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On the Geometry of External Spectral Factors and the Riccati Inequality

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Abstract
We study the geometric structure of the spectral factors of a given spectral density Φ. We show that these factors can be associated to a set of invariant subspaces and we exhibit the manifold structure of this set, providing also an explicit parametrization for it. We also make some connection with the set of solutions to the Riccati Inequality.

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1 Introduction

The characterization of all the spectral factors of a given $p \times p$ spectral density Φ of rank m_0 is a problem which has been widely studied in connection with the lattice of solutions to the Riccati inequality associated to Φ. In particular Anderson and Faurre gave a precise characterization of this set of solution (see e.g. [3]). More recently the problem has been studied by Lindquist and Picci [8] within the framework of stochastic realization. We are not aware, though, of a geometric characterization of the set of all the spectral factors of Φ of a given dimension $p \times m$. We exhibit here the manifold structure of this set, and we indicate how the corresponding set $\mathcal{P}$ of solutions to the Riccati inequality can be obtained as quotient modulo a group of unitary transformations. The basic idea is to characterize a spectral factor by means of its zeros. We show that in general there is a "natural"

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set of r stable zeros for a factor where r is invariant. Moreover, these zeros can be constructed as functions of the zeros of W_+ , the maximum-phase spectral factor. The notion of zero we use here is an adaptation to Hilbert space setting of the zero module (see [9]). An interesting feature of such a representation lies in the simplicity of the structure of the set of spectral factors (a product of spheres), which can give some insight into the more complex structure of the set $\mathcal{P}$.

2 Preliminaries and notation

We work with the Hardy spaces of the disk $\mathbb{D}$; we define [6] $L^2_m(\mathbb{T})$ to be the set of the square integrable $m-$*dimensional* vector valued functions on the unit circle $\mathbb{T}$, and H^2_m ($\overline{H}^2_m$) to be the subspace of $L^2_m(\mathbb{T})$ whose strictly positive (strictly negative) Fourier coefficients vanish. The functions in H^2_m ($\overline{H}^2_m$) are defined on $\mathbb{T}$, but they can be extended to analytic functions on on the open complement $\mathbf{E}$ of the closure of the disk $\overline{\mathbb{D}}$ (to the disk $\mathbb{D}$), by taking for instance the Cauchy integral:

$$f(z) := \frac{1}{2\pi}\int_{\mathbb{T}} \frac{f(e^{-i\omega})}{e^{i\omega}-z^{-1}} de^{i\omega} \qquad\qquad f \in H^2_p$$

$$\left(g(z) := \frac{1}{2\pi}\int_{\mathbb{T}} \frac{g(e^{i\omega})}{e^{i\omega}-z} de^{i\omega} \qquad\qquad g \in \overline{H}^2_p \right)$$

The transposed of the function $(f^*(z)) := \overline{f(\bar{z}^{-1})}^T$, where ^{T} denotes transposition, represents the extension in $\overline{H}^2_p$ of $\overline{f(e^{-i\omega})}$. The inner product on H^2_p thus becomes:

$$\langle f, g\rangle = \overline{\tfrac{1}{2\pi i}\textstyle\int_{\mathbb{T}} f^*(z)g(z)\tfrac{dz}{z}}$$

A $p \times m$ matrix function F with columns in H^2_p is said to be *rigid* if $F(e^{i\omega})F^*(e^{i\omega}) = I_p$ *a.e.* (I_p is the identity) This clearly entails $p \leq m$. It is *inner* (or stable all-pass) if $p = m$.

Given a rigid function Q, we can construct the space $H^2_m \ominus QH^2_m$, denoted in the following by $H(Q)$.

The orthogonal projection onto a subspace X or onto the span of a vector b will be denoted by $\mathbf{P}^X$ and $\mathbf{P}^b$ respectively; in particular, projections onto H^2_m and $\overline{H}^2_m$ will be denoted by $\mathbf{P}^-$and $\mathbf{P}^+$, respectively. If B is an inner function in H^2_m , we define $H(B) := H^2_m \ominus BH^2_m$. The shift on H^2_m is defined as $Uf := M_z f$. Clearly U is a unitary operator, i.e. $U^* = U^{-1}$, and $U^*H^2_m \subset H^2_m$. A space X in H^2_m is *invariant* (for U^*, or backward invariant), if $U^*X \subset X$. A subspace Y is said to be invariant for the forward shift, or *forward invariant*, if $\mathbf{P}^- UY \subset Y$. Similarly, we say that $X \subset Z$ is invariant in Z if $\mathbf{P}^X U^* Z \subset X$. Forward invariance in Z is defined analogously. A subspace $X \subset H^2_m$ is called *semiinvariant* if there exists a subspace Y, such that $Z := X \vee Y$ is forward invariant, and X is invariant in Z. It can be shown (see [1]) that the subspace Y is not unique, but there is only one such space which is orthogonal to X, and it will be forward invariant. In fact it is also easy to see this fact directly: define $Z := span\{\mathbf{P}^- U^n X; n \geq 0\}$. If X is, according to the definition, invariant for U^* in Z, its orthogonal complement in Z will be invariant for the adjoint $\mathbf{P}^- U^*$ of U, i.e. it is forward invariant.

We define now minimum-phase functions. An element $f \in H^2_m$ is *generating* an invariant subspace X, if X is the minimal invariant subspace containing f. An invariant subspace in H^2_m has multiplicity p if it can be generated by no less than p vectors. It is maximal if there is no subspace $Y \supset X$ of the same multiplicity p. A matrix function is *minimum-phase* in H^2_m if its columns generate a maximal subspace.

Let Φ be a rational spectral density, i.e. a symmetric $m \times m$ rational function of rank m_0 which is nonegative definite on the circle $\mathbb{T}$. It is well known that for any $m > m_0$ there are infinitely many factorizations W of dimension $p \times m$, i.e. $\Phi = WW^*$, but there exists an essentially unique factorization $\Phi = W_- W_-^*$ with W_- minimum-phase in H^2_p. Moreover this induces a decomposition of H^2_p as the space generated by its columns $H = \overline{span}\{w^i_- z^n; w^i_- \ columns\ of\ W_-\ and\ n \geq 0\}$ and its orthogonal in H^2_p, $H^\perp$.

Similarly there is a unique factorization, $\Phi = \overline{W}_+ \overline{W}_+$ where $\overline{W}_+ \in \overline{H}^2_p$, with $\overline{W}_+$ conjugate minimum-phase. Both this factorizations have dimension $p \times m_0$. Moreover, it can be shown (see [4]) that there exists a unique inner function K_+ of dimension $m_0 \times m_0$ of minimal degree such that the columns of $W_+ := \overline{W}_+ K_+$ are in H^2_p; the matrix function W_+ is called the *maximum-phase spectral factor*

We recall that a *scalar* zero of a rational transfer function W is a complex number z such that the matrix

$$\begin{bmatrix} \zeta I - A & B \\ C & D \end{bmatrix}$$

associated to any minimal realization of W has a rank drop in z.

We say that $\mathbf{z}_i = b_i \frac{z}{z - z_i}$ an unstable zero of W_+ if $b_i \in \mathbb{C}^{m_0}$ is a scalar zero of W_+ and the rows of W_+ are orthogonal to $\mathbf{z}$.

This is not a new definition of zero; it is simply an adaptation to the Hilbert space setting of the standard definitions exisiting in the literature (see e.g. [9]). In particular, the set of unstable zeros generates the equivalent of the *zero module* for W_+.

We also say that $\mathbf{z}_i$ is an unstable zero of algebraic multiplicity k of W_+, if $b_i \frac{z}{z - z_i} \left(\frac{1 - z\bar{z}_i}{z - z_i} \right)^{(k-1)}$ is orthogonal to the rows of W_+. In the sequel we assume, for sake of simplicity, that all zero of W_+ have multiplicity one. Results for arbitratry algebraic and geometric multiplicity are not intrinsically more difficult, but involve a non uniqueness problem which will not be discussed here. Finally we make a normalization assumption, i.e. $\left\|b_i \frac{z}{z-z_i}\right\| = 1$, i.e. $b_i b_i^* = 1 - z_i z_i^*$

If W is any other spectral factor, it is well known (see [8]) that there exists a rigid function Q_W of dimension $m_0 \times m$ such that $WQ_W = W_+$.

This function Q is used to define the zeros of W as functions of those of W_+ . Let G and Q_1 be matrix functions with entries in H^2, with the same number of columns (of rows) and with Q_1 rigid. Then Q_1 *divides* G on the right if $GQ_1^* \in H^2$ (on the left if $Q^*G \in H^2$).

Let now $\mathbf{d} = d\frac{z}{z-a}$, and define the inner matrix

$$B_\mathbf{d} = \left[\mathbf{P}^d \frac{1-za}{z-a} + I - \mathbf{P}^d\right]$$

The matrix $B_\mathbf{d}$ is often called elementary Blascke factor. We say that:

$\mathbf{d}$ is a stable internal zero of W if $B_\mathbf{d}$ has dimension $m_0 \times m_0$ and it divides Q_W on the left.

$\mathbf{d}$ is a stable external zero of W if $B_\mathbf{d}$ divides Q_W on the left, but its dimension is $m \times m$ with $m > m_0$.

$Q\mathbf{d}$ is a unstable external zero of W if $B_\mathbf{d}$ divides W_+ on the right, but its dimension is $m \times m$ with $m > m_0$.

$Q\mathbf{d}$ is an unstable internal zero if there exists an inner function Q_0 dividing W_+ such that d is an internal zero of $W_+ Q_0^*$.

Again, the above definitions are adapted from standard theory (see [9]): in particular what we call external zeros are a finite subset of the extended zero of [9].

3 Main Results

By *state space* for a given spectral factor W we mean a semiivariant subspace X in H^2_m such that $W \in X$. It can be shown that in general (in the external case) there exists a unique minimal state space (in the sense that it has dimension as small as possible: it can be shown [8] that this dimension is always n). By *minimal* spectral factor we therefore mean a factor of degree n. Denote by X_+ the *state space* of the maximum-phase spectral factor, i.e.

$$X_+ = span\{\mathbf{P}^- U^n W_+\}$$

Let $\mathbf{z}_i \in H^2_m$ $i = 1, \ldots, k$ be such that

- each $\mathbf{z}_i$ is a forward invariant vector
- the $\mathbf{z}_i$ are linearly independent

- $\mathbf{z}_i \perp W_+$

We set $Z = span\{\mathbf{z}_i; i = 1, \ldots, k\}$ and $X_Z := (X_+ \vee Z) \ominus Z$.

Lemma 3.1 *X_Z is a seminvariant minimal subspace containing W_+*

Proof: $X_+ \vee Z$ is forward invariant (since both X_+ and Z are. The elements $z_i \in Z$ are forward invariant and thus Z is invariant in $X_+ \vee Z$. Thus its orthogonal complement X_Z is seminvariant in $X_+ \vee Z$. Since $Z \perp W_+$, $W_+ \in X_Z$, and in view of the dimension, X_Z is minimal. ■

Let now $\{\mathbf{z}_i; i = 1, \ldots, r\}$ be the zeros of W_+ . We then set $Z_0 := span\{\mathbf{z}_i; i = 1, \ldots, r\}$.

Suppose $Z = span\langle \mathbf{x}_1, \ldots, \mathbf{x}_k \rangle$ is such that

$$\mathbf{P}^H Z = \hat{Z} \subset Z_0$$

and $\hat{Z}$ is forward invariant in Z_0. Then $X_Z = (X^+ \vee Z) \ominus Z$ is seminvariant. In fact, $\hat{Z} \subset Z_0$ implies $Z \perp W^+$. Conversely, if $\mathbf{x}$ is an external unstable zero of W, then $\mathbf{P}^H \mathbf{x} \in Z_0$ and is also forward invariant. As a consequence, if Z is forward invariant, so is also $\mathbf{P}^H Z$. We need to clarify the link between W and the space Z. We will assume, from now on, that Z is a *minimal* set of zeros, i.e. $dim Z = dim \mathbf{P}^H Z$

Lemma 3.2 *There a one to one correspondence between minimal spectral factors W and minimal spaces Z of stable zeros such that $\mathbf{P}^H Z \subset Z_0$.*

Proof: given $W = W_+ Q^*$, we can set by definition $Z_W = H(Q_W)$. Conversely, given Z such that $\mathbf{P}^H Z \subset Z_0$, we know that there exists a rigid function Q such that $Z = H(Q)$. Moreover, the elements of H(Q) are orthogonal to the rows of W_+ , and therefore Q divides W_+ on the right. The desired factor is then given by $W = W_+ Q^*$. ■

The space Z in the above lemma is called *error space* (see [7]) and will be denoted by $Z_W^{\triangleright}$

Lemma 3.3 *There exists a unique semiinvariant subspace $Z_W^{\triangleleft}$, orthogonal to $Z_W^{\triangleright}$ such that $Z_W^{\diamond} = Z^{\triangleleft} \oplus Z_W^{\triangleright}$ is a forward invariant subspace and $\mathbf{P}^{Z_W^{\diamond}} = Z_0$.*

Proof: This fact is simple to see: define $Z_W^{\diamond}$ to be the smallest invariant subspace containing both Z_0 and $Z_W^{\triangleright}$; it can also be written as

$$Z_W^{\diamond} = \overline{span}\{U^n(Z_0 \vee Z_W^{\triangleright})\}; n \geq 0\}$$

Then $Z_W^{\triangleleft} = Z_W^{\diamond} \ominus Z_W^{\triangleright}$ is the desired subspace. The only thing which is not entirely trivial is that $\mathbf{P}^{Z_W^{\diamond}} = Z_0$. But both Z_0 and $Z_W^{\triangleright}$ are orthogonal to $H \ominus Z_0$; but this is a subspace invariant for $M_{z^{-1}}$, and therefore orthogonal also to $U^n(Z_0 \vee Z_W^{\triangleright})$. ■

Corollary 3.1 *Suppose* $\mathbf{z} \in Z_W^{\triangleright}$ *is an external zero and* $\hat{\mathbf{z}} = \mathbf{P}^H \mathbf{z}$. *Then there exists a* $\mathbf{z}' \in Z_W^{\triangleleft}$ *such that* $\mathbf{P}^H \mathbf{z}' = \hat{\mathbf{z}}$

Proof: suppose $Z_W^{\triangleleft} \perp \hat{\mathbf{z}}$: then $\hat{\mathbf{z}} \in Z_W^{\triangleright} \cap Z_0$, and thus $\mathbf{z} - \hat{\mathbf{z}} \in Z_W^{\triangleright}$. But this implies that $dim Z_W^{\triangleright} > dim \mathbf{P}^H Z_W^{\triangleright}$, which contradicts minimality. ■

In other words, external zeros have both a stable and an unstable component in H.

Lemma 3.4 *Let* W *be a minimal spectral factor. Then* W *induces a triivariant decomposition of* Z_0 *as*

$$Z_0 = Z_s \oplus \mathbf{P}^{Z_0} Z_e \oplus Z_u$$

where Z_s, Z_e *and* Z_u *are, respectively, the stable, external and unstable zeros of* W

Proof: let Z_W as above. Then $Z_s = Z_0 \cap (Z_W^{\triangleright})^{\perp}$ is invariant for the backward shift in Z_0, $Z_u = Z_W^{\triangleright} \cap Z_0$ is forward invariant and $Z_e = Z \ominus Z_s$ is the middle term. By definition, Z_u contains all the unstable internal zeros of W, and remembering that $(Z_W^{\triangleright})^{\perp} = Q_W H_m^2$, we see that Z_s contains the stable ones.

The above decomposition is related with the tightest local frame $X_{tlf}(W)$ of [8]. In fact, it can be shown that $X_{tlf}(W) = Z_e \vee X_{Z_u}$. Observe that $Z_e = \mathbf{P}^H Z_W^{\triangleleft} \cap \mathbf{P}^H Z_W^{\triangleright}$

The next lemma shows how the inner factors relative to $\mathbf{z}$ and $\mathbf{z}'$ in Lemma (3.1) are related.

Lemma 3.5 *Let* $\hat{\mathbf{b}} = \hat{b}\frac{1}{z-a}$, *with* $\hat{b} \in \mathbb{C}^m$. *Then, for any* $\tilde{b} \in \mathbb{C}^{p-m}$,

$$B_{\hat{\mathbf{b}}} = [I_m, 0] B_{\mathbf{b}} B_{\mathbf{b}^{\perp}} \begin{bmatrix} I_m \\ 0 \end{bmatrix}$$

where $\mathbf{b} = \begin{bmatrix} \alpha \hat{b} \\ (1-\alpha^2)^{1/2} \tilde{b} \end{bmatrix} \frac{1}{z-a}$ *and* $\mathbf{b}^{\perp} = \begin{bmatrix} (1-\alpha^2)^{1/2} \hat{b} \\ -\alpha \tilde{b} \end{bmatrix} \frac{1}{z-a}$

Proof: computing the product

$$[I_m, 0] B_{\mathbf{b}} B_{\mathbf{b}^{\perp}} \begin{bmatrix} I_m \\ 0 \end{bmatrix} =$$

$$[I_m, 0][\mathbf{P}^{\mathbf{b}} \frac{1-za}{z-a} + I_p - \mathbf{P}^{\mathbf{b}}][\mathbf{P}^{\mathbf{b}^{\perp}} \frac{1-za}{z-a} + I_p - \mathbf{P}^{\mathbf{b}^{\perp}}] \begin{bmatrix} I_m \\ 0 \end{bmatrix}$$

$$[I_m, 0][\mathbf{P}^{b} \frac{1-za}{z-a} + \mathbf{P}^{b^{\perp}} \frac{1-za}{z-a} + I_p - \mathbf{P}^{b^{\perp}} - \mathbf{P}^{b}] \begin{bmatrix} I_m \\ 0 \end{bmatrix}$$

Now, since b and $b^{\perp}$ are orthogonal, $\mathbf{P}^{b} + \mathbf{P}^{b^{\perp}} = \mathbf{P}^{b_1} + \mathbf{P}^{b_2}$, where $b_1 = \begin{bmatrix} \hat{b} \\ 0 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 0 \\ \tilde{b} \end{bmatrix}$. Therefore

$$
\begin{aligned}
&[I_m, 0][\mathbf{P}^{b}\frac{1-za}{z-a} + \mathbf{P}^{b^{\perp}}\frac{1-za}{z-a} + I_p - \mathbf{P}^{b^{\perp}} - \mathbf{P}^{b}]\begin{bmatrix} I_m \\ 0 \end{bmatrix} \\
&= [I_m, 0][(\mathbf{P}^{b_1} + \mathbf{P}^{b_2})\frac{1-za}{z-a} + +I_p - \mathbf{P}^{b_1} - \mathbf{P}^{b_2}]\begin{bmatrix} I_m \\ 0 \end{bmatrix} \\
&= [\mathbf{P}^{\hat{b}}\frac{1-za}{z-a} + I_m - \mathbf{P}^{\hat{b}}]
\end{aligned}
$$

as wanted. ■

How do we characterize all the W? From the above lemmas we need to characterize all the $\hat{Z}$ invariant in Z_0. But that classification, in the generic case, is very simple. There exist, in the case of distinct zeros $\hat{\mathbf{z}}_1, ..., \hat{\mathbf{z}}_r$ of the maximum-phase spectral factor, $\begin{pmatrix} r \\ l \end{pmatrix}$ different subspaces of dimension l. For each subset of indeces $\mathcal{I} = \{i_1, i_2, ..., i_l\}$, define a space by picking a basis $\hat{\mathbf{z}}_{i_1}, \hat{\mathbf{z}}_{i_2}, ..., \hat{\mathbf{z}}_{i_l}$ in H^2_{mo} which will be of the form

$$
\hat{\mathbf{z}}_i = \hat{b}_i \frac{z}{z - z_i}, \qquad \hat{b}_i \in \mathbb{C}^{\,mo},\ i \in \mathcal{I}
$$

and extend, for $i \in \mathcal{I}$, the vector $\hat{b}_i$ as

$$
b_i = \begin{bmatrix} \hat{b}_i \\ \tilde{b}_i \end{bmatrix}
$$

let $\mathbf{z}_i = b_i \frac{1}{z - z_i}$ and

$$
B_i(z) = \left[b_i^T b_i \cdot \frac{1}{\|b_i\|^2} \frac{1 - z\bar{z}_i}{z - z_i} + I - \mathbf{P}^{b_i} \right]
$$

As usual, $H^2_p\, B_i \perp b_i \frac{z}{z - z_i}$

Set then $Z = span\{\mathbf{z}_{i_1}, \mathbf{z}_{i_2}, \ldots, \mathbf{z}_{i_l}\}$. To compute the correseponding Q, we will need to work with orthonormalized versions of the zeros. Therefore we make the following definitions:

$\sigma \in S_r$ is the a permutation on the set of r integers

$b^1_\sigma = b_{\sigma_1}$

$b^k_\sigma = b_{\sigma_k} \prod_{j=1}^{k-1} (B^j_\sigma)^*(z_{\sigma_k})$

$B^k_\sigma = \left[\mathbf{P}^{b^k_\sigma} \frac{1 - z\bar{z}_{\sigma_k}}{z - z_{\sigma_k}} + I - \mathbf{P}^{b^k_\sigma} \right]$

ζ_{σ_k} is the orthonormalization of $\mathbf{z}_{\sigma_k}$

Lemma 3.6 *The basis*

$$\zeta_{\sigma_k} = b_\sigma^k \frac{zN_\sigma^k}{z - z_{\sigma_k}} \cdot \prod_{j=1}^{k-1} B_\sigma^{k-j} \tag{1}$$

where $N_\sigma^k = \left(\frac{1-z_{\sigma_k}\bar{z}_{\sigma_k}}{b_\sigma^k (b_\sigma^k)^*}\right)^{1/2}$*, represents the orthonormalization of the vectors* $\mathbf{z}_{\sigma_1}, \ldots, \mathbf{z}_{\sigma_n}$

Proof we show that $\zeta_{\sigma_k} - \mathbf{z}_{\sigma_k} \in H(\prod_{j=1}^{k-1} B_\sigma^{k-j})$. In fact

$$\begin{aligned}
&\int_{\mathbb{T}} (\zeta_{\sigma_k} - \mathbf{z}_{\sigma_k}) \prod_{j=1}^{k} (B_\sigma^j)^* \frac{dz}{z} \\
= &\int_{\mathbb{T}} \frac{z}{z - z_{\sigma_k} k} b_{\sigma_k} \left[N_\sigma^k \prod_{j=1}^{k-1} (B_\sigma^j)^*(z_{\sigma_j}) \prod_{j=1}^{k-1} B_\sigma^{k-j}(z) - I\right] \prod_{j=1}^{k-1} (B_\sigma^j(z))^* \frac{dz}{z} \\
= &\; b_{\sigma_k} \left[N_\sigma^k \prod_{j=1}^{k-1} (B_\sigma^j)^*(z_{\sigma_j}) - \prod_{j=1}^{k-1} (B_\sigma^j)^*(z_{\sigma_k})\right] = 0
\end{aligned}$$

Returning now to the set of indeces $\mathcal{I}$, and choosing σ so that $\sigma_j = i_j$, it then easily seen that $Q = \prod_{j=0}^{l-1} B_\sigma^{l-j}$.

It is quite clear that the above construction defines parametrization of the external spectral factors. It is also possible to obtain the internal factors which are not maximum-phase, but other charts are needed and we refer to [5] for the complete atlas. We state now some other results on the topological structure of the set of spectral factors.

Let $\hat{\mathbf{b}} = \hat{b}\frac{1}{z-a}$ be a zero of W_+ of dimension m. Denote by $\mathcal{W}_{\hat{\mathbf{b}}}^m$ the set of minimal spectral factors W which have only unstable zeros of which at most one, $\mathbf{b}$, is external and such that $\mathbf{P}^H\mathbf{b} \subset span\{\hat{\mathbf{b}}\}$.

Lemma 3.7 *The set* $\mathcal{W}_{\hat{\mathbf{b}}}^m$ *is a* $2(m - m_0)$*-sphere.*

For the proof see [5]

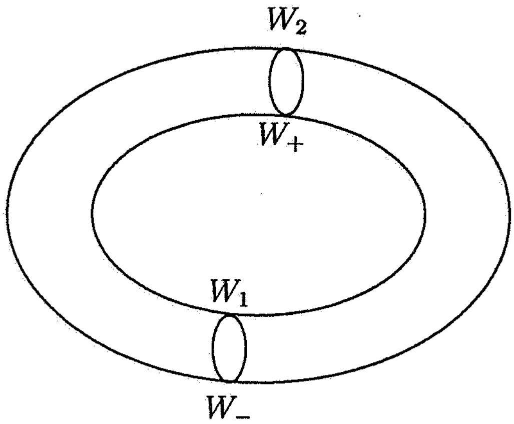
We recall that we are working over the complex domain, and therefore we have the factor 2 in the dimension. The above result can be extended quite naturally to any set of zeros.

Theorem 3.1 *Let* Z_0 *have only zeros with geometric multiplicity* 1*. Then, for each* $\hat{Z} \subset Z_0$ *of dimension* k *invariant in* Z_0*, the set* $\mathcal{W}_{\hat{Z}}^m := \{W : \; dim\; W = p \times m,\; \mathbf{P}^H Z_W \subset \hat{Z}\}$ *is a smooth manifold of dim*$2(m - m_0) \cdot k + (m - m_0)^2$.

For the proof we refer again to [5]. In fact we can show that this manifold is the product of n $(p - m_0)$-dimensional spheres.

The result has an intuitive explanation: usually a spectral factor W is considered modulo a unitary transformation in H_m^2. By fixing W_+ and thus Z_0 we have actually chosen a fixed representative of this class. But the external component of the noise, and therefore the external part of the generic W will still be parametrized by $U(n-m_0)$. This explains the term $(m-m_0)^2$ in the dimension of $\mathcal{W}_{\hat{Z}}$.

The above results hold, with minor modifications, also for the real case. For example, if$W_+ = \frac{(1-zz_1)(1-zz_2)}{(z-p_1)(z-p_2)}$, then $\mathcal{W}_{\hat{Z}_0}$, will look like the torus in the figure,



where

$$W_- = \frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)}$$
$$W_1 = \frac{(z-z_1)(1-zz_2)}{(z-p_1)(z-p_2)}$$
$$W_2 = \frac{(1-zz_1)(z-z_2)}{(z-p_1)(z-p_2)}$$

The inner matrix relating W_+ and W_- is $Q_+ = \frac{(1-zz_1)(1-zz_2)}{(z-p_1)(z-p_2)}$. The two degree one elementary divisors of $[W_+, 0]$ are

$$B_i = \left(\frac{1-zz_i}{z-z_i}\mathbf{P}^{b_i} + I - \mathbf{P}^{b_i}\right) \qquad i = 1, 2$$

where $b_i = \begin{bmatrix} \hat{b}_i \\ \tilde{b}_i \end{bmatrix}$ Then we can easily generate two familes of spectral factors $W_{b_i} = [W_+, 0]B_i^*$, corresponding, respectively, to the upper small circle and to the inner circle of the torus. For example

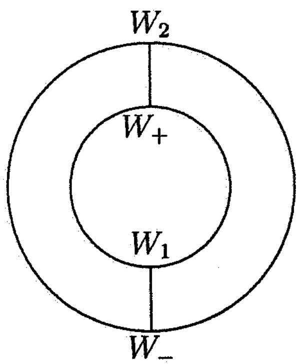
$$\begin{aligned} W_{b_1} &= [W_+, 0]B_1^* = \frac{(1-zz_1)(1-zz_2)}{(z-p_1)(z-p_2)}[I, 0]\left(\frac{z-z_1}{1-zz_1}\mathbf{P}^{b_1} + I - \mathbf{P}^{b_1}\right) \\ &= \frac{(z-z_1)(1-zz_2)}{(z-p_1)(z-p_2)}[I, 0]\mathbf{P}^{b_1} + \frac{(1-zz_1)(1-zz_2)}{(z-p_1)(z-p_2)}[I, 0]\left(I - \mathbf{P}^{b_1}\right) \end{aligned}$$

This result can also give some information on the structure of the set $\mathcal{P}$ of solutions to the Riccati inequality. This set is well known and we refer to [3] for details. What intersts us here is an important result of Anderson and Faurre (see [3]) stating that to each element W of $\mathcal{W}_Z^m$ we can associate a unique matrix $P(W) \in \mathcal{P}$, and that this map is, in some sense, invertible. To be precise, it determines a fibration on an open dense subset of $\mathcal{W}_Z^m$, with group $U(m-m_0)$ (of course, we have to attach a particular realization to each spectral factor: this is done by a *uniform choice of basis*, see [2] for details). Using this fact and the above result, we have the following:

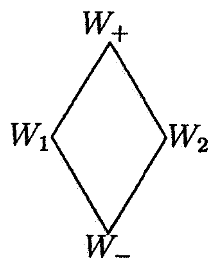
Corollary 3.2 *The interior of $\mathcal{P}$ is homeomorphic to an open dense set $\mathcal{W}_{Z_0}^{n+m_0}/U(n)$ and is therefore a $2n \cdot k$-dimensional manifold, where $2n$ is the degree of Φ. For $m - m_0 < n$, the interior (in the induced topology) of the subset of $\mathcal{P}$ corresponding to the set of $p \times m$ spectral factors is a submanifold of dimension $2(m-m_0) \cdot k$*

In fact, $\mathcal{P}$ is a manifold with boundary; since $\mathcal{W}_{Z_0}^{n+m_0}$ has no boundary, it means that the rank of $P(W)$ drops at some points: in fact, the preimage of the boundary of $\mathcal{P}$ is made of the submanifolds $\mathcal{W}_{Z_0}^m$, and the group generating the fibers on these submanifold collapse to a subgroup of $U(m)$. This gives account of the different topological structures of $\mathcal{W}_{Z_0}^m$ and $P(W)$.

For instance, going back to the above example, we see that, in the real case, the quotient has to be taken by $O(1)$, but we have to remove from the manifold the points W_-, W_+, W_1, W_2, where there is no external part, and where the quotient is the identity map. Thus to obtain the quotient manifold induced by $P(W)$ we have to identify points symmetric with respect to the vertical axis on the two generating circles. By identifying the points on the smaller circle we get



Identifying eventually points on the concentric circles we get



which is the usual lattice of solutions to the Riccati inequality.

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