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Closed G^1-continuous Cubic Bézier Surfaces

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Abstract

This report first presents an analysis of the sufficient and necessary polynomial degree for several scattered data interpolation problems. The attention is then focussed on the construction of a piecewise triangular cubic Bézier surface that interpolates given triangle vertices with prescribed normal vectors, and is G^1-continuous everywhere. In order to get enough degrees of freedom to define the Bézier control points, a triangle three-split, a two-split and a six-split scheme are developed. The split into six sub-triangles results in a surface that is G^1-continuous as well as visually pleasing.

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1 Introduction

We can distinguish several scattered data interpolation problems, depending on the input data and the continuity requirements. In each case our purpose is to construct a smooth closed surface in 3D through the vertices. The input consists at least of a triangulation of a set of vertices. This is equivalent to a closed polyhedron of triangular facets, for example obtained by a reconstruction algorithm [Veltkamp, 91]. Additional data at the vertices can be the (unit) surface normal, tangent vectors (direction of derivative) of the patch edges, or derivatives vectors (tangent and magnitude) of the edges. Data along the edges are for example surface normals, cross edge derivatives and curvatures. Usual continuity requirements are G^1- or G^2-continuity everywhere at the surface. I will refer to the interpolation of vertex positions as the P-interpolation problem, position and surface normal at the vertices as the PN-interpolation problem, position and edge tangent vector as the PT-, and position and edge derivative vector as the PD-interpolation problem.

Section 2 introduces Bézier surfaces and G^1-continuity. Section 3 presents an analysis of the required polynomial degree for the various interpolation problems in 3D, which is not found in the literature before. Section 4 gives an overview of surface normal estimation methods, and Section 5 gives an overview of existing local methods for the PN-interpolation problem. Section 6 introduces a new solution that is cubic and based on a splitting a triangle into three sub-triangles, and Section 7 describes how the triangle splitting can be made adaptive, that is, dependent of the geometry of the triangulation. Section 8 presents a scheme that split a triangle into six sub-triangles, and Section 9 gives some concluding remarks.

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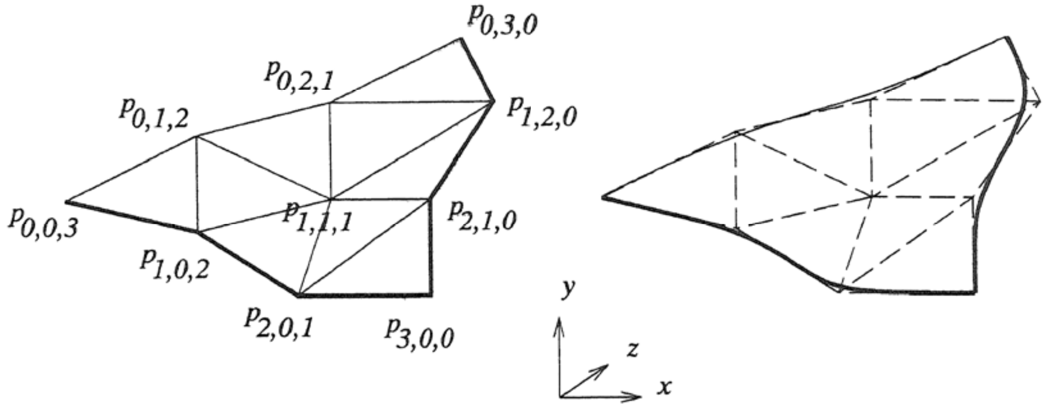

Figure 1: Left: Bézier control polyhedron. Right: corresponding cubic Bézier patch.

2 Preliminaries

This section gives some preliminary information that is needed in the rest of this report. The following subsections introduce Bézier surfaces, G^1-continuity, and the G^1-continuity conditions for two adjacent Bézier triangles.

2.1 Bezier patches

Any point D in the plane can be uniquely expressed in so-called barycentric coordinates (t, u, w) with $t + u + w = 1$, relative to three ordered points A, B, and C that are not collinear: $D = tA+uB+wC$. Note that this is equivalent to $D = A+u(B-A)+w(C-A)$. Barycentric coordinates are treated in more detail by [Farin, 86] and [Farin, 90].

Bernstein polynomials of degree n over a non-degenerate triangle (A, B, C) are defined by

$$B^n_{i,j,k}(t, u, w) = \frac{n!}{i!j!k!} t^i u^j w^k, \; i + j + k = n, \; i, j, k \geq 0,$$

where t, u, and w, with $t + u + w = 1$, are barycentric coordinates with respect to (A, B, C). The Bernstein polynomials form a basis for all polynomials of total degree n over that triangle. That is, every polynomial function $f : (A, B, C) \to \mathbb{R}$ of degree n can be written in the form

$$f(t, u, w) = \sum_{i+j+k=n} p_{i,j,k} B^n_{i,j,k}(t, u, w).$$

Here and in the following it is assumed that $i, j, k \geq 0$ when $i + j + k = n$. Function f describes a surface over the domain triangle (A, B, C). A surface patch in this form is called a Bézier patch, and the scalars $p_{i,j,k}$ are called Bézier ordinates.

Such a patch is functional and cannot have arbitrary shape in 3D. In particular, it cannot be used for scattered data interpolation or form a closed surface. A parametric Bézier patch in arbitrary dimension is defined component-wise:

$$P(t, u, w) = \sum_{i+j+k=n} p_{i,j,k} B^n_{i,j,k}(t, u, w),$$

where $p_{i,j,k}$ are points in the embedding space, and are called control points, forming an open control polyhedron. See Figure 1 for a control polyhedron and the corresponding *cubic* Bézier patch. Note that a parametric patch is defined without explicit reference to a domain triangle. An extensive presentation of triangular Bézier patches is given by [Farin, 86].

2.2 G^1-continuity

Parametric continuity for surfaces is based on the equality of derivatives: surfaces $P(s,t)$ and $Q(u,w)$ are C^n-continuous at (s_0,t_0) and (u_0,w_0), if and only if $P^{(i,j)}(s_0,t_0) = Q^{(i,j)}(u_0,w_0)$, $i+j=0,\ldots,n$. The surfaces are C^n along a common curve if they are C^n at each point on that curve.

Note that two surfaces need not have the same first order partial derivatives in order to have the same tangent plane. The derivatives depend on the parameterization while the tangent plane does not. Moreover, on closed surfaces singularities occur where the derivative of the surface is not defined [Veltkamp, 92]. Geometric continuity is based on the surface tangent plane and curvatures. The tangent plane at $P(s_0,t_0)$ is spanned by the derivative vectors $P^{(1,0)}(s_0,t_0)$ and $P^{(0,1)}(s_0,t_0)$. The tangent plane is normal to the surface normal vector

$$N(s_0,t_0) = \frac{P^{(1,0)}(s_0,t_0) \times P^{(0,1)}(s_0,t_0)}{\|P^{(1,0)}(s_0,t_0) \times P^{(0,1)}(s_0,t_0)\|}, \tag{1}$$

where '$\times$' denotes the vector or cross product. The tangent planes at $P(s_0,t_0)$ and $Q(u_0,w_0)$ coincide if and only if $P^{(1,0)}(s_0,t_0)$, $P^{(0,1)}(s_0,t_0)$, $Q^{(1,0)}(u_0,w_0)$, and $Q^{(0,1)}(u_0,w_0)$ are coplanar.

However, a common tangent plane is not sufficient for first order geometric, or G^1-continuity, since the surfaces must have the same orientation, i.e. the same unit normal vector. Otherwise they join with a sharp ridge.

DEFINITION 1 (G^1-CONTINUITY) *Two surfaces are G^1-continuous at a point if and only if their unit normal vectors coincide at that point.*

Second order geometric continuity is based on curvature. For any direction d in the tangent plane at $P(s_0,t_0)$, the plane through d and $N(s_0,t_0)$ intersects $P(s,t)$ in a curve. The normal curvature of this curve is the normal curvature of the surface in the direction of d: $\kappa_d(s_0,t_0)$. Unless $\kappa_d(s_0,t_0)$ is the same in all directions, there are two directions d_1 and d_2 in which $\kappa_d(s_0,t_0)$ takes the maximum and minimum values: the principal curvatures $\kappa_1(s_0,t_0)$ and $\kappa_2(s_0,t_0)$ respectively.

DEFINITION 2 (G^2-CONTINUITY) *Two surfaces are G^2-continuous if and only if their principal curvatures coincide.*

In the rest of this report we are concerned with G^1-continuity between triangular Bézier patches.

The first order partial derivatives are a special case of a *directional* derivative. The directional derivative of an arbitrary surface $P(s_0,t_0)$ in the direction $d=(d_s,d_t)$ in the parameter space, is

$$\nabla_d P(s_0,t_0) = \lim_{h\to 0} \frac{P(s_0+hd_s, t_0+hd_t) - P(s_0,t_0)}{h}.$$

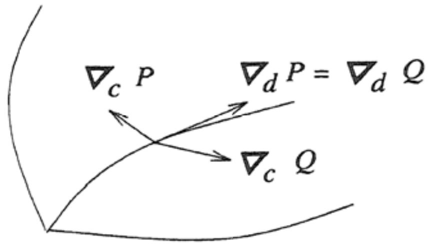

Figure 2: Tangent plane continuity is achieved if $\nabla_c P$, $\nabla_d P = \nabla_d Q$, and $\nabla_c Q$ are coplanar.

The partial derivatives are obtained in the direction of axes of the parameter space: $\nabla_{(1,0)}P(s_0,t_0) = P^{(1,0)}(s_0,t_0)$, and $\nabla_{(0,1)}P(s_0,t_0) = P^{(0,1)}(s_0,t_0)$. The derivative of a Bézier patch in the direction $d = (d_1, d_2, d_3)$ is given by

(2) $$\nabla_d P(t,u,w) = n \sum_{i+j+k=n-1} (d_1 p_{i+1,j,k} + d_2 p_{i,j+1,k} + d_3 p_{i,j,k+1}) B^{n-1}_{i,jk}(t,u,w).$$

We are interested in the continuity of spline surfaces $P(s,t)$ and $Q(u,w)$, in particular along a common curve or edge. Let us consider two patches P with control points $p_{i,j,k}$ and Q with control points $q_{i,j,k}$, having a common edge. Without loss of generality we may assume that $P(t,u,0) = Q(t,u,0)$. The tangent plane of P along $P(t,u,0)$ is spanned by any two derivative vectors having different directions, for example the directions $d = (0,1,0)-(1,0,0) = (-1,1,0)$ and $c = (0,0,1)-(1,0,0) = (-1,0,1)$. The derivative vector $\nabla_d P$ is actually the tangent vector of $P(t,u,0)$. Since $P(t,u,0) = Q(t,u,0)$, we also have $\nabla_d P = \nabla_d Q$. So, the tangent plane of Q spanned by $\nabla_d Q$ and $\nabla_c Q$ coincides with the tangent plane of P if and only if $\nabla_c Q$, $\nabla_d Q = \nabla_d P$, and $\nabla_c P$ lie in the same plane. That is,

(3) $$determinant[\nabla_d P, \nabla_c P, \nabla_c Q] = 0.$$

Such a situation is depicted Figure 2. Necessary and sufficient conditions on the control points are derived from this constraint by [DeRose, 90].

Using Equation (2) we get $\nabla_d P = \nabla_d Q = n(K-M)$, $\nabla_c P = n(L-M)$, and $\nabla_c Q = n(R-M)$, where

(4) $$\begin{aligned} M &= \sum_{i+j=n-1} p_{i+1,j,0} B^{n-1}_{i,j,0}(t,u,0), \\ K &= \sum_{i+j=n-1} p_{i,j+1,0} B^{n-1}_{i,j,0}(t,u,0), \\ L &= \sum_{i+j=n-1} p_{i,j,1} B^{n-1}_{i,j,0}(t,u,0), \ and \\ R &= \sum_{i+j=n-1} q_{i,j,1} B^{n-1}_{i,j,0}(t,u,0). \end{aligned}$$

M, K, L, and R are functions of t, since $u = 1-t$. We see that $\nabla_d P$, $\nabla_c P$, and $\nabla_c Q$ are all of degree $(n-1)$.

The requirement that the determinant in Equation (3) equals zero amounts to

(5) $$(R-M) = \alpha(t)(K-M) + \beta(t)(L-M).$$

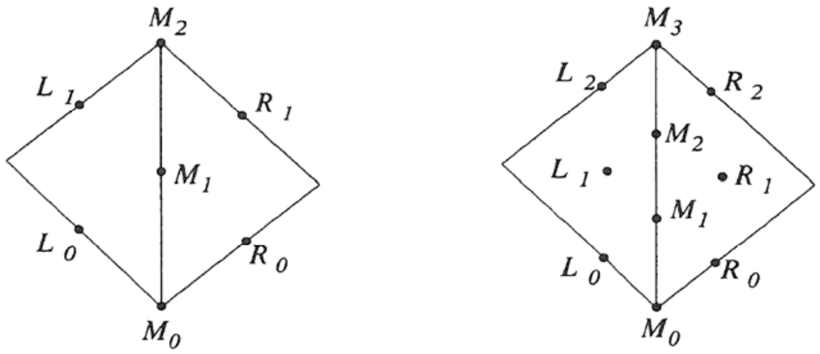

Figure 3: Involved control points for G^1 connection of two quadratic (left) and cubic (right) patches.

Solving $\alpha(t)$ and $\beta(t)$ shows that they are rational polynomial functions having a numerator and denominator of degree $n-1$ at most. An equivalent formulation for Equation (5) is thus:

$$E(t,u)(K-M) + F(t,u)(L-M) + G(t,u)(R-M) = 0, \ t+u=1, \tag{6}$$

where E, F, and G are polynomials having at most degree $n-1$.

The edge that P and Q have in common may be degenerate, that is, of lower degree than the patch itself. Also the two patches may be of different degree. So, the degrees of $\nabla_d P$, $\nabla_c P$, and $\nabla_c Q$ can all be different. The degrees of E, F, and G, and the necessary and sufficient conditions on the control points are derived by [Liu and Hoschek, 89].

3 Analysis of surface degree

A polyhedron is a linear interpolation of the vertices with only C^0-continuity. One may wonder what polynomial degree is necessary for the various G^1 interpolation problems PN, PT, and PD. It has been shown by [Piper, 87] that degree four is sufficient for the PD problem. He has further shown by means of a counterexample, but not by analysis, that degree three is not always sufficient. In this section I present an analysis of the required polynomial degree for the various interpolation problems in 3D.

Let us consider the G^1-continuity conditions for two Bézier patches P and Q that have a common edge, say $P(u,v,0) = Q(u,v,0)$. The involved control points for the G^1 connection of two patches are shown in Figure 3. To simplify notation, we denote $p_{i,j,0} = q_{i,j,0}$ by M_i (middle column of control points), $p_{i,j,1}$ by L_i (left), and $q_{i,j,1}$ by R_i (right).

Quadratic case. For quadratic patches M, K, L, and R, given by Equation (4), are $M = tM_0 + uM_1$, $K = tM_1 + uM_2$, $L = tL_0 + uL_1$, and $R = tR_0 + uR_1$. The functions E, F, and G are at most linear: $E(t,u) = e_0t + e_1u$, $F(t,u) = f_0t + f_1u$, and $G(t,u) = g_0t + g_1u$. Since $u = 1-t$, we see that $E(t,u) = t(e_0 - e_1) + e_1$ reduces to a constant if $e_0 = e_1$, and likewise for F and G.

Substitution of M, K, L, R, E, F, and G into the tangent plane continuity condition Equation (6), yields $t^2C_0 + tuC_1 + u^2C_2 = 0$, with coefficients C_i as given below. Since this equation must hold for all $t+u=1$, C_0, C_1, and C_2 must all be zero:

$$
\begin{aligned}
C_0 &= e_0(M_1-M_0)+f_0(L_0-M_0)+g_0(R_0-M_0)=0,\\
C_1 &= e_1(M_1-M_0)+e_0(M_2-M_1)+f_1(L_0-M_0)+f_0(L_1-M_1)+\\
&\quad g_1(R_0-M_0)+g_0(R_1-M_1)=0,\\
C_2 &= e_1(M_2-M_1)+f_1(L_1-M_1)+g_1(R_1-M_1)=0.
\end{aligned}
\tag{7}
$$

If not $e_0=e_1$, $f_0=f_1$, and $g_0=g_1$, that is, if E, F, and G do not degenerate to constant functions, then this set of equations is independent.

Equation (7) is a set of three vector equations, comprising nine scalar equations. For a whole surface of N_e edges we therefore have $9N_e$ equations. In the P-problem, only the control points M_0 and M_2 of each edge are known. Control point M_1 and the coefficients e_i, f_i, g_i, $i=0,1$, are then unknown. Since the control points consist of three coordinates, there are $9N_e$ unknowns. So there is in general a solution to the P-problem, and if E, and F, and G do not degenerate to constant functions, this solution is unique. In that case however, there are no degrees of freedom left to interpolate prescribed normals, tangents, or derivatives. We conclude that degree 2 is necessary and in general sufficient for the P-problem, but not sufficient for the PN-, PT-, and the PD-problem.

Cubic case. For cubic patches M, K, L, and R, given by Equation (4), are $M=t^2M_0+2tuM_1+u^2M_2$, $K=t^2M_1+2tuM_2+u^2M_3$, $L=t^2L_0+2tuL_1+u^2L_2$, and $R=t^2R_0+2tuR_1+u^2R_2$. The functions E, F, and G are at most quadratic: $E(t,u)=e_0t^2+e_1tu+e_2u^2$, etc. Since $u=1-t$, we see that $E(t,u)=t^2(e_0-e_1+e_2)+t(e_1-2e_2)+e_2$ reduces to a linear function if $e_0-e_1+e_2=0$, and to a constant if additionally $e_1-2e_2=0$.

Substitution of M, K, L, R, E, F, and G into the tangent plane continuity condition Equation (6) now gives $t^4C_0+t^3uC_1+t^2u^2C_2+tu^3C_3+u^4C_4=0$, with

$$C_0 = e_0(M_1-M_0)+f_0(L_0-M_0)+g_0(R_0-M_0)=0, \tag{8a}$$

$$
\begin{aligned}
C_1 &= e_1(M_1-M_0)+2e_0(M_2-M_1)+f_1(L_0-M_0)+\\
&\quad 2f_0(L_1-M_1)+g_1(R_0-M_0)+2g_0(R_1-M_1)=0,
\end{aligned}
\tag{8b}
$$

$$
\begin{aligned}
C_2 &= e_2(M_1-M_0)+2e_1(M_2-M_1)+e_0(M_3-M_2)+\\
&\quad f_2(L_0-M_0)+2f_1(L_1-M_1)+f_0(L_2-M_2)+\\
&\quad g_2(R_0-M_0)+2g_1(R_1-M_1)+g_0(R_2-M_2)=0,
\end{aligned}
\tag{8c}
$$

$$
\begin{aligned}
C_3 &= 2e_2(M_2-M_1)+e_1(M_3-M_2)+2f_2(L_1-M_1)+\\
&\quad f_1(L_2-M_2)+2g_2(R_1-M_1)+g_1(R_2-M_2)=0,
\end{aligned}
\tag{8d}
$$

$$C_4 = e_2(M_3-M_2)+f_2(L_2-M_2)+g_2(R_2-M_2)=0. \tag{8e}$$

If not $e_0-e_1+e_2=0$, $f_0-f_1+f_2=0$, and $g_0-g_1+g_2=0$, i.e. if E, F, and G do not degenerate to linear functions, then this set of equations is independent.

Equation (8) is a set of five vector equations, or fifteen scalar equations. For a whole surface of N_e edges and N_t triangles we therefore have $15N_e$ equations. Considering first the P-problem, only the control points M_0 and M_3 of each edge are known. Control points M_1 and M_2 and the coefficients e_i, f_i, g_i, $i=0,1,2$, are unknown, and for each triangle the control point L_1 is also unknown. So, there are $15N_e+3N_t$ unknowns. However, for a closed triangulation $3N_t=2N_e$. This results in a total of $15N_e$ equations in $17N_e$ unknowns, so that there is in general more than one solution.

The PN-problem prescribes that the control points M_1 and M_2 lie in given tangent planes at the vertices M_0 and M_3 respectively. This results in $2N_e$ additional equations, giving a total of $17N_e$ equations in $17N_e$ unknowns. So there is in general a solution to the PN-problem, and if E, F, and G do not degenerate to linear functions, this solution is unique. In that case however, there are no degrees of freedom left to interpolate prescribed tangents or derivatives. We conclude that degree three is generally necessary and sufficient for the PN-problem, but not sufficient for the PT- and PD-interpolation problem.

[Piper, 87] shows that degree 4 is necessary and sufficient for the PD-interpolation problem of *two* patches, but analogous to the previous analysis it is easily verified that it also applies to a whole surface. Consequently, also for the PT-problem degree 4 is sufficient, and necessary as well, as shown above.

The results of this section are summarized in the following table, giving the necessary and sufficient polynomial degrees for the considered interpolation problems:

	2	3	4
P	+		
PN		+	
PT			+
PD			+

4 Surface normal estimation

In the rest of this chapter only the PN-interpolation problem is considered. First, the surface normals at the vertices must be estimated, then we we construct a G^1 surface that interpolates the vertices and the surface normals. Analogous to the tangent vector of a curve, the normal vector at V_i can be estimated by weighting the unit normals of all the incident triangles, and normalize the sum:

1. Weight by area. For each triangle, take the cross-product of two different vectors between its vertices. This vector is normal to the triangle, and its magnitude is twice the triangle area. Sum the vectors, and normalize to unit length. The idea of this method is that the larger triangles correspond to a larger part of the surface, and should affect the orientation of the tangent plane most.
2. Weight uniformly. Divide the cross-products by twice the area of the triangle so as to give unit normals. Then normalize the sum.
3. Weight by inverse area. Divide each unit normal by the area of the triangle, then normalize the sum. The reasoning behind this method is that a close neighboring vertex knows more about the local surface normal and should have a larger weight than far away vertices.

5 Local schemes

Although degree 3 is sufficient for a solution to the PN problem to exist, it is a global solution, resulting from a large set of equations involving all the control points. Local schemes are preferred because they are simpler, computationally cheaper, and allow local changes of vertex position and normal.

In order to achieve local solutions to the PN-interpolation problem we need more degrees of freedom to choose the control points so as to obtain G^1-continuity. There are

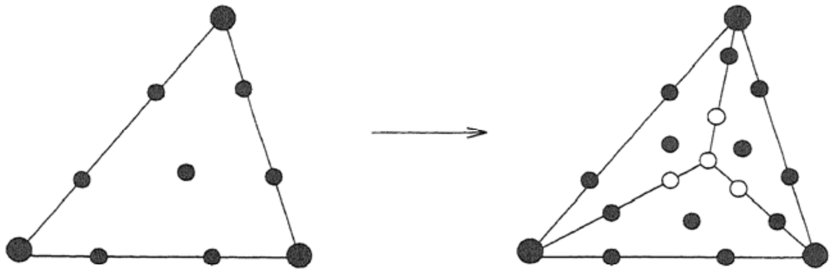

Figure 4: Schematic representation of a three-split of a cubic Bézier triangle.

three well known strategies to get more degrees of freedom: blending of patches, raising the degree of the polynomial patch from 3 to 4, and patch subdivision. Blended patches are composed of a sum of patches giving an interpolating result, and a correction term to make the patch G^1. However, the correction term is a rational polynomial. Blending methods are applied by for example [Herron, 85], [Nielson, 87], and [Hagen and Pottmann, 88].

Raising the polynomial degree of the patch is performed by [Farin, 83], [Piper, 87], [Jensen, 87], [Pfluger and Neamtu, 91], and [Schmitt et al., 91]. Some of these methods introduce a degeneracy: [Farin, 83] and [Jensen, 87] let the patch edge be of actual degree 3 (as a result the connection to cubic rectangular patches is straightforward, see [Farin, 82]), while [Pfluger and Neamtu, 91], and [Schmitt et al., 91] contract some of the interior control points into one.

General patch subdivision splits a patch into several patches that together have the same shape as the original one. Subdivision algorithms for Bézier triangles are given by [Goldman, 83]. The so-called Clough-Tocher triangle splitting scheme (named after [Clough and Tocher, 65]) subdivides a triangular patch $P(t, u, w)$ at the surface point $P(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ into three new triangles. The parent triangle is referred to as the macro triangle, the three new ones as micro triangles. See Figure 4 for a schematic representation of a three-split of a cubic Bézier triangle.

The Clough-Tocher split has been used for cubic functional surfaces in finite-element analysis, see [Strang and Fix, 73], and later for quartic parametric surfaces in scattered data interpolation by [Farin, 83]. The triangle split has two effects: the number of control points is increased, and the interpolation constraints apply to only one edge of the micro triangle, the one that coincides with macro triangle edge. At the interior edges no interpolation data is prescribed, and only G^1-continuity is required. Consequently, the open control points in Figure 4 can be moved without affecting interpolation and continuity along the macro triangle edges.

[Farin, 83], [Piper, 87], and [Jensen, 87] all use quartic patches as well as the triangle three-split to achieve a local solution. In the following section I show that a *cubic* three-split scheme generally provides enough degrees of freedom to solve the PN-problem locally.

6 A cubic three-split scheme

Let a set of vertices be given, as well as their topology (a triangulation) and surface normals at the vertices. The notation used and the control point lay-out and naming is illustrated in Figure 5. Let us consider the macro triangle edge between M_0 and M_3, and let E, F, and G be linear: $E(t, u) = e_0t + e_1u$, $F(t, u) = f_0t + f_1u$, $G(t, u) = g_0t + g_1u$. The tangent

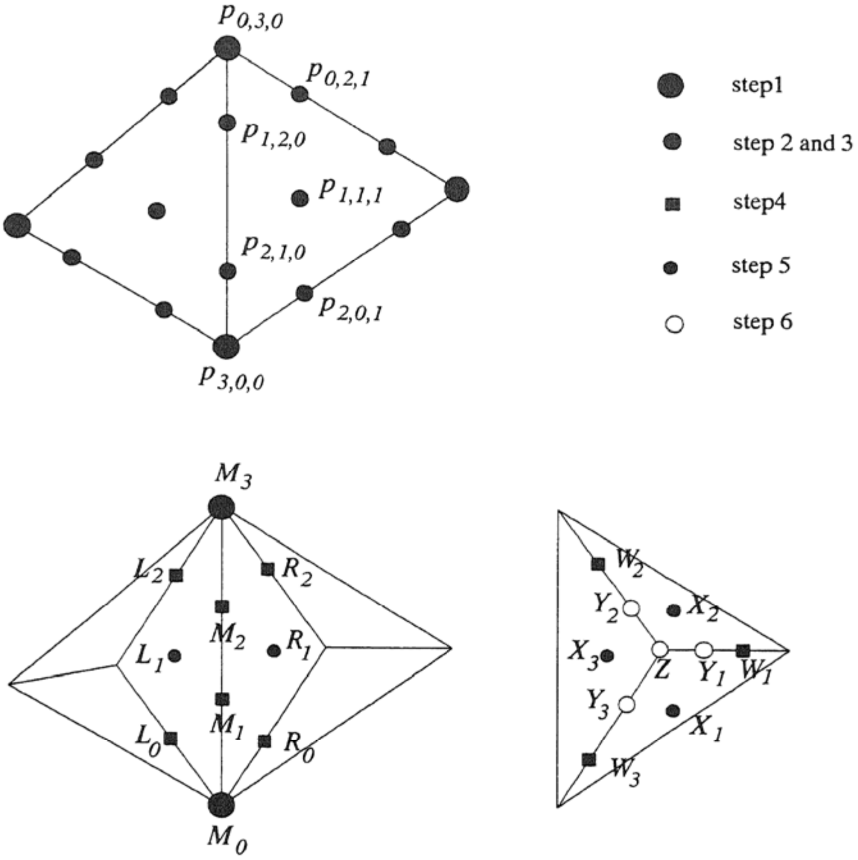

Figure 5: Control points used in the three-split scheme.

plane continuity condition Equation (6) then becomes $t^3C_0 + t^2uC_1 + tu^2C_2 + u^3C_3 = 0$, with

(9a) $$C_0 = e_0(M_1 - M_0) + f_0(L_0 - M_0) + g_0(R_0 - M_0) = 0,$$

(9b) $$C_1 = e_1(M_1 - M_0) + 2e_0(M_2 - M_1) + f_1(L_0 - M_0) + 2f_0(L_1 - M_1) + g_1(R_0 - M_0) + 2g_0(R_1 - M_1) = 0,$$

(9c) $$C_2 = 2e_1(M_2 - M_1) + e_0(M_3 - M_2) + 2f_1(L_1 - M_1) + f_0(L_2 - M_2) + 2g_1(R_1 - M_1) + g_0(R_2 - M_2) = 0,$$

(9d) $$C_3 = e_1(M_3 - M_2) + f_1(L_2 - M_2) + g_1(R_2 - M_2) = 0.$$

The algorithm consisting of the following six steps splits each macro triangle into three cubic micro triangles, and sets the Bézier control points so as to satisfy the tangent plane continuity condition along all the generated Brezier patch edges, that is, all the micro triangle edges.

Step 1. First, we are going to construct cubic macro triangles P, interpolating the given vertices. To this end, $p_{3,0,0}$, $p_{0,3,0}$, and $p_{0,0,3}$ are set to the vertices of a given triangle.

Step 2. Control point $p_{2,1,0}$ is set as follows: $p_{0,3,0}$ is projected onto the tangent plane at $p_{3,0,0}$ giving a point $p^*_{0,3,0}$, then

$$p_{2,1,0} = p_{3,0,0} + (p^*_{0,3,0} - p_{3,0,0})/3.$$

Control points $p_{1,2,0}$, $p_{0,2,1}$, $p_{0,1,2}$, $p_{2,0,1}$, and $p_{1,0,2}$ are set in a symmetrical way.

Step 3. Following [Farin, 83], $p_{1,1,1}$ is set to as follows:

$$p_{1,1,1} = (p_{2,1,0} + p_{1,2,0} + p_{2,0,1} + p_{1,0,2} + p_{0,2,1} + p_{0,1,2})/4 - (p_{3,0,0} + p_{0,3,0} + p_{0,0,3})/6.$$

We now have a surface interpolating the vertices and normals, but it is not yet tangent plane continuous.

Step 4. Each macro triangle is split at $P(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ into three micro triangles by Bézier patch subdivision. For each macro triangle edge the control points M_i, $i = 0, 1, 2, 3$, are computed and remain fixed. R_0 is set to $\frac{1}{3}(p_{3,0,0} + p_{2,1,0} + p_{2,0,1})$, and L_0, L_2, and R_2 are set analogously.

Step 5. Control points L_1 and R_1 have to be set such that the tangent plane continuity condition Equation (6) is satisfied. Since M_i, $i = 0, 1, 2, 3$, and L_i, R_i $i = 0, 2$ are known by now, Equations (9a) and (9d) determine the coefficients e_i, f_i, and g_i, $i = 0, 1$, up to a constant factor, so that we can arbitrarily set $e_0 = e_1 = 1$. The unknowns L_1, and R_1 are constrained by (9b) and (9c). Rewriting gives:

$$\begin{aligned} 2f_0L_1 + 2g_0R_1 &= 2f_0M_1 + 2g_0M_1 - e_1(M_1 - M_0) - 2e_0(M_2 - M_1) \\ &\quad - f_1(L_0 - M_0) - g_1(R_0 - M_0), \\ 2f_1L_1 + 2g_1R_1 &= 2f_1M_1 + 2g_1M_1 - 2e_1(M_2 - M_1) - e_0(M_3 - M_2) \\ &\quad - f_0(L_2 - M_2) - g_0(R_2 - M_2)). \end{aligned} \tag{10}$$

If this pair of equations is independent, it uniquely determines L_1, and R_1. Tangent plane continuity is then achieved across the macro triangle edges.

Step 6. In order to get tangent plane continuity across the edges of the micro triangles we set the control points Y_1, Y_2, Y_3, and Z from Figure 5 as follows (see [Farin, 83]):

$$\begin{aligned} Y_1 &= (W_1 + X_1 + X_2)/3, \\ Y_2 &= (W_2 + X_2 + X_3)/3, \\ Y_3 &= (W_3 + X_3 + X_1)/3, \\ Z &= (Y_1 + Y_2 + Y_3)/3. \end{aligned}$$

Correctness of this algorithm is proved by the following theorem.

THEOREM 1 *Steps 1 to 6 above construct an overall tangent plane continuous surface.*

Proof. By construction (Step 5), the macro triangle edges satisfy Equation (9), and are thus tangent plane continuous. In Step 4, W_3 $(= R_0)$ is set to $\frac{1}{3}(p_{3,0,0} + p_{2,1,0} + p_{2,0,1})$, and in Step 6, Y_3 is set to $\frac{1}{3}(W_3 + X_1 + X_3)$, and Z to $\frac{1}{3}(Y_0 + Y_1 + Y_2)$. This makes the patch tangent plane continuous along the micro triangle edge between $p_{3,0,0}$ and Z, since Equation (6), when applied to this edge, is satisfied for constant values of $E(t,u)$, $F(t,u)$,

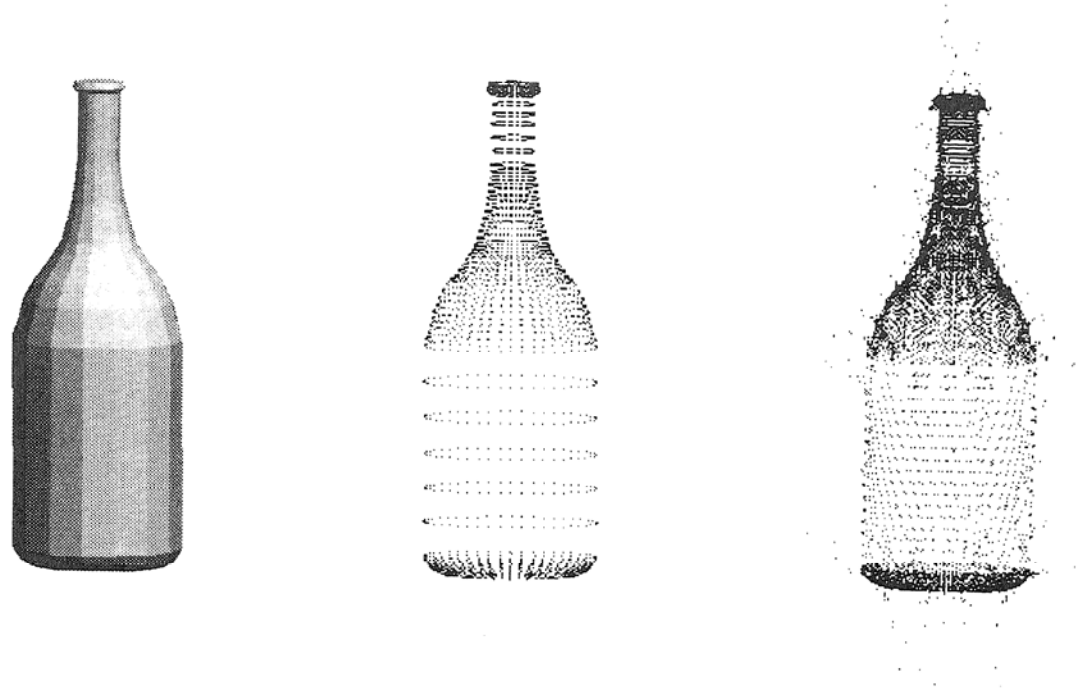

Figure 6: Result of the three-split scheme. Left: macro triangle Bézier control points before splitting. Right: final control points of all micro triangles.

$G(t, u)$. The other two edges are treated in the symmetrical way in Steps 4 and 6. Because the same constants are used for the other two edges, there is no conflict in setting Z.

The whole surface is thus tangent plane continuous along all triangle edges. Since cubic Bézier triangles are internally C^2-continuous, the whole surface is tangent plane continuous everywhere. □

Essentially the same three-split scheme is described by [Cottin and van Damme, 90], but independently derived here.

Analogous to the construction of cubic G^1 curves, in addition to the tangent plane continuity condition the orientation of the tangent plane must be properly defined at every point on the common edge of two adjacent triangles, in order to achieve a continuously changing unit normal vector, in other words, a G^1-continuous surface. Only when both conditions are met, the surface is G^1-continuous. The tangent plane orientation will be properly defined if f_0 and g_0 in Equation (9a) (and also f_1 and g_1 in (9d)) have opposite signs. This condition is satisfied for many sets of vertices whose neighboring normal vectors direction do not change wildly. In other cases, the normal vector at a vertex should be altered in order for the projection in Step 2 to result in opposite signs of f_0 and g_0. A detailed analysis on this is presented by [Cottin and van Damme, 90].

We see that a three-split gives sufficient degrees of freedom to construct a tangent plane continuous surface of cubic patches. The tangent plane continuity along the macro triangle edges is enforced by setting the control points L_1 and R_1 so as to satisfy Equation (10). It turns out, however, that the resulting position of L_1 and R_1 is often far away from their previous position. Apparently L_1 and R_1 often have enough room to satisfy Equation (10) only when the plane through M_1, M_2, L_1, and R_1 is very tilted (with respect to its previous orientation). Although the surface tangent plane is than continuous along the common edge, the surface oscillates wildly in such cases. Even if the surface tangent plane

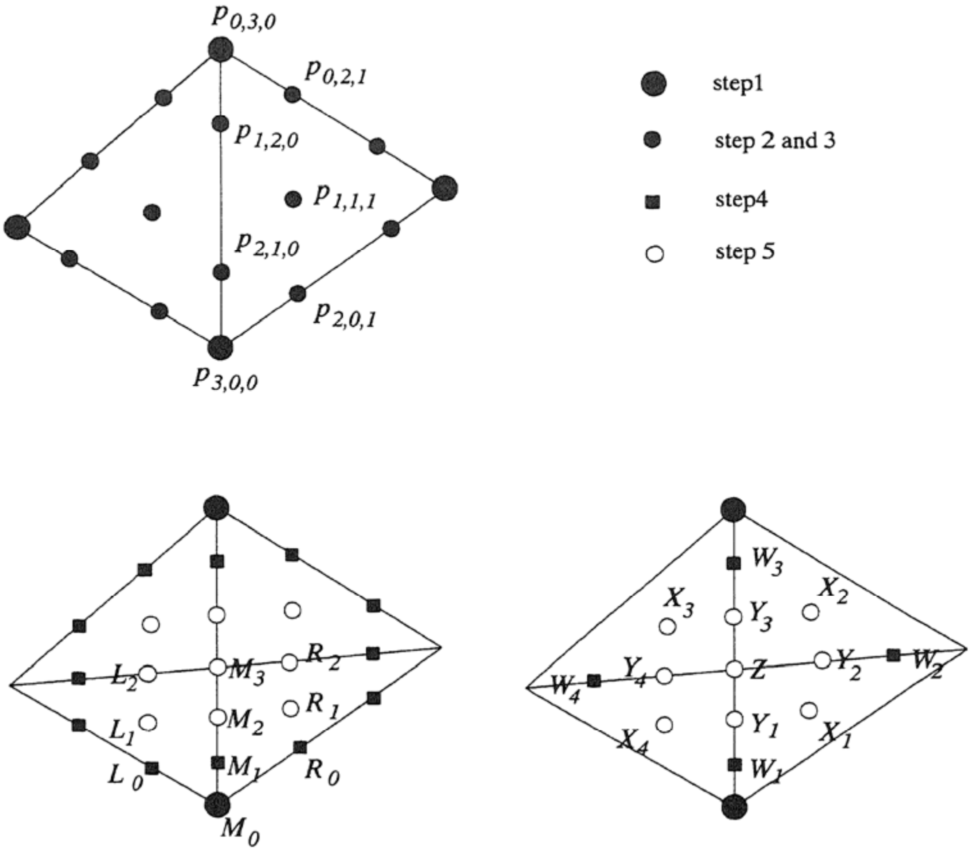

Figure 7: Control points used in the two-split scheme.

additionally has a continuous orientation, i.e. the surface is G^1-continuous, it does not give a visually smooth impression.

This is illustrated in Figure 6, showing the result of the algorithm applied to the triangular polyhedral surface of the bottle shown at the left. The image in the middle shows the Bézier control points of the macro triangles before splitting. The right image depicts the final control points, i.e. when the resulting surface is tangent plane continuous.

The reason for this unsatisfactory result is that Equation (10) imposes too strict constraints on L_1 and R_1. This could be avoided if there were additional degrees of freedom which could be used to select a solution that is optimal in some sense. Such a solution is presented in Section 8.

7 Towards an adaptive splitting scheme

Another disadvantage of the three-split is that very thin triangles can emerge. This may cause numerically instable computations.

We can subdivide a thin macro triangle into *two* micro triangles instead of three, by creating a new edge from one vertex to a point on the opposite edge. The neighboring macro triangle at the side of the split edge must also be subdivided. Figure 7 shows a schematic representation of a two-split of two cubic Bézier triangles.

Let us consider two patches P and Q such that $P(t,u,0) = Q(t,u,0)$, that are each split into two micro triangles, while the four neighboring macro triangles incident to the other edges are split into three micro triangles. Steps 1 to 3 are the same as in the preceding section. The patches then interpolate the (macro triangle) vertices and normals, but are not yet tangent plane continuous. That is achieved by a two-split scheme in the following steps.

Step 4. The two macro triangles are split at $P(\frac{1}{2},\frac{1}{2},0)$ and $Q(\frac{1}{2},\frac{1}{2},0)$ into four micro triangles by Bézier patch subdivision. The control points M_0, M_1, L_0, R_0 are computed and remain fixed. By symmetry there are four sets of these control points: one for each inner micro triangle edge.

Step 5. Let E, F, and G be constant: $E(t,u) = e$, $F(t,u) = f$, and $G(t,u) = g$. The tangent plane continuity condition, Equation (6), then yields $t^2C_0 + tuC_1 + u^2C_2 = 0$, with

$$C_0 = e(M_1 - M_0) + f(L_0 - M_0) + g(R_0 - M_0) = 0, \quad (11a)$$

$$C_1 = e(M_2 - M_1) + f(L_1 - M_1) + g(R_1 - M_1) = 0, \quad (11b)$$

$$C_2 = e(M_3 - M_2) + f(L_2 - M_2) + g(R_2 - M_2) = 0. \quad (11c)$$

Since M_0, M_1, L_0, and R_0 are known, Equation (11a) determines the constants e, f, and g.

Let us call the unknown control points X_i, Y_i, $i = 1, \ldots, 4$ and Z, as illustrated in Figure 7. We need to set these control points so as to achieve tangent plane continuity across the four inner micro triangle edges. Equation (11b) and (11c) for these four edges together is:

$$e_i(Y_i - W_i) + f_i(X_{i-1} - W_i) + g_i(X_i - W_i) = 0, \quad (12a)$$

$$e_i(Z - Y_i) + f_i(Y_{i-1} - Y_i) + g_i(Y_{i+1} - Y_i) = 0, \quad (12b)$$

for $i = 1, \ldots, 4$, and $i-1$ and $i+1$ taken modulo 4. This results in eight equations in nine unknowns. This set of equations in general has a solution. There are even enough degrees of freedom to choose a 'best' solution, by optimizing a suitable object function. For example, minimizing

$$\sum_{i=1}^{4} \|X_i - X_i^{old}\| + \sum_{i=1}^{4} \|Y_i - Y_i^{old}\| + \|Z - Z^{old}\|,$$

where Z^{old} is the value of Z just before Step 5, and likewise for X_i and Y_i, gives a solution to Equation (12) that is close to the old configuration of control points. This is considered good, since the old values were chosen in a sensible way.

The four micro triangles are now tangent plane continuously connected to each other, but must also be G^1-continuously connected to the four neighboring patches at the four outer edges of the two macro triangles. However, the inner control points of the micro triangles have already been fixed in the previous step. Therefore Step 5 of the three-split scheme must be adapted when used in combination with the two-split scheme. Let us consider one such outer edge, and switch again to the notation of Figure 3: We have seen that choosing E, F, and G to be linear uniquely determines control points L_1 and R_1 by Equation (10). If, say, L_1 has already been fixed by a two split step (L_1 thus corresponds to a control point X_i above), there are not enough degrees of freedom to solve R_1, since it is over-determined by Equation (10). Instead, we let E, F, and G be quadratic, which gives the most general constraints of Equation (8). We let $e_0 = e_1 = e_2$, $f_0 = f_1 = f_2$, and $g_0 = g_1 = g_2$, so that Equation (8) reduces to:

$$
\begin{aligned}
&(13a) \quad & C_0 &= e_0(M_1 - M_0) + f_0(L_0 - M_0) + g_0(R_0 - M_0) = 0, \\
&(13b) \quad & C_1 &= e_0(M_1 - M_0) + 2e_0(M_2 - M_1) + f_0(L_0 - M_0) + \\
& & & \quad 2f_0(L_1 - M_1) + g_0(R_0 - M_0) + 2g_0(R_1 - M_1) = 0, \\
&(13c) \quad & C_2 &= e_0(M_1 - M_0) + 2e_0(M_2 - M_1) + e_0(M_3 - M_2) + \\
& & & \quad f_0(L_0 - M_0) + 2f_0(L_1 - M_1) + f_0(L_2 - M_2) + \\
& & & \quad g_0(R_0 - M_0) + 2g_0(R_1 - M_1) + g_0(R_2 - M_2) = 0, \\
&(13d) \quad & C_3 &= 2e_0(M_2 - M_1) + e_0(M_3 - M_2) + 2f_0(L_1 - M_1) + \\
& & & \quad f_0(L_2 - M_2) + 2g_0(R_1 - M_1) + g_0(R_2 - M_2) = 0, \\
&(13e) \quad & C_4 &= e_0(M_3 - M_2) + f_0(L_2 - M_2) + g_0(R_2 - M_2) = 0.
\end{aligned}
$$

Equations (13a) and (13e) imply:

$$\frac{area(M_0, M_1, L_0)}{area(M_0, M_1, R_0)} = \frac{area(M_3, M_2, L_2)}{area(M_3, M_2, R_2)} = \frac{g_0}{f_0}.$$

An algorithm to set M_i, $i = 0, \ldots, 4$, L_0, L_2, R_0, and R_1 so as to satisfy this ratio is given by [Farin, 83]. Coefficients e_0, f_0, and g_0 are then determined up to a common factor, so that we can arbitrarily set $e_0 = 1$. Subtracting Equation (13a) from (13b) (or (13e) from (13d)) gives:

$$2e_0(M_2 - M_1) + 2f_0(L_1 - M_1) + 2g_0(R_1 - M_1) = 0,$$

from which we can determine R_1 because all other variables are known. Note that Equation (13c) is automatically satisfied: (13c)=(13a)+(13d)=(13b)+(13e).

So far, it has been essential that the two two-split triangles have four three-split triangles as neighbors. The triangles that are split into two can therefore not be chosen arbitrarily. Furthermore, the two-split triangles must be constructed before the three-split triangles, because L_1 in Step 6 (corresponding to a X_i in Step 5) must be known first.

To be able to adaptively choose which pairs of macro triangles to split into two, independently of the neighboring macro triangles, it must also be possible to achieve tangent plane continuity when two or three sides of a triangle are split. This is indeed possible, and in fact gives even more degrees of freedom, but symmetry is lost and the determination of the control points gets a bit involved.

8 A cubic six-split scheme

In the previous section we saw that the two-split scheme gives enough degrees of freedom to apply an optimization criterion to the solution of the tangent plane continuity constraints on the control points. We have also seen in Section 6 that such an optimization is important in order to avoid very distorted control points configurations, which result in tangent planes that are too tilted to be aesthetically pleasing. However, the two-split as presented in Section 7 cannot be applied to all macro triangles.

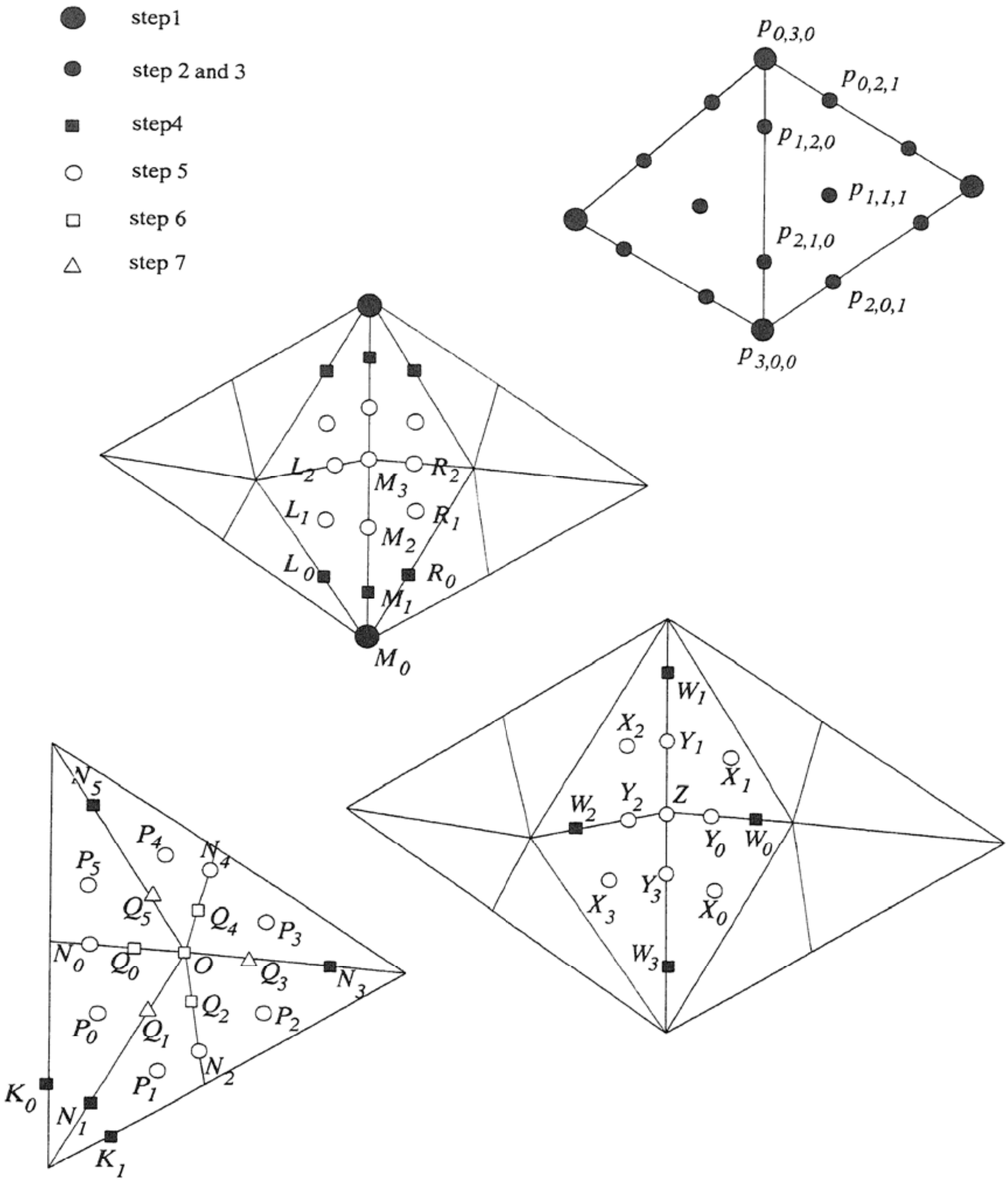

Figure 8: Control points used in the six-split scheme.

A triangle split that provides sufficient degrees of freedom for optimization to be applied to all macro triangles is a split into six micro triangles. Figure 8 shows a schematic representation of a six-split of two adjacent cubic Bézier triangles, and the control point naming used in this section.

Let us consider the micro triangle edge between M_0 and M_3, and choose E, F, and G constant: $E = e$, etc. The tangent plane continuity condition, Equation (6), then is $t^2C_0 + tuC_1 + u^2C_2 = 0$, with C_0, C_1, and C_2 given by Equation (11), i.e.

$$
\begin{aligned}
&\text{(14a)} && C_0 = e(M_1 - M_0) + f(L_0 - M_0) + g(R_0 - M_0) = 0,\\
&\text{(14b)} && C_1 = e(M_2 - M_1) + f(L_1 - M_1) + g(R_1 - M_1) = 0,\\
&\text{(14c)} && C_2 = e(M_3 - M_2) + f(L_2 - M_2) + g(R_2 - M_2) = 0.
\end{aligned}
$$

An alternative formulation but equivalent condition is the following:

$$
\begin{aligned}
&\text{(15a)} && \tilde{e}_1(M_0 - M_1) + \tilde{f}_1(L_0 - M_1) + \tilde{g}_1(R_0 - M_1) = 0,\\
&\text{(15b)} && \tilde{e}_1(M_1 - M_2) + \tilde{f}_1(L_1 - M_2) + \tilde{g}_1(R_1 - M_2) = 0,\\
&\text{(15c)} && \tilde{e}_1(M_2 - M_3) + \tilde{f}_1(L_2 - M_3) + \tilde{g}_1(R_2 - M_3) = 0.
\end{aligned}
$$

Equations (15b) and (15c) in terms of X_i, Y_i, and Z, and applied to the four edges together become

$$
\begin{aligned}
&\text{(16a)} && \tilde{e}_i(W_i - Y_i) + \tilde{f}_i(X_i - Y_i) + \tilde{g}_i(X_{i+1} - Y_i) = 0,\\
&\text{(16b)} && \tilde{e}_i(Y_i - Z) + \tilde{f}_i(Y_{i-1} - Z) + \tilde{g}_i(Y_{i+1} - Z) = 0,
\end{aligned}
$$

with $i = 0, \dots, 3$. For $i = 0$, Equation (16a) can written as

$$
\text{(17a)} \qquad \tilde{e}_i(Q_i - N_i) + \tilde{f}_i(P_{i-1} - N_i) + \tilde{g}_i(P_i - N_i) = 0.
$$

For tangent plane continuity along the whole micro triangle edge between Z and O, the following condition must also hold:

$$
\text{(17b)} \qquad \tilde{e}_i(O - Q_i) + \tilde{f}_i(Q_{i-1} - Q_i) + \tilde{g}_i(Q_{i+1} - Q_i) = 0.
$$

Similar conditions apply to the edges corresponding to N_2, Q_2 and N_4, Q_4. For $i = 0, 2, 4$ the $\tilde{e}_i$ will be given the same value, say h, and likewise $\tilde{f}_i = k$, and $\tilde{g}_i = \ell$ for $i = 0, 2, 4$. Equation (17) thus is

$$
\begin{aligned}
&\text{(18a)} && h(Q_i - N_i) + k(P_i - N_i) + \ell(P_{i+1} - N_i) = 0,\\
&\text{(18b)} && h(O - Q_i) + k(Q_{i-1} - Q_i) + \ell(Q_{i+1} - Q_i) = 0.
\end{aligned}
$$

For $i = 1, 3, 5$ similar conditions must be satisfied, but the constants may be different, say a, b, and c:

$$
\begin{aligned}
&\text{(19a)} && a(Q_i - N_i) + b(P_i - N_i) + c(P_{i+1} - N_i) = 0,\\
&\text{(19b)} && a(O - Q_i) + b(Q_{i-1} - Q_i) + c(Q_{i+1} - Q_i) = 0.
\end{aligned}
$$

With $i = 1$, Equation (19) applies to the edge between M_0 and O. In order to achieve tangent plane continuity along this whole micro triangle edge, the following condition must also hold:

$$a(N_1 - M_0) + b(K_0 - M_0) + c(K_1 - M_0) = 0. \tag{20}$$

For the edges associated with N_3 and N_5 analogous conditions must hold.

The algorithm consisting of the following seven steps splits each macro triangle into six cubic micro triangles, and sets the Bézier control points so as to satisfy the tangent plane continuity condition along all the generated Brezier patch edges, i.e. all the micro triangle edges.

Step 0. Before calculating anything, choose a suitable value for b, for example $b = 1/2$, and set $a = 1$ and $c = b$. Set h $(= \tilde{e}_0 = \tilde{e}_2 = \tilde{e}_4)$ to 1, and k $(= \tilde{f}_0 = \tilde{f}_2 = \tilde{f}_4)$ and ℓ $(= \tilde{g}_0 = \tilde{g}_2 = \tilde{g}_4)$ to $-(2b+1)/(3b+2)$.

Steps 1 to 3 of the construction algorithm are the same as in Sections 6 and 7. The patches then interpolate the macro triangle vertices and normals, but are not yet tangent plane continuous. That is achieved in the following steps.

Step 4. All macro triangles $P(t, u, w)$ are split at $P(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $P(\frac{1}{2}, \frac{1}{2}, 0)$, $P(\frac{1}{2}, 0, \frac{1}{2})$, and $P(0, \frac{1}{2}, \frac{1}{2})$ into six micro triangles as illustrated in Figure 8. Now M_0, M_1 $(= K_0)$, and K_1 are known. Compute N_1 by Equation (20), and N_3 and N_5 analogously.

Step 5. Compute $\tilde{e}_1$, $\tilde{f}_1$, and $\tilde{g}_1$ from Equation (15), and $\tilde{e}_i$, $\tilde{f}_i$, and $\tilde{g}_i$, $i = 3, 5$, analogously. All $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{g}_i$, $i = 0, \dots, 5$ are now known. Calculate Z and $Y_0, \dots, Y_3$, by minimizing

$$\sum_{i=0}^{3} \|Y_i - Y_i^{old}\| + \|Z - Z^{old}\|,$$

under the constraints of Equation (16b) with $i = 0, \dots, 3$ (Y_i^{old} and Z^{old} are the values of Y_i and Z resulting from Step 4). Compute $X_0, \dots, X_3$ by minimizing

$$\sum_{i=0}^{3} \|X_i - X_i^{old}\|,$$

under the constraints of Equation (16a) with $i = 1, 3$ (X_i^{old} is the value of X_i resulting from Step 4).

Step 6. All P_i and N_i are known by now. Compute Q_0, Q_2, and Q_4 by Equation (18a) with $i = 0, 2, 4$. Set $O = (Q_0 + Q_2 + Q_4)/3$.

Step 7. Compute Q_1, Q_3, and Q_5 by Equation (19a) with $i = 1, 3, 5$.

It is not immediately clear that the above algorithm gives an overall tangent plane continuous surface. This is proved by the following theorem:

THEOREM 2 *Steps 0 to 7 above construct an overall tangent plane continuous surface.*

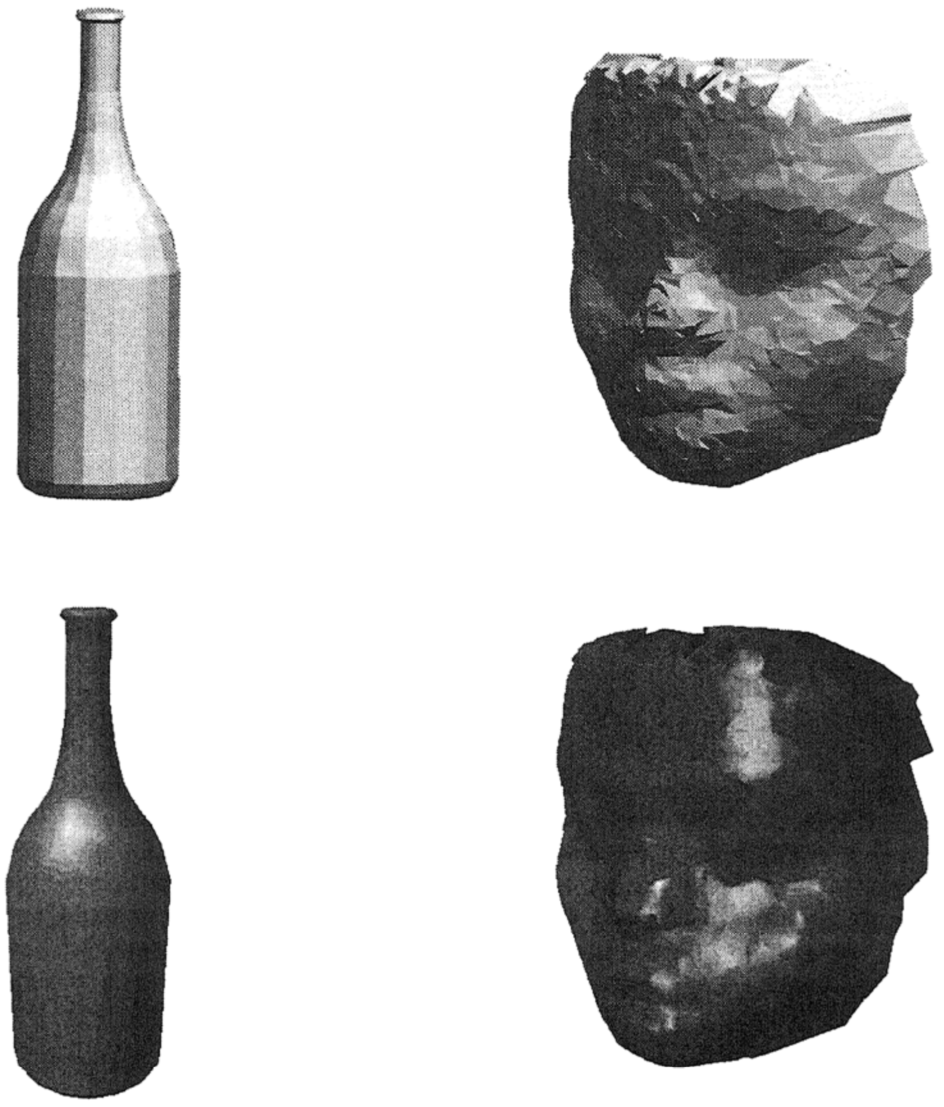

Figure 9: Two polyhedral surfaces and the corresponding G^1-continuous surfaces resulting from the six-split scheme.

Proof. By construction (Step 5) the micro triangle edges between $p_{3,0,0}$ and Z, and between $p_{0,3,0}$ and Z satisfy Equation (14). The surface is thus tangent plane continuous along these edges.

The edges incident to O satisfy the tangent plane continuity condition Equation (6) if there is no conflict around O, that is, if both Equation (18b) for $i = 0, 2, 4$, and Equation (19b) for $i = 1, 3, 5$ are satisfied. The crux of the algorithm is that both conditions are automatically satisfied by setting O to $(Q_0 + Q_2 + Q_4)/3$ and the special relation between b and k: $k = -(2b+1)/(3b+2)$. Incidentally, O is also equal to $(Q_1 + Q_3 + Q_5)/3$.

The whole surface is thus tangent plane continuous along all triangle edges. Since cubic Bézier triangles are internally C^2-continuous, the whole surface is tangent plane continuous everywhere. □

Figure 9 shows the result of the above algorithm on two polyhedral surfaces. The six-split scheme has extra degrees of freedom compared to the three-split scheme of Section 6. This allows to minimize the displacement of control vertices in Step 5 above, so that the surface does not oscillate as wildly as with the three-split scheme, which was demonstrated in Figure 6.

9 Concluding remarks

The algorithms presented in Sections 6, 7, and 8 all have a time complexity $\mathcal{O}(N_e) = \mathcal{O}(N_t)$. This is clear from the iterations over the macro triangles and their edges, and the relation between the number of triangles and edges: $3N_t = 2N_e$.

A G^1-continuous surface generally looks smooth, but its appearance depends on the illumination. As has been derived by [Veltkamp, 92], the reflection of a linear light source on a G^1-continuous surface need not be G^1. For G^2-continuity more control points are needed then we have used here. This means that patches of higher degree must be used in order to keep the scheme local.

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