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Mathematical Morphology as a Tool for Shape Description *

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Abstract

Mathematical morphology is an approach in image processing based on geometrical concepts such as transformation groups and metric spaces. As such it is well-suited for the extraction of information about the shape of the various parts in a scene. This paper presents an overview of some better or less known morphological techniques (e.g. skeletonization, granulometric analysis) for the description and decomposition of shape.

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1. Introduction

At many instances in the literature it is claimed that mathematical morphology is an approach well-suited for the extraction of shape information from a scene. However, 'shape' is one of these notions of which everyone has an intuitive idea what it means, but which, nevertheless is hard to catch in a definition which is not ad-hoc. We shall not attempt to fill this gap. Rather do we take the opportunity to show in which sense mathematical morphology can offer useful tools for the description of shape. In Sect. 2 we briefly recall the basic concepts from mathematical morphology and discuss some elementary morphological operators for binary images. In Sect. 3 we explain how such operators can be extended to the grey-scale case by means of the umbra transform.

Two geometrical concepts lie at the heart of mathematical morphology, namely (i) *geometrical transformations* such as translations, rotations, reflections, perspective transformations, and (ii) *metric spaces and convexity*. Geometrical transformations form the basis for Sect. 4, in particular Subsect. A where we discuss transformation-based morphology in a rather general context. We show how an arbitrary transformation group can be used as the basis for a family of morphological operators invariant under these transformations.

An alternative way to construct morphological operators, discussed in Subsect. B, is based on the notion of distance. On any metric space one can define morphological operators like dilations, erosions, openings, closings, etc. A class of metric spaces particularly important in the context of mathematical morphology is formed by the so-called Minkowski spaces. It turns out that for such spaces the transformation-based approach and the distance-based approach are closely related.

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In Sect. 5 we discuss granulometries. These can be viewed as the mathematical formalization of a sieving process, and have been applied with success to many practical image analysis problems.

Probably one of the most popular tools in image processing is formed by the skeleton (and its variants). The skeleton can be defined conveniently in terms of morphological operators. The morphological definition makes it rather easy to define skeletons based on a distance other than the Euclidean distance. As in the previous section, 'convexity' is the magic word here.

In Sect. 7 we explain how the morphological opening can be used as a tool for shape decomposition. Actually, we discuss two decomposition algorithms here, the first due to Pitas and Venetsanopoulos [22] and the second to Ronse [27, 29]. The exposition presented here is largely detracted from the paper by Ronse [27] where a more general approach has been discussed.

In Sect. 8 we conclude with some additional remarks.

2. Basic notions

There is an enormous literature on mathematical morphology. Basic references are the monographs by Matheron [19] and Serra [30]. A basic account can also be found in [7]. A second volume, edited by Serra [31] treats a number of theoretical issues; a substantial part of this book is devoted to the theory of morphological filters. In this section we recall some of the basic material for the convenience of the reader. More details can be found in the references listed above.

The central idea of mathematical morphology is to examine the geometrical structure of an image by matching it with small patterns at various locations in the image. By varying the size and the shape of the matching patterns, called *structuring elements*, one can obtain useful information about the shape of the image. In general such a procedure results in nonlinear image operators which are well-suited for the analysis of the geometrical and topological structure of an image.

Originally, mathematical morphology has been developed for binary images which can be represented mathematically as sets. The corresponding morphological operators essentially use only four ingredients from set theory, namely set intersection, union, complementation and translation.

Let $\mathcal{P}(\mathbb{R}^d)$ be the space of subsets of $\mathbb{R}^d$. We choose a structuring element $A \subseteq \mathbb{R}^d$. The Minkowski addition and subtraction are resp. defined as

$$X \oplus A = \bigcup_{a \in A} X_a \tag{2.1}$$

$$X \ominus A = \bigcap_{a \in A} X_{-a}. \tag{2.2}$$

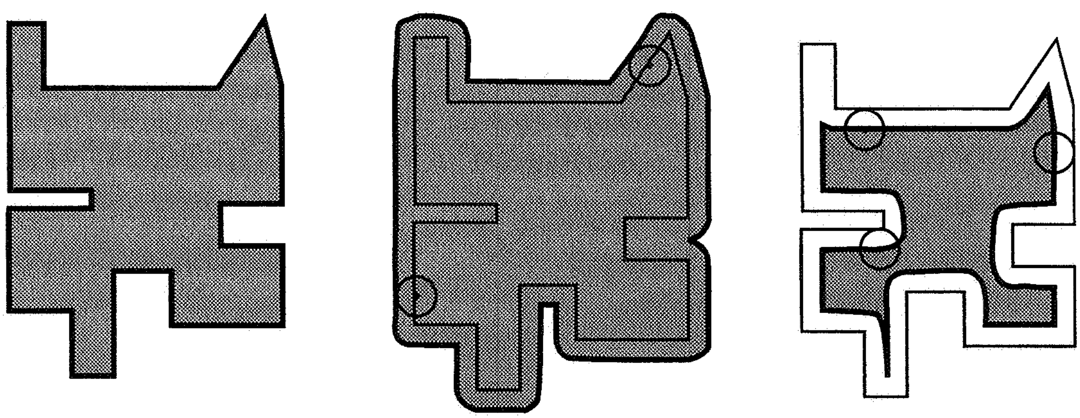

FIGURE 1. *A set (left), and its dilation (middle) and erosion (right) with a disk.*

Here X_a is the translate of X along the vector a. Instead of (2.1) we can also write

$$X \oplus A = \{h \in \mathbb{R}^d \mid \check{A}_h \cap X \neq \emptyset\} \tag{2.3}$$
$$X \ominus A = \{h \in E \mid A_h \subseteq X\}. \tag{2.4}$$

Here $\check{A}$ is the reflection of A with respect to the origin, that is $\check{A} = \{-a \mid a \in A\}$. Usually, one refers to the Minkowski addition (2.1) as the *dilation* by A, and to the Minkowski subtraction (2.2) as the *erosion* by A. Dilation and erosion are illustrated in Fig. 1. We introduce the notation $\delta_A(X) = X \oplus A$ and $\varepsilon_A(X) = X \ominus A$. The operators δ_A and ε_A are not each other's inverses in general, that is, $(X \oplus A) \ominus A \neq X \neq (X \ominus A) \oplus A$. We define

$$X \circ A = (X \ominus A) \oplus A \tag{2.5}$$

as the *opening* by A. It is easy to show that

$$X \circ A = \bigcup \{A_h \mid h \in \mathbb{R}^d \text{ and } A_h \subseteq X\}.$$

In other words, $X \circ A$ is the union of all translates of the structuring element A which are contained in X. Since the opening plays such an important role in this paper we present an example in Fig. 2.

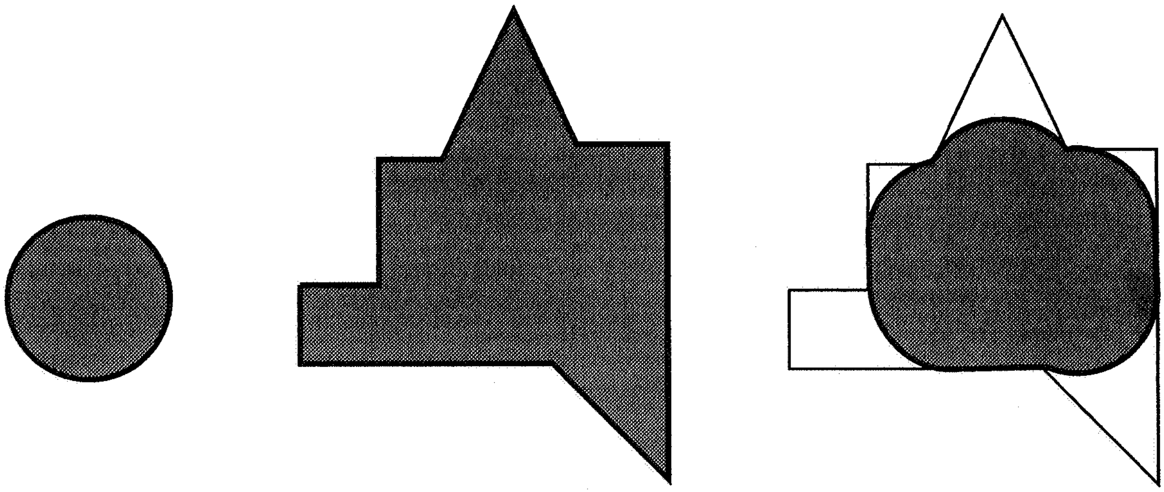

FIGURE 2. *The opening $X \circ A$ (right) of the set X (middle) with the disk A (left).*

The opening has the following properties: it is

- increasing, i.e., $X \subseteq Y$ implies that $X \circ A \subseteq Y \circ A$;
- translation invariant, i.e., $X_h \circ A = (X \circ A)_h$;
- anti-extensive, i.e., $X \circ A \subseteq X$;
- idempotent, i.e., $(X \circ A) \circ A = X \circ A$.

Note that the first two properties also hold for dilations and erosions. Every operator $\alpha : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$ which is increasing, anti-extensive and idempotent is called an opening. If α_1, α_2 are openings on $\mathcal{P}(\mathbb{R}^d)$, then

$$\alpha_1 \leq \alpha_2 \iff \alpha_1\alpha_2 = \alpha_1 \iff \alpha_2\alpha_1 = \alpha_1.$$

Here '$\alpha_1 \leq \alpha_2$' means that $\alpha_1(X) \subseteq \alpha_2(X)$ for every $X \in \mathcal{P}(\mathbb{R}^d)$, and '$\alpha_1\alpha_2$' is the composition of α_1 and α_2, i.e., $\alpha_1\alpha_2(X) = \alpha_1(\alpha_2(X))$. For the openings $X \to X \circ A$ it can be shown that $X \circ A \subseteq X \circ B$ for every X if and only if A is B-open. The latter means that $A \circ B = A$. For example, this condition holds if A is a square with sides 1 and B a line segment with length ≤ 1.

The operator given by

$$X \bullet A = (X \oplus A) \ominus A$$

is called the *closing* by A and has the same properties as the opening apart from the third: it is extensive instead of anti-extensive. The latter means that $X \subseteq X \bullet A$ for every $X \in \mathcal{P}(\mathbb{R}^d)$.

The observation that Minkowski addition and subtraction are not each other's inverses motivated Ghosh to adress the problem whether it is possible to extend $\mathcal{P}(\mathbb{R}^d)$ with so-called *negative shapes* so that the space becomes a group under Minkowski addition. An account of his findings can be found in [6].

Recently, mathematical morphology has been extended to the framework of complete lattices. We recall that a complete lattice is a partially ordered set in which every subset has an infimum (greatest lower bound) $\bigwedge$ and supremum (smallest upper bound) $\bigvee$; see [1]. The space $\mathcal{P}(\mathbb{R}^d)$ with the inclusion order is a complete lattice. For a comprehensive account of the extension of mathematical morphology to complete lattices we refer to [13, 28, 31] and [11].

2.1. Definition. Let $\mathcal{L}$ be a complete lattice. An operator $\delta : \mathcal{L} \to \mathcal{L}$ is called a *dilation* if it distributes over arbitrary suprema, that is,

$$\delta(\bigvee_{i\in I} X_i) = \bigvee_{i\in I} \delta(X_i),$$

for any family $\{X_i \mid i \in I\}$. Dually, an operator $\varepsilon : \mathcal{L} \to \mathcal{L}$ is called an *erosion* if it distributes over arbitrary infima, that is,

$$\varepsilon(\bigwedge_{i\in I} X_i) = \bigwedge_{i\in I} \varepsilon(X_i),$$

for any family $\{X_i \mid i \in I\}$.

A pair of operators (ε, δ), both mapping $\mathcal{L}$ into $\mathcal{L}$, is called an adjunction on $\mathcal{L}$ if for every $X, Y \in \mathcal{L}$,

$$\delta(X) \le Y \iff X \le \varepsilon(Y). \tag{2.6}$$

If (ε, δ) is an adjunction, then δ is a dilation and ε an erosion. Moreover, with every dilation δ one can associate a unique erosion ε so that (ε, δ) forms an adjunction. We say that ε and δ are *adjoint operators.* If (ε, δ) is an adjunction on $\mathcal{L}$ then $\delta\varepsilon$ is an opening and $\varepsilon\delta$ a closing.

The pair $(\varepsilon_A, \delta_A)$ introduced above forms an adjunction on $\mathcal{P}(\mathbb{R}^d)$. In Sect. 4 below we will discuss some other examples.

3. Grey-scale morphology

Many binary morphological operators can be extended to grey-scale images (modeled mathematically as functions). We denote by $\mathsf{Fun}(E)$ the space of functions mapping E into $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. It is easy to check that $\mathsf{Fun}(E)$ is a complete lattice. If $E = \mathbb{R}^d$, the Minkowski addition and subtraction of two functions F and G can be defined as

$$(F \oplus G)(x) = \bigvee_{h\in E} [F(x-h) + G(h)], \tag{3.1}$$

and

$$(F \ominus G)(x) = \bigwedge_{h\in E} [F(x+h) - G(h)]. \tag{3.2}$$

The opening is given by $F \circ G = (F \ominus G) \oplus G$. G is called the *structuring function.*

A general approach to extend binary morphological operators to functions is provided by the *umbra transform.* For an extensive discussion we refer to [10]. The key idea is to represent a function F on the space E by the set of points in $E \times \mathbb{R}$ on and below the graph of E. The resulting set is called an umbra.

3.1. Definition. Let E be an arbitrary set.

(a) A set $U \subseteq E \times \mathbb{R}$ is called an *umbra* if $(x, t) \in U$ if and only if $(x, s) \in U$ for every $s < t$.

(b) A subset $U \subseteq E \times \mathbb{R}$ is called a *pre-umbra* if $(x, t) \in U$ implies that $(x, s) \in U$ for every $s < t$.

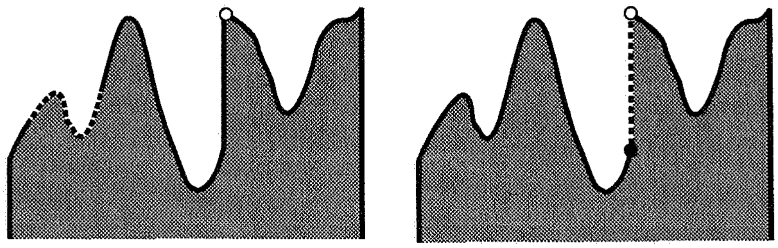

FIGURE 3. *A pre-umbra (left) and an umbra (right).*

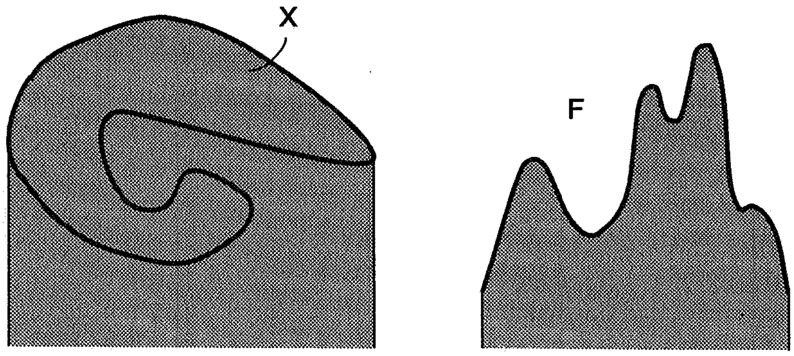

FIGURE 4. *The umbras* $\mathcal{U}_s(X)$ *of a set* X *(left) and* $\mathcal{U}_f(F)$ *of a function* F *(right).*

For an illustration we refer to Fig. 3.

The set of all umbras is denoted by $\mathsf{Umbra}(E)$. For a subset $X \subseteq E \times \mathrm{I\!R}$ we define $\mathcal{U}_s(X)$ as the smallest umbra containing X. In other words, $\mathcal{U}_s(X)$ is the intersection of all umbras containing X; see Fig. 4. If F is a function mapping a set E (usually $\mathrm{I\!R}^d$ or $\mathbb{Z}^d$) into $\overline{\mathrm{I\!R}}$, then we define the umbra $\mathcal{U}_f(F)$ of F as

$$\mathcal{U}_f(F) = \{(x,t) \in E \times \mathrm{I\!R} \mid t \leq F(x)\}; \tag{3.3}$$

see Fig. 4. The subscripts $\underline{\mathrm{s}}$ and $\underline{\mathrm{f}}$ refer to set and function respectively. The mapping $\mathcal{U}_f : \mathsf{Fun}(E) \to \mathsf{Umbra}(E)$ is called the *umbra transform*.

To every umbra U there corresponds a unique function given by

$$[\mathcal{F}_u(U)](x) = \bigvee \{t \in \mathrm{I\!R} \mid (x,t) \in U\}.$$

From now on we write $\mathcal{F}$ and $\mathcal{U}$ for $\mathcal{F}_u$ and $\mathcal{U}_f$ respectively.

3.2. Proposition. $\mathsf{Umbra}(E)$ *with the set inclusion as partial order is a complete lattice with infimum and supremum of* U_i, $i \in I$, *respectively given by* $\bigwedge_{i\in I} U_i = \bigcap_{i\in I} U_i$ *and* $\bigvee_{i\in I} U_i = \mathcal{U}_s(\bigcup_{i\in I} U_i)$. *This lattice is isomorphic to* $\mathsf{Fun}(E)$ *with the isomorphism and its inverse respectively given by* $\mathcal{F}$ *and* $\mathcal{U}$.

Given a scalar $v \in \mathrm{I\!R}$, the vertical translate of a set $X \subseteq E \times \mathrm{I\!R}$ and a function $F \in \mathsf{Fun}(E)$ are respectively defined as $X^v = \{(x, t+v) \mid (x,t) \in X\}$ and $F^v(x) = F(x) + v$. It is obvious that $X \to X^v$ maps a (pre-)umbra onto a (pre-)umbra. Furthermore $\mathcal{F}(U^v) = [\mathcal{F}(U)]^v$ and $\mathcal{U}(F^v) = [\mathcal{U}(F)]^v$.

3.3. Lemma. *Let* $U \subseteq E \times \mathrm{I\!R}$.

(a) *U is a pre-umbra if and only if* $U \subseteq U^v$ *for every* $v > 0$.

(b) *U is an umbra if and only if* $U = \bigcap_{v>0} U^v$.

(c) *If U is a pre-umbra then* $\mathcal{U}_s(U) = \bigcap_{v>0} U^v$.

For a proof we refer to [10].

We use the umbra transform to extend operators on $\mathcal{P}(E \times \mathrm{I\!R})$ to operators on $\mathsf{Fun}(E)$. Assume that ψ is an increasing operator on $\mathcal{P}(E \times \mathrm{I\!R})$ which is invariant under vertical translations. The latter means that $\psi(X^v) = [\psi(X)]^v$ for $X \subseteq E \times \mathrm{I\!R}$. If U is a pre-umbra, then $U \subseteq U^v$ for $v > 0$ and hence

$$\psi(U) \subseteq \psi(U^v) = [\psi(U)]^v;$$

in other words, $\psi(U)$ is a pre-umbra as well. We define the operator $\tilde{\psi}$ on $\mathcal{P}(E \times \mathbb{R})$ as

$$\tilde{\psi}(X) = \bigcap_{v>0} [\psi(X)]^v = \bigcap_{v>0} \psi(X^v). \tag{3.4}$$

From Lemma 3.3 it follows that $\tilde{\psi}(U) = \mathcal{U}_s(\psi(U))$ if U is a pre-umbra, and thus $\tilde{\psi}(U)$ is an umbra. This yields in particular that $\tilde{\psi}$ leaves $\mathsf{Umbra}(E)$ invariant.

3.4. Theorem. *Given an increasing operator ψ on $\mathcal{P}(E \times \mathbb{R})$ which is invariant under vertical translations, the operator Ψ given by*

$$\Psi = \mathcal{F} \circ \tilde{\psi} \circ \mathcal{U} \tag{3.5}$$

defines an increasing operator on $\mathsf{Fun}(E)$ invariant under vertical translations.

Obviously, if ψ is invariant under translations in E, then the same holds for Ψ.

As an example we consider the Minkowski addition for functions. Given a function G on $\mathbb{R}^d$. The operator $F \to F \oplus G$ derives from the above construction with $\psi(X) = X \oplus \mathcal{U}(G)$.

If we take for G a function which assumes the value 0 on its domain A and $-\infty$ elsewhere, then the resulting dilation (erosion, etc.) is called a *flat dilation* (erosion, etc.). Recall that the domain dom(F) of a function F consists of all $x \in \mathbb{R}^d$ for which $F(x) > -\infty$. More generally, one can define flat function operators as follows. Given an increasing binary operator ψ_0 on $\mathcal{P}(E)$, we can define an increasing operator ψ on $\mathcal{P}(E \times \mathbb{R})$ by putting

$$\pi_t \psi(X) = \psi_0(\pi_t X)$$

for $X \subseteq E \times \mathbb{R}$ and $t \in \mathbb{R}$. Here π_t is the operator given by $\pi_t X = \{x \in E \mid (x,t) \in X\}$. In other words, $\psi(X)$ is the set obtained by applying ψ_0 to every cross section $\pi_t X$. The extension of ψ given by (3.5) yields a grey-scale operator Ψ. If we define the threshold sets of F as $X(F,t) = \{x \in E \mid F(x) \geq t\}$ then $\Psi(F)$ is given by

$$\Psi(F)(x) = \sup\{t \in \mathbb{R} \mid x \in \psi_0(X(F,t))\}.$$

We call Ψ the *flat extension* of ψ_0 to $\mathsf{Fun}(\mathbb{R}^d)$. Ψ inherits most properties of ψ_0; for instance, if ψ_0 is an opening then Ψ is such as well. We refer to [9] for more details.

4. Morphology versus geometry

Somewhat simplifying one might say that morphological operators can be defined by moving a small test pattern over the image, checking at all positions how it relates to the image and using the outcome to define an output image. In classical translation morphology "moving" means "translating". But one can think of situations where translation is not appropriate, or even worse, not possible. We mention some examples.

In certain applications (e.g. radar imaging) rotation symmetry comes in naturally. In such cases one has to include rotations in the group of permitted motions. Similar remarks apply to situations where perspective transformations play a role; think, for instance, of the problem of monitoring the traffic on a highway with a camera at a fixed position. It is apparent that in this case the detection algorithms must take the distance between the camera and the cars into account.

If the underlying support space is not just the Euclidean space $\mathbb{R}^d$ or some regular grid, but rather a manifold (the sphere, e.g), translation has to be understood in the sense of translation along geodesics as Roerdink explains in [25]. In general the motion of a pattern along geodesics is a troublesome matter. However, for some specific examples such as the sphere, it is nevertheless possible to obtain concrete results [24, 26].

Another class of images which requires reflections about possible ways to move around a pattern is formed by the graph-based images. In a number of applications a graph provides the appropriate mathematical structure to model an image. This occurs when the image contains a large amount of relatively small objects (e.g. cells in an electron microscopy image). In such cases the edges of the graph can be used to model the spatial relationships between the objects. Heijmans et al. [12, 14] use the notion of a structuring graph to define morphological operators on such graphs. A more direct approach, based on distance, was given by Vincent [35]. In the latter approach the central idea is to define a pattern at every position (a ball with given radius) rather than moving around a given pattern using a given group of transformations.

In this section we will discuss both approaches in a more general context. For the sake of exposition we restrict to binary images, or, to stay within mathematical terms, to the space $\mathcal{P}(E)$, where E is the support space. This can be the Euclidean space $\mathbb{R}^d$, the discrete space $\mathbb{Z}^d$, a manifold, a graph, etc. In the transformation approach it is assumed that we are given a transformation group on E. In practical cases the choice of this group is often determined by the underlying mathematical structure of E. In the distance approach we merely assume that E is a metric space.

A. Transformation-based morphology

In this subsection we outline how basic morphological operators as dilation, erosion, opening and closing can be extended to general geometric spaces. A first observation is that translation morphology is rather special since translations define a simply transitive abelian group (definitions below).

Consider the Minkowski addition $X \to X \oplus A$ where $A \subseteq \mathbb{R}^2$. In general, this operation is not invariant under rotations, that is $(R_\varphi X) \oplus A \neq R_\varphi(X \oplus A)$. Here R_φ is the rotation around 0 over an angle φ. In fact, one can show that the operation is rotation invariant if and only if A is, that is $R_\varphi A = A$ for every $\varphi \in [0, 2\pi]$. We put this simple example into a more general algebraic framework.

The following account is based on older work of Heijmans and Ronse [8, 13, 28] and Roerdink [24, 26]. For a comprehensive account we refer to the forthcoming book of the author [11]. Assume that E is a set and that $\mathbb{T}$ is a transformation group on E. By a transformation we mean a bijective mapping. We say that $\mathbb{T}$ is *transitive* on E if for every $x, y \in E$ there exists a transformation $\tau \in \mathbb{T}$ such that $\tau x = y$. If this transformation is unique for every pair x, y, then we call $\mathbb{T}$ *simply transitive* on E. It is easy to show that for an abelian transformation group, transitivity implies simple transitivity. The group of isometries on $\mathbb{R}^2$ is transitive but not simply transitive. Recall that an isometry is a transformation which preserves distances [23]. The translations form an abelian, simply transitive transformation group on $\mathbb{R}^2$. The rotations around 0 yield an abelian group which, however, is not transitive.

We say that $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is a $\mathbb{T}$-operator if

$$\psi \circ \tau = \tau \circ \psi, \qquad \text{for every } \tau \in \mathbb{T}.$$

A $\mathbb{T}$-operator which is a dilation will be called a $\mathbb{T}$-dilation, etc. For $X \subseteq E$ and $\tau \in \mathbb{T}$ we define $\tau X = \{\tau x \mid x \in X\}$.

Following the expression $X \oplus A = \bigcup_{a \in A} X_a$ we may attempt to define a $\mathbb{T}$-dilation as follows:

$$\delta_{\mathbb{A}}(X) = \bigcup_{\tau \in \mathbb{A}} \tau X; \tag{4.1}$$

here $\mathbb{A}$ is an arbitrary subset of $\mathbb{T}$. It is apparent that $\delta_{\mathbb{A}}$ is a dilation in the sense of Def. 2.1. The adjoint erosion is given by

$$\varepsilon_{\mathbb{A}}(X) = \bigcap_{\tau \in \mathbb{A}} \tau^{-1} X. \tag{4.2}$$

If the transformation group $\mathbb{T}$ is abelian then $\delta_{\mathbb{A}}$ is a $\mathbb{T}$-dilation and $\varepsilon_{\mathbb{A}}$ a $\mathbb{T}$-erosion. If, additionally, $\mathbb{T}$ is transitive, then every $\mathbb{T}$-dilation and $\mathbb{T}$-erosion on $\mathcal{P}(E)$ are of this form. Let us consider this case in more detail. First we fix an origin $o \in E$. With every $x \in E$ there corresponds a unique transform $\tau_x \in \mathbb{T}$ which carries o to x, $\tau_x o = x$ (compare this with the relation between affine spaces and vector spaces). For $x \in E$ we define $A(x) \subseteq E$ as

$$A(x) = \{\tau x \mid \tau \in \mathbb{A}\}.$$

Obviously, $A(x) = \tau_x A(o)$, so we can interpret $A(x)$ as the 'translate' of the structuring element $A(o)$. Then

$$\delta_{\mathbb{A}}(X) = \bigcup_{x \in X} A(x), \quad \varepsilon_{\mathbb{A}}(X) = \{x \in E \mid A(x) \subseteq X\}. \tag{4.3}$$

The opening $\delta_{\mathbb{A}}\varepsilon_{\mathbb{A}}$ is given by

$$\delta_{\mathbb{A}}\varepsilon_{\mathbb{A}}(X) = \bigcup \{A(h) \mid h \in E \text{ and } A(h) \subseteq X\}. \tag{4.4}$$

In fact, in this case we get all the results which we also know from the translation invariant case.

4.1. Proposition. *Let $\mathbb{T}$ be an abelian, (simply) transitive transformation group on E. Fix $o \in E$ and let, for $x \in E$, τ_x be the unique transformation in $\mathbb{T}$ which carries o to x. For every $A \subseteq E$ the pair*

$$\delta(X) = \bigcup_{x \in X} \tau_x(A), \quad \varepsilon(X) = \{h \in E \mid \tau_h(A) \subseteq X\}$$

forms a $\mathbb{T}$-adjunction on $\mathcal{P}(E)$. Moreover, every $\mathbb{T}$-adjunction is of this form.

If the translation group $\mathbb{T}$ is not abelian then neither $\delta_{\mathbf{A}}$ nor $\varepsilon_{\mathbf{A}}$ are $\mathbb{T}$-invariant in general. To achieve $\mathbb{T}$-invariance we rewrite $\delta_{\mathbf{A}}$ in the following way:

$$\delta_{\mathbf{A}}(X) = \bigcup_{x \in X} A(x), \tag{4.5}$$

where $A(x) = \{\tau x \mid \tau \in \mathbb{A}\} = \delta_{\mathbf{A}}(\{x\})$. This expression for $\delta_{\mathbf{A}}$ corresponds to the intuitive idea of moving a structuring element A over all points $x \in X$. The operator given by (4.5) is a dilation for every mapping $A : E \to \mathcal{P}(E)$. It can easily be shown that $\delta_{\mathbf{A}}$ is $\mathbb{T}$-invariant if and only if

$$A(\tau x) = \tau A(x) \tag{4.6}$$

for every $x \in E$ and $\tau \in \mathbb{T}$. Assume from now on that $\mathbb{T}$ is transitive on E. Fix an origin $o \in E$. Let Σ be the subgroup of $\mathbb{T}$ containing all τ which leave o invariant,

$$\Sigma = \{\tau \in \mathbb{T} \mid \tau o = o\}.$$

Σ is sometimes called the *stabilizer* of o. For example, if $\mathbb{T}$ is the group on $\mathbb{R}^2$ consisting of all rotations and translations, then the stabilizer of a point consists of all rotations around that point. If $\mathbb{T}$ is simply transitive then Σ contains only the identity mapping. Clearly, (4.6) implies that

$$A(o) = \Sigma A(o), \tag{4.7}$$

where $\Sigma X = \{\tau x \mid \tau \in \Sigma,\ x \in X\}$. It follows with (4.6) that

$$A(x) = \tau_x A(o),$$

where τ_x is a transformation carrying o to x.

4.2. Proposition. *Let $\mathbb{T}$ be a transitive transformation group on E and let $o \in E$ be fixed. The dilation δ given by*

$$\delta_A(X) = \bigcup_{x \in X} A(x), \tag{4.8}$$

where $A : E \to \mathcal{P}(E)$, is $\mathbb{T}$-invariant if and only if $A(\tau x) = \tau A(x)$ for every $\tau \in \mathbb{T}$ and $x \in E$. This implies in particular that $A(o) = \Sigma A(o)$. Moreover, every $\mathbb{T}$-dilation is of this form under the given assumptions. The adjoint erosion is given by

$$\varepsilon_A(X) = \{x \in E \mid A(x) \subseteq X\}. \tag{4.9}$$

Essentially, this result says that the only way to obtain $\mathbb{T}$-adjunctions is the following. Take a structuring element $A \subseteq E$ and define $A(x) = \tau_x \Sigma A$. Then the pair $(\varepsilon_A, \delta_A)$, where δ_A, ε_A are given by (4.8) and (4.9) respectively, defines a $\mathbb{T}$-adjunction. In fact, in the introductory example of this subsection dealing with the translation-rotation invariant case, we reached a similar conclusion.

We conclude with some remarks about the corresponding openings. The opening $\delta_A \varepsilon_A$ given by

$$\delta_A \varepsilon_A(X) = \{\tau \Sigma A \mid \tau \in \mathbb{T} \text{ and } \tau \Sigma A \subseteq X\},$$

is $\mathbb{T}$-invariant. However, the opening

$$\alpha_A(X) = \bigcup \{\tau A \mid \tau \in \mathbb{T} \text{ and } \tau A \subseteq X\}$$

is $\mathbb{T}$-invariant as well. It is easy to see that

$$\alpha_A \geq \delta_A \varepsilon_A,$$

where the equality holds if and only if A is Σ-symmetric, that is $A = \Sigma A$. We illustrate the difference between these two openings for the translation-rotation invariant case. Let $X \circ A$ be given by (2.5), with $A \subseteq \mathbb{R}^2$. Then

$$\delta_A \varepsilon_A(X) = X \circ \Big(\bigcup_{0 \leq \varphi \leq 2\pi} R_\varphi A \Big),$$

(observe that in this case $\Sigma A = \bigcup_{0 \leq \varphi \leq 2\pi} R_\varphi A$), and

$$\alpha_A(X) = \bigcup_{0 \leq \varphi \leq 2\pi} X \circ R_\varphi A.$$

For instance if A is a line segment with length l and centre 0, then α_A preserves all line segments with length $\geq l$ no matter their orientation, whereas $\delta_A \varepsilon_A$ preserves only disks with diameter $\geq l$; see Fig. 5.

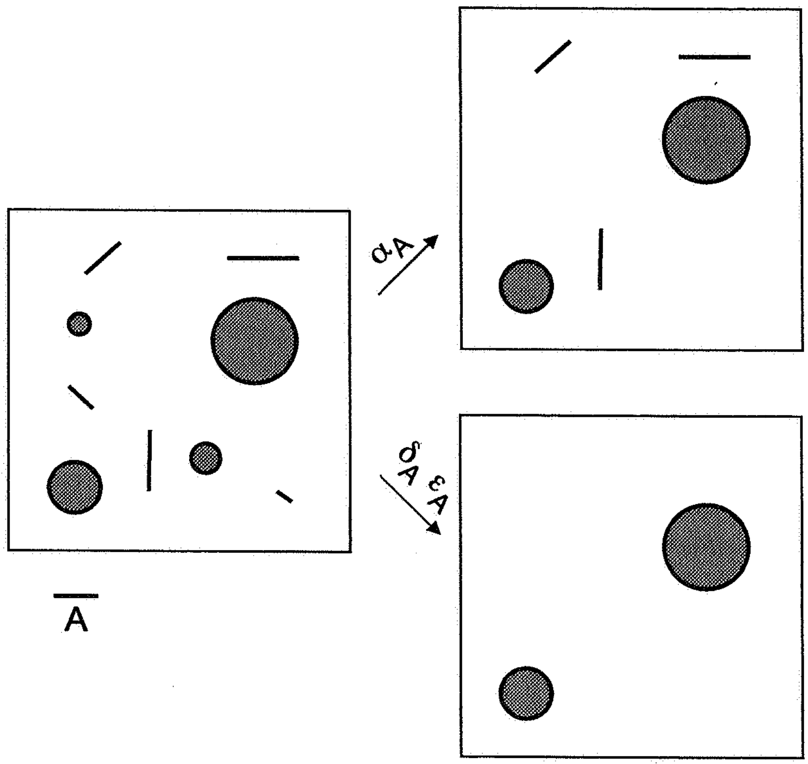

FIGURE 5. *The opening α_A preserves all line segments with length $\geq l$, whereas the opening $\delta_A \varepsilon_A$ preserves only disks with diameter $\geq l$.*

B. Distance-based morphology

In this subsection we explain how to build morphological operators on $\mathcal{P}(E)$ when E is equipped with a notion of distance. Recall that a function $d : E \times E \to \mathbb{R}_+$ is called a *metric* or *distance function* if for $x, y, z \in E$

(D1) $d(x,y) = 0 \iff x = y$;

(D2) $d(x,y) = d(y,x)$;

(D3) $d(x,z) \leq d(x,y) + d(y,z)$.

The last property is called the *triangle inequality.* If d is a metric on E then we say that (E,d) is a *metric space.* The ball with radius r centered at x is given by

$$B(x,r) = \{y \in E \mid d(x,y) \leq r\}.$$

We can define morphological operators on $\mathcal{P}(E)$ by taking these balls as structuring elements. More specifically, we can define a family of dilations δ^r, $r \geq 0$, as follows:

$$\delta^r(X) = \bigcup_{x \in X} B(x, r). \tag{4.10}$$

The adjoint erosion is given by

$$\varepsilon^r(X) = \{h \in E \mid B(h, r) \subseteq X\}. \tag{4.11}$$

One can easily show that

$$\delta^r \delta^s \leq \delta^{r+s}, \quad r, s \geq 0. \tag{4.12}$$

In fact this relation is a consequence of the triangle inequality.

An important instance of a metric space is the so-called *Minkowski space*; this can be defined as a finite-dimensional normed vector space [23]. If $\|\cdot\|$ is a norm on E then $d(x, y) = \|x - y\|$ defines a metric. Besides the axioms (D1)–(D3) this metric satisfies

(D4) $d(x + h, y + h) = d(x, y)$

(D5) $d(\lambda x, \lambda y) = |\lambda| d(x, y)$,

for $\lambda \in \mathbb{R}$, and $x, y, h \in E$. The best known example of a Minkowski space is of course the Euclidean space. In a Minkowski space the balls are of the special form $B(x, r) = (rB)_x$, the translate of rB along the vector x. Here B is the unit ball centered at the origin. This unit ball is compact, convex, contains 0 in its interior and is reflection symmetric with respect to 0. Since the equality

$$rA \oplus sA = (r + s)A, \quad r, s \geq 0 \tag{4.13}$$

holds if and only if A is convex, we find that the dilations δ^r given by (4.10) satisfy

$$\delta^r \delta^s = \delta^{r+s}, \quad r, s \geq 0, \tag{4.14}$$

if E is a Minkowski space. We shall see below that this semigroup relation is of great importance for the construction of granulometries.

Before we conclude this section we take the opportunity to point out that the distance-based approach and the transformation-based approach are not complementary but rather alternative formulations of the same idea. To make this point clear we consider the case $E = \mathbb{R}^d$. Let the structuring element $A \subseteq \mathbb{R}^d$ have the same properties as the unit ball in a Minkowski space. That is, A is compact, convex, contains 0 in the interior and is reflection symmetric with respect to 0. Then there is a norm $\|\cdot\|_A$ on $\mathbb{R}^d$ for which the unit ball is A, namely,

$$\|x\|_A = \inf\{t > 0 \mid \frac{1}{t} x \in A\}.$$

In this case the dilation δ^r is given by

$$\delta^r(X) = X \oplus rA.$$

This identity says that the transformation-based dilation $X \to X \oplus rA$ corresponds to a distance-based dilation with a suitably choosen metric.

For a comprehensive discussion of the role of metric spaces in mathematical morphology we refer to [11]; see also [31, Sections 1.6 and 2.4].

5. Granulometries

In many image analysis problems one wants to measure a size distribution of the various objects in the scene. Many sizing techniques are somehow based on the intuitive notion of a sieving process: suppose we have a binary image consisting of a finite number of isolated particles. We put these particles through a stack of sieves with decreasing mesh widths and measure the number or the total volume of the particles remaining on a particular sieve. This results in a histogram which may be interpreted as a size distribution. Such an intuitive approach immediately raises a number of questions. A first objection one could make is that objects are not classified according to their volume but rather according to the property that they can or cannot pass a certain mesh opening. Furthermore, one has to decide which motions (translation, rotation, reflection) should be allowed in order to force a particle through a certain sieve. Another problem is that particles in an image may overlap and will be classified as one large particle by the system. Thus we are led to the conclusion that the intuitive characterization of a size distribution as the outcome of a sieving process is too vague and too restricted.

It was Matheron [19] who first realized that the concept of an opening in the morphological sense should underly a formal definition of a size distribution which is not only very general but also very attractive from a mathematical point of view. In this section we shall present Matheron's definition of what he called a *granulometry* (meaning a tool to "measure the grains"). In the first part of this section we restrict to the binary image space $\mathcal{P}(\mathbb{R}^d)$. At the end we briefly discuss some recent result for grey-scale images.

5.1. Definition. By a granulometry on $\mathcal{P}(\mathbb{R}^d)$ we mean a one-parameter family of openings $\{\alpha_r \mid r > 0\}$ such that

$$\alpha_s \leq \alpha_r \text{ if } s \geq r. \tag{5.1}$$

It is easy to show that property (5.1) is equivalent to

$$\alpha_r \alpha_s = \alpha_s \alpha_r = \alpha_s, \text{ if } s \geq r. \tag{5.2}$$

If we define $\alpha_r(X)$ as the union of all connected components of X whose volume is not less than Cr^d, where $C > 0$ is a constant, then $\{\alpha_r\}$ is a granulometry. This granulometry satisfies the additional properties

- α_r is translation invariant;
- $\alpha_r(X) = r\alpha_1(r^{-1}X)$, $r > 0$.

A granulometry on $\mathcal{P}(\mathbb{R}^d)$ with these properties will be called a *Minkowski granulometry.* We point out that the terminology "Euclidean granulometry" is more common in the morphological literature [19]; however, we reserve this name for a more specific example (see below).

If $\{\delta^r \mid r > 0\}$ is a family of dilations which have the semigroup property

$$\delta^r \delta^s = \delta^{r+s}, \quad r, s > 0, \tag{5.3}$$

then the openings $\alpha_r = \delta^r \varepsilon^r$ form a granulometry. Here ε^r is the erosion adjoint to δ^r. In particular, if d is a metric on $\mathbb{R}^d$ then the balls with radius r, $B(r) = B(0, r)$ satisfy

$$B(r) \oplus B(s) \subseteq B(r + s). \tag{5.4}$$

This is a direct consequence of the triangle inequality. If, by some argument, we know that equality in (5.4) holds, that is

$$B(r) \oplus B(s) = B(r + s), \tag{5.5}$$

then the dilations $\delta^r(X) = X \oplus B(r)$ have the semigroup property (5.3) and in this case the openings $\alpha_r(X) = X \circ B(r)$ form a granulometry. By the fact that every opening involves only one structuring element, this granulometry is called a *structural granulometry.* To avoid confusion we point out the following two facts: (i) in general the family $B(r)$ satisfies only (5.4) and not (5.5); (ii) in order that $\alpha_r(X) = X \circ B(r)$ is a granulometry (5.5) is sufficient but by no means necessary. More specifically, $X \circ B(r)$ defines a granulometry if and only if $B(s)$ is $B(r)$-open for $s \geq r$. There are many families $B(\cdot)$ which satisfy this condition but not (5.5). However, this is no longer true if we assume in addition that the resulting granulometry is of Minkowski-type. Namely, suppose that α_r is of Minkowski-type and that $\alpha_1(X) = X \circ B$. Then $\alpha_r(X) = r\alpha_1(r^{-1}X) = r(r^{-1}X \circ B) = X \circ rB$. This yields that $B(r) = rB$ for some $B \subseteq \mathbb{R}^d$. In order that the openings $X \circ rB$ define a granulometry the structuring element must satisfy the condition that sB is rB-open for $s \geq r$, or equivalently

$$rB \text{ is } B\text{--open for } r \geq 1.$$

The following result is due to Matheron [19].

5.2. Theorem. *Let $B \subseteq \mathbb{R}^d$ be compact. Then rB is B-open for $r \geq 1$ if and only if B is convex.*

We point out that for convex B the relation $rB \oplus sB = (r+s)B$ holds.

5.3. Corollary. *Assume that $B(r)$ is compact for $r \geq 0$. The openings $\alpha_r(X) = X \circ B(r)$ define a Minkowski granulometry if and only if $B(r) = rB$ for some B which is compact and convex.*

5.4. Remark. For completeness we point out that the openings

$$\alpha_r(X) = \bigcup_{s \geq r} X \circ sB$$

define a Minkowski granulometry for any structuring element $B \subseteq \mathbb{R}^d$. In applications the infinite union is undesirable. To get rid of it we have to assume that B is convex.

In the previous section we have seen that for a compact, convex subset B of $\mathbb{R}^d$ which contains 0 in the interior and is reflection symmetric with respect to 0, there is a unique norm $\|\cdot\|_B$ on $\mathbb{R}^d$ such that the unit ball $\{x \in \mathbb{R}^d \mid \|x\|_B = 1\}$ coincides with B. The corresponding metric space is a Minkowski space; otherwise stated, every Minkowski space is uniquely determined by a set B with the given properties.

To close the circle of arguments let δ^r be the dilation given by $\delta^r(X) = X \oplus rB$, where B is convex. Then δ^r obeys the semigroup property (5.3) and $\delta^r \varepsilon^r(X) = X \circ rB$.

As said, granulometries which are translation invariant and scale-compatible are usually called "Euclidean granulometries". However, we reserve the adjective "Euclidean" for the granulometry given by the openings $X \circ rB$ where B is the unit ball in the Euclidean metric (obviously, this is also a Minkowski granulometry).

The granulometries mentioned in Cor. 5.3 and Remark 5.4 are not the only ones which are of Minkowski-type. It is not hard to show that the union of an arbitrary collection of Minkowski granulometries is again a Minkowski granulometry. In fact, one can prove the following result.

5.5. Theorem. *Every Minkowski granulometry on $\mathcal{P}(\mathbb{R}^d)$ is of the form*

$$\alpha_r(X) = \bigcup_{B \in \mathcal{B}} \bigcup_{s \geq r} X \circ sB,$$

where $\mathcal{B}$ is an arbitrary collection of subsets of $\mathbb{R}^d$.

Let us have some closer look at the granulometry given by the openings $X \circ rB$, where B is convex. Suppose that Y is a component of X which contains at least one translate of rB, that is, $Y \circ rB \neq \emptyset$. In $\alpha_r(X) = X \circ rB$ not the whole component Y will be preserved, but only the subset $Y \circ rB$, which may be much smaller in practice. In some applications (e.g. in the case where one measures the total area of $\alpha_r(X)$) one would prefer to retain the whole particle Y if its opening by rB is non-void. This can be achieved by defining the following modification:

$$\check{\alpha}_r(X) = \rho(X \circ rB; X), \tag{5.6}$$

where $\rho(X; M)$ is the reconstruction of X within the mask set M, i.e., the union of all connected components of M which intersect X; see Fig. 6.

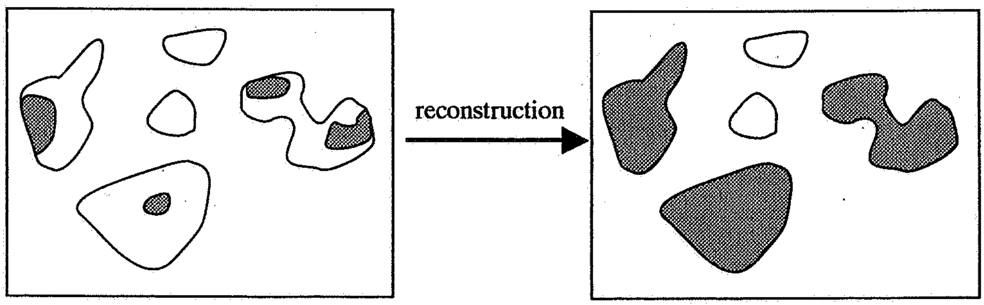

FIGURE 6. *Geodesic reconstruction.*

In fact, we can prove a much more general result.

5.6. Proposition. *If $\{\alpha_r\}$ is a granulometry and if $\breve{\alpha}_r$ is given by $\breve{\alpha}_r(X) = \rho(\alpha_r(X); X)$, then $\{\breve{\alpha}_r\}$ defines a granulometry as well.*

We conclude this section with some results for grey-scale granulometries recently obtained by Kraus et al. [15]. As in the binary case, a granulometry on $\mathrm{Fun}(\mathbb{R}^d)$ is defined as a collection of openings α_r on $\mathrm{Fun}(\mathbb{R}^d)$ which satisfy (5.1), or equivalently (5.2). It is easy to show that the extension of binary granulometries to grey-scale functions by thresholding yields grey-scale granulometries. If we wish to extend the notion of a Minkowski granulometry we must define translations and scalings for grey-scale functions. Concerning translations we may either restrict to translations in the domain (H-translations) or allow translations in the grey-level space as well (together called T-translations). A T-translation of F is given by $(F_h^v)(x) = F(x-h) + v$, where $h \in \mathbb{R}^d$ and $v \in \mathbb{R}$. Here we restrict to the second alternative. A similar choice has to be made when we want to introduce scalings. Either we can work with the so-called *umbral scaling*

$$(\lambda \cdot F)(x) = \lambda F(x/\lambda) \tag{5.7}$$

(here the adjective 'umbral' expresses that this operation scales the umbra, the points on and below the graph of the function) or with the *spatial scaling*

$$(\lambda \cdot F)(x) = F(x/\lambda). \tag{5.8}$$

Both cases are illustrated in Fig. 7.

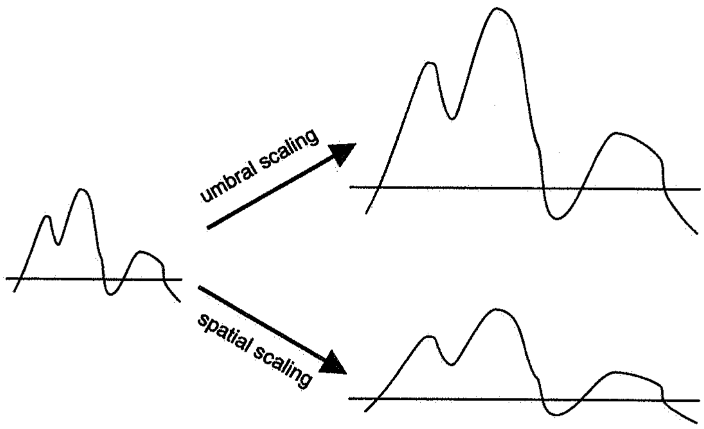

FIGURE 7. *Umbral scaling versus spatial scaling.*

It goes without saying that the choice between either of the two scalings has an enormous impact on the kind of shape information extracted by the granulometry. Here we choose the spatial scaling (also referred to as *H-scaling*).

We call a granulometry α_r on $\mathrm{Fun}(\mathbb{R}^d)$ a *(T,H)-Minkowski granulometry* if it is invariant under T-translations (i.e., $\alpha_r(F_h^v) = [\alpha_r(F)]_h^v$) and compatible with H-scalings (i.e., $\alpha_r(F) = r \cdot \alpha_1(r^{-1} \cdot F)$) where '$\cdot$' denotes the scaling defined by (5.8). It is easy to show that, for every structuring function G, the openings

$$\alpha_r(F) = \bigvee_{s \geq r} F \circ (s \cdot G)$$

define a (T,H)-Minkowski granulometry. To get rid of the outer supremum we must find functions G such that

$$(r \cdot G) \circ G = r \cdot G \text{ for } r \geq 1. \tag{5.9}$$

If we assume that G is upper semi-continuous and has compact domain, then condition (5.9) is satisfied if and only if the domain of G is convex, and G is constant there. If we denote the domain of G by B, then $\alpha_r(F) = F \circ (r \cdot G)$ is the flat extension of the binary Minkowski granulometry $X \to X \circ rB$. For more details we refer to [15]. Related results can be found in [5].

6. Skeletons

Skeletonization algorithms have become of paramount importance in image processing. A first systematic study of skeletons was undertaken by Blum [2, 3] in the context of models for visual perception; essentially however, the underlying ideas can be traced back to the work of Motzkin [20, 21, 32]. Blum introduced the prairie fire model to visualize his ideas. Think of the set X as a dry prairie and suppose that the grass at the edge of X is set afire at the same moment. The resulting fires propagate at constant speed according to Huygen's principle. The *skeleton* $\Sigma(X)$ (or *medial axis* as Blum called it; later he introduced the term *symmetric axis* [3]) is the set of quench points where fire fronts coming from different directions extinguish each other. Shortly after the publication of Blum's first paper [2] there appeared an influential paper by Calabi and Hartnett [4] which carried the mathematical theory of skeletons much further. Lantuejoul [16] was the first to write down an explicit expression for the skeleton using morphological transformations; cf. [30, Chapter XI] and [17]. Recently, Matheron [31, Chapter 11,12] derived a number of interesting topological results about the skeleton. We point out that both Serra and Matheron restrict to open sets.

In the original work of Blum it was assumed that the fire spreads at a constant speed in all directions. In the formal discussion below we do not make this restriction, but allow that the speed is non-isotropic.

Assume that $B \subseteq \mathbb{R}^d$ is a compact convex set. We assume moreover that B contains more than one point. We define the *regular* and *singular* part of a set X (with respect to B) as

$$\mathrm{reg}_B(X) = \bigcup_{r>0} X \circ rB \tag{6.1}$$

$$\mathrm{sing}_B(X) = X \setminus \mathrm{reg}(X). \tag{6.2}$$

The first expression means that a point lies in the regular part of X if it is contained in $(rB)_h$ for some $r > 0$ which lies completely inside X. Apparently,

$$X = \mathrm{reg}_B(X) \cup \mathrm{sing}_B(X).$$

It is easy to show that $\mathrm{reg}_B(\cdot)$ defines a translation invariant opening on $\mathcal{P}(\mathbb{R}^d)$, and that $X^\circ \subseteq \mathrm{reg}_B(X)$. Here X° denotes the interior of X. Consequently, $\mathrm{sing}_B(X) \subseteq \partial X$, the boundary of X.

6.1. Definition. Assume that $(rB)_h$ is contained in X. We say that $(rB)_h$ is a *maximal B-shape* in X if $(rB)_h \subseteq (r'B)_{h'} \subseteq X$ implies that $r' = r$ and $h' = h$.

Furthermore we define the r'th B-skeleton subset

$$\Sigma_{B,r}(X) = \mathrm{sing}_B(X \ominus rB). \tag{6.3}$$

6.2. Lemma. *$(rB)_h$ is a maximal B-shape in X if and only if $h \in \Sigma_{B,r}(X)$.*

PROOF. We use that $h \in \Sigma_{B,r}(X)$ if and only if
(i) $h \in X \ominus rB$, and
(ii) $h \notin (X \ominus rB) \circ \varepsilon B$, for every $\varepsilon > 0$.
'if': assume that (i) and (ii) are satisfied. We show that $(rB)_h$ is a maximal B-shape in X. Suppose that $(rB)_h \subseteq [(r+\varepsilon)B]_k \subseteq X$. Then $h \in (\varepsilon B)_k \subseteq X \ominus rB$, which yields that $h \in (X \ominus rB) \circ \varepsilon B$, a contradiction.
'only if': assume that $(rB)_h$ is a maximal B-shape in X. Then $h \in X \ominus rB$, i.e. (i) holds. Assume that $h \in (X \ominus rB) \circ \varepsilon B$ for some $\varepsilon > 0$. Then $h \in [(r+\varepsilon)B]_k \ominus rB = (\varepsilon B)_k \subseteq (X \ominus rB)$ which implies that $(rB)_h \subseteq [(r+\varepsilon)B]_k \subseteq X \circ rB \subseteq X$. But this means that $(rB)_h$ is not a maximal B-shape, a contradiction. ∎

It is obvious that $\Sigma_{B,r}(X) \cap \Sigma_{B,s}(X) = \emptyset$ if $r \neq s$. We define the *B-skeleton* $\Sigma_B(X)$ of X as the (disjoint) union of all the $\Sigma_{B,r}(X)$,

$$\Sigma_B(X) = \bigcup_{r \geq 0} \Sigma_{B,r}(X). \tag{6.4}$$

6.3. Theorem. *The B-skeleton of a set has empty interior.*

Proof. Assume $h \in \Sigma_{B,r}(X)$. We show that h does not lie in the interior of $\Sigma_{B,r}(X)$. If $r = 0$ then $h \in \partial X$ and therefore it cannot lie in the interior of $\Sigma_B(X)$. We assume that $r > 0$. The set $(rB)_h$ must intersect the boundary ∂X in a point $y \neq h$.

We restrict to the 2-dimensional case; for higher dimensions one can use similar arguments. Suppose first that $B^\circ \neq \emptyset$. If $0 \in B^\circ$ then the assertion is trivial; otherwise choose a point p so that $0 \in (B_p)^\circ$ and use (6.6) below. Suppose next that $B^\circ = \emptyset$. Then B is a line segment. It is obvious that a maximal line segment in X must intersect ∂X in at least two points. This proves the assertion.

We show that a point $k \in (h, y]$ cannot be contained in $\Sigma_B(X)$. Suppose namely that $k \in \Sigma_{B,s}(X)$. Then, by Lemma 6.2, $(sB)_k$ is a maximal B-shape inside X. There is a $\lambda \in [0, 1)$ so that $k = \lambda h + (1-\lambda)y$. It is easy to check that $s \leq s_0$ where s_0 is the solution of $k + s_0/r(y-h) = y$. A straightforward calculation shows that $s_0 = \lambda r$ and we conclude that $s \leq \lambda r$. But it is not difficult to show that $(\lambda r B)_k \subseteq (rB)_h$ which contradicts our assumption that $(sB)_k$ is a maximal B-shape. ∎

The skeleton contains information about the shape of an object. In a sense it expresses how the shape of a set X relates to the shape of the structuring element B. Although it is tempting to make this assertion more concrete we do not pursue this matter any further, but confine ourselves to the example depicted in Fig. 8. Here we compute the B-skeleton of a rectangle and a disk when B is a disk, respectively a square. Note that in the first case the speed of the prairie fire is uniform in all directions, whereas in the case that B is a square the prairie fire has the largest speed ($\sim \sqrt{2}$) in the diagonal directions and the smallest speed (~ 1) in the horizontal and vertical directions.

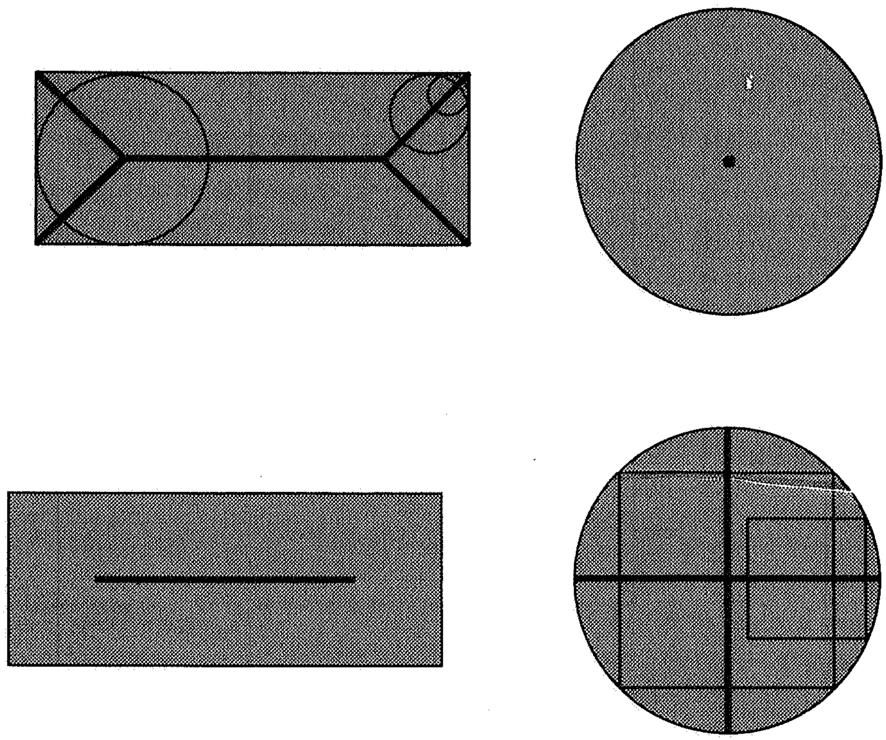

FIGURE 8. *The B-skeleton of a rectangle and a disk when B is a disk (top) respectively a square (bottom).*

The *quench function* q_B is defined as

$$q_B(X, h) = r \text{ if } h \in \Sigma_{B,r}(X).$$

Note that $q_B(X, \cdot)$ has domain $\Sigma_B(X)$ for every set X.

If $h \in \Sigma_B(X)$ and $r = q_B(X, h)$ then $(rB)_h \subseteq X$. This yields that

$$\bigcup_{h \in \Sigma_B(X)} \left[q_B(X, h)B\right]_h = \bigcup_{r \geq 0} \Sigma_{B,r}(X) \oplus rB \subseteq X.$$

If we are able to show that equality holds then we may conclude that the original set X can be reconstructed from the data of the B-skeleton and the associated quench function. It is not difficult to see that this holds if every $x \in X$ is contained in at least one maximal B-shape.

6.4. Theorem. *Let X be closed. The equality*

$$\bigcup_{r \geq 0} \Sigma_{B,r}(X) \oplus rB = X \tag{6.5}$$

holds in each of the following cases:

(i) *X is bounded;*

(ii) *B is the closed unit ball and X contains no half-spaces;*

(iii) *B is a finite line segment and X contains no half-lines with the same orientation;*

(iv) *B is a square and X contains no quarter-spaces with the same orientation.*

The B-skeleton depends on the position of the origin (which we always assume to be contained within B). Moreover, a translation of B does not only induce a translation of the skeleton. In fact, one can show that a translation $B \to B_p$ has the following effect:

$$\Sigma_{B_p}(X) = \{h - q_B(X,h)p \mid h \in \Sigma_B(X)\} \tag{6.6}$$

$$q_{B_p}(X, h - q_B(X,h)p) = q_B(X,h). \tag{6.7}$$

We find the following expression for the erosion $X \ominus tB$:

$$\begin{aligned} \Sigma_r(X \ominus tB) &= \mathsf{sing}((X \ominus tB) \ominus rB) \\ &= \mathsf{sing}(X \ominus (r+t)B) \\ &= \Sigma_{r+t}(X). \end{aligned}$$

This yields that

$$\Sigma_r(X \ominus tB) = \Sigma_{r+t}(X) \quad \text{and} \quad X \ominus tB = \bigcup_{r \geq 0} \Sigma_{r+t}(X) \oplus rB,$$

and hence that

$$\Sigma(X \ominus tB) = \bigcup_{r \geq t} \Sigma_r(X).$$

We consider the opening $X \circ tB$. If $r \geq t$ then

$$\begin{aligned} (X \circ tB) \ominus rB &= \Big(((X \ominus tB) \oplus tB) \ominus tB\Big) \ominus (r-t)B \\ &= (X \ominus tB) \ominus (r-t)B \\ &= X \ominus rB. \end{aligned}$$

From this identity we get that

$$\Sigma_r(X \circ tB) = \Sigma_r(X) \text{ if } r \geq t.$$

Unfortunately it is not possible to make general statements about $\Sigma_r(X \circ tB)$ for $r < t$. Even though $X \circ tB$ is a union of B-shapes of radius $\geq t$ it may occur that $X \circ tB$ contains maximal B-shapes of radius $\leq t$. However, it is possible to reconstruct $X \circ tB$ from the r'th skeleton subsets $\Sigma_r(X)$, $r \geq t$. Thereto we use the expression for $X \ominus tB$ derived above. We get that

$$\begin{aligned} X \circ tB &= (X \ominus tB) \oplus tB \\ &= \Big[\bigcup_{r \geq 0} \Sigma_{r+t}(X) \oplus rB\Big] \oplus tB \\ &= \bigcup_{r \geq 0} \Sigma_{r+t}(X) \oplus (r+t)B, \end{aligned}$$

that is

$$X \circ tB = \bigcup_{r \geq t} \Sigma_r(X) \oplus rB. \tag{6.8}$$

This observation suggests a family of morphological transformations, called *quench function transformations* by Serra [30, Excercise XI-I.8]. Let $f : \mathbb{R}_+ \to \mathbb{R}$; define the transformation ψ_f on $\mathcal{P}(\mathbb{R}^d)$ as

$$\psi_f(X) = \bigcup_{r \geq 0} \Sigma_{B,r}(X) \oplus f(r)B.$$

Here we take $f(r)B = \emptyset$ if $f(r) < 0$. Dilation, erosion and opening by tB are examples of such transformations.

We conclude this section by pointing out a relation between the B-skeleton and the Minkowski granulometry $\alpha_r(X) = X \circ rB$ discussed in the previous section. As we have seen, a granulometry provides useful information about the distribution of grains in an image. If B is convex then $X \circ sB \subseteq X \circ rB$ if $s \geq r$. If we observe that $X \circ rB$ is substantially larger than $X \circ (r + \varepsilon)B$, where $\varepsilon > 0$ is small, then we may conclude that X has components with B-size r; here the phrase "Y has B-size r" means that Y contains a B-shape with radius r but not with radius $> r$.

6.5. Theorem. *For any set X we have*

$$\Sigma_{B,r}(X) = \emptyset \iff X \circ rB = \bigcup_{\varepsilon > 0} X \circ (r + \varepsilon)B$$

if $r \geq 0$.

Proof. "$\Rightarrow$": assume that $\Sigma_{B,r}(X) = \mathrm{sing}_B(X \ominus rB) = \emptyset$. This means that $X \ominus rB = \bigcup_{\varepsilon>0}(X \ominus rB) \circ \varepsilon B$. Then

$$\begin{aligned} X \circ rB &= (X \ominus rB) \oplus rB \\ &= [\bigcup_{\varepsilon>0} (X \ominus rB) \circ \varepsilon B] \oplus rB \\ &= \bigcup_{\varepsilon>0} [((X \ominus rB) \circ \varepsilon B) \oplus rB] \\ &= \bigcup_{\varepsilon>0} ((X \ominus rB) \ominus \varepsilon B) \oplus \varepsilon B \oplus rB \\ &= \bigcup_{\varepsilon>0} X \circ (r + \varepsilon)B. \end{aligned}$$

"$\Leftarrow$": assume that $X \circ rB = \bigcup_{\varepsilon>0} X \circ (r + \varepsilon)B$. We get that

$$\begin{aligned} X \ominus rB &= (X \circ rB) \ominus rB \\ &= [\bigcup_{\varepsilon>0} X \circ (r + \varepsilon)B] \ominus rB \\ &= [\bigcup_{\varepsilon>0} (X \ominus (r + \varepsilon)B) \oplus \varepsilon B] \oplus rB \ominus rB \\ &\subseteq [\bigcup_{\varepsilon>0} (X \ominus (r + \varepsilon)B) \oplus \varepsilon B] \\ &= \bigcup_{\varepsilon>0} (X \ominus rB) \circ \varepsilon B. \end{aligned}$$

The reverse inclusion is trivial. We conclude that $\Sigma_{B,r}(X) = \mathrm{sing}_B(X \ominus rB) = \emptyset$, which had to be proved. ∎

7. Morphological shape decomposition

An important problem in image processing is the decomposition of an object into simpler parts. Such decompositions can e.g. be used for object recognition tasks. In the literature one can find a multitude of techniques for shape decomposition. In this section we briefly describe two approaches based on morphological openings. A first approach, described in Subsect. A, was given by Pitas and Venetsanopoulos [22]. In their approach the simplest possible shape is a disk B (or any other convex structuring element). Starting with an object X they look for the largest radius r_1 for which $X \circ r_1 B \neq \emptyset$ and define

$$X_1 = X \circ r_1 B,$$

the first order approximation in the shape decomposition. Subsequently, they compute the largest radius r_2 for which

$$(X \setminus X_1) \circ r_2 B \neq \emptyset.$$

The second order approximation is given by

$$X_2 = X_1 \cup (X \setminus X_1) \circ r_2 B.$$

Thus their decomposition algorithm is described by the recursion formula

$$\begin{cases} X_0 = \emptyset \\ X_{t+1} = X_t \cup (X \setminus X_t) \circ r_{t+1} B, \ t \geq 0, \end{cases}$$

where r_t is the radius of the maximal inscribable ball rB in $X \setminus X_{t-1}$. Note that $(X \setminus X_{t-1}) \circ r_t B$ is of the form $L \oplus r_t B$, where L is the part of the skeleton of $X \setminus X_{t-1}$ where the quench function is maximal. A set of the form $L \oplus rB$, where L is an arc, is called *Blum ribbon* [22] An example is depicted in Fig. 9. Here we start with a binary hexagonal image X and compute X_t for $t \geq 1$. The structuring element is the elementary hexagon consisting of 7 points. In this figure we have depicted respectively the original image X and its decompositions X_1 ($r_1 = 29$), X_6 ($r_6 = 17$), X_{11} ($r_{11} = 11$), X_{15} ($r_{15} = 7$), X_{19} ($r_{19} = 3$). In this example we need 21 iterations to arrive at the original image.

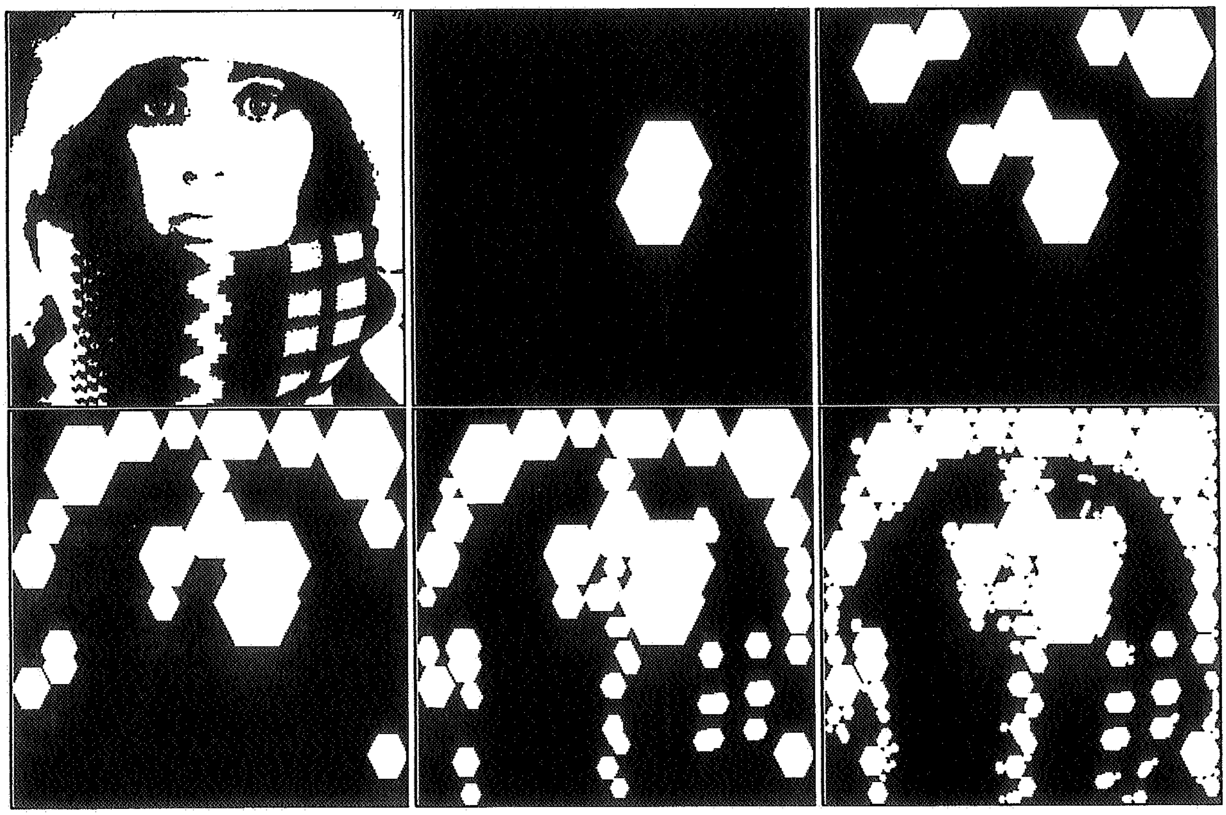

FIGURE 9. *Pitas–Venetsanopoulos decomposition.*

Recently Ronse [27] has developed a very general theory for morphological shape description and decomposition; see also [29]. His theory, which applies to a large class of partially ordered sets, is based upon such notions as *toggles of openings, choice functions* and *open-condensations.* The latter concept will turn up again below. Besides the Pitas–Venetsanopoulos decomposition, which yields a union of disjoint components, Ronse also studies a decomposition in which the building components are not necessarily disjoint.

In Subsect. A below we give a more detailed description of the Pitas–Venetsanopoulos algorithm, and in Subsect. B we consider the algorithm due to Ronse. Both subsections are based on Ronse's work, in particular on his notion of open-condensation which we now briefly discuss. Throughout the remainder of this section we restrict to the binary case. However, the results can easily be extended to gray-scale images.

7.1. Definition. An operator ψ on $\mathcal{P}(E)$ is said to be *condensing* if $X \subseteq Y \subseteq Z$ and $\psi(X) = \psi(Z)$ implies that $\psi(X) = \psi(Y) = \psi(Z)$. If ψ is anti-extensive, idempotent and condensing then it is called an *open-condensation.*

The condensation property is slightly more general than monotonicity: every increasing or decreasing operator is condensing. Similarly the concept of open-condensation extends that of an opening; in particular, every opening is an open-condensation. We refer to [27] for a number of basic results.

A. Pitas–Venetsanopoulos decomposition

In this subsection we discuss a morphological decomposition algorithm originally due to Pitas and Venetsanopoulos [22]. Our treatment, however, is based on the work of Ronse [27].

Suppose we are given a finite collection of openings

$$\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n.$$

In the original theory of Pitas and Venetsanopoulos these openings correspond to openings with balls with decreasing radius.

Let $X \subseteq E$ and define $i(X)$ as the smallest index i such that $\alpha_i(X) \neq \emptyset$; if such index does not exist then we put $i(X) = n + 1$. We define

$$\theta(X) = \alpha_{i(X)}(X)$$

with $\alpha_{n+1}(X) = \emptyset$. In general, θ is not increasing. However we can prove the following result.

7.2. Lemma. *θ is an open-condensation.*

PROOF. It is obvious that θ is anti-extensive and idempotent. We show that θ has the condensation property. Suppose $X \subseteq Y \subseteq Z$ and $\theta(X) = \theta(Z)$. It is obvious that $i(X) \geq i(Z)$. On the other hand,

$$\alpha_{i(Z)}(Z) = \alpha_{i(Z)}(\alpha_{i(Z)}(Z)) = \alpha_{i(Z)}(\alpha_{i(X)}(X)) \subseteq \alpha_{i(Z)}(X),$$

which yields that $i(X) \leq i(Z)$. Therefore equality holds. But then also $i(Y) = i(X)$ and the result follows. ∎

Consider the following decomposition algorithm:

$$\begin{cases} X_0 = \emptyset \\ X_{t+1} = X_t \cup \theta(X \setminus X_t), \ t \geq 0. \end{cases}$$

It is not difficult to see that the corresponding index sequence $i(X \setminus X_t)$ is increasing. As soon as $i(X \setminus X_t)$ reaches the value $n + 1$ the recursion is stopped. We define the operator γ_t as

$$\gamma_t(X) = X_t.$$

7.3. Proposition. *The operator γ_t is an open-condensation.*

To prove this we need the following result.

7.4. Lemma. *Assume that ψ is an open-condensation. Then γ defined by $\gamma(X) = \psi(X) \cup \theta(X \setminus \psi(X))$ is an open-condensation as well.*

PROOF. It is evident that γ is anti-extensive. We show that γ is idempotent. Since $\psi(X) \subseteq \gamma(X) \subseteq X$, $\psi(\psi(X)) = \psi(X)$ and ψ is an open-condensation, we obtain that $\psi\gamma(X) = \psi(X)$. Therefore,

$$\gamma^2(X) = \psi(X) \cup \theta(\gamma(X) \setminus \psi(X)).$$

We use that $\gamma(X) \setminus \psi(X) = \theta(X \setminus \psi(X)) \setminus \psi(X) = \theta(X \setminus \psi(X))$ and obtain that $\gamma^2(X) = \gamma(X)$.

Finally we show that γ is condensing. Suppose that $X \subseteq Y \subseteq Z$ and $\gamma(X) = \gamma(Z)$. Since $\psi\gamma = \psi$ we get that $\psi(X) = \psi(Z)$. Since ψ is an open-condensation this yields that $\psi(X) = \psi(Y) = \psi(Z)$. Then

$$\theta(X \setminus \psi(X)) = \gamma(X) \setminus \psi(X) = \gamma(Z) \setminus \psi(Z) = \theta(Z \setminus \psi(Z))$$

and

$$X \setminus \psi(X) \subseteq Y \setminus \psi(Y) \subseteq Z \setminus \psi(Z).$$

But θ is an open-condensation (cf. Lemma 7.2) and we may conclude that $\theta(X \setminus \psi(X)) = \theta(Y \setminus \psi(Y))$. This yields that $\gamma(X) = \gamma(Y)$. ∎

Prop. 7.3 follows from this result by induction. Namely $\gamma_0(X) = \emptyset$ defines an open-condensation. Suppose that γ_t is an open-condensation. Then $\gamma_{t+1}(X) = \gamma_t(X) \cup \theta(X \setminus \gamma_t(X))$, and by Lemma 7.4 this is an open-condensation as well.

To conclude this subsection we point out that the decomposition above is quite different from the "classical" morphological multiscale description given by the sequence $X_t = \alpha_t(X)$.

B. Ronse decomposition

In this subsection we discuss a variant of the decomposition algorithm given by Ronse in [27]. An application can be found in [29]. The underlying idea is captured by the following example. Suppose that we are given a collection of structuring elements $B_1, B_2, \cdots, B_n$, and that we want to approximate an object X by the openings $X \circ B_1, X \circ B_2, \cdots X \circ B_n$. It depends on the shape of X which B_i suits best. Consider the convex polygon X which has 8 equal edges of length l as depicted in Fig. 10, let D be a disk and S a square both with are A. If $l^2 \geq A$ then $X_S = X$ whereas X_D is a strict subset of X. Therefore X_S is a better approximation than X_D for such values of A. This changes if we increase A. In particular, if $A > 4l^2$ then $X_S = \emptyset$; however $X_D = \emptyset$ if and only if $A > \frac{\pi}{4}(3 + 2\sqrt{2})l^2$. Inside X always fits a disk with radius $l(2\tan\frac{\pi}{8})^{-1} = \frac{1}{2}l(\sqrt{2}+1)$. Note that $\frac{\pi}{4}(3+2\sqrt{2}) > 4$.

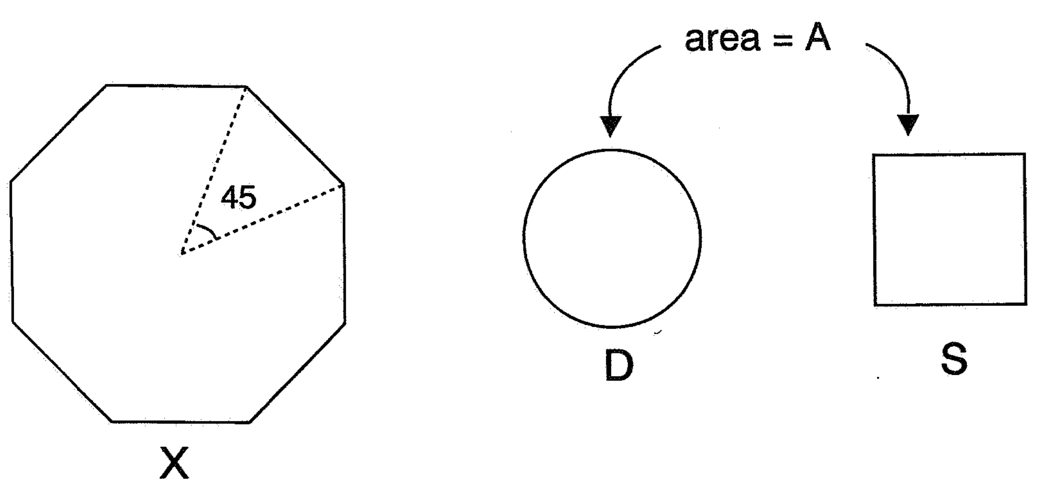

FIGURE 10. *See text.*

How do we put such a simple observation into a formal mathematical framework. For that purpose Ronse [27] introduced the general notion of a choice function. In our discussion we shall try to be more specific and assume the existence of a distance function on the object space. This distance function is used to choose, at every step, between the different alternatives.

7.5. Assumption. Let $\mathcal{L} \subseteq \mathcal{P}(E)$ and let D be a function which maps a pair $X, Y \in \mathcal{L}$ with $X \subseteq Y$ onto a positive real; assume moreover that D has the following properties:

- $D(X,Y) = 0$ iff $X = Y$;
- $D(X,Y) \leq D(X,Y')$ if $X \subseteq Y \subseteq Y'$.

Here $\mathcal{L}$ denotes the object space. We assume that the openings introduced below map $\mathcal{L}$ into $\mathcal{L}$ and that $\mathcal{L}$ is closed under finite unions. As a concrete example we mention the case where $\mathcal{L}$ is the space of compact sets in $\mathbb{R}^d$, and D is the Hausdorff metric.

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be an arbitrary collection of openings. Furthermore, let ψ be an open-condensation. For $X \subseteq Y$ we define $i(X,Y)$ as the smallest index i such that $D(X, X \cup \alpha_i(Y))$ is maximal. If, however, this expression is 0 for every i (which is the case iff $\alpha_i(Y) \subseteq X$ for every i) then we put $i(X,Y) = n+1$. Furthermore we define $\alpha_{n+1}(X) = \emptyset$ for every X.

7.6. Lemma. *Given an open-condensation ψ, the operator γ given by*

$$\gamma(X) = \big(\psi \vee \alpha_{i(\psi(X),X)}\big)(X)$$

is an open-condensation as well.

PROOF. Assume that $\gamma(X) \subseteq Y \subseteq X$. We show that $\gamma(X) = \gamma(Y)$. It is clear that $\psi(X) \subseteq Y \subseteq X$. Since ψ is an open-condensation we get that $\psi(X) = \psi(Y)$. Let $i_0 = i(\psi(X), X)$, then $\alpha_{i_0}(X) \subseteq Y \subseteq X$. The fact that α_{i_0} is an opening yields that

$$\alpha_{i_0}(X) \subseteq \alpha_{i_0}(Y) \subseteq \alpha_{i_0}(X),$$

and therefore $\alpha_{i_0}(X) = \alpha_{i_0}(Y)$. Since $\alpha_j(Y) \subseteq \alpha_j(X)$ for all j we conclude that

$$D(\psi(Y), \psi(Y) \cup \alpha_j(Y)) = D(\psi(X), \psi(X) \cup \alpha_j(Y))$$

is maximal if $j = i_0$. This yields that $i(\psi(Y), Y) = i_0$, and hence that $\gamma(X) = \gamma(Y)$.

It is obvious that γ is anti-extensive. Thus, substituting $Y = \gamma(X)$, we find that γ is idempotent. We show that γ is condensing. Assume $X \subseteq Y \subseteq Z$ and $\gamma(X) = \gamma(Z)$. Then $\gamma(Z) = \gamma(X) \subseteq X \subseteq Y \subseteq Z$. From the considerations above it follows that $\gamma(Y) = \gamma(Z)$. This concludes the proof. ∎

Now we introduce the following algorithm:

$$\begin{cases} X_0 = \emptyset \\ X_{t+1} = X_t \cup \alpha_{i(X_t,X)}(X). \end{cases}$$

It is obvious that $X_0 \subseteq X_1 \subseteq \cdots \subseteq X$. In fact, this sequence is strictly increasing until it reaches the final result $\bigcup_{i=1}^n \alpha_i(X)$ (after at most n iterations). Putting

$$\gamma_t(X) = X_t$$

we draw the following conclusion from Lemma 7.6

7.7. Proposition. *γ_t is an open-condensation for every $t \geq 0$.*

The approach outlined above is less general than the original approach of Ronse [27]. The algorithm described by Ronse and Macq [29] uses structuring elements of different scales. It first checks if there is a structuring element in the largest size scale which yields an increment in the approximation; within that scale class it chooses the shape giving the largest increment. If there do no longer exists stucturing elements which give an increment, only then the scale is decreased and structuring elements in the next, smaller, size class are being considered.

To conclude this section, we point out that in both approaches the decomposition operators γ_t are translation invariant if every α_i is such.

8. Final remarks

Mathematical morphology is a branch in image processing particularly suited for shape description. In fact, the concept of a structuring element enables one to infer shape-related information from a scene. We have explained that there are essentially two different ways to conceive of a structuring element. Firstly, it can be regarded as a subset of a certain transformation group. This point of view is quite flexible in that one may choose those transformations which are best suited for the given application. Alternatively, one can use the notion of distance to define structuring elements. In this approach, the notion of convexity plays an important role.

In this paper we have presented a bird's eye view of the different morphology-based methods for shape description, such as granulometric analysis, skeletonization techniques, and morphological decomposition algorithms.

A concept very closely related to the morphological granulometry is the so-called *pattern spectrum* introduced by Maragos [18]. For a compact set $X \subseteq \mathbb{R}^d$ the pattern spectrum relative to a convex structuring element $B \subseteq \mathbb{R}^d$ is defined as

$$\begin{aligned} PS_X(r, B) &= -\frac{d}{dr}\mathsf{Area}(X \circ rB), \quad r \geq 0, \\ PS_X(-r, B) &= \frac{d}{dr}\mathsf{Area}(X \bullet rB), \quad r > 0. \end{aligned}$$

For discrete images one defines $nB = B \oplus \cdots \oplus B$ (n times) and

$$\begin{aligned} PS_X(n, B) &= \mathsf{Area}(X \circ nB \setminus X \circ (n+1)B), \quad n \geq 0, \\ PS_X(-n, B) &= \mathsf{Area}(X \bullet nB \setminus X \bullet (n-1)B), \quad n \geq 1. \end{aligned}$$

Analogous definitions can be given for grey-scale images. Maragos uses the pattern spectrum as a shape-size descriptor. Furthermore he points out connections between the pattern spectrum and the skeleton transform (cf. Thm. 6.5).

Recently, van den Boomgaard and Smeulders [33, 34] have initiated a theory which can be regarded as the morphological analogue of the Gaussian scale space. The basic idea is to dilate (or erode) an image with a parametrized family of quadratic structuring elements. They show that resulting images satisfy a nonlinear PDE related to Burger's equation.

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