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A. Gombani

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On some Results by Fuhrmann on Hankel Operators

Andrea Gombani
LADSEB-CNR
Corso Stati Uniti 4, 35020 Padova, ITALY
and
CWI
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

Abstract
We present here some generalizations and some simpler derivations of results by Fuhrmann which appeared in [1].

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1 Introduction

In spite of the fact that Hankel operators have become, after the seminal paper of Adamjan, Arov and Krein [2], a central tool in systems theory and in interpolation theory, some of its features have been studied in detail only recently in [1]. Among the relevant results are the connection between restricted shift realizations and between the singular values of the original operator and those of its Nehari extension. We present a simple derivation of these results and some generalizations.

2 Preliminaries and Notation

We work in the Hilbert space setting of the plane; we define $L^2(\Im)$ to be the set of the square integrable functions on the imaginary axis, and H^2_+ to be the subspace of L^2 of functions analytic in the right half-plane and s. t.

$$\sup_{x>0} \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x+iy)|^2 \, dy < \infty$$

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(H^2_- is defined similarly on the left hand-plane).

The inner product in H^2_+ is

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(i\omega)\,\overline{g(i\omega)}\,d\omega \tag{2.1}$$

where bar denotes conjugate. If $p(s)$ is a polynomial with real coefficients, we set

$$p^*(s) := p(-s).$$

Let $f = \frac{n}{d}$ be a stable rational function of L^2: we denote by $\mathbf{P}^+(\mathbf{P}^-)$ the orthogonal projection of L^2 onto $H^2_+(H^2_-)$. The Hankel operator with symbol f is defined as

$$H_f h = \mathbf{P}^- f h \quad h \in H^2_+ \tag{2.2}$$

A Schmidt pair (ξ, η) of H_f is a pair of vectors $\xi \in H^2_+$, $\eta \in H^2_-$ so to

$$H_f\xi = \sigma\eta, \quad H^*_f\eta = \sigma\xi. \tag{2.3}$$

for a convenient positive number σ, called singular value. It can be shown (see [1] or [2]) that if $f \in H^2_-$ and it is rational of degree n, then there exist n different singular values.

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$$

and n pair $(\xi_1, \eta_1)\ldots(\xi_n, \eta_n)$ (Schmidt pairs) satisfying (2.3).

Moreover, there exist polynomials $p_1, \ldots, p_n$ of degree at most $n-1$ such that $\xi_i = \frac{p_i^*}{d^*}$, $\eta_i = \varepsilon_i\frac{p_i}{d}$ where ε_i is a constant of modulus 1. In view of the definition (2.2), (2.3) becomes

$$\frac{n}{d}\frac{p_i^*}{d^*} = \varepsilon_i\sigma_i\frac{p_i}{d} + \frac{\pi_i}{d^*} \tag{2.4}$$

where π_i are convenient polynomials of degree at most $n-1$.

The polynomials p_i are in general determined up to a (constant) factor by $\frac{n}{d}$, as well as the values σ_i and ε_i. We set $\lambda_i = \varepsilon_i\sigma_i$ and assume in the sequel that $\sigma_1 > \sigma_2 > \ldots > \sigma_n$. In this case the p_i are unique (up to a constant factor) and $\frac{p_i}{d}$ is orthogonal to $\frac{p_j}{d}$ for $i \neq j$ (see [1]).

The equation (2.4) is called *fundamental equation* (see [1]). In what follows we will also assume d monic.

3 Main Results

The fundamental equation (2.4) can be written (see [1], [2]) as

$$\lambda_1 \frac{p_1 d^*}{p_1^* d} \cdot \frac{p_i^*}{d^*} = \lambda_i \frac{p_i}{d} + \frac{\pi_i^*}{p_1^*} \tag{3.1}$$

Therefore

$$\begin{aligned}
\lambda_1 \frac{p_1^* d}{p_1 d^*} \cdot \frac{\pi_i^*}{p_1^*} &= \lambda_1^2 \frac{p_i^*}{d^*} - \lambda_i \lambda_1 \frac{p_i}{d} \frac{p_1^* d}{p_1 d^*} \\
&= \lambda_1^2 \frac{p_i^*}{d^*} - \lambda_1^2 \frac{p_i^*}{d^*} - \lambda_i \frac{\pi_i}{p_1} \\
&= -\lambda_i \frac{\pi_i^*}{p_1} + (\lambda_1^2 - \lambda_i^2) \frac{p_i^*}{d^*}
\end{aligned}$$

i.e. $\frac{p_1^* d}{p_1 d^*}$ has signed singular values $\lambda_2, \ldots, \lambda_n$ and singular vectors $\frac{\pi_2^*}{p_1^*}, \ldots, \frac{\pi_n^*}{p_1^*}$. This result has been obtained first by Fuhrmann, and it admits a simple generalization to the all pass functions generated by any Schmidt pair.

Theorem 3.1. Let $p_k = u_k s_k$, with u_k unstable and s_k stable and let $\varphi_k = \frac{s_k u_k d^*}{s_k^* u_k^* d}$; if φ_k has s-vectors $\frac{\alpha_k^*}{d^* u}$ and signed singular values $\mu_i, (\mu_1 = \mu_2 \ldots = \mu_{2k-1} > \mu_{2k} > \ldots > \mu_{n+k-1}$, then φ_k^* has signed singular vectors $\bar{\mu}_i = -\mu_{2k+i-1}$, $1 \leq i \leq n-k$, and s-vectors $\frac{\beta_i^*}{s_k^*}$, where the $\frac{\beta_i^*}{s_k^*}$ satisfy the fundamental equation.

Proof. From the fundamental equation we get

$$\mu_k \varphi_k \frac{\alpha_i^*}{d^* u} = \mu_i \frac{\alpha_i}{d u^*} + \frac{\beta_i^*}{s^*} \quad i = 2k, \ldots, n+k-1$$

and

$$\begin{aligned}
\mu \varphi_k^* \frac{\beta_i^*}{s^*} &= \mu_k^2 \frac{\alpha_i^*}{d^* u} - \mu_k \mu_i \varphi^* \frac{\alpha_i}{d u^*} = \\
&= \mu_k^2 \frac{\alpha_i^*}{d^* u} - \mu_i^2 \frac{\alpha_i^*}{d^* u} - \mu_i \frac{\beta_i}{s} \\
&= -\mu_i \frac{\beta_i}{s} + (\mu_k^2 - \mu_i^2) \frac{\alpha_i^*}{d^* u}
\end{aligned}$$

and hence the result. Q.E.D.

Another result in [1] admitting a simple generalization is the following: the function $\frac{1}{\lambda_n} \frac{d^*}{d} \frac{p_n^*}{p_n}$ has s-values $\frac{1}{\lambda_i}$, and s-vectors $\frac{p_i}{d^*}$.

To see this, from the fundamental equation for $i = n$ we obtain

$$\lambda_1 \frac{p_1 d^*}{p_1^* d} = \lambda_n \frac{p_n}{d} \frac{d^*}{p_n^*} + \frac{\pi_n^*}{p_1^*} \frac{d^*}{p_n^*}$$

applying this equality to the fundamental equation again we get

$$\begin{aligned}\lambda_1 \frac{p_1 d^*}{p_1^* d} \frac{p_j^*}{d^*} &= \lambda_j \frac{p_j}{d} + \frac{\pi_j}{p_1^*} = \\ &= \lambda_n \frac{p_n}{p_n^*} \frac{d^*}{d} \frac{p_j^*}{d^*} + \frac{\pi_n^*}{p_1^*} \frac{d^*}{p_n^*} \frac{\pi_j^*}{d^*}\end{aligned}$$

Multiplying by $\frac{p_n^*}{p_n}$ we get

$$\lambda_j \frac{d^*}{d} \frac{p_n^*}{p_n} \frac{p_j}{d^*} = \lambda_n \frac{p_j^*}{d} + \frac{d^*}{p_n} \frac{\pi_n^*}{p_1^*} \frac{\pi_j^*}{d^*} - \frac{p_n^*}{p_n} \frac{\pi_j^*}{p_1^*}$$

division by $\lambda_j \lambda_n$ yields the result.

The same idea is behind the next theorem:

Theorem 3.2 Let φ_k have s-vectors $\xi_i = \frac{\alpha_i^*}{d^* u_k}$, $2k \leq i \leq k+n-1$ and SSV $\mu_1 = \ldots = \mu_{2k-1}, \mu_{2k}, \ldots, \mu_{k+n-1}$. Then $\frac{\alpha_{k+n-1}}{\alpha_{k+n-1}} \frac{d^* u}{du^*}$ has SSV $\bar{\mu}_1 = \mu_{k+n-1}$, $\bar{\mu}_2 = \mu_{k+n-1}, \ldots, \bar{\mu}_{n-k} = \ldots = \bar{\mu}_{k+n-1} = \mu_1^{-1}$ and s-vectors $\frac{\alpha_i}{\alpha^* u_k}$

Proof. Write the fundamental equation for φ_k

$$\varphi_k \frac{\alpha_i^*}{ud^*} = \mu_i \frac{\alpha_i}{u^* d} + \frac{\beta_i}{s^*}$$

yielding

$$\varphi_k = \mu_n \frac{\alpha_{k+n-1} u_k d^*}{\alpha_{k+n-1}^* u_k^* d} + \frac{\beta_{n+k-1}}{s_k^*} \frac{u_k d^*}{\alpha_{n+k-1}^*}$$

applying the above to $\frac{\alpha_i^*}{u_k d^*}$ we obtain

$$\begin{aligned}\mu_n \frac{\alpha_{n+k-1} u_k d^*}{\alpha_{n+k-1}^* u_k^* d} \frac{\alpha_i^*}{u_k d^*} + \frac{\beta_{n+k-1}}{s_k^*} \frac{u_k d^*}{\alpha_{n+k-1}^*} \frac{\alpha_i^*}{ud^*} &= \\ = \mu_i \frac{\alpha_i}{u_k^* d} + \frac{\beta_i}{s_k^*}\end{aligned}$$

multiplication by $\frac{\alpha_{n+k-1}^*}{\alpha_{n+k-1}}$ gives

$$\mu_i \frac{\alpha_{n+k-1}^*}{\alpha_{n+k-1}} \frac{ud^*}{u^* d} \frac{\alpha_i}{ud^*} = \mu_n \frac{\alpha_i^*}{u_k^* d} + \frac{\beta_{n+k-1} \alpha_i^*}{s_k^* \alpha_{n+k-1} ud^*} - \frac{\alpha_{n+k-1}^* \beta_i}{\alpha_{n+k-1} s_k^*}$$

Division by $\mu_i \mu_n$ yields the result. Q.E.D.

Duality: From the above result we obtain a duality structure for the generic s-vector which is slightly more complex than the one presented in [1].
Letting

$$\varphi_k = \frac{p_k}{p_k^*}\frac{d^*}{d} \tag{3.2}$$

$$\tilde{\varphi}_k = \frac{p_k^* d^*}{p_k d}$$

$$\varphi_{k,n} = \frac{\alpha_{n+k-1}^*}{\alpha_{n+k-1}}\frac{u_k d^*}{u^* d}$$

$$\tilde{\varphi}_{k,n} = \frac{\beta_{2n-k-1}^* s_k^* d}{\beta_{2n-k-1} s_k d}$$

where $\frac{\alpha_{n+k-1}}{u^* d}$ is the last s-vector of φ_k and $\frac{\beta_{2n-k-1}^*}{s_k d}$ is the last s-vector of $\tilde{\varphi}_k$. Then the above results can be summarized as follows:

$$\sigma_j\left(\varphi_k\right) = \sigma_{n+k-j}^{-1}\left(\varphi_{k,n}\right. \tag{3.3}$$

$$\sigma_j\left(\tilde{\varphi}_k\right) = \sigma_{2n-k-j+1}^{-1}\left(\tilde{\varphi}_{k,n}\right)$$

We have seen that a Hankel operator is uniquely determined by its first or its last Schmidt vectors. Is this true for any Schmidt vector? The answer is positive, as the next result shows.

Theorem. Let $\frac{p_k}{d}$ be a stable strictly proper rational function of degree n with k stable zeros. Then there is a unique (up to a constant factor) Hankel operator of degree n with $\frac{p_k^*}{d^*}$ as its k-th s-vector.

Proof. We want to find a stable p_1 and λ_1, λ_k such that.

$$\lambda_1 \frac{p_1}{p_1^*}\frac{d^*}{d} = \lambda_k \frac{p_k d^*}{p_k^* d} + \frac{\pi_k d^*}{p_1^* p_k^*}$$

Multiplying by $\frac{d}{d^*}\,\lambda_k^{-1}$ we obtain

$$\frac{\lambda_1}{\lambda_k}\frac{p_1}{p_1^*} = \frac{p_k}{p_k^*} + \frac{\pi_k d}{p_1^* p_k^*}$$

Computing in z_i (zeros of d) we get

$$\alpha\,\frac{p_1}{p_1^*}(z_i) = \frac{p_k}{p_k^*} \quad i = 1, \ldots, n$$

Now this is a Pick-Navanlinna problem and it admits a unique solution of minimal norm $|\hat{\alpha}|$. Thus the conclusion.

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References

[1] Fuhrmann, P.A. (1991) A Polynomial Approach to Hankel Norm Balanced Approximations. *Linear Algebra and its Applications* **146**, pp. 133-220.

[2] Adamjan, V.M., D.A. Arov, and M.G. Krein. (1971) Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takagi problem. *Math. USSR Sbornik* **15**, pp. 31-73.

[3] Gombani, A. (1991) On Balancing and Hankel Operators, in *Modeling, Estimation and Control of Systems with Uncertainty.* G.B. Di Masi, A. Gombani, and A. Kurzhanski (Eds.). Birkhäuser Boston.