# 1992

A. Gombani

On the Schmidt pairs of multivariable Hankel operators
and robust control

Department of Operations Research, Statistics, and System Theory Report BS-R9226 October

CWI is the research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a non-profit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands organization for scientific research (NWO).

# On the Schmidt Pairs of Multivariable Hankel Operators and Robust Control

*Andrea Gombani*
LADSEB-CNR
Corso Stati Uniti 4, 35020 Padova, ITALY
and
CWI
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

**Abstract**
We consider here the multivariable version of a result appeared in Fuhrmann and Ober [6]. We show that the singular values and the Schmidt vectors of the Hankel operator with symbol a normalized coprime factorization of a plant can be given an explicit representation in terms of the plant, of its superoptimally robust controller and of the Schmidt pairs of another Hankel operator. Our derivation is obtained using techniques of superoptimal Nehari extension developed by Young.

*1991 Mathematics Subject Classification:* 93B36, 93B28.
*Keywords:* Schmidt vectors, normalized coprime factorization, Nehari extension, optimally robust stabilizing controller.
*Note:* This research was supported in part by a CNR-NATO fellowship.

## 1 Introduction

The relevance of Hankel operators in the theory of robust control has been apparent since the seminal paper of Adamjan, Arov and Krein [1]. Nevertheless, some of its features have been studied in detail only recently, in [6] for relations between Schmidt pairs and robust control, and in Young (see [9]) for a generalization of the basic thinking of [1] to the multivariable case. The work of Young, in particular, has never been exploited very much. Still, we believe that there is some insight to be gained by using Young's approach to extend the results of [6] to the multivariable case. This is, in fact, the basic idea of this work.



Report BS-R9226
ISSN 0924-0659
CWI
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

The paper is structured as follows: section two begins with some notation, and then gives a brief account of the results of [9] adjusted to our setting; section three contains the main result (Theorem 3), and some corollaries.

## 2 Preliminaries and notation

We work in the Hilbert space setting of the plane; we define [7] $L^2(I\!I)$ to be the set of the vector or matrix valued (the proper dimension will be clear from the context) square integrable functions on the imaginary axis, and $H^2_+$ to be the subspace of $L^2$ of functions analytic in the right half-plane and s.t.

$$\sup_{x>0} \frac{1}{2\pi} \int_{-\infty}^{\infty} tr(F^*(x+iy)F(x+iy))\, dy < \infty$$

where $^*$ denotes transposed conjugate. If $F$ and $G$ are vectors, the inner product in $H^2_+$ is

$$\langle F, G\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(i\omega)\, F(i\omega)\, d\omega$$

where bar denotes conjugate. Analogously, $H^\infty_+$is the subspace of $L^2$ of functions analytic in the right half-plane and s. t.

$$\sup_{x>0} (\operatorname{ess\,sup}_{y\in R} \|F(x+iy)\|\ ) < \infty$$

where $\|F(x+iy)\|$ denotes the usual matrix norm. ($H^2_-$ and $H^\infty_-$are defined similarly on the left half-plane).

Let $F$ be function of $L^\infty$: we denote by $\mathbf{P}_+(\mathbf{P}_-)$ the orthogonal projection of $L^2$ onto $H^2_+(H^2_-)$. The Hankel operator with symbol $F$ is defined as

$$H_F h = \mathbf{P}_- F h \quad h \in H^2_+ \tag{1}$$

A Schmidt pair $(\xi, \eta)$ of $H_F$ is a pair of vectors $\xi \in H^2_+$, $\eta \in H^2_-$ such that

$$H_F \xi = \sigma\eta, \quad H^*_f \eta = \sigma\xi. \tag{2}$$

for a convenient positive number $\sigma$, called singular value. It can be shown (see [1] or [5]) that if $F \in H^2_-$ and is rational of degree $n$, then there exist $n$ singular values.

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$$

and $n$ pair $(\xi_1, \eta_1)\ldots(\xi_n, \eta_n)$ (Schmidt pairs) satisfying (2).

In a few places we will also need the *conjugate* Hankel operator $\hat{H}_F$ with symbol $F$:

$$\hat{H}_F h = \mathbf{P}_+ F h \quad h \in H^2_- \tag{3}$$

Let a matrix $G \in L^\infty$ of dimension $p \times m$ be given. To fix notation we will assume $p \leq m$. We say that a factorization $NM^{-1}$ of $G$ is a right normalized coprime factorization (NRCF) if

$$M^*M + N^*N = I \tag{4}$$

and $G = \overline{M}^{-1}\overline{N}$ is a normalized left coprime factorization (NLCF) for $G$ if

$$\overline{M}\overline{M}^* + \overline{N}\overline{N}^* = I \tag{5}$$

It is well known that internally stabilizing controllers $K = UV^{-1} = \overline{V}^{-1}\overline{U}$ of $G$ satisfy the Bezout equations $\overline{V}M - \overline{U}N = I$ and $\overline{M}V - \overline{N}U = I$, and that if $(U_0, V_0)$ is one solution, then any other controller is obtained by the Youla parametrization

$$K = (U_0 + MQ)(V_0 + NQ)^{-1} \qquad Q \in H^\infty_+ \tag{6}$$

The matrices $M, N, \overline{M}, \overline{N}, U, V, \overline{U}, \overline{V} \in H^\infty_+$ are called a doubly coprime factorization of a plant $G$ if

$$\begin{bmatrix} \overline{M} & \overline{N} \\ \overline{U} & \overline{V} \end{bmatrix} \begin{bmatrix} V & -N \\ -U & M \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{7}$$

Let $\sigma_i(G)$ denote the i-th singular value of a matrix $G$. A function $R^*$ is called superoptimal Nehari extension of a function $F^* \in L^\infty$ if

- $\sigma_i(R^*(\imath\omega))$ is constant for $1 \leq i \leq p$
- the Hankel operators with symbols $F^*$ and $R^*$ coincide

the second property is equivalent, as is well known, to the fact that the strictly proper antistable parts of the functions coincide.

Define now $\lambda_i$ for $1 \leq i \leq p$ as

$$\lambda_i = \sup_{s \in \mathbb{C}^+} \sigma_i(R^*(s))$$

and set $\Lambda = diag\{\lambda_i\}$, and denote by $F^\#$ the pseudoinverse of a matrix. We can now quote a theorem from [9]:

**Theorem 1** *Let $R_0^*$ be a $m \times p$ matrix-valued function in $H_-^\infty$ such that $\lambda_p > 0$. Then there exists a unique function $R^* \in L^\infty$ of minimal degree which is a superoptimal Nehari extension of $R_0^*$ and it is given by*

$$R^* = \sum_{i=1}^{p} \lambda_i \eta_i \xi_i^\# \tag{8}$$

*where $\xi_i$ and $\eta_i$ are the Schmidt vectors of $H_{R_0^*}$ corresponding to the singular values $\lambda_i$. Therefore*

$$R^* \xi_i = \lambda_i \eta_i \qquad R \eta_i = \lambda_i \xi_i$$

*Moreover, the vectors $\xi_i$ and $\xi_j$ are pointwise orthogonal for $i \neq j$. The same holds for $\eta_i$ and $\eta_j$.*

**Remark** There are two facts which need to be mentioned about the above formulation, which differs slightly from the one in [9], to which we refer for the details we discuss in the next few lines: the first is that our setting is continuous and not discrete, and therefore a Cayley transform argument has to be used to transpose the result; the second is that our formulation corresponds to a one sided Pick-Nevanlinna problem, and therefore the statement of the theorem simplifyes considerably. In particular, all spaces that in the proof of Theorem 2 in [9] are in $L^2$, are in our case in $H_+^2$; in particular, the vectors $\xi_i$ are in $H_+^2$, and hence they are trivially Schmidt vectors of $R^*$. Since $H_{R^*} = H_{R_0^*}$ the $\xi_i$ are also Schmidt vectors of the last operator.

We say that the singular values $\lambda_i$ and the Schmidt vectors $\xi_i$ occurring in Theorem 1 are *Young singular values* and *Young vectors* for $H_{R^*}$. Note that, togheter with the values $\lambda_i$ they uniquely determine the Hankel operator $H_{R^*}$. The pairs $\{\xi_i, \eta_i\}$ are likewise called *Young pairs.* Setting $X = [\xi_1, \xi_2, ..., \xi_p]$ and $Y = [\eta_1, \eta_2, ..., \eta_p]$ we can write (8) as $R^* = Y \Lambda X^{-1}$.

**Corollary 1** *Let $X$ and $Y$ as above. The superoptimal Nehari extension of $F$ can then be written as:*

$$R^* = (Y^\#)^* \Lambda X^* \tag{9}$$

*Proof*: the claim follows directly from Theorem 1 applied to the adjoint operator $(H_{R_0^*})^*$, which coincides with the (conjugate) Hankel operator with symbol $R_0$ and whose (conjugate) Young pairs are $\{\eta_i, \xi_i\}$. Then the superoptimal (conjugate) Nehari extension of $R_0$ is $S = X \Lambda Y^\#$. But from the definition it follows that $S^*$ is a superoptimal Nehari extension of $R_0^*$ and thus, in view of uniqueness, $S^* = R^*$. ∎

**Lemma 1** *Let $X = [\xi_1, \xi_2, ..., \xi_p]$ and $Y = [\eta_1, \eta_2, ..., \eta_p]$. Then there exist $Q, \overline{Q}$ inner and $D$ diagonal outer in $H_+^2$ such that*

$$X = QD \qquad Y = \overline{Q}^* \begin{bmatrix} D^* \\ 0 \end{bmatrix} = \overline{Q}_1^* D^* \tag{10}$$

*Moreover,* $RR^* = Q\Lambda^2 Q^*$ *and* $R^*R = \overline{Q}\Lambda^2\overline{Q}^*$

**Proof:** since the columns of $X$ are pointwise orthogonal in $H^2_+$, $X^*X$ is diagonal, and has therefore a diagonal outer spectral factor $D$. Obviously $Q = XD^{-1}$ is inner.

About $Y$ we follow a quite standard argument(see e.g. [3]). Again in view of pointwise orthogonality in $H^2_-$, there exists an outer $p \times p$ diagonal function $E$ such that $Y^*Y = EE^*$, and therfore $\overline{Q}_1 = E^{-1}Y^*$ is a $p \times m$ in $H^\infty_+$such that $\overline{Q}_1^*\overline{Q}_1$ is idempotent, and therefore a projection, *a.e.*. Thus so is $I - \overline{Q}_1^*\overline{Q}_1$ which can be factored (in an essetially unique manner) as $\overline{Q}_2^*\overline{Q}_2$ where $\overline{Q}_2$ is an $(m-p) \times m$ matrix in $H^\infty_+$. In conclusion, the function $\overline{Q} = \begin{bmatrix} \overline{Q}_1 \\ \overline{Q}_1 \end{bmatrix}$ is inner. Moreover, it is $\overline{Q}Y = \begin{bmatrix} E^* \\ 0 \end{bmatrix}$, by construction.

To see that $D = E$, observe that we have just shown that

$$R^* = \overline{Q}^* \begin{bmatrix} E^* \\ 0 \end{bmatrix} \Lambda D^{-1} Q^*$$

and in view of corollary 1, it is

$$R = X\Lambda Y^\# = QD\Lambda[(E^{-1})^*, 0]\overline{Q}$$

and so, taking the conjugate,

$$\overline{Q}^* \begin{bmatrix} E^{-1} \\ 0 \end{bmatrix} \Lambda D^* Q^* = \overline{Q}^* \begin{bmatrix} E^* \\ 0 \end{bmatrix} \Lambda D^{-1} Q^*$$

which implies $E^{-1}\Lambda D^* = E^*\Lambda D^{-1}$, i.e. $D = E$, since $D$ and $E$ are both outer. ∎

What we have said so far is based on the properties of some particular Schmidt pairs of $H_{R^*}$, the Young pairs. We now turn to a generic Schmidt pair: let $\lambda_k$ and $\xi_k, \eta_k$ denote, respectively, the $k-th$ singular value and Schmidt pair of $H_{R^*}$, for $1 \le k \le n$, and denote the Young pairs with the index $k_i$ for $1 \le i \le p$. First of all, it is obvious that there exists unique $\psi_k \in H^2_+$ and $\phi_k \in H^2_-$ such that

$$R^*\xi_k = \lambda_k\eta_k + \psi_k \tag{11}$$

$$R\eta_k = \lambda_k\xi_k + \phi_k \tag{12}$$

The above are called, following [5], *fundamental equations* for $H_{R^*}$. They clearly characterize the Schmidt pairs. Then the following lemma is immediate:

**Lemma 2** *The Hankel operator* $H_R$ *has singular values* $-\lambda_k$*, with* $k \neq k_i$*, and Schmidt vectors* $\{\phi_k, \psi_k\}$*, again with* $k \neq k_i$

**Proof**: multiplying (11) by $R$ and using (12), we obtain:

$$R\psi_k = -\lambda_k \phi_k + (RR^* - \lambda_k^2)\xi_k \qquad (13)$$

where, in view of corollary 1, $(RR^* - \lambda_k^2)\xi_k \in H_+^2$. Analgously,

$$R^*\phi_k = -\lambda_k \psi_k + (R^*R - \lambda_k^2)\eta_k \qquad (14)$$

and now the last term is in $H_-^2$. Therefore (13) and (14) are fundamental equations for $H_R$, as wanted.

We will define now our function $R^*$ using the following result from [6] :

**Theorem 2** *:let $(M,N)$ and $(\overline{M},\overline{N})$ be coprime factorizations of $G$, and let $(U,V)$ satisfy the Bezout equation*

$$\overline{M}V - \overline{N}U = I$$

*Then the strictly proper antistable part of $R_0^*$ of $M^*U + N^*V$ is independent of the choice of $U$ and $V$. Similarly, let $(\overline{U},\overline{V})$ be a solution to the Bezout equation $\overline{V}M - \overline{U}N = I$. Then the strictly proper antistable part $\overline{R}_0^*$ of $\overline{U}\overline{M}^* + \overline{V}\overline{N}^*$ is independent of the choice of $\overline{U}$ and $\overline{V}$. Moreover, $R_0^* = \overline{R}_0^*$, and there exist unique pairs $U_0, V_0, \overline{U}_0, \overline{V}_0$ such that*

$$R_0^* = M^*U_0 + N^*V_0 = \overline{U}_0\overline{M}_0^* + \overline{V}_0\overline{N}_0^* \qquad (15)$$

Still a direct consequence from [6] is the following:

**Proposition 1** *Let $R^* \in L^\infty$ be a function whose stable strictly proper part is $R_0^*$. Then there exist unique pairs $(U,V)$ and $(\overline{U},\overline{V})$ such that*

$$R^* = M^*U + N^*V = \overline{U}\overline{M}^* + \overline{V}\overline{N}^*$$

**Proof**: from (6) we get

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} + \begin{bmatrix} M \\ N \end{bmatrix} Q$$

Multipling on the left by $[M^*, N^*]$ we get

$$M^*U + N^*V = M^*U_0 + N^*V_0 + (M^*M + N^*N)Q = R_0^* + Q$$

Therefore choosing $Q = R^* - R_0^*$ will yield the desired $(U,V)$ ■

The following well known result (it is a direct consequence of the Youla parametrization), will be needed:

**Corollary 2** *The following equalities hold:*

$$\begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix} R = \begin{bmatrix} \overline{U}^* \\ \overline{V}^* \end{bmatrix} + \begin{bmatrix} N \\ -M \end{bmatrix} \tag{16}$$

*Similarly,*

$$\begin{bmatrix} N \\ -M \end{bmatrix} R^* = \begin{bmatrix} U \\ V \end{bmatrix} + \begin{bmatrix} \overline{N}^* \\ -\overline{M}^* \end{bmatrix} \tag{17}$$

**Proof**: multiplication of (16) by the matrix $\begin{bmatrix} \overline{M} & \overline{N} \\ -N^* & M^* \end{bmatrix}$ yields immediately the result. Similarly for (17)

# 3 Main results

We can now make a particular choice of $R^*$ and $(U, V)$ in view of Theorem 1. Given a function $G = NM^{-1} = \overline{M}^{-1}\overline{N}$, we can define $R^*_\Lambda$ to be the superoptimal Nehari extension of $M^*U + N^*V$, where $(U, V)$ is any solution of the Bezout equation. Clearly, in view of Theorems (1) and (2), $R^*_\Lambda$ is well defined and does not depend on the choice of the controller. Define next $U_\Lambda$, $V_\Lambda$, $\overline{U}_\Lambda$, $\overline{V}_\Lambda \in H^\infty_+$ as the solutions (existing and unique in view of Proposition (1)) to:

$$M^*U_\Lambda + N^*V_\Lambda = R^*_\Lambda \tag{18}$$

$$\overline{U}_\Lambda\overline{M}^* + \overline{V}_\Lambda\overline{N}^* = R^*_\Lambda \tag{19}$$

Then we have the following:

**Lemma 3** *Let $(M, N)$ and $(\overline{M}, \overline{N})$ be coprime factorizations of $G$, and let $U_\Lambda$, $V_\Lambda$, $\overline{U}_\Lambda$, $\overline{V}_\Lambda$ be defined from (18) and (19) for $R^*_\Lambda$ as in theorem (1). Then they constitute a doubly coprime factorization of $G$ which implies $R = \overline{R}$ and $UV^{-1} = \overline{V}^{-1}\overline{U}$*

**Proof:** we have

$$\begin{bmatrix} M & \overline{N} \\ N & -\overline{M} \end{bmatrix} \begin{bmatrix} M^* & N^* \\ \overline{N}^* & -\overline{M}^* \end{bmatrix} = \begin{bmatrix} M^* & N^* \\ \overline{N}^* & -\overline{M}^* \end{bmatrix} \begin{bmatrix} M & \overline{N} \\ N & -\overline{M} \end{bmatrix} = I$$

Then

$$\begin{aligned} \overline{V}_\Lambda U_\Lambda - \overline{U}_\Lambda V_\Lambda &= [\overline{V}_\Lambda, -\overline{U}_\Lambda] \begin{bmatrix} M & \overline{N} \\ N & -\overline{M} \end{bmatrix} \begin{bmatrix} M^* & N^* \\ \overline{N}^* & -\overline{M}^* \end{bmatrix} \begin{bmatrix} U_\Lambda \\ V_\Lambda \end{bmatrix} \\ &= [I, R^*_\Lambda] \begin{bmatrix} R^*_\Lambda \\ -I \end{bmatrix} = 0 \qquad \blacksquare \end{aligned}$$

The controller $K = UV^{-1} = \overline{V}^{-1}\overline{U}$ is called *superoptimal robust stabilizing controller* of $G$.

The next theorem extends to the multivariable case a result of [6].

**Theorem 3** *Let $(\xi_k, \eta_k)$ be the Schmidt pairs of $H_{R_\Lambda^*}$. Then the Hankel operator $H_{\left[\begin{smallmatrix}\overline{M}^*\\ \overline{N}^*\end{smallmatrix}\right]}$ has singular values $\sigma_k = \frac{\lambda_k}{\sqrt{1+\lambda_k^2}}$ and Schmidt pairs*

$$\left\{\xi_k, \frac{1}{\sqrt{1+\lambda_k^2}}\left(\begin{bmatrix}\overline{U}_\Lambda^{\ *}\\ \overline{V}_\Lambda^{\ *}\end{bmatrix}\eta_k - \begin{bmatrix}\overline{M}^*\\ \overline{N}^*\end{bmatrix}\phi_k\right)\right\}$$

*where the $\phi_k$ are as in (12)*

**Proof**: in view of (7) (19) and (12), we can write

$$\begin{bmatrix}\overline{M} & \overline{N}\\ -N^* & M^*\end{bmatrix}\begin{bmatrix}\overline{U}^*\\ \overline{V}^*\end{bmatrix}\eta_k = \begin{bmatrix}R_\Lambda\\ I\end{bmatrix}\eta_k = \begin{bmatrix}\lambda_k\xi_k + \phi_k\\ \eta_k\end{bmatrix}$$

and

$$\begin{bmatrix}\overline{M} & \overline{N}\\ -N^* & M^*\end{bmatrix}\begin{bmatrix}\overline{M}^* - V\\ \overline{N}^* + U\end{bmatrix}\xi_k = \begin{bmatrix}0\\ R_\Lambda^*\end{bmatrix}\xi_k = \begin{bmatrix}0\\ \lambda_k\eta_k + \psi_k\end{bmatrix}$$

Multiply on both sides by $\begin{bmatrix}\overline{M}^* & -N\\ \overline{N}^* & M\end{bmatrix}$ and subtract from the second equation the first multiplied by $\lambda_k$:

$$\begin{bmatrix}\overline{M}^* - V\\ \overline{N}^* + U\end{bmatrix}\xi_k - \lambda_k\begin{bmatrix}\overline{U}^*\\ \overline{V}^*\end{bmatrix}\eta_k = \begin{bmatrix}\overline{M}^* & -N\\ \overline{N}^* & M\end{bmatrix}\begin{bmatrix}-\lambda_k^2\xi_k - \lambda_k\phi_k\\ \psi_k\end{bmatrix}$$

or

$$\begin{bmatrix}\overline{M}^*\\ \overline{N}^*\end{bmatrix}\xi_k(1+\lambda_k^2) = \lambda_k\begin{bmatrix}\overline{U}^*\\ \overline{V}^*\end{bmatrix}\eta_k - \lambda_k\begin{bmatrix}\overline{M}^*\\ \overline{N}^*\end{bmatrix}\phi_k + \begin{bmatrix}V\\ -U\end{bmatrix}\xi_k + \begin{bmatrix}-N\\ M\end{bmatrix}\psi_k$$

where the first two terms in the second member are in $H^2_-$and the other two are in $H^2_+$. Hence we have shown that

$$H_{\left[\begin{smallmatrix}\overline{M}^*\\ \overline{N}^*\end{smallmatrix}\right]}\xi_k = \frac{\lambda_k}{1+\lambda_k^2}\left(\begin{bmatrix}\overline{U}^*\\ \overline{V}^*\end{bmatrix}\eta_k - \begin{bmatrix}\overline{M}^*\\ \overline{N}^*\end{bmatrix}\phi_k\right)$$

To conclude the claim, observe that

$$[\overline{M},\overline{N}]\left(\begin{bmatrix}\overline{U}^*\\ \overline{V}^*\end{bmatrix}\eta_k-\begin{bmatrix}\overline{M}^*\\ \overline{N}^*\end{bmatrix}\phi_k\right)=R\eta_k-\phi_k=\lambda_k\xi_k$$

and thus

$$H_{[\overline{M},\overline{N}]}\frac{1}{\sqrt{1+\lambda_k^2}}\left(\begin{bmatrix}\overline{U}^*\\ \overline{V}^*\end{bmatrix}\eta_k-\begin{bmatrix}\overline{M}^*\\ \overline{N}^*\end{bmatrix}\phi_k\right)=\frac{\lambda_k}{\sqrt{1+\lambda_k^2}}\xi_k$$

which is exactly what we wanted. ∎

The obvious dual of the previous result, with a similar proof, is the following:

**Theorem 4** *The Hankel operator with symbol $H_{[M^*,N^*]}$ has Schmidt pairs*

$$\left\{\frac{1}{\sqrt{1+\lambda_k^2}}\left(\begin{bmatrix}U\\ V\end{bmatrix}\xi_k-\begin{bmatrix}M\\ N\end{bmatrix}\psi_k\right),\eta_k\right\}$$

*with singular values* $\sigma_k=\dfrac{\lambda_k}{\sqrt{1+\lambda_k^2}}$

**Corollary 3** *Let $(\xi_i,\eta_i)$ be the Young pairs of $H_{R_\Lambda^*}$. Then the Hankel operator $H_{\begin{bmatrix}\overline{M}^*\\ \overline{N}^*\end{bmatrix}}$ has* Young pairs

$$\left\{\xi_i,\frac{1}{\sqrt{1+\lambda_i^2}}\begin{bmatrix}\overline{U_\Lambda}^*\\ \overline{V_\Lambda}^*\end{bmatrix}\eta_i\right\}$$

*with singular values* $\sigma_i=\dfrac{\lambda_i}{\sqrt{1+\lambda_i^2}}$

Again, a similar result holds for the operator $H_{[M^*,N^*]}$

**Corollary 4** *The following equalities hold:*

$$\begin{bmatrix}\overline{M}^*\\ \overline{N}^*\end{bmatrix}\xi_i=\frac{\lambda_i}{1+\lambda_i^2}\begin{bmatrix}\overline{U_\Lambda}^*\\ \overline{V_\Lambda}^*\end{bmatrix}\eta_i-\frac{1}{1+\lambda_i^2}\begin{bmatrix}-V_\Lambda\\ U_\Lambda\end{bmatrix}\xi_i \tag{20}$$

*and*

$$\begin{aligned}\begin{bmatrix}M\\ N\end{bmatrix}\eta_i &= \frac{1}{\lambda_i}\begin{bmatrix}U_\Lambda\\ V_\Lambda\end{bmatrix}\xi_i+\frac{1}{\lambda_i}\begin{bmatrix}\overline{N}^*\\ -\overline{M}^*\end{bmatrix}\xi_i \\ &= \frac{\lambda_i}{1+\lambda_i^2}\begin{bmatrix}U_\Lambda\\ V_\Lambda\end{bmatrix}\xi_i-\frac{1}{1+\lambda_i^2}\begin{bmatrix}-\overline{V_\Lambda}^*\\ \overline{U_\Lambda}^*\end{bmatrix}\eta_i\end{aligned} \tag{21}$$

Another consequence is the following:

**Corollary 5** *The superoptimal Nehari extension* $\begin{bmatrix} A \\ B \end{bmatrix}$ *of* $\begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix}$ *is given by:*

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -V_\Lambda \\ U_\Lambda \end{bmatrix} Q(I+\Lambda^2)^{-1}Q^* \tag{22}$$

*and*

$$\begin{bmatrix} \overline{M}^* + A \\ \overline{N}^* + B \end{bmatrix} = \begin{bmatrix} \overline{U}_\Lambda{}^* \\ \overline{V}_\Lambda{}^* \end{bmatrix} \overline{Q}^* \begin{bmatrix} D^* \\ 0 \end{bmatrix} \Lambda(I+\Lambda^2)^{-1}D^{-1}Q^* \tag{23}$$

**Proof**: the second equality follows from Theorem 1 and Lemma 1. The first is obtained subtracting $\begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix}$ from the second, and using (20).

**Lemma 4** *the fuctions* $U_\Lambda$,$V_\Lambda$,$\overline{U}_\Lambda$,$\overline{V}_\Lambda$ *satisfy the conditions:*

$$U_\Lambda{}^*U_\Lambda + V_\Lambda{}^*V_\Lambda = Q(I+\Lambda^2)Q^* \tag{24}$$
$$[\overline{U}_\Lambda\overline{U}_\Lambda{}^* + \overline{V}_\Lambda\overline{V}_\Lambda{}^*]\overline{Q}_1^* = \overline{Q}_1^*(I+\Lambda^2) \tag{25}$$

*where* $Q$ *and* $\overline{Q}$ *are as in (10)*

*Proof*: multiplying (21) by $[U_\Lambda{}^*, V_\Lambda{}^*]$ we obtain

$$R_\Lambda\eta_i = \frac{\lambda_i}{1+\lambda_i^2}(U_\Lambda{}^*U_\Lambda + V_\Lambda{}^*V_\Lambda)\xi_i$$

and since $R_\Lambda\eta_i = \lambda_i\xi_i$, we have

$$(U_\Lambda{}^*U_\Lambda + V_\Lambda{}^*V_\Lambda)\xi_i = (1+\lambda_i^2)\xi_i$$

In view of (10), we get the result. A similar argument holds for the other equation.

We turn now to the inverse problem, that is to characterize all plants stabilized by a given controller ia superoptimal manner.

Define $S \in H_+^\infty$ as the unique outer solution to

$$S^*S = U_\Lambda{}^*U_\Lambda + V_\Lambda{}^*V_\Lambda$$

in view of (24), it is

$$Q^*S^*SQ = (I+\Lambda^2) \tag{26}$$

which implies that $S$ is coercive and therefore a unit in $H_+^\infty$and the vector $\begin{bmatrix} U_N \\ V_N \end{bmatrix} = \begin{bmatrix} U_\Lambda \\ V_\Lambda \end{bmatrix} S^{-1}$ clearly constitutes a normalized coprime factorization of the controller. Simlarly, for $[\overline{U}_\Lambda, \overline{V}_\Lambda]$, we can define the outer spectral factor $\overline{S} \in H_+^\infty$ as the solution to

$$\overline{S}\overline{S}^* = \overline{U}_\Lambda \overline{U}_\Lambda^* + \overline{V}_\Lambda \overline{V}_\Lambda^*$$

and obtain a left normalized coprime factorization of the controller:

$$[\overline{U}_N, \overline{V}_N] = \overline{S}^{-1}[\overline{U}_\Lambda, \overline{V}_\Lambda]$$

Again, in view of (25)

$$\overline{Q}_1 \overline{S}\overline{S}^* \overline{Q}_1^* = (I + \Lambda^2) \tag{27}$$

So the equations relating plant and superoptimal controller now write, in view of (10), (20), (21), as

$$\begin{aligned} \begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix} &= \begin{bmatrix} \overline{U}_\Lambda^* \\ \overline{V}_\Lambda^* \end{bmatrix} Y\Lambda(I+\Lambda^2)^{-1}X^{-1} + \begin{bmatrix} V_\Lambda \\ -U_\Lambda \end{bmatrix} X(I+\Lambda^2)^{-1}X^{-1} \\ &= \begin{bmatrix} \overline{U}_\Lambda^* \\ \overline{V}_\Lambda^* \end{bmatrix} \overline{Q}_1^* D^* \Lambda(I+\Lambda^2)^{-1} D^{-1} Q^* + \begin{bmatrix} V_\Lambda \\ -U_\Lambda \end{bmatrix} Q(I+\Lambda^2)^{-1}Q^* \\ &= \begin{bmatrix} \overline{U}_N^* \\ \overline{V}_N^* \end{bmatrix} \overline{S}^* \overline{Q}_1^* D^* \Lambda(I+\Lambda^2)^{-1} D^{-1} Q^* + \begin{bmatrix} V_N \\ -U_N \end{bmatrix} (S^*)^{-1} \end{aligned} \tag{28}$$

(we recall that $Q_1 = [I_p, 0]Q$). Multiplying by $S^*$ and setting

$$T^* = \overline{S}^* \overline{Q}_1^* D^* \Lambda(I+\Lambda^2)^{-1} D^{-1} Q^* S^*$$

we get finally

$$\begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix} S^* = \begin{bmatrix} \overline{U}_N^* \\ \overline{V}_N^* \end{bmatrix} T^* + \begin{bmatrix} V_N \\ -U_N \end{bmatrix} \tag{29}$$

The dual equation (21) yields:

$$\begin{bmatrix} M \\ N \end{bmatrix} Y = \begin{bmatrix} U_\Lambda \\ V_\Lambda \end{bmatrix} X\Lambda(I+\Lambda^2)^{-1} - \begin{bmatrix} \overline{V}_\Lambda^* \\ \overline{U}_\Lambda^* \end{bmatrix} Y(I+\Lambda^2)^{-1}$$

Multiplying by $D^{-1}(I+\Lambda^2)$ and using (27),

$$\begin{aligned} \begin{bmatrix} M \\ N \end{bmatrix} Y &= \begin{bmatrix} U_N \\ V_N \end{bmatrix} SQ\Lambda\overline{Q}_1(\overline{S}^*)^{-1}\overline{S}^*\overline{Q}_1^* - \begin{bmatrix} \overline{V}_N^* \\ \overline{U}_N^* \end{bmatrix} \overline{S}^*\overline{Q}_1^* \\ &= \begin{bmatrix} U_N \\ V_N \end{bmatrix} T\overline{S}^*\overline{Q}_1^* - \begin{bmatrix} \overline{V}_N^* \\ \overline{U}_N^* \end{bmatrix} \overline{S}^*\overline{Q}_1^* \end{aligned}$$

**Lemma 5** *$T^*T = \overline{S}^*\overline{S} - I$ and $TT^* = SS^* - I$*

**Proof**: from (26) we have

$$
\begin{aligned}
T^*T &= \overline{S}^*\overline{Q}_1{}^*D^*\Lambda(I+\Lambda^2)^{-1}D^{-1}Q^*S^*SQ(D^{-1})^*(I+\Lambda^2)^{-1}\Lambda D\overline{Q}_1\overline{S} \\
&= \overline{S}^*\overline{Q}_{11}^{\,*}D^*\Lambda D^{-1}(D^{-1})^*(I+\Lambda^2)^{-1}\Lambda D\overline{Q}_1\overline{S} \\
&= \overline{S}^*\overline{Q}_{11}^{\,*}\Lambda^2(I+\Lambda^2)^{-1}\overline{Q}_1\overline{S} \\
&= \overline{S}^*\overline{S} - I
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
TT^* &= SQ(D^{-1})^*(I+\Lambda^2)^{-1}\Lambda D\overline{Q}_1\overline{S}\overline{S}^*\overline{Q}_1{}^*D^*\Lambda(I+\Lambda^2)^{-1}D^{-1}Q^*S^* \\
&= SQ(D^{-1})^*\Lambda DD^*\Lambda(I+\Lambda^2)^{-1}D^{-1}Q^*S^* \\
&= SQ\Lambda^2(I+\Lambda^2)^{-1}Q^*S^* \\
&= SS^* - I
\end{aligned}
$$

Since, from (26), $L = SQ(I+\Lambda^2)^{-1}$ is inner, we obtain

$$TT^* = L\Lambda^2L^* \tag{30}$$

So $T^*L$ has Young values $\Lambda$, and thus so does $T^*$ (apply $H_{T^*}$ to $LH^2_+$, see [9] for details). Therefore, given normalized coprime factorizations of the plant, the superoptimal controller and all the functions $T, S, \overline{S}$ are uniquely determined.

If we now consider the inverse problem, i.e. given normalized coprime factorizations of the controller, what can be said about all the other functions occuring in (29)? Clearly the key point is to find the function $T^*$, since everything else is then uniquely determined. Now we want also (30) to be satisfied for a given $\Lambda$. Two questions arise: for which $\Lambda$ does (29) possibly have a solution, and how to compute it. The first question finds a simple answer in the following

**Lemma 6** *(29) has a solution only if $\Lambda \geq \Lambda_K$, where $\Lambda_K$ are the Young values of $H_{U_N^*M+V_N^*N}$*

**Proof**: it is clear that there exists, in view of theorem (2) applied to $(U, V)$ instead of $(M, N)$, a unique superoptimal Nehari extension $R_K$ of $\mathbf{P}_-(U_N{}^*M + V_N{}^*N)$ with Young values $\Lambda_K$. Therefore, from the very definition of superoptimal Nehari extension, $\sigma_i(R(i\omega) \geq \sigma_i(R_K(i\omega)$, and therefore we reach the conclusion. ∎

The next result is about the reduction of the 2 block interpolation problem (29) to a one block problem.

**Lemma 7** *let $T^*$ satisfy*

$$\overline{N}S^* = \overline{V}_N{}^* T^* - U_N \tag{31}$$

*for some $\overline{M}, S \in H_+^\infty$ with $S$ outer. Then $T^*$ also satisfies*

$$\overline{M}S^* = \overline{U}_N{}^* T^* + V_N \tag{32}$$

*for some $\overline{N} \in H_+^\infty$, and therefore satisfies (29). Moreover, $\overline{M}, \overline{N}, S$ can be chosen so that $(\overline{M}, \overline{N})$ are normalized coprime.*

**Proof**: let $P$ be the minimal degree inner function (denote this degree by $n_K$) such that $U_N P^*, V_N P^* \in H_-^\infty$, and denote by $(z_i, v_i), i = 1, \ldots, n_K$ the zeros of $P^*$ (i.e. the pairs $(z_i, v_i)$ for which $P^*(z_i)v_i = 0$. Then (29) is equivalent to

$$\begin{array}{ll} \left[(\overline{U}_N{}^* T^* + V_N)P^*\right](z_i)v_i = 0 & \\ & i = 1, \ldots, n \\ \left[(\overline{V}_N{}^* T^* - U_N)P^*\right](z_i)v_i = 0 & \end{array}$$

Deriving $T^* P^*(z_i)v_i$ from the second equation, we get

$$T^* P^*(z_i)v_i = \left[(\overline{V}_N{}^*)^{-1} U_N P^*\right](z_i)v_i$$

substitution in the first yields

$$\left[\overline{U}_N{}^* (\overline{V}_N{}^*)^{-1} U_N P^*\right](z_i)v_i = -\left[V_N P^*\right](z_i)v_i$$

and thus

$$\left[(V_N{}^*)^{-1} U_N{}^* U_N P^*\right](z_i)v_i = -\left[V_N P^*\right](z_i)v_i$$

or

$$\left[(U_N{}^* U_N + V_N{}^* V_N)P^*\right](z_i)v_i = 0$$

But this is always verified, and thus so is (32). ∎

In conclusion, to solve (29), we just need to look at (31). The interpolation problem we have obtained can in principle be solved, but we have to do it throught a double parametrization: in fact, given $\Lambda$, for each fixed $L$, we have to parametrize all solutions $T$ to (31) satisfying (30).

This problem drops if we make the simple choice $\Lambda = \lambda I$. In this case (31) becomes a standard Pick-Nevanlinna problem (see e.g. [2] or [3]), and we reobtain the standard results about optimal controllers (see e.g. [5] or [8]). Define

$$\mathcal{T}_{K,\lambda} = \{T \in L^\infty; \|T\| \leq \lambda \text{ and } T \text{ satisfies (31)}\}$$

Then we obtain the characterization of all plants stabilized by a given controller with prescribed stability margin.

**Theorem 5** *let $T \in \mathcal{T}_{K,\lambda}$. Then the pairs $(\overline{M}, \overline{N})$ defined by (29), and the pair $(M, N)$ given by*

$$\left[\begin{array}{c} M \\ N \end{array}\right] \overline{S} = \left[\begin{array}{c} U_N \\ V_N \end{array}\right] T - \left[\begin{array}{c} \overline{V_N}^* \\ \overline{U_N}^* \end{array}\right] \tag{33}$$

*are normalized coprime factors of the same plant $G$ and $K$ stabilizes $G$ with stability margin not smaller than* $(1 + \lambda^2)^{-1}$

The proof is standard and will not be given here.

# References


[1] Adamjan, V.M., D.A. Arov, and M.G. Krein. (1971) Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takagi problem. *Math. USSR sbornik* **15**, pp. 31-73.

[2] J. Ball, I. Gohberg, and L. Rodman *Interpolation of rational matrix functions.* Birkhäuser Verlag, Basel, 1990.

[3] H.Dym, $J$-contractive matrix functions, reproducing kernel spaces and interpolation, CBMS lecture notes, Vol 71, 1989, *American mathematical society,* Rhodes Island.

[4] B.Francis, *A Course in $H^\infty$ Control Theory,* Springer-Verlag, New York, 1987.

[5] Fuhrmann, P.A. (1991) A Polynomial Approach to Hankel Norm Balanced Approximations. *Linear Algebra and its Applications* **146**, pp. 133-220.

[6] P.A.Fuhrmann, R.Ober, A functional approach to LQG balancing, to appear.

[7] Hoffman, K. (1962) *Banach Spaces of Analytic Functions.* Prentice-Hall, Englewood Cliffs.

[8] McFarlane D.C., Glover K., (1990) *Robust Controller Design using Normalized Coprime Factor Plant Description*, Springer-Verlag

[9] N. Young, On the Nevanlinna-Pick Problem for Matrix-valued functions, *J.Operator theory,* **15**, 1986, 239-265