $\delta(X)$ the "C-obstacle" generated by X and

$$\mathcal{C}_{\text{free}} = \bigcap_{i=1}^{m} (\mathcal{C} \setminus \delta(X_i)) = \mathcal{C} \setminus \delta(\bigcup_{i=1}^{m} X_i)$$

the "free space" of paths avoiding obstacles $X_1, \ldots, X_m$. The robot motion planning problem can then be defined as the task of finding paths v joining specified states $q_0, q_1 \in C_{\text{free}}$ and satisfying $v(t) \in C_{\text{free}}$.

Latombe [14, p. 89 ff.] proves topological and algebraic properties of δ in the case of a 'rigid robot' where $\mathcal{C} = \mathbb{R}^n$ and $\mathcal{A}(c) = A_c$ is the translation of a fixed set A. Note that in this case we get $\delta(X) = X \oplus \check{A}$, $\epsilon(Y) = Y \ominus \check{A}$, $\delta^*(X) = Y \ominus A$ and $\epsilon^*(Y) = Y \oplus A$, so that such results can be obtained from existing results in mathematical morphology.

The local knowledge principle Proposition 7 states that (in the general case) for $X, Z \in \mathcal{L}$

$$\delta(X) \cap \delta^*(Z) = \delta(X \cap Z) \cap \delta^*(Z)$$

i.e. that the robot avoids an obstacle X while remaining inside a space Z iff it avoids $X \cap Z$ while remaining inside Z. The conditional closing operator can also be used to restrict attention to a subclass of permissible states of the robot.

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