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Summary
Pollaczek's queueing-oriented research dates mainly from the 2nd quarter of the present century. His analysis of the GI/G/1 queueing model was and still is significant from a methodological as well as an applicability point of view. Techniques from the theory of Complex Functions constitute the core of his analysis.
The advances in Probability Theory and in the Theory of Boundary Value Problems for analytic functions, which became known in the third quarter of our century have provided a wider context for the analysis of the GI/G/1 queueing model. The present study exposes the analysis out from these backgrounds.
For the time dependent as well as the stationary case this approach leads to the formulation of Riemann Boundary Value Problems. The analysis of it for the stationary case is exposed and its solution is derived under slightly weaker conditions than originally considered by Pollaczek.

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1. Introduction

The title of this contribution to the Pollaczek memorial volume is derived from a publication of Pollaczek [13] in 1946. In the title of that paper we read "... application de la théorie de functions ...". Before the second World War "théorie de functions" in French, or "Funktionen theorie" in German indicated the theory of complex functions. Among queueing theorists it is well known that Pollaczek's unweary efforts have shown that the theory of complex functions provides tools and techniques which are of basic importance for the analysis of waiting time phenomena. From a methodological as well as a technical view point Pollaczek's work has been critisised rather sharply, see [9], and the discussion in [26], p.28; the commentator is a young promising scientist educated in the English school of probabilists, the critisised a man already in his seventies well versed in classical hard analysis and matured in prewar continental circles of applied mathematicians. The critique not without elements of admiration and appreciation for Pollaczek's achievements in queueing analysis was, and rightly so, not appreciated by him as the present author learned from one of Pollaczek's letters in 1969. Their conflicting opinions concerned the appropriate approaches to the analysis of queueing models, and after twenty five years it is of some interest to consider again these opinions. Kingman, cf.[9], p.16, states that Pollaczek's approach "disguises simple algebra as complicated and deep analysis" and objects the exclusion of probabilistic arguments. Pollaczek, cf.[16] p.38, deems the potentiality of such arguments too weak and considers analysis to be the appropriate tool. In [9] Kingman outlines an algebraic approach.

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For a good understanding of Pollaczek's research one should know that his ultimate strive has been the analytic description of the congestion phenomena in a many server queueing model with unspecified distributions of service and interarrival times. This model was and still is of great interest for the performance analysis of trunk groups in telephone systems. Pollaczek has spent a huge effort on the analysis of this GI/G/s model, and that under often very difficult personal circumstances. After having published his monograph [15] in 1961 he continued the analysis of this model for many years. His monograph [14] dating from 1957 concerns the single server model GI/G/1.

Pollaczek's first teletraffic study stems from 1930, and he belongs together with Kosten, Palm and Vaulot to the second generation of queueing theorists; Erlang and Molina are the prominent members of the first generation. The studies of the first generation concern mainly models with negatively exponentially distributed service and interarrival times. Pollaczek has to be credited for the development of the analytical techniques for models with unspecified distributions. Quite some basic thinking was required to formulate the queueing models with unspecified distributions as well posed mathematical problems. Once formulated the mathematical problem was not easily classified within the then existing literature on theoretical physics or applied mathematics; much pioneering work had to be done.

Queueing phenomena are usually phrased in terms of waiting times, queue lengths and server loads. These characteristics are modelled as nonnegative stochastic variables, and so their distribution functions possess Laplace-Stieltjes transforms. Let $f(s)$ be the Laplace-Stieltjes transform of a probability distribution of a nonnegative stochastic variable. Then $f(s)$ is well defined for Re $s \geq 0$, it is a uniformly bounded continuous function in Re $s \geq 0$ and, a regular function in Re $s > 0$, so it is here continuously differentiable, moreover $f(s)$ has a finite limit for $|s| \to \infty$, $|\arg s| < \frac{1}{2}\pi$. Such a L.S.-transform is a very manipulable function. Frequently, its domain of definition can be extended into Re$s < 0$ by analytic continuation, apart from points where the extension may have isolated singularities, usually poles and branching points. These properties of L.S.-transforms are the main reasons to analyse queueing characteristics in terms of their L.S.-transforms rather then directly in terms of their distributions, and as such the theory of complex functions enters queueing analysis. Experimental analysis and, presently, simulation analysis by using high spead computing machinery are approaches in which only the distribution functions are needed, these techniques have their own merits and difficulties.

The theory of complex functions is well developed and with its introduction in queueing analysis a very sharp and efficient tool became available. However, a price has to be paid. The determination of the distribution from its L.S.-transform requires the evaluation of a contour integral of a complex function if detailed information concerning the distribution function is needed; its moments can usually be obtained by simple differentiations. Presently, unfortunately, the theory of complex functions is not very popular and complex integration seems to be feared. In this book Feller [7] avoids the presentation of the standard and most powerful inversion formula expressed as a contour integral. It should be stressed that presently the numerical evaluation of contour integrals is in general not a difficult problem, cf.[1].

Pollaczek handles the theorems on complex functions masterly and his analysis of the single server model [14] is effective and beautiful, the more so when considered in the context of the literature of his time. His studies are written with great care for the mathematical details. The complexity of the GI/G/s model, cf.[15], requires an intricate notation and leads to complicated formulas and long chained repeated contour integrals (characterised in [6] p.15 as a dazzling series of manipulations). Despite of these difficulties his research has led to results which are presently well accessible for numerical evaluation, as it has been shown by de Smit [17], [18]. The inherent difficulties of the GI/G/s model, Pollaczek's endeavours for reaching calculable results, and also his rather arduous style of presentation make his analysis difficult to read, and have unfortunately discouraged many a reader. The technical critique in [9] originates from these points. Concerning the methodological critique see section 4. We first discuss in the next section the derivation of the functional equation for the w_m-process; in section 3 that equation, which actually formulates a Riemann Boundary Value

Problem, will be solved. Its solution leads to a new formula for the L.S.-transform of the stationary waiting time distribution valid under very weak conditions and as such it is a refinement of Pollaczek's classical result, cf.[14], p. 80; section 3 concludes with a derivation of Pollaczek's integral equation. In the appendices some background concerning principal value integrals is exposed, and an asymptotic formula is derived, which is needed in the analysis of the boundary value problem.

2. The functional equation of the waiting time process

In this section we shall present the derivation of the functional equation for the waiting time process. The structure of this derivation is essentially based on the new insights in the theory of stochastic processes which have been developed and obtained since the late fifties. Pollaczek's famous integral equation follows readily from this functional equation, see section 3, and as such the analysis to be presented will confront the reader with the basic ideas behind this integral equation and also with its classification in the context of the theory of integral equations.

For the GI/G/1 single server model denote by:

i. τ_m the service time of the mth arriving cumtomer; (2.1)

ii. σ_{m+1} the interarrival time of the mth and $(m+1)$th arriving customer;

iii. $\mathbf{w}_m$ the waiting time of the mth arriving customer.

Here $\{\tau_m, m = 1, 2, \ldots\}$ and, similarly, $\{\sigma_{m+1}, m = 1, 2, \ldots\}$ are sequences of independent, identically distributed, nonnegative stochastic variables, and these sequences are independent families.

The L.S.-transforms of the probability distributions of τ_m and σ_{m+1} are defined by:

$$\beta(\rho) := \mathrm{E}\{e^{-\rho\tau_m}\}, \quad \alpha(\rho) := \mathrm{E}\{e^{-\rho\sigma_{m+1}}\}, \quad \mathrm{Re}\,\rho \geq 0, \tag{2.2}$$

$$\gamma(\rho) := \alpha(-\rho)\beta(\rho) = \mathrm{E}\{e^{-\rho(\tau_m-\sigma_{m+1})}\}, \qquad \mathrm{Re}\,\rho = 0.$$

In the following derivations it will always be assumed that

i. $\beta := \mathrm{E}\{\tau_m\} < \infty, \quad \alpha := \mathrm{E}\{\sigma_{m+1}\} < \infty,$ (2.3)

ii. $\mathbf{w}_1 = 0;$

note that (2.3)ii implies that the first arriving customer meets an empty system.

Denote by $\mathbf{n}$ the *the number* of *customers served during the first busy cycle*, so that

$$\begin{aligned} \mathbf{n} \quad &:= \min\{m: \quad \mathbf{w}_{m+1} = 0, \quad m = 1, 2, \ldots\}, \\ &:= \infty \quad \text{if} \quad \mathbf{w}_{m+1} > 0, \quad m = 1, 2, \ldots, \end{aligned} \tag{2.4}$$

With the notation

$$[x]^+ := \max(0, x), \quad x \text{ real}, \tag{2.5}$$

Pollaczek formulates the structure of the waiting time process $\{\mathbf{w}_m, m = 1, 2, \ldots\}$ as

$$\mathbf{w}_{m+1} = [\mathbf{w}_m + \tau_m - \sigma_{m+1}]^+, \quad m = 1, 2, \ldots . \tag{2.6}$$

So from (2.4) and (2.6),

i. $\mathbf{w}_1 = 0,$ (2.7)

$$\mathbf{w}_{m+1} = \mathbf{w}_m + \boldsymbol{\tau}_m - \boldsymbol{\sigma}_{m+1} \quad \text{for} \quad m = 1, 2, \dots, \mathbf{n}-1,$$

$$\mathbf{w}_{\mathbf{n}+1} = 0,$$

ii. $\mathbf{i} := -(\mathbf{w}_\mathbf{n} + \boldsymbol{\tau}_\mathbf{n} - \boldsymbol{\sigma}_{\mathbf{n}+1}) \geq 0;$

here $\mathbf{i}$ is obviously the first *idle period* of the queueing process.
With

$$\boldsymbol{\gamma}_m := \boldsymbol{\tau}_m - \boldsymbol{\sigma}_{m+1}, \quad m = 1, 2, \dots, \tag{2.8}$$

it follows from (2.7): for Re$\rho \geq 0$, $|r| < 1$,

$$\sum_{m=1}^{\mathbf{n}} r^m \mathrm{e}^{-\rho \mathbf{w}_m} = r + \sum_{m=2}^{\mathbf{n}} r^m \mathrm{e}^{-\rho \mathbf{w}_m} = r + r \sum_{m=1}^{\mathbf{n}-1} r^m \mathrm{e}^{-\rho \mathbf{w}_{m+1}} = r + r \sum_{m=1}^{\mathbf{n}-1} r^m \mathrm{e}^{-\rho(\mathbf{w}_m + \boldsymbol{\gamma}_m)} =$$

$$r + r \sum_{m=1}^{\mathbf{n}} [r^m \mathrm{e}^{-\rho(\mathbf{w}_m + \boldsymbol{\gamma}_m)}] - r^{\mathbf{n}+1} \mathrm{e}^{-\rho(\mathbf{w}_\mathbf{n} + \boldsymbol{\gamma}_\mathbf{n})} = r \sum_{m=1}^{\mathbf{n}} [r^m \mathrm{e}^{-\rho(\mathbf{w}_m + \boldsymbol{\gamma}_m)}] + r[1 - r^{\mathbf{n}} \mathrm{e}^{\rho \mathbf{i}}],$$

or

$$\sum_{m=1}^{\mathbf{n}} r^m \mathrm{e}^{-\rho \mathbf{w}_m} (1 - r \mathrm{e}^{-\rho \boldsymbol{\gamma}_m}) = r[1 - r^{\mathbf{n}} \mathrm{e}^{\rho \mathbf{i}}]. \tag{2.9}$$

The relation (2.9) is an identity, it holds for every sample function of the $\mathbf{w}_m$-process. From this identity we obtain the *functional equation* (2.10) for the waiting time process.

THEOREM 2.1. For the waiting time process $\{\mathbf{w}_m, m = 1, 2, \dots\}$ with $\mathbf{w}_1 = 0$, the functional equation reads: for $|r| < 1$, Re $\rho = 0$,

$$[1 - r\gamma(\rho)]\, \mathrm{E}\{\sum_{m=1}^{\mathbf{n}} r^m \mathrm{e}^{-\rho \mathbf{w}_m}\} = r[1 - r^{\mathbf{n}} \mathrm{e}^{\rho \mathbf{i}}]. \tag{2.10}$$

PROOF. For the indicator function of an event A we shall use the notation (A), i.e.,

$$\begin{aligned} (A) &:= 1 \quad \text{if} \quad A \quad \text{occurs,} \\ &:= 0 \quad ,, \quad A \quad \text{does not occur.} \end{aligned} \tag{2.11}$$

All the expectations in (2.10) exist for $|r| < 1$, Re $\rho = 0$, and so from (2.9): for $|r| < 1$, Re $\rho = 0$,

$$r[1 - \mathrm{E}\{r^{\mathbf{n}} \mathrm{e}^{\rho \mathbf{i}}\}] = \mathrm{E}\{\sum_{m=1}^{\mathbf{n}} r^m \mathrm{e}^{-\rho \mathbf{w}_m} [1 - r \mathrm{e}^{-\rho \boldsymbol{\gamma}_m}]\} = \tag{2.12}$$

$$\mathrm{E}\{\sum_{m=1}^{\infty} r^m \mathrm{e}^{-\rho \mathbf{w}_m} (\mathbf{n} \geq m)[1 - r \mathrm{e}^{-\rho \boldsymbol{\gamma}_m}]\} = \sum_{m=1}^{\infty} \mathrm{E}\{r^m \mathrm{e}^{-\rho \mathbf{w}_m} (\mathbf{n} \geq m)[1 - r \mathrm{e}^{-\rho \boldsymbol{\gamma}_m}]\} =$$

$$[1 - r\mathrm{E}\{\mathrm{e}^{-\rho \boldsymbol{\gamma}_m}\}] \sum_{m=1}^{\infty} \mathrm{E}\{r^m \mathrm{e}^{-\rho \mathbf{w}_m} (\mathbf{n} \geq m)\} = [1 - r\gamma(\rho)] \mathrm{E}\{\sum_{m=1}^{\mathbf{n}} \mathrm{e}^{-\rho \mathbf{w}_m}\}.$$

Here the first equality follows from (2.9); the second equality in (2.12) follows from the definition of

$\mathbf{n}$, cf.(2.4) and (2.11). Because for $\mathrm{Re}\,\rho = 0$,

$$|r^m \mathrm{e}^{-\rho \mathbf{w}_m}(\mathbf{n} \geq m)[1 - r\mathrm{e}^{-\rho \boldsymbol{\gamma}_m}]| \leq 2|r|^m, \quad m = 1, 2, \ldots,$$

the sum in the third member of (2.12) converges absolutely for $|r| < 1$, $\mathrm{Re}\,\rho = 0$, i.e., summation and expectation can be interchanged, and this motivates the third equality. The fourth one results from the independence of $\mathbf{w}_m$ and $\boldsymbol{\gamma}_m$, cf.(2.1), (2.7)i and (2.8); the last one follows again from (2.4) and (2.11), and so (2.10) has been proved. □

Pollaczek considers in his research the function $\sum_{m=0}^{\infty} r^m \mathrm{E}\{\mathrm{e}^{-\rho \mathbf{w}_m}\}$. Put

$$\Phi(r, \rho) := \sum_{m=1}^{\infty} r^m \mathrm{E}\{\mathrm{e}^{-\rho \mathbf{w}_m}(\mathbf{w}_1 = 0)\}, \quad |r| \leq 1, \quad \mathrm{Re}\,\rho \geq 0. \tag{2.14}$$

The functional equation for $\Phi(r, \rho)$, is formulated in the following theorem, cf. (2.16).

THEOREM 2.2. For the $\mathbf{w}_m$-process with $\mathbf{w}_1 = 0$:

$$\Phi(r, \rho) = \frac{1}{1 - \mathrm{E}\{r^{\mathbf{n}}\}} \mathrm{E}\{r^m \mathrm{e}^{-\rho \mathbf{w}_m}\}, \quad |r| < 1, \ \mathrm{Re}\,\rho \geq 0. \tag{2.15}$$

$$[1 - r\gamma(\rho)]\Phi(r, \rho) = \frac{1}{1 - \mathrm{E}\{r^{\mathbf{n}}\}}[1 - \mathrm{E}\{r^{\mathbf{n}} \mathrm{e}^{\rho \mathbf{i}}\}], \quad |r| < 1, \ \mathrm{Re}\,\rho = 0. \tag{2.16}$$

PROOF. Obviously (2.16) follows immediately from (2.10) and (2.15), so it suffices to prove (2.15).

When the first idle time expires the then arriving customer meets an idle system, similarly, as the first one did, cf.(2.3)ii. Because the distributions of service and interarrival times are independent of the arrival instants, i.e. of m, it follows: for $|r| < 1$, $\mathrm{Re}\,\rho \geq 0$,

$$\begin{aligned} &\mathrm{E}\{\sum_{m=1}^{\infty} r^m \mathrm{e}^{-\rho \mathbf{w}_m}(\mathbf{w}_1 = 0)\} - \mathrm{E}\{\sum_{m=1}^{\mathbf{n}} r^m \mathrm{e}^{-\rho \mathbf{w}_m}\} = \mathrm{E} \sum_{m=\mathbf{n}+1}^{\infty} r^m \mathrm{e}^{-\rho \mathbf{w}_m}\} = \\ &\mathrm{E}\{\sum_{m=1}^{\infty} r^{m+\mathbf{n}} \mathrm{e}^{-\rho \mathbf{w}_{m+\mathbf{n}}}\} = \mathrm{E}\{r^{\mathbf{n}}\}\mathrm{E}\{\sum_{\mathrm{m}=1}^{\infty} r^{\mathrm{m}} \mathrm{e}^{-\rho \mathbf{w}_{m+\mathbf{n}}}\} = \\ &\mathrm{E}\{r^{\mathbf{n}}\}\mathrm{E}\{\sum_{m=1}^{\infty} r^m \mathrm{e}^{-\rho \mathbf{w}_m}(\mathbf{w}_1 = 0)\}. \end{aligned} \tag{2.17}$$

Here the last but one equality follows from the independence of $\mathbf{n}$ and $\mathbf{w}_{m+\mathbf{n}}$, cf.(2.1), (2.4) and (2.7). The last equality follows from the fact that of the two sequences

$$\{\mathbf{w}_m, m = 1, 2, \ldots\} \quad \text{and} \quad \{\mathbf{w}_{m+\mathbf{n}}, m = 1, 2, \ldots\}$$

any two subsets $\{\mathbf{w}_m, m \in M\}$ and $\{\mathbf{w}_{m+\mathbf{n}}, m \in M\}$ of stochastic variables, with M a finite subset of $\{1, 2, \ldots\}$, have the same $|M|$-dimensional joint distribution for every M with $|M| < \infty$, $|M|$ being the number of elements of M. The absolute convergence of the sums in (2.17) follows as in the proof of theorem 2.1, cf.(2.13), and it is seen that (2.15) follows from (2.17). □

An important point in the analysis of the $\mathbf{w}_m$-process is the investigation of the distribution of $\mathbf{w}_m$ for $m \to \infty$. In investigating this limit it is necessary to distinguish between the cases where $\boldsymbol{\gamma}_m = \boldsymbol{\tau}_m - \boldsymbol{\sigma}_{m+1}$, cf.(2.8), has or has not a lattice distribution, cf.[2], p.284. Here we shall only discuss the case:

γ_m is not a lattice variable; (2.18)

it includes the case with τ_m or σ_{m+1} having a continuous distribution, as considered by Pollaczek, cf.[14], p.121.

By using *renewol theory* it may be shown, cf.[2], p.363, and [3], that (2.18) implies:

if $\mathrm{E}\{\mathbf{n}\} < \infty$ *then* $\mathrm{W}(w) := \lim_{m\to\infty} \Pr\{\mathbf{w}_m < w\}$ *exists for all* $w \in (0,\infty)$ *and* $\mathrm{W}(w)$ *is a true probability distribution, i.e.* $\mathrm{W}(w) \to 1$ *for* $w \to \infty$. (2.19)

Hence if $\mathbf{n}$ has a finite first moment (so that then $\mathbf{n}$ is necessarily finite with probability one) we may define a stochastic variable $\mathbf{w}$ of which the distribution $\mathrm{W}(\cdot)$ is the limiting distribution of the sequence $\{\mathbf{w}_m, m = 1,2,\ldots\}$; and from Feller's convergence theorem for L.S.-transforms, cf.[7], [2], it then follows that

$$\begin{aligned}\Phi(\rho) &:= \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\} = \lim_{m\to\infty} \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}_m}\}, \quad \mathrm{Re}\rho \geq 0,\\ \Phi(0) &= 1.\end{aligned} \tag{2.20}$$

The following theorem formulates the functional equation, see (2.22), for the limiting distribution $\mathrm{W}(\cdot)$ of the waiting time process.

THEOREM 2.3. If $\mathrm{E}\{\mathbf{n}\} < \infty$ and (2.18) applies then:

$$\Phi(\rho) = \frac{1}{\mathrm{E}\{\mathbf{n}\}}\mathrm{E}\{\sum_{m=1}^{\mathbf{n}} \mathrm{e}^{-\rho\mathbf{w}_m}\}, \quad \mathrm{Re}\,\rho \geq 0. \tag{2.21}$$

$$[1-\gamma(\rho)]\Phi(\rho) = \frac{1}{\mathrm{E}\{\mathbf{n}\}}[1 - \mathrm{E}\{\mathrm{e}^{\rho\mathbf{i}}\}], \quad \mathrm{Re}\,\rho = 0. \tag{2.22}$$

PROOF. From: for $|r| < 1$, $\mathrm{Re}\,\rho \geq 0$,

$$|\mathrm{E}\{\sum_{m=1}^{\mathbf{n}} r^m \mathrm{e}^{-\rho\mathbf{w}_m}\}| \leq \mathrm{E}\{\sum_{m=1}^{\mathbf{n}} |r^m \mathrm{e}^{-\rho\mathbf{w}_m}|\} \leq \mathrm{E}\{\sum_{m=1}^{\mathbf{n}} 1\} = \mathrm{E}\{\mathbf{n}\}, \tag{2.23}$$

it follows that if $\mathrm{E}\{\mathbf{n}\} < \infty$ then we may take in (2.10) the limit for $r \uparrow 1$ and interchange this limit and E, and so we obtain: for $\mathrm{Re}\,\rho = 0$.

$$[1-\gamma(\rho)]\mathrm{E}\{\sum_{m=1}^{\mathbf{n}} \mathrm{e}^{-\rho\mathbf{w}_m}\} = 1 - \mathrm{E}\{\mathrm{e}^{\rho\mathbf{i}}\}. \tag{2.24}$$

An Abelian theorem, cf.[2], implies that if $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}_m}\}$, $\mathrm{Re}\,\rho \geq 0$, has a limit for $m \to \infty$ then the following limit exists and for $\mathrm{Re}\,\rho \geq 0$,

$$\Phi(\rho) = \lim_{r\uparrow 1}(1-r)\Phi(r,\rho). \tag{2.25}$$

Because

$$\lim_{r\uparrow 1} \frac{1-\mathrm{E}\{r^{\mathbf{n}}\}}{1-r} = \mathrm{E}\{\mathbf{n}\},$$

the relation (2.21) follows from (2.25), and (2.22) follows from (2.21) and (2.24), and so the theorem has been proved. □

3. The boundary value problem

In section 2 we have derived the functional equation (2.22), i.e., if $\mathrm{E}\{\mathbf{n}\} < \infty$ and (2.18) holds then: for $\mathrm{Re}\,\rho = 0$,

$$[1 - \gamma(\rho)]\Phi(\rho) = \frac{1}{\mathrm{E}\{\mathbf{n}\}}[1 - \mathrm{E}\{e^{\rho \mathbf{i}}\}]. \tag{3.1}$$

From the definition of $\Phi(\rho)$ and $\mathrm{E}\{e^{\rho \mathbf{i}}\}$, cf. (2.20) and (2.7)ii, it follows that:

i. $\Phi(\rho) = \mathrm{E}\{e^{-\rho \mathbf{w}}\}$ is a regular function of ρ for $\mathrm{Re}\,\rho > 0$, which is continuous for $\mathrm{Re}\,\rho \geq 0$; (3.2)

ii. $\mathrm{E}\{e^{\rho \mathbf{i}}\}$ is a regular function of ρ for $\mathrm{Re}\,\rho < 0$, which is continuous for $\mathrm{Re}\,\rho \leq 0$.

The relation (3.1) together with the conditions (3.2) formulate a boundary value problem for the unknown functions $\Phi(\rho), \mathrm{Re}\,\rho \geq 0$, and $\mathrm{E}\{e^{\rho \mathbf{i}}\}$, $\mathrm{Re}\,\rho \leq 0$. It is actually a Riemann Boundary Value Problem with the line $\mathrm{Re}\,\rho = 0$ as the line of discontinuity, cf.[8], [12].

Remark 3.1. Similarly, the relation (2.16) formulates a Riemann Boundary Value Problem, in the conditions (3.2) $\Phi(\rho)$ and $\mathrm{E}\{e^{\rho \mathbf{i}}\}$, should then be replaced by

$$\Phi(r, \rho) \quad \text{and} \quad \mathrm{E}\{r^{\mathbf{n}} e^{\rho \mathbf{i}}\}$$

for fixed r with $|r| < 1$. □

The boundary value problem formulated in remark 3.1 has actually a somewhat simpler structure than the one described by (3.1), and (3.2), this being due to the fact that in (3.1) the coefficient of $\Phi(\rho)$ is zero for $\rho = 0$, whereas that of $\Phi(r, \rho)$ in (2.16) differs always from zero for $\mathrm{Re}\,\rho = 0$, $|r| < 1$, and as such the boundary value problem formulated in remark 3.1 can be analysed by the standard technique, cf.[6].

In the present study we shall analyse the Riemann Boundary Value Problem formulated by (3.1) and (3.2) under the following conditions:

i. $\mathrm{E}\{\mathbf{n}\} < \infty$; (3.3)

ii. $0 \leq \delta_1 := \lim\limits_{\substack{|\xi| \to \infty \\ \mathrm{Re}\xi = 0}} |\gamma(\xi)| \leq 1,$

iii. $\mathrm{E}\{|\sim_m|^{1+\delta_2}\} < \infty \quad \text{for a} \quad \delta_2 > 0;$

iv. $|\gamma(\xi_1) - \gamma(\xi_2)| \leq A|\frac{1}{|\xi_1|} - \frac{1}{|\xi_2|}|^{\delta_3}, |\xi_j| \to \infty, \quad \mathrm{Re} \quad \xi_j = 0, \quad j = 1, 2,$

$0 < \delta_3 \leq 1, \quad A$ a constant.

Remark 3.2. Note that (3.1) is based on (3.3)i, cf. theorem 2.3. The condition (3.3)ii implies that γ_m is not a lattice variable, cf.[11], so (3.3)i, ii, imply the validity of (3.1). The conditions (3.3)ii, iv are introduced to guarantee the existence of several principal value integrals, see lemma A.1 and the validity of the Plemelj-Sokhotski formulas, cf.(3.6). Note that (3.3)iv implies the existence of the limit in (3.3)ii and hence excludes the case that $\Pr\{\gamma_m = \mathrm{K}\} > 0$ for a finite $\mathrm{K} \neq 0$; concerning (3.3)iv, see also [7], section XV.4. □

Lemma 3.1. *The conditions (3.3)i, ii imply*

i. $\mathrm{E}\{\mathbf{i}\} = (\alpha - \beta)\mathrm{E}\{\mathbf{n}\}$, (3.4)

ii. $\alpha > \beta$.

PROOF. From (2.20) we have $\Phi(0) = 1$, and from (2.2) and (3.1) it follows that

$$\lim \frac{1}{\rho}[1 - \mathrm{E}\{\mathrm{e}^{\rho \mathbf{i}}\}] \quad \text{for} \quad \rho \to 0, \ \mathrm{Re}\,\rho = 0,$$

exists and so a simple calculation leads to (3.4)i. Because $\mathbf{i} \geq 0$, cf.(2.7), it follows that $\alpha \geq \beta$. Because γ_m is not a lattice variable, cf. remark 3.2, it follows from (3.3) that $\Pr\{\gamma_m \neq 0\} > 0$, and so $\Pr\{\mathbf{i} = 0\} < 1$, hence $\mathrm{E}\{\mathbf{i}\} > 0$, i.e. $\alpha > \beta$. □

REMARK 3.3. The relation (3.4) also follows from, cf.(2.7),

$$\mathbf{i} = -\sum_{m=1}^{\mathbf{n}} (\boldsymbol{\tau}_m - \boldsymbol{\sigma}_{m+1}),$$

by applying Wald's theorem, note that $\mathbf{n}$ is a stopping time for the sequence $\{\boldsymbol{\tau}_m - \boldsymbol{\sigma}_{m+1}, m = 1, 2, 3 \ldots\}$.

We next start with the analysis of the boundary value problem formulated by (3.1) and (3.2). Put

$$H(\rho) := \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Re}\,\xi=0} \{\log \frac{1-\gamma(\xi)}{(\beta-\alpha)\xi}\} \frac{\rho}{(\xi-\rho)\xi} \mathrm{d}\xi, \tag{3.5}$$

for arbitrary finite ρ. From lemma A.2 it is seen that this integral is well defined if (3.3) holds, and that its integrand satisfies the Hölder conditions on $\mathrm{Re}\,\xi = 0$, cf.(3.3)iv and [8], section 4.6. Hence the following limits exist and the Plemelj-Sokhotski formulas hold, i.e., for $\mathrm{Re}\,\rho = 0$, $|\rho| < \infty$,

$$H^-(\rho) := \lim_{\substack{t \to \rho \\ \mathrm{Re}\,t > 0}} H(t) = -\frac{1}{2} \log \frac{1-\gamma(\rho)}{(\beta-\alpha)\rho} + H(\rho), \tag{3.6}$$

$$H^+(\rho) := \lim_{\substack{t \to \rho \\ \mathrm{Re}\,t < 0}} H(t) = \frac{1}{2} \log \frac{1-\gamma(\rho)}{(\beta-\alpha)\rho} + H(\rho),$$

$$H^+(\rho) - H^-(\rho) = \log \frac{1-\gamma(\rho)}{(\beta-\alpha)\rho},$$

and $H(\rho)$ is regular for $\mathrm{Re}\,\rho > 0$ and for $\mathrm{Re}\,\rho < 0$, continuous for $\mathrm{Re}\,\rho \leq 0$ and for $\mathrm{Re}\,\rho \geq 0$.

REMARK 3.2. Actually the principal branch for the logarithm in (3.5) and (3.6) should be specified, but from the following analysis, it is readily seen that the choice of this principal value is irrelevant for the results to be derived, see (3.11), □

From (3.4)i and (3.6) it is seen that (3.1) may be rewritten as: for $\mathrm{Re}\,\rho = 0$,

$$\Phi(\rho)\mathrm{e}^{-H^-(\rho)} = \frac{1 - \mathrm{E}\{\mathrm{e}^{\rho \mathbf{i}}\}}{-\rho \mathrm{E}\{\mathbf{i}\}} \mathrm{e}^{-H^+(\rho)}. \tag{3.7}$$

Put

$$Z(\rho) := \Phi(\rho)\mathrm{e}^{-H(\rho)}, \qquad \mathrm{Re}\,\rho > 0,$$
$$:= \frac{1-\mathrm{E}\{\mathrm{e}^{\rho \mathbf{i}}\}}{-\rho \mathrm{E}\{\mathbf{i}\}}\mathrm{e}^{-H(\rho)}, \qquad \mathrm{Re}\,\rho < 0. \tag{3.8}$$

From the definition of $\Phi(\rho) = \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}$, cf.(2.20), and from (3.5) and (3.6) it is seen that $Z(\rho)$ is regular in $\mathrm{Re}\,\rho > 0$, continuous in $\mathrm{Re}\,\rho \geq 0$, similarly, it is regular in $\mathrm{Re}\,\rho < 0$, continuous in $\mathrm{Re}\,\rho \leq 0$. Hence it follows from (3.8) that $Z(\rho)$ for $\mathrm{Re}\,\rho \leq 0$ and for $\mathrm{Re}\,\rho \geq 0$ are each other's analytic continuations. We next consider $Z(\rho)$ for $|\rho| \to \infty$.

From (2.20), (3.6) and lemma A.2, we have

$$\lim_{\substack{|\rho|\to\infty \\ \mathrm{Re}\,\rho>0}} \Phi(\rho) = \Pr\{\mathbf{w}=0\}, \qquad \lim_{\substack{|\rho|\to\infty \\ \mathrm{Re}\,\rho>0}} |H(\rho)| < \infty, \tag{3.9}$$

$$\lim_{\substack{\rho\to\infty \\ \mathrm{Re}\,\rho<0}} \left|\frac{1-\mathrm{E}\{\mathrm{e}^{\rho \mathbf{i}}\}}{-\rho \mathrm{E}\{\mathbf{i}\}}\mathrm{e}^{\log\rho}\right| < \infty, \quad \lim_{\substack{|\rho|\to\infty \\ \mathrm{Re}\,\rho=0}} |\Phi(\rho)\mathrm{e}^{-H^-(\rho)}| < \infty,$$

$$\lim_{\substack{\mathrm{Re}\,\rho\to\infty \\ \mathrm{Re}\,\rho=0}} \left|\frac{1-\mathrm{E}\{\mathrm{e}^{\rho \mathbf{i}}\}}{-\rho \mathrm{E}\{\mathbf{i}\}}\mathrm{e}^{-H^+(\rho)}\right| < \infty.$$

Consequently, $Z(\rho)$ is uniformly bounded in ρ and hence by using Liouville's theorem, cf.[19], it is a constant, say, D so

$$\text{i.} \quad \Phi(\rho) = \mathrm{e}^{H(\rho)}D, \qquad \mathrm{Re}\,\rho > 0,$$
$$= \mathrm{e}^{H^-(\rho)}D, \qquad \mathrm{Re}\,\rho = 0; \tag{3.10}$$

$$\text{ii.} \quad \frac{1-\mathrm{E}\{\mathrm{e}^{\rho \mathbf{i}}\}}{-\rho \mathrm{E}\{\mathbf{i}\}} = \mathrm{e}^{H(\rho)}D, \qquad \mathrm{Re}\,\rho > 0,$$
$$= \mathrm{e}^{H^+(\rho)}D, \qquad \mathrm{Re}\,\rho = 0,$$

and

$$D = 1,$$

because $\Phi(0) = 1$, cf.(2.20), and $H^-(\rho) = 0$ for $\rho = 0$.

THEOREM 3.1. *For the conditions (3.3) holds,*

$$\text{i.} \quad \Phi(\rho) = \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\} = \mathrm{e}^{\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{Re}\,\xi=0}\{\log\frac{1-\gamma(\xi)}{(\beta-\alpha)\xi}\}\frac{\rho}{(\xi-\rho)\xi}\mathrm{d}\xi}, \qquad \mathrm{Re}\,\rho > 0,$$

$$\text{ii.} \quad \frac{1-\mathrm{E}\{\mathrm{e}^{\rho \mathbf{i}}\}}{-\rho \mathrm{E}\{\mathbf{i}\}} = \mathrm{e}^{\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{Re}\,\xi=0}\{\log\frac{1-\gamma(\xi)}{(\beta-\alpha)\xi}\}\frac{\rho}{(\xi-\rho)\xi}\mathrm{d}\xi}, \qquad \mathrm{Re}\,\rho < 0,$$

PROOF. The relations (3.11) follow immediately from (3.5), (3.10) and (3.11). □

REMARK 3.4. Theorem 3.1 represents the solution of the Riemann Boundary Value Problem formulated by (3.1) and (3.2) with $\Phi(0) = 1$. From the derivations above it is readily seen that (3.11) is the unique solution of this boundary value problem. The importance of the present result is the fact that the relations (3.11) represent under fairly weak conditions the Laplace-Stieltjes transforms of the distributions of the waiting time $\mathbf{w}$ and the idle time $\mathbf{i}$ in terms of an integral of $\gamma(\rho)$, the L.S.-transform of the distribution of $\tau_m - \sigma_{m+1}$, and as such the derived formulas are new. □

REMARK 3.5. Because $\Phi(\rho) \to \Pr\{\mathbf{w}=0\}$ for $|\rho| \to \infty$, $\arg|\rho| \leq \frac{1}{2}\pi$ we can obtain an expression for it from (3.11); however, it is in general not permitted to exchange in the integrals of (3.11) integration and the limit for $|\rho| \to \infty$. □

REMARK 3.6. Suppose that $\beta(\rho)$, cf.(2.2), is regular for $|\rho| < \epsilon$ for some $\epsilon > 0$. Actually, this implies that all moment of τ_m are finite, and that $\beta(\rho)$ is regular for $\mathrm{Re}\,\rho > -\epsilon$, cf.[11], so we have that $\gamma(\rho)$ is regular for $-\epsilon < \mathrm{Re}\,\rho < 0$, which implies that

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Re}\,\xi=0} \{\log \frac{1-\gamma(\xi)}{(\beta-\alpha)\xi}\} \frac{\rho\, \mathrm{d}\xi}{(\xi-\rho)\xi} = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Re}\,\xi=0-} \{\log \frac{1-\gamma(\xi)}{(\beta-\alpha)\xi}\} \frac{\rho\, \mathrm{d}\xi}{(\xi-\rho)\xi}, \tag{3.12}$$

as it is seen by using Cauchy's integral theorem.

By using again Cauchy's theorem with a contour in the half plane $\mathrm{Re}\,\xi < 0$ it follows that

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Re}\,\xi=0-} \{\log(\beta-\alpha)\xi\} \frac{\rho}{(\xi-\rho)\xi} \mathrm{d}\xi = 0 \quad \text{for} \quad \mathrm{Re}\,\rho > 0, \tag{3.13}$$

From (3.11)i, (3.12) and (3.13) it follows that: for $\mathrm{Re}\,\rho > 0$,

$$\Phi(\rho) = \mathrm{e}^{\frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Re}\,\xi=0-} \{\log(1-\gamma(\xi)\} \frac{\rho}{(\xi-\rho)\xi} \mathrm{d}\xi}. \tag{3.14}$$

This relation (3.27) has originally been obtained by Pollaczek [14], p.80. He bases his derivation on the assumptions that the distributions of τ_m and σ_{m+1} are both continuous, that they have L.S.-transforms which are both regular at $\rho = 0$, and that $\alpha - \beta > 0$; note that these assumptions imply the conditions (3.3). □

REMARK 3.7. For the case that $\alpha(\rho)$ or $\beta(\rho)$ have rational L.S.-transforms, see [2], section II.5.10, II.5.11. □

We shall conclude this section with the formulation of Pollaczek's integral equation.

THEOREM 3.2. POLLACZEK'S INTEGRAL EQUATION.

If $\alpha(\rho)$ is regular at $\rho = 0$ then: for $|r| < 1$, $\mathrm{Re}\,\rho > 0$,

$$\Phi(r,\rho) = \frac{r}{2\pi \mathrm{i}} \int_{\mathrm{Re}\,\xi=0+} \gamma(\xi)\Phi(r,\xi) \frac{\rho}{(\xi-\rho)\xi} \mathrm{d}\xi + r. \tag{3.15}$$

PROOF. If $\alpha(\rho)$ is regular of $\rho = 0$ then a $\epsilon > 0$ exists such that $\alpha(-\xi)$ is regular for $\mathrm{Re}\,\xi < \epsilon$, and so $\gamma(\xi)$ is here also regular. Consequently, it is seen that the lefthand side of (2.16) can be continued analytically into $0 < \mathrm{Re}\,\rho < \epsilon$, which implies that the righthand side of (2.16) can be continued analytically into $0 < \mathrm{Re}\,\rho < \epsilon$, so: for $0 < \mathrm{Re}\,\xi < \epsilon$, $|r| < 1$,

$$[1 - r\gamma(\xi)]\Phi(r,\xi) = \frac{r}{1 - \mathrm{E}\{r^{\mathbf{n}}\}}[1 - \mathrm{E}\{r^{\mathbf{n}} \mathrm{e}^{\xi \mathbf{i}}\}]. \tag{3.17}$$

From (3.17) it follows for $0 < \mathrm{Re}\,\xi < \epsilon$, $\mathrm{Re}\,\xi < \mathrm{Re}\,\rho$, $|r| < 1$,

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Re}\,\xi=0+} [1 - r\gamma(\xi)]\Phi(r,\xi) \frac{\rho}{(\xi-\rho)\xi} \mathrm{d}\xi =$$

$$\frac{r}{1-\mathrm{E}\{r^{\mathbf{n}}\}}\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{Re}\,\xi=0+}[1-\mathrm{E}\{r^{\mathbf{n}}\mathrm{e}^{\xi\mathbf{i}}\}]\frac{\rho}{(\xi-\rho)\xi}\mathrm{d}\xi. \tag{3.18}$$

Because $\mathrm{E}\{\mathrm{e}^{\xi\mathbf{i}}\}$ is regular for $\mathrm{Re}\,\xi < 0$, it follows by using Cauchy's theorem to evaluate the second integral in (3.18), that it is equal to the residue of its integrand at $\xi = 0$ and so for $0 < \mathrm{Re}\,\rho$,

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{Re}\,\xi=0+}[1-r\gamma(\xi)]\Phi(r,\xi)\frac{\rho}{(\xi-\rho)\xi}\mathrm{d}\xi = r;$$

and this leads again by using Cauchy's theorem to (3.15) for $0 < \mathrm{Re}\,\rho$. □

4. EPILOGUE

In the two preceding sections the present approach of the waiting time process for the GI/G/1 single server model has been exposed. The results of the theory of Regenerative Processes, of Fluctuation Theory and those of the theory of Riemann Boundary Value Problems are here of basic importance and lead to a fairly simple set up of the analysis of the waiting time process. Around the mid sixties the significance of Regenerative Processes and Fluctuation Theory had been generally acknowledged in probabilistic circles. About ten years earlier the theory of Riemann Boundary Value Problems became known, mainly as a result of the research in continuum mechanics by the Russian school. In the late seventies it was discovered that problems in Applied Probability could be formulated as such boundary value problems,

Pollaczek started his research of queueing problems with unspecified dirtributions of service and interarrival times in the early thirties and at that time the theories mentioned above were not available, at best only in a very rudimentary form. By using Heaviside's formula for the representation of the unit step function by means of a contour integral, Pollaczek succeeded in transforming the sample function relations of the $\mathbf{w}_m$-process, cf.(2.6), into a functional equation, and so complex functions and contour integrals entered in the analysis, cf. theorem 3.2. He further succeeded in solving this integral equation under fairly mild conditions, conditions, which are fully acceptable from a view point of application in engineering. In his comments on the discussion of his paper [16] he expresses his opinion that the analysis of his integral equation should be fully carried out within the theory of complex functions. Only so the subtle points can be understood which occur in the search for analytic results accessible for numerical evaluation; here he deemed probabilistic arguments not to be of much help.

An approach as outlined in the two preceding sections was not within reach of the research of the sixties for the reasons outlined above; the methodological critique by Kingman [9] in 1966 on Pollaczek's work of 1957 and 1961 cannot be justified. The more so since the algebraic approach suggested in [6], although elegant, did not appear very efficient, as time has learned, neither does it contribute to an improvement of Pollaczek's results, i.e. a weakening of the conditions for their validity, such as the regularity conditions of the L.S.-transforms at the origin. Kingman's algebraic approach does not reveal clearly the technical difficulties encountered in the analysis of the many server model GI/G/s, again the use of the theory of complex functions is here a more appropriate tool. Pollaczek has shown how to handle and apply these tools. He has done here much pioneering work, work of great benefit for the third generation of queueing analysists and work of merit for engineering.

APPENDIX A.

In the discussion of section 3 we use several concepts from the theory of complex integration, in particular principal value integrals. We shall briefly discuss them here; for details the reader is referred to [6], [8], [12].

Let $\phi(z)$ be a bounded, continuous function of the complex variable $z = u + \mathrm{i}v$, $a < v < b$. Define:

for Re $z=u$,

$$\Lambda(z) \equiv \frac{1}{2\pi \mathrm{i}} \int_{u+\mathrm{i}a}^{u+\mathrm{i}b} \frac{\phi(\xi)}{\xi - z} \mathrm{d}\xi := \lim_{\xi \downarrow 0} \frac{1}{2\pi \mathrm{i}} \{ \int_{u+\mathrm{i}a}^{u+\mathrm{i}(v-\xi)} + \int_{u+\mathrm{i}(v+\xi)}^{u+\mathrm{i}b} \} \frac{\phi(\xi)}{\xi - z} \mathrm{d}\xi, \tag{a.1}$$

if the limit exists. Then the singular integral in the lefthand side of (a.1) is said to be defined as a Cauchy principal value integral at $z = u + \mathrm{i}v$; for Re $z \neq u$ this integral is called the Cauchy integral of $\phi(\cdot)$.

It may be shown, cf. [6], [11], [12], that the limit in (a.1) exists if $u(z)$ satisfies on $[u + \mathrm{i}a, u + \mathrm{i}b]$ the *Hölder condition*, i.e. there exist positive constants A_1 and μ_1 with $\mu_1 \leq 1$ such that

$$|\phi(z_1) - \phi(z_2)| \leq A|z_1 - z_2|^{\mu_1}, \tag{a.2}$$

for all $z_1 = u + \mathrm{i}v_1$, $z_2 = u + \mathrm{i}v_2$, with $v_1, v_2 \in [a, b]$. In particular the Hölder condition holds if $\phi(z)$ is differentiable in this interval.

For the case: $a = -\infty$, $b = \infty$, and Re $z \neq u$, define

$$\frac{1}{2\pi \mathrm{i}} \int_{u-\mathrm{i}\infty}^{u+\mathrm{i}\infty} \frac{\phi(\xi)}{\xi - z} \mathrm{d}\xi := \lim_{R \to \infty} \frac{1}{2\pi \mathrm{i}} \int_{u-\mathrm{i}R}^{u+\mathrm{i}R} \frac{\phi(\xi)}{\xi - z} \mathrm{d}\xi, \tag{a.3}$$

if this limit exists. Then the integral in the lefthand side of (a.3) is said to be defined as a Cauchy *principal* value integral *at infinity.*

For Re $z = u$ the integral in (a.3) is defined by:

$$\frac{1}{2\pi \mathrm{i}} \int_{u-\mathrm{i}\infty}^{u+\mathrm{i}\infty} \frac{\phi(\xi)}{\xi - z} \mathrm{d}\xi := \lim_{R \to \infty} \lim_{\epsilon \downarrow 0} \frac{1}{2\pi \mathrm{i}} \{ \int_{u-\mathrm{i}R}^{u+\mathrm{i}(v-\epsilon)} + \int_{u+\mathrm{i}(v+\epsilon)}^{u+\mathrm{i}R} \} \frac{\phi(\xi)}{\xi - z} \mathrm{d}\xi, \tag{a.4}$$

if it exists.

Suppose that $\phi(z)$ satisfies the Hölder condition then the following limits exist and relations hold: for $z = u + \mathrm{i}v$, $a < v < b$,

$$\Lambda^+(z) \equiv \lim_{\substack{t \to z \\ \mathrm{Re}\, t < u}} \Lambda(t) = \frac{1}{2}\phi(z) + \Lambda(z), \tag{a.5}$$

$$\Lambda^-(z) \equiv \lim_{\substack{t \to z \\ \mathrm{Re}\, t > u}} \Lambda(t) = -\frac{1}{2}\phi(z) + \Lambda(z),$$

$$\Lambda^+(z) - \Lambda^-(z) = \phi(z);$$

they are the so called Plemelj-Sokhotski formulas. These relations also apply for the integral defined in (a.3) if for $|z_1|, |z_2|$, sufficiently large

$$|\phi(z_1) - \phi(z_2)| \leq A_2 |\frac{1}{|z_1|} - \frac{1}{|z_2|}|^{\mu_2}, \tag{a.6}$$

with A_2 and μ_2 positive constants and $0 < \mu_2 \leq 1$, cf.[8], p.37.

LEMMA A.1. *If* $\mathrm{E}\{|\gamma_m|^{1+\delta_2}\} < \infty$ *for a* $\delta_2 > 0$ *then*

$$f(\xi) := \log \frac{1 - \gamma(\xi)}{\xi}, \quad \mathrm{Re}\, \xi = 0,$$

satisfies the Hölder condition on any segment on Re $\xi = 0$ *with finite endpoints.*

PROOF. From [10] p.199, we have for $0 < \delta_2 < 1$, $\mathrm{Re}\,\xi = 0$,

$$\frac{1-\gamma(\xi)}{\xi} \approx 1 - 2^{1-\delta_2}\theta' \frac{\mathrm{E}\{|\tau_m - \sigma_{m+1}|^{1+\delta_2}\}}{\beta-\alpha} \frac{|\xi|^{\delta_2}}{1+\delta_2} \quad \text{for} \quad |\xi| \to 0,$$

so

$$f(\xi) \approx \xi^{\delta_2} A_3 \quad \text{for} \quad \xi \to 0, \mathrm{Re}\,\xi = 0, \tag{a.7}$$

here θ' and A_3 are constants. We have, e.g. cf.[8], p. 57,

$$|{\xi_1}^{\delta_2} - {\xi_2}^{\delta_2}| < |\xi_1 - \xi_2|^{\delta_2},$$

and so (a.7) implies that on $\mathrm{Re}\,\xi = 0$ a neighbourhood of $\xi = 0$ exists where $f(\xi)$ satisfies the Hölder condition, cf.(a.2). Further, the differentiability of $f(\xi)$ at every finite point $\xi \neq 0$ with $\mathrm{Re}\xi = 0$ implies that it satisfies the Hölder condition on every segment of $\mathrm{Re}\,\xi = 0$, $\xi \neq 0$, with finite endpoints. Since $f(\xi)$ also satisfies the Hölder condition in a neighbourhood of $\xi = 0$, $\mathrm{Re}\,\xi = 0$, the statement has been proved, the Hölder exponent μ being equal to δ_2. □

LEMMA A.2. *Under the condition (3.3)i,*
i. the integral

$$H_1(\rho) := \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Re}\,\xi=0} \{\log \frac{1-\gamma(\xi)}{\xi}\} \frac{\rho}{(\xi-\rho)\xi} \mathrm{d}\xi \tag{a.8}$$

exists;

ii. if condition (3.3)iv holds then: for $|\rho| \to \infty$,

$$\begin{aligned} H_1(\rho) &= \quad -\log\rho + \Lambda_1(\rho), \quad \mathrm{Re}\,\rho < 0, \\ &= -\tfrac{1}{2}\log\rho + \Lambda_2(\rho), \quad\quad\;\; = 0, \\ &= \qquad\qquad\quad \Lambda_3(\rho), \quad\quad\;\; > 0. \end{aligned}$$

here $\Lambda_j(\rho)$ are functions regular at $|\rho| = \infty$, i.e. $\Lambda_j(\frac{1}{r})$ is regular at $r = 0$;

iii. if conditions (3.3)ii and iv hold then for any finite $\Delta > 0$, with δ_1 as defined in (3.3)ii,

$$\begin{aligned} \frac{1}{2\pi\mathrm{i}} \int_{\substack{\mathrm{Re}\xi=0 \\ |\xi|>\Delta}} \{\log(1-\gamma(\xi))\} \frac{\mathrm{d}\xi}{\xi-\rho} &\to \frac{1}{2}\log(1-\delta_1) \quad \text{for} \quad |\rho| \to \infty, \quad \mathrm{Re}\,\rho > 0, \\ &\to 0 \quad \text{for} \quad |\rho| \to \infty, \quad \mathrm{Re}\,\rho = 0, \\ &\to -\frac{1}{2}\log(1-\delta_1) \quad \text{for} \quad |\rho| \to \infty, \quad \mathrm{Re}\,\rho < 0. \end{aligned}$$

REMARK a.1. Note that, cf. (3.5),

$$
\begin{aligned}
H(\rho) &= H_1(\rho) - \log(\beta-\alpha)\frac{1}{2\pi \mathrm{i}}\int_{\mathrm{Re}\,\xi=0}\frac{\rho}{(\xi-\rho)\xi}\mathrm{d}\xi \quad \text{for} \quad \mathrm{Re}\,\rho>0\\
&= H_1(\rho) - \log(\beta-\alpha) \qquad ,, \quad \mathrm{Re}\,\rho>0\\
&= H_1(\rho) \qquad ,, \quad =0,\\
&= H_1(\rho) + \frac{1}{2}\log(\beta-\alpha) \qquad ,, \quad <0.
\end{aligned}
$$

□

PROOF OF LEMMA A.2. For $|\rho|<\infty$ and finite $R>0$ the integral

$$
\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}R}^{+\mathrm{i}R}\{\log\frac{1-\gamma(\xi)}{\xi}\}\frac{\rho}{(\xi-\rho)\xi}\mathrm{d}\xi, \tag{a.9}
$$

exists as a Cauchy integral for $\mathrm{Re}\,\rho\neq 0$, and as a principal value singular Cauchy integral for $\mathrm{Re}\,\rho=0$, since the integrand satisfies the Hölder condition, cf. lemma A.1. Because for $|\xi|\to\infty$, $\mathrm{Re}\,\xi=0$,

$$
|\frac{1}{\xi-\rho}|\to 0 \quad \text{and} \quad |\frac{\log\xi}{\xi}|\to 0, \tag{a.10}
$$

the integrals

$$
\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}R_1}^{\mathrm{i}R_2}\frac{\mathrm{d}\xi}{\xi(\xi-\rho)} \quad \text{and} \quad \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}R_1}^{\mathrm{i}R_2}\frac{\log\xi}{\xi}\frac{\mathrm{d}\xi}{\xi-\rho}, \tag{a.11}
$$

with $R_1>0$, $R_2>0$, both converge to zero for $R_1\to\infty$, $R_2\to\infty$, independently of each other, and so the first statement has been proved.

To prove the third statement write with $\Delta>0$,

$$
\begin{aligned}
H_1(\rho) &= \frac{1}{2\pi\mathrm{i}}\int_{\xi=-\mathrm{i}\Delta}^{\mathrm{i}\Delta}\{\log\frac{1-\gamma(\xi)}{\xi}\}\frac{\rho\mathrm{d}\xi}{(\xi-\rho)\xi} + \frac{1}{2\pi\mathrm{i}}\int_{\substack{|\xi|>\Delta\\ \mathrm{Re}\,\xi=0}}\{\log(1-\gamma(\xi))\}\frac{\rho\mathrm{d}\xi}{(\xi-\rho)\xi}\\
&-\frac{1}{2\pi\mathrm{i}}\int_{\substack{|\xi|>\Delta\\ \mathrm{Re}\,\xi=0}}\{\log\}\frac{\rho\mathrm{d}\xi}{(\xi-\rho)\xi}
\end{aligned} \tag{a.12}
$$

The arguments used above show that all these integrals are well defined for $|\rho|<\infty$. It is readily verified that the following limit exists and

$$
|\lim_{|\rho|\to\infty}\frac{1}{2\pi\mathrm{i}}\int_{\xi=-\mathrm{i}\Delta}^{\mathrm{i}\Delta}\{\log\frac{1-\gamma(\xi)}{\xi}\}\frac{\rho\mathrm{d}\xi}{(\xi-\rho)\xi}| = |-\frac{1}{2\pi\mathrm{i}}\int_{\xi=-\mathrm{i}\Delta}^{\mathrm{i}\Delta}\{\log\frac{1-\gamma(\xi)}{\xi}\}\frac{\mathrm{d}\xi}{\xi}|<\infty, \tag{a.13}
$$

for finite Δ.

The condition (3.3)iv implies that for $\mathrm{Re}\,\xi_j=0$,

$$
|\log(1-\gamma(\xi_1))-\log(1-\gamma(\xi_2))|\leq A_4|\frac{1}{|\xi_1|}-\frac{1}{|\xi_3|}|^{\delta_3}, \quad 0<\delta_3\leq 1,
$$

for $|\xi_j|\to\infty$ and with A_4 a positive constant. Hence, cf.[8] p.33, [12] p.110, for $|\rho|<\infty$ the integral

$$
\frac{1}{2\pi\mathrm{i}}\int_{\substack{|\xi|>\Delta\\ \mathrm{Re}\,\xi=0}}\{\log(1-\gamma(\xi))\}\frac{\mathrm{d}\xi}{\xi-\rho} \tag{a.14}
$$

exists as principal value integral at infinity if $\mathrm{Re}\,\rho\neq 0$, and if $\mathrm{Re}\,\rho=0$ also as a principal value singular integral at ρ if $\Delta < \mathrm{Im}\,\rho < \infty$. It is readily seen that the integral has a limit for $|\rho| \to \infty$ and that for $\Delta > 0$, and δ_1 as defined in (3.3)ii,

$$\frac{1}{2\pi i}\int_{\substack{\mathrm{Re}\,\xi=0\\|\xi|>\Delta}}\{\log(1-\gamma(\xi))\}\frac{d\xi}{\xi-\rho} = \frac{1}{2\pi i}\int_{\substack{\mathrm{Re}\,\xi=\Delta\\|\xi|=0}}\{\log\frac{1-\gamma(\xi)}{1-\delta_1}\}\frac{d\xi}{\xi-\rho}+ \\ \{\frac{1}{2\pi i}\int_{\substack{\mathrm{Re}\,\xi=0\\|\xi|>\Delta}}\frac{d\xi}{\xi-\rho}\}\log(1-\delta_1) \tag{a.15}$$

$\to \frac{1}{2}\log(1-\delta_1)$ for $|\rho| \to \infty$ with $\mathrm{Re}\,\rho > 0$,

$\to 0$,, $|\rho| \to \infty$,, $= 0$,

$\to -\frac{1}{2}\log(1-\delta_1)$,, $|\rho| \to \infty$,, < 0,

and so the third statement has been proved; and further we have from (a.15) for $\Delta > 0$,

$$\frac{1}{2\pi i}\int_{\substack{|\xi|>\Delta\\\mathrm{Re}\,\xi=0}}\{\log(1-\gamma(\xi))\}\frac{\rho}{\xi(\xi-\rho)}d\xi + \frac{1}{2\pi i}\int_{\substack{|\xi|>\Delta\\\mathrm{Re}\,\xi=0}}\{\log(1-\gamma(\xi))\}\frac{d\xi}{\xi} = \tag{a.16}$$

$= \frac{1}{2}\log(1-\delta_1)$ for $|\rho| \to \infty$, $\mathrm{Re}\,\rho > 0$,

$= 0$,, $|\rho| \to \infty$, $= 0$,

$= -\frac{1}{2}\log(1-\delta_1)$,, $|\rho| \to \infty$, < 0.

We now complete the proof of the second statement.

To investigate the third integral in (a.12) write

$$\frac{1}{2\pi i}\int_{\substack{|\xi|>\Delta\\\mathrm{Re}\xi=0}}\{\log\xi\}\frac{\rho d\xi}{(\xi-\rho)\xi} = \frac{1}{2\pi i}\{\int_{i\Delta}^{iR} + \int_{-iR}^{-i\Delta} + \int_{iR}^{i\infty} + \int_{-i\infty}^{-iR}\}\{\log\xi\}\frac{\rho d\xi}{(\xi-\rho)\xi}, \tag{a.17}$$

with $R > 0$. The last two integrals in (a.17) can each be made arbitrarily small in absolute value for R sufficiently large, uniformly in ρ, cf.(a.10).

To analyse the first two integrals put

$$\eta = -\frac{1}{\xi}, \quad r = -\frac{1}{\rho}, \tag{a.18}$$

then

$$\frac{1}{2\pi i}\int_{i\Delta}^{iR}\{\log\xi\}\frac{\rho d\xi}{(\xi-\rho)\xi} = -\frac{1}{2\pi i}\int_{i/\Delta}^{i/R}\{\log\eta - \log(-1)\}\frac{d\eta}{\eta - r}, \tag{a.19}$$

$$\frac{1}{2\pi i}\int_{-iR}^{-i\Delta}\{\log\xi\}\frac{\rho d\xi}{(\xi-\rho)\xi} = -\frac{1}{2\pi i}\int_{-i/R}^{-i/\Delta}\{\log\eta - \log(-1)\}\frac{d\eta}{\eta - r}.$$

The asymptotic expressions for $r \to 0$ of the integrals in (a.19) follow from appendix B, cf.(b.12) and (b.21). Because in (a.19) and (b.21) the integration directions are opposite, "right" and "left" have to be interchanged when using (b.21) in (a.19), and so it follows

$$\lim_{\Delta\to 0}\lim_{R\to\infty}\frac{1}{2\pi \mathrm{i}}\{\int_{\mathrm{i}/\Delta}^{\mathrm{i}/R}+\int_{-\mathrm{i}/R}^{-\mathrm{i}/\Delta}\}\frac{\log\eta}{\eta-r}\mathrm{d}\eta = \tag{a.20}$$

$$\begin{array}{lrclll} = & -\log r & + & K_1(r), & r\in\dot{\mathcal{N}}(0) & \text{and } \operatorname{Re} r>0,\\ = & -\frac{1}{2}\log r & + & K_2(r), & r\in\dot{\mathcal{N}}(0) & \quad ,, \qquad =0,\\ = & 0 & + & K_3(r), & r\in\dot{\mathcal{N}}(0) & \quad ,, \qquad <0,\end{array}$$

here $\dot{\mathcal{N}}(a)$ is a punctured neighbourhood of the point a, i.e., $a\notin\dot{\mathcal{N}}(a)$, and $K_j(r)$, $j=1,2,3$, are regular functions of r in a neighbourhood $\mathcal{N}(0)$ of $z=0$. Note that the contributions of $\log(-1)$ in the integrands (a.9) are incorporated in $K_j(r)$. From (a.12), (a.13), (a.16), (a.17), (a.18), (a.19) and (a.20) the relations of the second statement follow. □

Appendix B.

We consider in this appendix the integral

$$\Omega_c(z) := \frac{1}{2\pi\mathrm{i}}\int_L \frac{\log(\tau-c)}{\tau-z}\mathrm{d}\tau, \tag{b.1}$$

i. $L := \{\xi : \operatorname{Im} a < \operatorname{Im}\xi < \operatorname{Im} b, \operatorname{Re}\xi = 0\}$,

$$\operatorname{Im} a \le \operatorname{Im} c \le \operatorname{Im} b, \quad \operatorname{Re} a = \operatorname{Re} b = \operatorname{Re} c = 0; \tag{b.2}$$

ii. the z-plane is cut along the imaginary axis from c via b to infinity, and this cutted z-plane defines the logarithm as a single valued function.

The problem concerns the asymptotic behaviour of $\Omega_c(z)$ for z in the vicinity of c. The analysis of this problem is based on the discussion in Gakbov [8], sections 8.2, ..., 8.5, 45.3. We first state results to be found in [8]. By

$$\mathcal{N}(z) \quad \text{and} \quad \dot{\mathcal{N}}(z),$$

we shall denote a neighbourhood and a punctured neighbourhood of the point z, so $z\in\mathcal{N}(z)$, $z\notin\dot{\mathcal{N}}(z)$. We successively discuss the cases, $c=a$, $c=b$ and $\operatorname{Im} a<\operatorname{Im} c<\operatorname{Im} b$.

i. The case $c=a$.
For this case we have: cf.[8], (8.27),

$$\Omega_a(z) = \omega_2(z,a) + \Lambda_0^{(a)}(z), \qquad z\in\dot{\mathcal{N}}(0)\backslash L, \tag{b.3}$$

$$\Omega_a(t) = \frac{1}{2}[\omega_2^+(t,a)+\omega_2^-(t,a)] + \Lambda_0^{(a)}(t), \quad t\in\dot{\mathcal{N}}(a)\cap L,$$

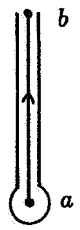
with

$$\Lambda_0^{(a)}(z) \text{ a function regular for } z\in\mathcal{N}(a), \tag{b.4}$$

$$\omega_2(z,a) := -\frac{1}{4\pi\mathrm{i}}\log^2(z-a) + \frac{1}{2}\log(z-a), \quad z\in\dot{\mathcal{N}}\backslash L. \tag{b.5}$$

Note that because of the cut from a via b to ∞, we have: for $t\in\dot{\mathcal{N}}(a)\cap L$,

$$\omega_2^+(t,a) = -\frac{1}{4\pi\mathrm{i}}\log^2(t-a) + \frac{1}{2}\log(t-a), \tag{b.6}$$



$$\omega_2^-(t,a) = -\frac{1}{4\pi i}[\log(t-a)+2\pi i]^2 + \frac{1}{2}\log[(t-a)+2\pi i]$$

$$= -\frac{1}{4\pi i}\log^2(t-a) - \frac{1}{2}\log(t-a),$$

$$\frac{1}{2}\omega_2^+(t,a) + \frac{1}{2}\omega_2^-(t,a) = -\frac{1}{4\pi i}\log^2(t-a).$$

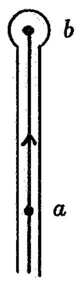
ii. The case $c = b$.

For this case with the cut along L from b via a to $-i\infty$, cf. [8], (8.8), (8.9), we have, cf. [8], (8.30), (8.31),

$$\Omega_b(z) = \phi_2(z,b) + \Lambda_1^{(b)}(z), \qquad z \in \dot{\mathcal{N}}(b)\backslash L,$$

$$\Omega_b(t) = \frac{1}{2}\phi_2^+(t,b) + \frac{1}{2}\phi_2^-(t,b) + \Lambda_2^{(b)}(t), \quad t \in \dot{\mathcal{N}}(b) \cap L, \tag{b.7}$$

with



$$\Lambda_1^{(b)}(z) \textit{ a function regular for } z \in \mathcal{N}(b), \tag{b.8}$$

$$\phi_2(z,b) := \frac{1}{4\pi i}\log^2(z-b) + \frac{1}{2}\log(z-b), \quad z \in \dot{\mathcal{N}}(b)\backslash L. \tag{b.9}$$

It follows

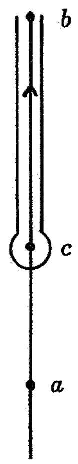
$$\phi_2^+(t,b) = \frac{1}{4\pi i}\log^2(t-b) + \frac{1}{2}\log(t-b), \tag{b.10}$$

$$\phi_2^-(t,b) = \frac{1}{4\pi i}[\log(t-b) - 2\pi i]^2 + \frac{1}{2}[\log(t-b) - 2\pi i]$$

$$= \frac{1}{4\pi i}\log^2(t-b) - \frac{1}{2}\log(t-b). \tag{b.11}$$

iii. The case $\operatorname{Im} a < \operatorname{Im} c < \operatorname{Im} b$.

Put



$$\Omega_{cb}(z) := \frac{1}{2\pi i}\int_{c+}^{b} \frac{\log(\tau - c)}{\tau - z}\,d\tau\,,$$

$$\Omega_{ac}(z) := \frac{1}{2\pi i}\int_{a}^{c-} \frac{\log(\tau - c)}{\tau - z}\,d\tau. \tag{b.12}$$

From case i with a replaced by c, cf.(b.3), (b.6),

$$\Omega_{cb}(z) = \omega_2(z,c) + \Lambda_0^{(c)}(z), \qquad z \in \dot{\mathcal{N}}(c)\backslash L, \tag{b.13}$$

$$\Omega_{cb}(t) = \frac{1}{2}\omega_2^+(t,c) + \frac{1}{2}\omega_2^-(t,c) + \Lambda_0^{(c)}(t),$$

$$= -\frac{1}{4\pi i}\log^2(t-c) + \Lambda_0^{(c)}(t), \qquad t \in \dot{\mathcal{N}}(c) \cap L.$$

In the present case $\log(z-c)$ has no discontinuity for $z = t \in (a,c)$, and the P.S.-formula yields, cf.[8] p. 59,

$$\Omega^+_{ac}(t) - \Omega^-_{ac}(t) = \log(t-c), \quad t \in L\backslash\{c\}. \tag{b.14}$$

Hence: for $t \in \dot{\mathcal{N}}(c) \cap (a,c)$,

$$\Omega^+_{ac}(t) - \Omega^-_{ac}(t) = \log(t-c). \tag{b.15}$$

From case ii with b replaced by c we have using (b.7) and (b.11) for the value of $\Omega_{ac}(t)$ with t on the contour, independently of the solution of the cut,

$$\Omega_{ac}(t) = \frac{1}{4\pi\mathrm{i}}\log^2(t-c) + \Lambda_1^{(c)}(t). \tag{b.16}$$

By applying the P.S.-formula for $t \in (a,c)$ we have

$$\Omega^+_{ac}(t) + \Omega^-_{ac}(t) = 2\Omega_{ac}(t) = \frac{2}{4\pi\mathrm{i}}\log^2(t-c) + 2\Lambda_1^{(c)}(t). \tag{b.17}$$

Hence, from (b.15) and (b.17),

$$\Omega^+_{ac}(t) = \frac{1}{4\pi\mathrm{i}}\log^2(t-c) + \frac{1}{2}\log(t-c) + \Lambda_1^{(c)}(t), \tag{b.18}$$

$$\Omega^-_{ac}(t) = \frac{1}{4\pi\mathrm{i}}\log^2(t-c) - \frac{1}{2}\log(t-c) + \Lambda_1^{(c)}(t).$$

Because $\log(z-c)$ is continuous to the left and to the right of ac, on (a,c), since ac is no part of the cut, we have from (b.18): for $z \in \dot{\mathcal{N}}(c)\backslash L$,

$$\begin{aligned} \Omega_{ac}(z) &= \frac{1}{4\pi\mathrm{i}}\log^2(z-c) + \frac{1}{2}\log(z-c) + \Lambda_1^{(c)}(z), \quad z\ \textit{left of}\ ac, \\ &= \frac{1}{4\pi\mathrm{i}}\log^2(z-c) + \frac{1}{2}\log(z-c) - \Lambda_1^{(c)}(z), \quad z\ \textit{right of}\ ac. \end{aligned} \tag{b.19}$$

From

$$\Omega_c(z) = \Omega_{ac-}(z) + \Omega_{c+,b}(z), \tag{b.20}$$

we obtain with $\Lambda_2^{(c)}(z)$ a function regular for $z \in \mathcal{N}(c)$:

$$\begin{aligned} &\text{i.} \quad \Omega_c(z) = \log(z-c) + \Lambda_2^{(c)}(z), && z \in \dot{\mathcal{N}}(c)\backslash L\ \textit{with}\ z\ \textit{left of}\ ac, \\ &\text{ii.} \quad\quad\quad\ \ = \Lambda_2^{(c)}(z), && z \in \dot{\mathcal{N}}(c)\backslash L\ \textit{with}\ z\ \textit{right of}\ ac, \\ &\text{iii.} \quad\quad\quad = \frac{1}{2}\log(t-c) + \Lambda_2^{(c)}(t), && t \in \dot{\mathcal{N}}(c) \cap L; \end{aligned} \tag{b.21}$$

Here (b.21)i, ii follows from (b.19), (b.20) and (b.3)-(b.6) in which a has to be replaced by c, whereas (b.21)iii follows from (b.20), (b.13) and (b.7) in which b has to be replaced by c.

The relations (b.21) describe the asymptotic behaviour of $\Omega_c(z)$ for z in a vicinity of c.

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