1992

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Department of Operations Research, Statistics, and System Theory Report BS-R9228 October

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On the Differential Structure of Matrix-valued Rational Inner Functions

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Abstract
We consider in this paper the set $\mathcal{I}_n^p$ of $p \times p$ complex matrix- valued rational inner functions of fixed McMillan degree n. We show that this set has a smooth manifold structure, and we exhibit two atlases for it; one is based on realization theory and exhibits $\mathcal{I}_n^p$ as the product of the manifold of equivalence classes of observable pairs and the unitary group$U(p)$; the second studies $\mathcal{I}_n^p$ as transfer functions and rests on Schur parameters. Similar results are obtained for the real-valued functions.

1991 Mathematics Subject Classification: 93B29, 93B30.

Keywords and Phrases: Schur functions, Schur algorithm, Nehari extension, inner functions.

Note: This paper is submitted for publication elsewhere. This research was performed in part while the third author visited the Cente for Mathematics and Computer Science, Amsterdam, The Netherlands, with the support of a NATO fellowship.

Report BS-R9228
ISSN 0924-0659
CWI
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

1 Introduction and preliminaries

1.1 Introduction

The present work is devoted to a differential-geometric study of rather classical objects in analysis namely inner matrix–valued functions. More precisely, we prove that the collection of such matrices with prescribed size and Mc–Millan degree is an embedded submanifold in the Hardy space, two parametrizations of which are derived. We do this with an eye on linear control systems not only because several methods from system–theory are relevant to such a study but also because the geometry of inner matrix–valued functions in itself impinges on the parametrization of systems *via* the Douglas–Shapiro–Shields factorization of transfer-functions introduced in [29].

First recall what Schur and inner functions of the disk are. A complex–valued function is called a Schur function if it is analytic in the open unit disk $\mathbb{D}$ and bounded by one in modulus there. Now, it is a theorem of Fatou that such a function has non-tangential boundary values almost everywhere on the unit circle $\mathbb{T}$; a Schur function is then called inner if these boundary values are of modulus one almost everywhere on $\mathbb{T}$. The multiplicative structure of Schur and inner functions is well–known (see e.g. [30] [25], [40]); in particular, rational inner functions, also called finite Blaschke products, are of the form:

$$b(z) = c\prod_{i=1}^{n} \frac{z - w_i}{1 - z\bar{w}_i} \tag{1.1}$$

where the w_i are in $\mathbb{D}$ and c is a complex number of modulus one. The number n is the degree of b, and we will denote by I_n^1 the set of Blaschke products of degree n. Considering I_n^1 as a subset of some Hardy space H_q where $1 \le q \le \infty$ (the actual value of q turns out to be irrelevant), we want to analyse the smoothness of this object. Expression (1.1), however explicit it may be, is not adequate for this purpose because the functions $b \to w_i$ are not differentiable at branching points. Of course, it is obvious how to remedy this: we simply set

$$p(z) = \prod_{i=1}^{n}(z - w_i) \tag{1.2}$$

and define the reciprocal polynomial of p by

$$\tilde{p}(z) = \prod_{i=1}^{n}(1 - z\bar{w}_i) = z^n \bar{p}(\frac{1}{z}) \tag{1.3}$$

so that

$$b(z) = c\frac{p(z)}{\tilde{p}(z)}. \tag{1.4}$$

Since p and $\tilde{p}$ are coprime, (1.4) defines a one-to-one correspondance between I_n^1 and pairs (c, p), so we can choose c (ranging over $\mathbb{T}$) and the coefficients of p except for the leading one (ranging over some open subset of $\mathbb{C}^n$) as coordinates.

There is, however, a more subtle way of describing I_n^1 which was introduced by Schur in his celebrated paper [49]: starting from a Schur function f, one defines recursively a sequence (f_k) of Schur functions by setting $f_0 = f$ and

$$f_{k+1}(z) = \frac{f_k(z) - f_k(0)}{z(1 - \overline{f_k(0)}\, f_k(z))}. \tag{1.5}$$

The process stops if, at some stage, $f_k(0)$ has modulus one. The sequence of numbers $\rho_k \overset{\text{def}}{=} f_k(0)$ are called the Schur coefficients of f, and they completely characterize the function. Moreover, f belongs to I_n^1 if and only if $|\rho_k| < 1$ for $k < n$ and $|\rho_n| = 1$. This leads to another proof of the smoothness of I_n^1 [12] by showing that it is diffeomorphic to the product of n copies of the open unit disk and of a copy of the unit circle. It should also be clear that everything in what precedes can be specialized to the subset RI_n^1 of I_n^1 consisting of real Blaschke products of degree n, namely those satisfying $b(\bar{z}) = \overline{b(z)}$. In this case, parameters c and p as above are real and so are the Schur coefficients. For more information on the Schur algorithm and some of its applications to signal processing, we refer to [32].

In this paper, our interest lies in matrix-valued functions rather than just scalar ones. A $\mathbb{C}^{p\times p}$-valued rational function Q is called inner if it is analytic in $\mathbb{D}$ and takes unitary values on the unit circle $\mathbb{T}$. The set of $\mathbb{C}^{p\times p}$-valued rational inner functions of degree n will be denoted by I_n^p, where the degree is now meant to be the McMillan degree. Our objective is to study the differential structure of I_n^p and also of RI_n^p which is the subset of I_n^p consisting of real functions (i.e. satisfying $Q(\bar{z}) = \overline{Q(z)}$). As we shall see, it is again true that I_n^p and RI_n^p are smooth, but it is more demanding to obtain effective parametrizations. A multiplicative decomposition into elementary factors still exists as was shown by Potapov [47], but again runs short of smoothness at branching points. Also, a direct analysis based on some explicit description of the matrix analogous to (1.4) is still possible when $p = 2$ [21] but runs into difficulties for $p > 2$. In contrast, the seemingly more involved approach using Schur parameters does carry over to the matrix case.

To state this more precisely, let us introduce the set $\mathcal{U}_p$ (resp. $\mathcal{O}_p$) of unitary (resp. orthogonal) $p \times p$ matrices and recall that $\mathcal{U}_p$ (resp. $\mathcal{O}_p$) is a manifold of dimension p^2 (resp. $p(p-1)/2$) (see e.g. [34]). Now, it will follow from the matrix version of the Schur algorithm that I_n^p, considered as a subset of some H_q with $1 \leq q \leq \infty$, is locally diffeomorphic to the product of n copies of the open unit ball in $\mathbb{C}^p$ and a copy of $\mathcal{U}_p$. Similarly, RI_n^p is locally diffeomorphic to the product of n copies of the open unit ball in $\mathbb{R}^p$ and a copy of $\mathcal{O}_p$. Hence the spaces I_n^p and RI_n^p are smooth (and even real analytic) manifolds of dimension $2np + p^2$ and $np + p(p-1)/2$ respectively. We shall say further that two members Q_1 and Q_2 of I_n^p are equivalent if there exists a unitary matrix U such that $Q_1 = Q_2 U$. A similar equivalence is defined in RI_n^p, U now being orthogonal. We will denote by $I_n^p/\mathcal{U}_p$ and $RI_n^p/\mathcal{O}_p$ respectively the associated quotient spaces, and we shall see that these spaces of "normalized" inner functions are also smooth manifolds.

Yet another way to proceed is to consider inverses of inner matrices rather than I_n^p itself. Indeed, the inverse of $Q \in I_n^p$ is a proper transfer function of McMillan degree n and we may resort to classical tools from system-theory like realizations and coprime factorizations. This time, however, charts will be obtained in terms of realizations showing in particular that $I_n^p/\mathcal{U}_p$ (resp. $RI_n^p/\mathcal{O}_p$) is diffeomorphic to the manifold of observable pairs (C, A) where $C \in \mathbb{C}^{p\times n}$ (resp. $\mathbb{R}^{p\times n}$) and $A \in \mathbb{C}^{n\times n}$ (resp. $\mathbb{R}^{n\times n}$). This, in some sense, can be expected from the Beurling-Lax theorem because members of $I_n^p/\mathcal{U}_p$ are in one-to-one correspondance with shift invariant subspaces of H_2^p of codimension n and the orthogonal complement of such a subspace, being n-dimensional and invariant under the left shift, is therefore the span of the columns of some $C(I_n - zA)^{-1}$. This system theoretic approach will be taken in section 2 and the Schur algorithm applied in section 3. We discuss in the final section a specific link to rational approximation in Hardy spaces and identification which is stressed in [12], [21] and [13].

1.2 Some preliminaries and notations

In this subsection, we fix notations and review a number of facts on matrix-valued functions which will be needed throughout the paper. Recall that the open unit disk and the unit circle are denoted by $\mathbb{D}$ and $\mathbb{T}$ respectively. The symbol $\mathbb{C}^{n\times m}$ stands for the space of $n \times m$ matrices with complex entries. When m is equal to 1, we will write $\mathbb{C}^n$ for short. We let $GL(n)$ denote the group of square $n \times n$ complex matrices with non zero determinant. We set $\mathcal{H}_p$ (resp. $\mathcal{S}_p$) to be the set of hermitian (resp. skew-hermitian) matrices of size p. The identity matrix of $\mathbb{C}^{n\times n}$ will be denoted by I_n, or I if n is understood from the context. The symbol A^* will designate the transposed conjugate of the matrix A as well as the adjoint of an operator between Hilbert spaces. In particular, when $z \in \mathbb{C}$, z^* is the conjugate of z. The transpose of the matrix A will be denoted by A^t. If H is a Hilbert space and M is a closed subspace of H, the orthogonal complement of M in H will be denoted by $H \ominus M$. The abbreviation $l.s.\{v_i\}$ is used to mean the linear span of the vectors v_i.

A complex scalar or matrix-valued function F, defined over a subset of $\mathbb{C}$ which is stable under conjugation, is said to be real if

$$F(z^*) = F(z)^*.$$

We let $L_q(\mathbb{T})$ stand for the usual Lebesgue space of the circle and $\|f\|_q$ means the norm of f in $L_q(\mathbb{T})$. The Hardy space with exponent q of the open unit disk will be denoted by H_q; recall that H_q is the space of functions f holomorphic in $\mathbb{D}$ and such that

$$\sup_{r<1} \|f(re^{i\theta})\|_{L_q(\mathbb{T})} < \infty. \tag{1.6}$$

Such a f has a nontangential limit at almost every point of $\mathbb{T}$ and defines in this way a function belonging to $L_q(\mathbb{T})$ whose Fourier coefficients of negative index do vanish. Conversely, any such function in $L_q(\mathbb{T})$ is the nontangential limit of some uniquely defined $f \in H_q$. It is then customary to consider H_q as a closed subspace of $L_q(\mathbb{T})$ by identifying f and its boundary function. The norm of the boundary function in $L_q(\mathbb{T})$ is, by definition, the norm of f in H_q and turns out to be equal to the *sup* in (1.6) (see e.g. [30], [25] or [40]). In particular, it follows from Parseval's theorem that

$$H_2 = \{f(z) = \sum_0^\infty a_k z^k \ , \ a_k \in \mathbb{C} \text{ and } \|f\|_2^2 = \sum_0^\infty |a_k|^2 < \infty\}.$$

Also, H_∞ is the space of bounded holomorphic functions on $\mathbb{D}$ endowed with the *sup* norm. We designate by $\mathcal{A}$ the disk algebra which is the closed subalgebra of H_∞ comprising functions that are continuous on $\bar{\mathbb{D}}$. The symbol RH_q will denote the real Hardy space of functions in H_q which are real or, equivalently, whose Fourier coefficients in $L_q(\mathbb{T})$ are real. The symbol $\bar{H}_q$ stands for the conjugate Hardy space of functions f analytic outside $\bar{\mathbb{D}}$ (including at infinity) and such that $f(1/z) \in H_q$. We define $R\bar{H}_q$ accordingly. Just like before, $\bar{H}_q$ identifies to a subspace of $L_q(\mathbb{T})$ but the Fourier coefficients vanish this time on positive indices.
When $1 \leq q < \infty$, the space $L_q^{p\times m}(\mathbb{T})$ of $p \times m$ matrices with entries in $L_q(\mathbb{T})$ will be endowed with the following norm: if M is such a matrix with entries $m_{i,j}$, we set

$$\|M\|_q = \left(\sum_{i,j} \|m_{i,j}\|_q^q\right)^{\frac{1}{q}}.$$

When $q = \infty$, we define

$$\|M\|_\infty = ess.\ \sup_{\mathbb{T}} \|M(e^{i\theta})\|,$$

where $\|A\|$ denotes the operator norm $\mathbb{C}^m \to \mathbb{C}^p$ of the complex $p\times m$ matrix A. Of course, we may work with many other equivalent norms in $L_q^{p\times m}(\mathbb{T})$ and the above choice is mainly for definiteness. When $q = \infty$, for instance, we may also take the *sup* of the $\|m_{i,j}\|_\infty$'s. The present definition, however, has the advantage of making $L_\infty^{p\times p}(\mathbb{T})$ into a Banach algebra and is the one usually adopted in control theory. The subspaces $H_q^{p\times m}$ and $\bar{H}_q^{p\times m}$ of $L_q^{p\times m}(\mathbb{T})$ are equipped with the induced norm and so are the real subspaces $RH_q^{p\times m}$ and $R\bar{H}_q^{p\times m}$.

For the convenience of the reader, we recall some basic facts from matrix-valued rational functions. If such a function W is analytic at infinity, it can be written as:

$$W(z) = D + C(zI_n - A)^{-1}B \tag{1.7}$$

where (A, B, C, D) are matrices of adequate sizes, D being merely the value at infinity. The expression (1.7) is called a realization of W. If the size n of A is minimal, the realization is said to be minimal. Two minimal realizations of a given function W are always similar, namely if

$$W(z) = D + C_i(zI_n - A_i)^{-1}B_i, \text{ for } i = 1, 2$$

are two minimal realizations, then there exists a unique invertible matrix S such that:

$$\begin{pmatrix} A_2 & B_2 \\ C_2 & D \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ C_1 & D \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & I \end{pmatrix}.$$

Recall a pair (C, A) of matrices in $\mathbb{C}^{p\times n} \times \mathbb{C}^{n\times n}$ is said to be observable if

$$\bigcap_{k=0}^{k=\infty} \mathrm{Ker}\{CA^k\} = \{0\}$$

and that a pair (A, B) is said to be reachable is (B^t, A^t) is observable, It is actually well-known that a realization (A, B, C, D) is minimal if and only if (C, A) is observable and (A, B) is reachable [42]. In this case, the poles of the rational function $W(z)$ are the eigenvalues of A.
Essentially equivalent to the above is the fact that any complex vector space $\mathcal{M}$ of germs of $\mathbb{C}^p$-valued analytic functions at 0 which is both finite-dimensional and invariant under R_0, the left shift at 0 defined by

$$R_0 f(z) = \frac{f(z) - f(0)}{z},$$

is in fact made of rational functions and can be described as the span of the columns of some matrix

(1.8) $$C(I_n - zA)^{-1}$$

where (C, A) is observable and n is the dimension of $\mathcal{M}$. Suppose now that W is real or equivalently that the rational matrix $W(z)$ has real coefficients. Since realization theory is valid over any field, the matrices A, B, C and D in (1.7) may be chosen so as to be real. Accordingly, we say that a space $\mathcal{M}$ of analytic germs as above is real if

$$\sum a_k z^k \in \mathcal{M} \Longrightarrow \sum \bar{a}_k z^k \in \mathcal{M}.$$

In this case, the matrices A and C in (1.8) may also be chosen so as to be real.

The minimal feasible n in (1.7) is called the McMillan degree of W. It is invariant under Moebius tansfomations of the argument and, since any W can be construed to be analytic at infinity by performing such a transformation, this provides a definition of the McMillan degree for general rational matrix functions. Also, if W is square and $\det W$ does not vanish identically, then

W and W^{-1} have same McMillan degree. We refer to the monographs [42], [16], [7], [29] and [33] for more details.

In the sequel, $\Sigma_{p,m}(n)$ denotes the set of $p \times m$ rational matrices of McMillan degree n that are analytic at infinity. We single out the subset $\Sigma_{p,m}^{-}(n)$ of $\Sigma_{p,m}(n)$ made of matrices having no poles outside $\mathbb{D}$. It is obvious that for any $q \geq 1$

$$\Sigma_{p,m}^{-}(n) = \Sigma_{p,m}(n) \cap \bar{H}_q^{p \times m}.$$

The subset $R\Sigma_{p,m}(n)$ of real elements in $\Sigma_{p,m}(n)$ is just the collection of transfer-functions of causal discrete-time linear dynamical systems with m inputs, p outputs and n-dimensional minimal state. The corresponding subset $R\Sigma_{p,m}^{-}(n)$ of $\Sigma_{p,m}^{-}(n)$ consists of stable transfer-functions.

Any $W \in \Sigma_{p,m}^{-}(n)$ can be factorized as

$$W = Q^{-1} R \tag{1.9}$$

where Q belongs to I_n^p while R is a rational matrix which is analytic in the closed disk. Moreover, such a pair (Q, R) is unique up to left multiplication by some unitary matrix. If, furthermore, W happens to be real, the pair (Q, R) can also be chosen so as to be real.
Expression (1.9) will be referred to as the left Douglas–Shapiro–Shields factorization of W and holds more generally for strictly noncyclic functions [24] [29]. An elementary account of the rational version above, which is all we shall need, and of the real case can be found in [13]. The fact that Q is, up to a unitary factor, uniquely determined by the property of having the same McMillan degree as W can be viewed as a consequence of Fuhrmann's realization theory. Working with the transpose allows one to define similarly a right Douglas–Shapiro–Shields factorization $W = R_1 Q_1^{-1}$, where this time $Q_1 \in I_n^m$.

For z and w two complex numbers, we set

$$\rho_w(z) = 1 - zw^* \tag{1.10}$$

$$b_w(z) = (z - w)/(1 - zw^*). \tag{1.11}$$

Given a matrix valued function $A(z)$, the function $A^\sharp(z)$ is defined to be $(A(\frac{1}{z^*}))^*$.

Throughout, the terms smooth and C^∞ are used interchangeably. If M is a manifold, modelled on some Banach space, the tangent space to M at x will be denoted by $\mathcal{T}_x(M)$ or simply by $\mathcal{T}_x$ if M is understood from the context. If $f : M_1 \to M_2$ is a smooth map between two manifolds, the symbol $\mathbf{D}f(x)$ is intended to mean the derivative of f at the point $x \in M_1$ which is a linear map $\mathcal{T}_x(M_1) \to \mathcal{T}_{f(x)}(M_2)$. The effect of $\mathbf{D}f(x)$ on the vector v will be denoted by $\mathbf{D}f(x).v$. If $f(x_1, x_2, ..., x_k)$ is a function of k arguments, $\mathbf{D}_j f(x)$ designates partial derivative with respect to x_j. For these and other basic notions in differential geometry (such as charts, submanifolds, embeddings and the like), we refer to [44], [1] and [34].

The symbol ■ will mark the end of a proof.

2 The differential structure of I_n^p

Let us define

$$(I_n^p)^\sharp = \{Q^\sharp;\ Q \in I_n^p\},$$

and observe that $Q^\sharp$ is then also Q^{-1} by definition of I_n^p. It is clear that $(I_n^p)^\sharp$ is the subset of $\Sigma_{p,p}^-(n)$ consisting of those matrices M satisfying $MM^\sharp = I$. In this section, we study I_n^p by applying to $(I_n^p)^\sharp$ some standard devices from system–theory. We first proceed with some preliminaries on the geometry of $\Sigma_{p,m}^-(n)$.

2.1 Embedding $\Sigma_{p,m}^-(n)$ in $\bar{H}_q^{p\times m}$

We denote by Ω the open subset of $\mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m} \times \mathbb{C}^{p\times n} \times \mathbb{C}^{p\times m}$ consisting of all minimal 4-tuples (A, B, C, D). Let

$$\Pi : \Omega \to \Sigma_{p,m}(n)$$

be given by

(2.1) $$(A, B, C, D) \longrightarrow D + C(zI_n - A)^{-1}B.$$

We obtain a topology on $\Sigma_{p,m}(n)$ by requiring Π to be a quotient map for the similarity relation. It is by now a standard procedure, originally due to Hazewinkel and Kalman, to make $\Sigma_{p,m}(n)$ into an abstract smooth manifold by constructing local sections of the map Π. More precisely, $\Sigma_{p,m}(n)$ can be covered by a finite collection of open sets $(\mathcal{V}_k)$ such that each $\mathcal{V}_k$ is the domain of a map $\phi_k : \mathcal{V}_k \to \Omega$ with the property that

$$\Pi \circ \phi_k = id$$

where id denotes the identity map. For $W \in \mathcal{V}_k$, a certain subcollection of the entries of $\phi_k(W)$ will serve as coordinates on $\Sigma_{p,m}(n)$ which is endowed in that way with a complex structure of dimension $mp + n(m+p)$, see for instance [41], [37], [22], [36]. With this definition Π becomes a smooth map. This complex structure in itself plays here no role since it does not carry over to $(I_n^p)^\sharp$. For this reason, we shall deal only with the underlying real structure of dimension $2mp+2n(m+p)$. However, many subsequent computations look more natural over $\mathbb{C}$ and we shall often find it convenient to use complex dimensions rather than real ones. In the sequel, dimension means real dimension unless the contrary is explicitely stated. Restricting ourselves to real-valued coordinates in the above process, we get a parametrization of $R\Sigma_{p,m}(n)$ which is thus a submanifold of $\Sigma_{p,m}(n)$ of dimension $mp+n(m+p)$.

Now, let Ω^- denote the open subset of Ω consisting of 4-tuples (A, B, C, D) such that the eigenvalues of A all belong to $\mathbb{D}$. It is easy to check that Π is an open map (see e.g. [10]), so that $\Sigma_{p,m}^-(n) = \Pi(\Omega^-)$ is an open subset of $\Sigma_{p,m}(n)$ hence a submanifold of the same dimension, charts being obtained by restriction. Let

$$j_q : \Sigma_{p,m}^-(n) \longrightarrow \bar{H}_q^{p\times m}$$

be the natural inclusion. It is proved in [11], using results from [22], that j_2 restricted to $R\Sigma_{p,m}^-(n)$ is an embedding, that is by definition a smooth immersion which is a homeomorphism onto its image. This we strengthen as follows.

Theorem 1 *For $1 \le q \le \infty$, the map j_q is an embedding. By restriction, j_q also induces an embedding $R\Sigma_{p,m}^-(n) \to R\bar{H}_q^{p\times m}$.*

Proof: we show smoothness first. Since the natural inclusion $\bar{H}_\infty \to \bar{H}_q$ is continuous, we may restrict ourselves to $q = \infty$. Because j_q can be locally expressed as $j_q \circ \Pi \circ \phi_k$ and ϕ_k is smooth by definition, we need only show that $j_\infty \circ \Pi : \Omega^- \to \bar{H}_\infty$, which is given by (2.1) is smooth. Being linear and continuous, the natural inclusions

$$\mathbb{C}^{p\times m} \to \bar{H}_\infty^{p\times m} \quad \mathbb{C}^{p\times n} \to \bar{H}_\infty^{p\times n} \quad \text{and} \quad \mathbb{C}^{n\times m} \to \bar{H}_\infty^{n\times m}$$

are smooth and so is the affine map

$$A \to (zI - A)$$

from $\mathbb{C}^{n\times n}$ to $L_\infty^{n\times n}(\mathbb{T})$. Taking the inverse being a smooth operation on the set of invertible elements in any Banach algebra, we have that

$$A \to (zI - A)^{-1}$$

is also smooth and it assumes values in $\bar{H}_\infty^{n\times n}$. Finally, since multiplication is a continuous 3-linear map

$$\bar{H}_\infty^{p\times n} \times \bar{H}_\infty^{n\times n} \times \bar{H}_\infty^{n\times m} \longrightarrow \bar{H}_\infty^{p\times m},$$

we get smoothness of (2.1) as desired.

For j_q to be an immersion, we still have to show that the derivative $\mathbf{D}j_q(W)$ is injective at every $W \in \Sigma_{p,m}^-(n)$ and that its image splits. This last condition arises because we are embedding $\Sigma_{p,m}^-(n)$ in an infinite–dimensional space [44] but is automatically satisfied here since $\operatorname{Im}\mathbf{D}j_q(W)$ is finite dimensional hence splits in the Banach space $\bar{H}_\infty^{p\times m}$ (see e.g. [48], [51]). Letting (A,B,C,D) be a minimal realization of W, the image of $\mathbf{D}j_q(W)$ obviously contains the image of the derivative of $j_q \circ \Pi$ at (A,B,C,D). If we differentiate this map, which is formally given by (2.1), with respect to the arguments, we find that

(2.2) $\mathbf{D}_4(j_q \circ \Pi) \ : \mathbb{C}^{p\times m} \to \bar{H}_\infty^{p\times m}$ is given by $U \to U$
(2.3) $\mathbf{D}_3(j_q \circ \Pi) \ : \mathbb{C}^{p\times n} \to \bar{H}_\infty^{p\times m}$ is given by $V \to V(zI-A)^{-1}B$
(2.4) $\mathbf{D}_2(j_q \circ \Pi) \ : \mathbb{C}^{n\times p} \to \bar{H}_\infty^{p\times m}$ is given by $W \to C(zI-A)^{-1}W$

and, using the standard rule to compute the derivative of the inverse in a Banach algebra, we see that

$$\mathbf{D}_1(j_q \circ \Pi) : \mathbb{C}^{n\times n} \to \bar{H}_\infty^{p\times m} \tag{2.5}$$

is given by

$$X \to -C(zI-A)^{-1}X(zI-A)^{-1}B.$$

Define $\mathcal{M}_{A,B,C}$ to be the subspace of $\bar{H}_\infty^{p\times m}$ spanned by the images of the derivatives (2.3), (2.4) and (2.5) so that

$$\operatorname{Im}\mathbf{D}[j_q \circ \Pi](A,B,C,D) = \operatorname{Im}\mathbf{D}_4(j_q \circ \Pi) + \mathcal{M}_{A,B,C}.$$

Observe the sum is direct since the first term is the space of constant matrix functions while each member of $\mathcal{M}_{A,B,C}$ vanishes at infinity. Therefore, counting complex dimensions leads to

$$\begin{aligned}\dim_{\mathbb{C}}(\operatorname{Im}\mathbf{D}[j_q\circ\Pi](A,B,C,D)) &= mp+\dim_{\mathbb{C}}\mathcal{M}_{A,B,C}\\ &\leq \dim_{\mathbb{C}}(\operatorname{Im}\mathbf{D}j_q(W))\\ &\leq \dim_{\mathbb{C}}\Sigma^-_{p,m}(n)=mp+n(m+p).\end{aligned}$$

To establish injectivity, we will prove that

$$\dim_{\mathbb{C}}\mathcal{M}_{A,B,C}\geq n(m+p) \tag{2.6}$$

by using nice selections, a tool originally due to Brunovsky and Kalman which is also instrumental in defining the manifold structure of $\Sigma_{p,m}(n)$. Specifically, we choose k column vectors $b_{i_1}, b_{i_2}, ..., b_{i_k}$ of B such that

$$b_{i_1}, Ab_{i_1}, ..., A^{\kappa_1}b_{i_1}, b_{i_2}, Ab_{i_2}, ..., A^{\kappa_2}b_{i_2}, ..., b_{i_k}, Ab_{i_k}, ..., A^{\kappa_k}b_{i_k}$$

is a basis of $\mathbb{C}^n$ and it is easy to see that controllability implies the existence of such vectors. This entails of course that $\sum_j \kappa_j = n-k$. Define $\mathcal{W}$ (resp. $\mathcal{X}$) to be the subspace of $\mathbb{C}^{n\times m}$ (resp. $\mathbb{C}^{n\times n}$) consisting of complex matrices whose kernel contains the elements e_{i_j}'s of the canonical basis of $\mathbb{C}^m$ (resp. contains the $A^\kappa b_{i_j}$'s whenever $\kappa<\kappa_j$). The complex dimensions of $\mathcal{W}$ and $\mathcal{X}$ are thus $n(m-k)$ and nk respectively.
Observe now that $\mathcal{M}_{A,B,C}$ is the collection of rational matrix functions of the form

$$V(zI-A)^{-1}B-C(zI-A)^{-1}X(zI-A)^{-1}B+C(zI-A)^{-1}W \tag{2.7}$$

where V, W and X range over $\mathbb{C}^{p\times n}$, $\mathbb{C}^{n\times m}$ and $\mathbb{C}^{n\times n}$ respectively. If we restrict W to $\mathcal{W}$ and X to $\mathcal{X}$, we claim that (2.7) cannot be zero unless V, W and X are all zero. This will show that $\mathcal{M}_{A,B,C}$ contains at least

$$\dim_{\mathbb{C}}\mathbb{C}^{p\times n}+\dim_{\mathbb{C}}\mathcal{W}+\dim_{\mathbb{C}}\mathcal{X}=n(m+p)$$

independent vectors over $\mathbb{C}$ so that (2.6) will hold.
To establish the claim, we assume that (2.7) is identically zero and we evaluate at e_{i_j} for some $j\in\{1,...,k\}$. We get from the definition of $\mathcal{W}$

$$V(zI-A)^{-1}b_{i_j}=C(zI-A)^{-1}X(zI-A)^{-1}b_{i_j}. \tag{2.8}$$

The Taylor expansion of $X(zI-A)^{-1}b_{i_j}$ at infinity is

$$\sum_{l=1}^{\infty} XA^{l-1}b_{i_j}\, z^{-l}$$

and has a zero of order at least κ_j+1 by definition of $\mathcal{X}$.
Because $C(zI-A)^{-1}$ vanishes at infinity, the right hand-side of (2.8) has a zero at infinity of order κ_j+2 at least. But then, this must hold for the left hand-side as well. Computing the Taylor expansion yields

$$VA^l b_{i_j} = 0 \text{ for } 0 \leq l \leq \kappa_j.$$

Since $j \in \{1, ..., k\}$ was arbitrary, we get from the definition of the b_{i_j}'s that $V = 0$ and (2.8) implies

$$C(zI-A)^{-1}X(zI-A)^{-1}b_{i_j} = 0 \quad \text{for} \quad 1 \leq j \leq k. \tag{2.9}$$

Now, the identity

$$X(zI-A)^{-1}A^l b_{i_j} = -XA^{l-1}b_{i_j} + zX(zI-A)^{-1}A^{l-1}b_{i_j}$$

and the definition of $\mathcal{X}$ shows that

$$C(zI-A)^{-1}X(zI-A)^{-1}A^l b_{i_j} = zC(zI-A)^{-1}X(zI-A)^{-1}A^{l-1}b_{i_j},$$

provided $1 \leq l \leq \kappa_{i_j}$. ¿From (2.9), we therefore get by induction

$$C(zI-A)^{-1}X(zI-A)^{-1}A^l b_{i_j} = 0 \quad \text{for } 1 \leq j \leq k \text{ and } 0 \leq l \leq \kappa_{i_j}.$$

The fact that the $A^l b_{i_j}$'s form a basis of $\mathbb{C}^n$ implies now

$$C(zI-A)^{-1}X(zI-A)^{-1} = 0 \quad \text{hence} \quad C(zI-A)^{-1}X = 0.$$

Computing the Taylor expansion at infinity and using observability yields $X = 0$. Since we assumed that (2.7) is zero, we finally conclude in the same manner that $W = 0$ thereby achieving the proof of the claim.

To see that j_q is an embedding, there remains for us to show that it is a homeomorphism onto its image, in other words that it has a continuous inverse. Let W_k be a sequence of rational matrix functions of McMillan degree n converging in $\bar{H}_q^{p\times m}$ to some W which is also of degree n. The Cauchy formula implies that the convergence is uniform over any compact subset of

$\{|z| > 1\}$, so that $W_k(\infty) \to W(\infty)$ and, say, the first $2n-1$ derivatives at infinity of W_k also converge to those of W. From this collection of derivatives, one can construct the associated $np \times nm$ block-Hankel matrices Ha_k and Ha of W_k and W and from them, knowing that degree of all the rational matrices involved is n, one can derive realizations $(A_k, B_k, C_k, W_k(\infty))$ of W_k and $(A, B, C, W(\infty))$ of W using for instance Ho's algorithm [42] which yields rational formulae for the coefficients of the realization in terms of the entries of Ha_k and Ha. The only arbitrary choice in the algorithm is that of a nonsingular submatrix of the Hankel matrix but, if such a choice is made on Ha, the corresponding submatrix of Ha_k will be nonsingular too when k is large enough since $\mathrm{Ha}_k \to \mathrm{Ha}$. In this manner, we may arrange things so that $(A_k, B_k, C_k, W_k(\infty))$ converges to $(A, B, C, W(\infty))$ in Ω^-. Since Π is continuous, we get that $W_k = \Pi(A_k, B_k, C_k, W_k(\infty))$ converges to $W = \Pi(A, B, C, W(\infty))$ in $\Sigma^-_{p,m}(n)$ as desired.
Since $R\Sigma^-_{p,m}(n)$ is a submanifold of $\Sigma^-_{p,m}(n)$, the assertion on the restriction of j_q to real matrix functions is obvious. ∎

Theorem 1 makes it possible to identify the tangent space $\mathcal{T}_W$ of $\Sigma^-_{p,m}(n)$ at the point W with

$$\mathrm{Im}\,\mathbf{D}j_q(W) = \mathrm{Im}\,\mathbf{D}\Pi(A, B, C, D))$$

which is a subspace of $\bar{H}^{p\times m}_q$ of complex dimension $mp + n(m+p)$. This subspace we shall now describe in more details. To this effect, we need some pieces of notation.

If $Q \in I^p_n$ and $Q_1 \in I^m_n$, we denote by $\mathcal{L}_Q$ (resp. $\mathcal{R}_{Q_1}$) the space of rational $p \times m$ matrix functions M such that M has no pole in the closed unit disk and $Q^{-1}M$ (resp. MQ_1^{-1}) has no pole in $\{|z| \geq 1\}$ and vanishes at infinity (resp. has no pole in $\{|z| \geq 1\} \cup \infty$). Note that $\mathcal{L}_Q$ may alternatively be defined as the space of matrices whose columns belong to $H^p_2 \ominus QH^p_2$ which is the prototype of the left shift–invariant subspace of complex dimension n that will appear in section 3. Consequently, $\mathcal{L}_Q$ has complex dimension nm. Similarly, we observe that M belongs to $\mathcal{R}_{Q_1}$ if and only if M^t belongs to $\mathcal{L}_{zQ_1^t}$. Since the McMillan degree of zQ_1^t is $m+n$, the complex dimension of $\mathcal{R}_{Q_1}$ is $p(m+n)$.

We now define $\mathcal{W}(Q, Q_1)$ to be the space of rational matrix functions M analytic in $\{|z| \leq 1\}$ such that $Q^{-1}MQ_1^{-1}$ has no poles in $\{|z| \geq 1\}$ including at infinity. It is plain that

$$\mathcal{L}_Q \subset \mathcal{W}(Q, Q_1) \text{ and } Q\mathcal{R}_{Q_1} \subset \mathcal{W}(Q, Q_1).$$

Therefore,

$$\mathcal{L}_Q + Q\mathcal{R}_{Q_1} \subset \mathcal{W}(Q, Q_1) \tag{2.10}$$

and the sum is direct since any member M in the intersection must satisfy

$$Q^{-1}M \in z^{-1}\bar{H}_2^{p\times m} \cap H_2^{p\times m} = 0.$$

Conversely, let M be an element of $\mathcal{W}(Q, Q_1)$. Then $Q^{-1}M$ belongs to $L_2^{p\times m}(\mathbb{T})$ and can be written as

$$Q^{-1}M = h_+ + h_- \quad \text{where } h_+ \in H_2^{p\times m} \text{ and } h_- \in \bar{H}_2^{p\times m}. \tag{2.11}$$

Multiplying (2.11) on the right by Q_1^{-1}, we find that

$$h_+ Q_1^{-1} = Q^{-1}MQ_1^{-1} - h_- Q_1^{-1} \in \bar{H}_2^{p\times m},$$

so that $h_+ \in \mathcal{R}_{Q_1}$. Multiplying now (2.11) to the left by Q, we obtain

$$Qh_- = M - Qh_+ \in H_2^{p\times m}.$$

Hence $Qh_- \in \mathcal{L}_Q$ and (2.10) is in fact an equality:

$$\mathcal{L}_Q \oplus Q\mathcal{R}_{Q_1} = \mathcal{W}(Q, Q_1). \tag{2.12}$$

In particular, we see that

$$dim_{\mathbb{C}}\mathcal{W}(Q, Q_1) = dim_{\mathbb{C}}\mathcal{L}_Q + dim_{\mathbb{C}}\mathcal{R}_{Q_1} = n(m+p) + pm. \tag{2.13}$$

We may now identify our tangent space as follows.

Proposition 1 *Let q satisfy $1 \leq q \leq \infty$. Let W belong to $\Sigma_{p,m}^-(n)$ and $W = Q^{-1}R = R_1Q_1^{-1}$ be left and right Douglas-Shapiro-Shields factorizations. Then, at W, the tangent space to $\Sigma_{p,m}^-(n)$ viewed as an embedded submanifold in $\bar{H}_q^{p\times m}$ is given by*

$$\mathcal{T}_W = Q^{-1}\mathcal{W}(Q, Q_1)Q_1^{-1}. \tag{2.14}$$

At $W \in R\Sigma_{p,m}^-(n)$, the tangent space to $R\Sigma_{p,m}^-(n)$ is still given by (2.14) provided Q and Q_1 are chosen real and we limit ourselves to real members of $\mathcal{W}(Q, Q_1)$.

Proof: let (A, B, C, D) be a minimal realization of W. From Theorem 1, the space $\mathcal{T}_W$ is made of matrix–valued functions of the form

$$(2.15) \qquad \begin{array}{l} U + V(zI - A)^{-1}B \\ \quad -C(zI - A)^{-1}X(zI - A)^{-1}B + C(zI - A)^{-1}W \end{array}$$

where U, V, W and X range over $\mathbb{C}^{p\times m}$, $\mathbb{C}^{p\times n}$, $\mathbb{C}^{n\times m}$ and $\mathbb{C}^{n\times n}$ respectively. Let

$$C(zI - A)^{-1} = \Xi^{-1}\Delta \ \text{ and } \ (zI - A)^{-1}B = \Delta_1\Xi_1^{-1}$$

be Douglas–Shapiro–Shields factorizations. It is obvious that (2.15) can be written as

$$(2.16) \qquad \Xi^{-1}\Lambda\Xi_1^{-1},$$

for some Λ which is rational and analytic in the closed disk. We claim that $\Xi = UQ$ where U is a unitary matrix. Indeed, since $C(zI - A)^{-1}$ has McMillan degree n,

$$W = \Xi^{-1}(\Delta B + \Xi D) \ \text{ and } \ W = Q^{-1}R$$

are two left Douglas–Shapiro–Shields factorizations of W. Since such a factorization is unique up to left multiplication by a unitary factor, we are done. A similar argument on the right shows that $\Xi_1 = Q_1U_1$ where U_1 is again unitary.
Therefore, functions of the form (2.16) are in $Q^{-1}\mathcal{W}(Q, Q_1)Q_1^{-1}$ so that $\mathcal{T}_W$ is included in the right hand-side of (2.14). But we must then have equality since complex dimensions coincide as follows from 2.13. The proof of the real case is *mutatis mutandis* the same. ∎

2.2 The differential structure of I_n^p

In this section, we prove that I_n^p, $I_n^p/\mathcal{U}_p$ and their real analogues are smooth manifolds. Since we now deal with square matrices, m is to be set equal to p in the preceding results.

Proposition 2 *The set $(I_n^p)^\sharp$ is an embedded submanifold in $\Sigma_{p,p}^-(n)$ of dimension $p^2 + 2np$. When embedded in $\bar{H}_q^{p\times p}$ with $1 \le q \le \infty$, the tangent space to $(I_n^p)^\sharp$ at $Q^\sharp$ is given by*

$$(2.17) \qquad \mathcal{T}_{Q^\sharp}(I_n^p)^\sharp = \{(S + F - F^\sharp)Q^{-1}\},$$

where $S \in \mathcal{S}_p$ and $F \in Q^{-1}\mathcal{L}_Q$. The subset $(RI_n^p)^\sharp$ is also an embedded submanifold of dimension $p(p-1)/2 + np$. Its tangent space is still given by (2.17) *provided S is restricted to range over skew-symmetric matrices and F over real elements of $\mathcal{L}_Q$.*

Proof: embed $\Sigma^-_{p,p}(n)$ into $\bar{H}^{p\times p}_\infty$ and define a map

$$\Upsilon : \Sigma^-_{p,p}(n) \longrightarrow L^{p\times p}_\infty(\mathbb{T})$$

by the formula

$$\Upsilon(W) = WW^\sharp.$$

Since $W \to W^\sharp$ is a linear continuous map $H^{p\times p}_\infty \to \bar{H}^{p\times p}_\infty$ and multiplication is bilinear and continuous in $L^{p\times p}_\infty(\mathbb{T})$, it is obvious that Υ is smooth, and its derivative at W

$$\mathbf{D}\Upsilon(W) : \mathcal{T}_W \longrightarrow L^{p\times p}_\infty(\mathbb{T})$$

is given by

(2.18) $$M \longrightarrow MW^\sharp + WM^\sharp.$$

Let $W = Q^{-1}R = R_1Q_1^{-1}$ be right and left Douglas–Shapiro–Shields factorizations of W. By Proposition 1, we have that

$$\mathcal{T}_W = Q^{-1}\mathcal{W}(Q, Q_1)Q_1^{-1}$$

and from (2.12) we can write every $M \in \mathcal{T}_W$ as

(2.19) $$M = Q^{-1}(\ell_Q + Qr_{Q_1})Q_1^{-1}$$

for some $\ell_Q \in \mathcal{L}_Q$ and some $r_{Q_1} \in \mathcal{R}_{Q_1}$. Plugging this into (2.18) and taking into account that $Q^{-1} = Q^\sharp$ and $Q_1^{-1} = Q_1^\sharp$, we get

(2.20) $$\mathbf{D}\Upsilon(W).M = Q^\sharp\ell_Q R_1^\sharp + r_{Q_1}R_1^\sharp + R_1\ell_Q^\sharp Q + R_1 r_{Q_1}^\sharp.$$

Note that $Q^\sharp\ell_Q R_1^\sharp$ belongs to $z^{-1}\bar{H}^{p\times p}_2$, so the Fourier coefficient of index 0 in (2.20) is $F_0 + F_0^*$ where F_0 is the zero–th Fourier coefficient of $r_{Q_1}R_1^\sharp$. If we choose $r_{Q_1} = KQ_1$ where K is a constant matrix, we get $F_0 = KW^\sharp(0) = KW(\infty)^*$. Let us first assume that $W(\infty)$ is invertible so that the linear map $\alpha : \mathcal{R}_{Q_1} \to \mathbb{C}^{p\times p}$ defined by $r_{Q_1} \to F_0$ is surjective. Then, the subspace $\mathcal{R}'_{Q_1} = \alpha^{-1}(\mathcal{S}_p)$ has codimension p^2 in $\mathcal{R}_{Q_1}$ hence dimension $p^2 + 2np$.

Suppose now that $M \in \ker \mathbf{D}\Upsilon(W)$ so that (2.20) is zero. Then $r_{Q_1} \in \mathcal{R}'_{Q_1}$ and ℓ_Q is completely determined by r_{Q_1}. Indeed, R_1 is invertible in $L_\infty^{p\times p}(\mathbb{T})$ by our assumption on $W(\infty)$, and if we denote by P_+ the orthogonal projection onto $H_2^{p\times p}$, we have

$$\ell_Q^\sharp = -R_1^{-1}\left[P_+(r_{Q_1}R_1^\sharp + R_1 r_{Q_1}^\sharp)\right] Q^\sharp.$$

Therefore,

(2.21) $$\dim \ker \mathbf{D}\Upsilon(W) \leq \dim \mathcal{R}'_{Q_1} = p^2 + 2np.$$

For $a \in \mathbb{D}$, the map $\varphi_a : H \to H \circ b_a$ (where b_a is defined by 1.11) is a linear isometry of $L_\infty^{p\times p}$ preserving $\Sigma_{p,p}^-(n)$ and it is immediately checked that φ_a commutes with Υ. Applying the chain rule, we get

$$\mathbf{D}\Upsilon(\varphi_a(W)) \circ \varphi_a = \varphi_a \circ \mathbf{D}\Upsilon(W)$$

so that

(2.22) $$\dim \ker \mathbf{D}\Upsilon(W) = \dim \ker \mathbf{D}\Upsilon(\varphi_a(W)).$$

Now, if we merely assume that W is invertible in $L_\infty^{p\times p}(\mathbb{T})$, it is possible to adjust a so that $W_a(\infty)$ is invertible. In view of (2.22), we conclude that (2.21) still holds in this case.

We claim that (2.21) is an equality whenever $W \in (I_n^p)^\sharp$. Since the rank cannot decrease locally, we will then conclude that the kernel of $\mathbf{D}\Upsilon$ has dimension $p^2 + 2np$ on a neighborhood of $(I_n^p)^\sharp$ and the infinite–dimensional version of the constant rank theorem (see e.g. [17] 5.10.5-6) will imply that $(I_n^p)^\sharp = \Upsilon^{-1}(I_p)$ is a smooth manifold whose tangent space at W is $\ker \mathbf{D}\Upsilon(W)$.

To prove the claim, set $W = Q^\sharp$. Plugging $R_1 = I_p$ and $Q_1 = Q$ in (2.20) yields

(2.23) $$\mathbf{D}\Upsilon(W).M = [r_Q + \ell_Q^\sharp Q] + [r_Q + \ell_Q^\sharp Q]^\sharp.$$

The first bracket in the right hand-side of (2.23) is analytic in the closed unit disk by the definition of ℓ_Q and r_Q so that (2.23) is zero if and only if

(2.24) $$r_Q + \ell_Q^\sharp Q = S$$

where S is a skew–Hermitian constant. Now, for any $\ell_Q \in \mathcal{L}_Q$ and any $S \in \mathcal{S}_p$, (2.24) determines a unique r_Q which is readily checked to belong to $\mathcal{R}_Q$. Moreover, r_Q can be zero only if ℓ_Q and S are both zero since $\ell_Q^\sharp Q$ vanishes at zero. The kernel of $\mathbf{D}\Upsilon(W)$ is thus of dimension $p^2 + 2np$ and this was the claim.

To establish (2.17), observe that $\ker \mathbf{D}\Upsilon(W)$ at $W = Q^\sharp$ is the set of matrix-valued functions of the form (2.19) where $Q_1 = Q$ and such that (2.24) holds. A simple computation shows that this set is just the collection

$$(S + (Q^{-1}\ell_Q) - (Q^{-1}\ell_Q)^\sharp)Q^{-1},$$

where $S \in \mathcal{S}_p$ and $\ell_Q \in \mathcal{L}_Q$. But it is easily checked from the definitions that $\mathcal{L}_Q = QP_-(Q^{-1}H_2^{p\times p})$ and this implies (2.17).

In the real case, the proof is identical provided a is chosen real. Note also that skew–hermitian has to be replaced by skew–symmetric so that $R\mathcal{R}'_{Q_1}$ has codimension $p^2 - p(p-1)/2$ in $R\mathcal{R}_{Q_1}$ hence dimension $p(p-1)/2 + np$. ■

We now introduce the set $NI_n^p \subset I_n^p$ of functions Q normalized by the condition $Q(1) = I_p$ and its real counterpart RNI_n^p. These functions will serve as representatives in $I_n^p/\mathcal{U}_p$ and we will study $(NI_n^p)^\sharp$.

Proposition 3 *The set $(NI_n^p)^\sharp$ is a smooth submanifold of $(I_n^p)^\sharp$ of dimension $2np$ and the product map*

$$(NI_n^p)^\sharp \times \mathcal{U}_p \longrightarrow (I_n^p)^\sharp \tag{2.25}$$

is a diffeomorphism. A similar statement holds in the real case replacing $\mathcal{U}_p$ by $\mathcal{O}_p$, the dimension of $(RNI_n^p)^\sharp$ being np.

Proof: let E be the evaluation map at 1. It is linear and continuous $\mathcal{A}^{p\times p} \to \mathbb{C}^{p\times p}$ (recall $\mathcal{A}$ is the disk algebra). Since $(I_n^p)^\sharp \subset \mathcal{A}^{p\times p}$, the restriction

$$E : (I_n^p)^\sharp \to \mathcal{U}_p$$

is smooth and its derivative at any $Q^\sharp$, which is defined from $\mathcal{T}_{Q^\sharp}(I_n^p)^\sharp$ into $\mathcal{T}_{Q^\sharp(1)}\mathcal{U}_p$ is again evaluation at 1. Since the tangent space to $\mathcal{U}_p$ at I_p is $\mathcal{S}_p$, it follows from (2.17) that E is submersive at every point of $E^{-1}(I_p) = (NI_n^p)^\sharp$ which is thus a submanifold of codimension p^2 hence of dimension $2np$.

Consider now the map (2.25). It is obviously smooth together with its inverse

$$Q^\sharp \to (Q^\sharp Q(1), Q^\sharp(1)).$$

The proof in the real case is similar. ∎

Recall that two members of $(I_n^p)^\sharp$ (resp. $(RI_n^p)^\sharp$) are equivalent if they are equal up to right multiplication by a unitary (resp. orthogonal) factor and that that the sets of equivalence classes are denoted by $(I_n^p)^\sharp/\mathcal{U}_p$ and $(RI_n^p)^\sharp/\mathcal{O}_p$ respectively. We endow them with the quotient topology and it is an immediate consequence of proposition 3 that the natural map $(NI_n^p)^\sharp \to (I_n^p)^\sharp/\mathcal{U}_p$ is a homeomorphism allowing us to carry over to $(I_n^p)^\sharp/\mathcal{U}_p$ the manifold structure of $(NI_n^p)^\sharp$. We proceed similarly in the real case.

Changing z into $1/z$, we are now able to restate the results of this section in terms of I_n^p:

Theorem 2 *The set I_n^p is a smooth manifold of dimension $p^2 + 2np$ embedded in $H_q^{p\times p}$ for $1 \leq q \leq \infty$. The set NI_n^p is a smooth submanifold of dimension $2np$ homeomorphic to $I_n^p/\mathcal{U}_p$. As well, the subset RI_n^p is a smooth manifold of dimension $p(p-1)/2+np$ embedded in $RH_q^{p\times p}$, and the set RNI_n^p is a smooth submanifold of dimension np homeomorphic to $RI_n^p/\mathcal{U}_p$.*

We were able so far to establish the smoothness of I_n^p and to identify its tangent space in $H_q^{p\times p}$. However, the above approach does not provide us with explicit charts since we did rely on the constant rank theorem. It is the purpose of the next subsection to fill this gap by making contact with the classical manifold of observable pairs.

2.3 Constructing charts from realizations

We need to introduce the set of observable pairs. Let Ω_o denote the open subset of $\mathbb{C}^{p\times n} \times \mathbb{C}^{n\times n}$ consisting of all observable pairs (C, A). Two pairs (C_1, A_1) and (C_2, A_2) will be said to be equivalent if there exists $T \in GL(n)$ such that

$$\begin{aligned} C_1 &= C_2 T \\ A_1 &= T^{-1} A_2 T. \end{aligned}$$

Let $Obs_p(n)$ be the set of equivalence classes for the above relation and

$$\Pi_o : \Omega_o \to Obs_p(n)$$

be the natural map. When $Obs_p(n)$ is equipped with the quotient topology, the Hazewinkel–Kalman method already mentioned in section 2.1 (see [37], [38], [10]) allows one to construct local sections of Π_o endowing $Obs_p(n)$ with a complex structure of complex dimension np. With this definition, Π_o is smooth. Again, we shall only work with the underlying real structure of dimension $2np$. The set $RObs_p(n)$ of equivalence classes of real pairs is in turn a submanifold of dimension np. We single out the subset Ω_o^- of Ω_o consisting of pairs (C, A) for which the spectrum of A is in $\mathbb{D}$. Being open in $Obs_p(n)$, the set $Obs_p^-(n) = \Pi_o(\Omega_o^-)$ is naturally a smooth manifold.

We first characterize realizations of members of $(I_n^p)^\sharp$. Representations similar to the one below can be found in [31], [5], [7] or [28] in the case of the half–plane. We give a simple proof for sake of completeness.

Proposition 4 *Let W be a rational $\mathbb{C}^{p\times p}$–valued function analytic at infinity and let $W(z) = D + C(zI_n - A)^{-1}B$ be a minimal realization of W. Then, the following statements are equivalent:*
(i) *The function W belongs to $(I_n^p)^\sharp$.*
(ii) *There exists a positive definite matrix P such that:*

$$A^*PA + C^*C = P \tag{2.26}$$

$$A^*PB + C^*D = 0 \tag{2.27}$$

$$B^*PB + D^*D = I \tag{2.28}$$

Proof: let us prove that (i) implies (ii). Since the given realization is minimal the spectrum of A is contained in a closed disk

$$E_\alpha = \{z \in \mathbb{C}; |z| \leq \alpha < 1\}.$$

Therefore the function $W(z)W(z)^*$ is in $L_2^{p\times p}(\mathbb{T})$ and, for $|z| = 1$ we can write the power series expansions:

$$\begin{aligned} W^*(z)W(z) &= (D^* + B^*(z^{-1}I_n - A^*)^{-1}C^*)(D + C(zI_n - A)^{-1}B) \\ &= D^*D + D^*C\left(z^{-1}\sum_{j=0}^{\infty}(z^{-1}A)^j\right)B + B^*\left(z\sum_{j=0}^{\infty}(zA^*)^j\right)C^*D \\ &\quad + B^*\left(z\sum_{j=0}^{\infty}(zA^*)^j\right)C^*C\left(z^{-1}\sum_{j=0}^{\infty}(z^{-1}A)^j\right)B. \end{aligned}$$

Equating Fourier coefficients in the equation $W^*W = I_p$, we get

$$(2.29)\quad \frac{1}{2i\pi}\int_{\mathbb{T}} W(z)^*W(z)\frac{dz}{z} = D^*D + B^*\sum_{j=0}^{\infty}\left(A^{*j}C^*CA^j\right)B = I_p$$

and for $k > 0$

$$\frac{1}{2i\pi}\int_{\mathbb{T}} z^{-k}W(z)^*W(z)\frac{dz}{z} =$$

$$(2.30)\quad B^*(A^*)^{k-1}C^*D + B^*A^{*k}\sum_{j=0}^{\infty}\left(A^{*j}C^*CA^j\right)B = 0.$$

Define

$$(2.31)\quad P = \sum_{j=0}^{\infty}\left((A^*)^jC^*CA^j\right).$$

Observability of the pair (A, C) implies that P is positive definite. ¿From the definition of P, (2.26) holds and (2.28) follows from (2.29). Rewrite (2.30) as

$$B^*A^{*(k-1)}(C^*D + A^*PB) = 0$$

to conclude that (2.27) holds, thanks to the controllability of the pair (A, B). To prove that (ii) implies (i), observe that the unique solution to (2.26) is given by (2.31) and reverse the above arguments. ∎

Note that equations (2.26)–(2.28) may be capsulized as

$$(2.32)\quad \begin{pmatrix} A^*P & C \\ B^*P & D^* \end{pmatrix}\begin{pmatrix} AP^{-1} & B \\ CP^{-1} & D \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_p \end{pmatrix}.$$

We now state the main result of this section.

Proposition 5 *Let $\nu : \Omega^- \longrightarrow \Omega_o^-$ be the natural projection*

$$(A, B, C, D) \longrightarrow (C, A).$$

The map $\Pi_o \circ \nu \circ \Pi^{-1}$ is a diffeomorphism from $(NI_n^p)^\sharp$ onto $Obs_p^-(n)$. By restriction, ν induces a diffeomorphism between $(RNI_n^p)^\sharp$ and $RObs_p^-(n)$.

Proof: set $\xi = \Pi_o \circ \nu \circ \Pi^{-1}$. This application is clearly well–defined and continuous thanks to the smoothness of the local sections ϕ_k of Π. To

construct a set–theoretic inverse, we first need to show that any pair $(C, A) \in \Omega_o^-$ can be completed into a minimal 4-uple (A, B, C, D) satisfying (2.26)–(2.28) fo some strictly positive matrix P (which is then given by (2.31). If A is invertible, this is done in [5, Theorem 3.10]. When A is not invertible, we approximate A by a sequence of invertible A_k such that $(C, A_k) \in \Omega_o^-$, and we let B_k and D_k denote completions of (C, A_k) satisfying (2.26)–(2.28). The corresponding matrices P_k are bounded thanks to equation (2.31) and have a convergent subsequence with limit, say, P. By a classical inertia theorem (see e.g. [5, Theorem 3.15]), P is invertible. Therefore, formula (2.28) implies that B_k and D_k are uniformly bounded and thus have convergent subsequences, with limit, say, B and D respectively, which realize the desired completion. Next, we claim that such a completion is unique up to right multiplication of B and D by the same unitary constant. Assume indeed that B_1 and D_1 is another one. Since P is uniquely determined by A and C, (2.32) shows that (B^*P, D^*) and (B_1^*P, D_1^*) have the same span of their rows. Thus there exists $P_1 \in GL(n)$ such that $B_1^* = P_1 B^*$ and $D_1^* = P_1 D^*$. Again from (2.32), P_1 is unitary.
Now, let W be defined by $W(z) = D + C(zI_n - A)^{-1}B$. The set–theoretic inverse to ξ is given by $WW(1)^\sharp$.
To complete the proof, it is enough to show that the above completion process $\Omega_o^- \to \Omega^-$ may be performed locally by a smooth function.
For $(C, A) \in \Omega_o^-$, let $P = P(A, C)$ be the smooth function of (C, A) defined by (2.31). Consider the map

$$\chi : \Omega^- \mapsto \mathbb{C}^{n \times p} \times \mathcal{H}_p$$

as

$$\chi(A, B, C, D) = \left\{ \begin{array}{c} A^*P(A, C)B + C^*D \\ B^*P(A, C)B + D^*D - I \end{array} \right\}$$

it follows from Proposition 4 that $\Pi^{-1}(I_n^p)^\sharp = \chi^{-1}(0)$.
Now χ is obviously smooth and we claim that the partial derivative with respect to (B, D) is surjective at every point of $\chi^{-1}(0)$. To this effect we compute

$$\mathbf{D}_2\chi(A, B, C, D) : V \mapsto \left[\begin{array}{c} A^*PV \\ V^*PB + B^*PV \end{array} \right] \tag{2.33}$$

$$\mathbf{D}_4\chi(A, B, C, D) : X \mapsto \left[\begin{array}{c} C^*X \\ X^*D + D^*X \end{array} \right] \tag{2.34}$$

Let $\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$ belong to $\mathbb{C}^{n\times p} \times \mathcal{H}_p$. Since P is invertible, (2.26) implies that

$$A^*CV_1 + C^*X_1 = S_1$$

is solvable. Define $V_2 = BS_2/2$ and $X_2 = DS_2/2$. Then we get:

$$\begin{aligned}
[\mathbf{D}_2\chi.V_2 + \mathbf{D}_4\chi.X_2] &= \begin{bmatrix} A^*PV_2 + C^*X_2 \\ V_2^*PB + B^*PV_2 + X_2^*D + D^*X_2 \end{bmatrix} \\
&= \begin{bmatrix} (A^*PB + C^*D)S_2/2 \\ S_2/2(B^*PB + D^*D) + (B^*PB + D^*D)S_2/2 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ S_2 \end{bmatrix}.
\end{aligned}$$

Setting $X = X_1 + X_2$ and $Y = Y_1 + Y_2$, we obtain

$$[\mathbf{D}_2\chi.V_2 + \mathbf{D}_4\chi.X_2] = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}, \tag{2.35}$$

thereby proving our claim.

It is now easy to construct a smooth local section of ν: start from (C, A) and take a completion (A, B, C, D). Extract from (B, D) a set of variables with respect to which the partial derivative is an isomorphism and apply the implicit function theorem. The proof in the real case is similar. ∎

Changing z into z^{-1}, we obtain at once

Corollary 1 *The space I_n^p is diffeomorphic to $I_n^p/\mathcal{U}_p \times \mathcal{U}_p$, and $I_n^p/\mathcal{U}_p$ is diffeomorphic to $Obs_p^-(n)$. Similar statements hold for RI_n^p, $\mathcal{O}_p$ and $RObs_p^-(n)$.*

Beside the fact that it expresses our spaces of inner matrix-valued functions in terms of familiar objects, this corollary has some topological consequences. For E a topological space, let us denote by $\mathbf{H}_k(E, G)$ the n-th homology group of E with coefficients in G. We further set $\mathcal{G}_{k,l}$ and $R\mathcal{G}_{k,l}$ to be the complex and real Grassmann manifolds of k–subspaces of an l–space.

Corollary 2 *$I_n^p/\mathcal{U}_p$, $RI_n^p/\mathcal{O}_p$, and I_n^p are connected while RI_n^p has two connected components. For $k \geq 0$, there are isomorphisms*

$$\mathbf{H}_k(I_n^p/\mathcal{U}_p, \mathbb{Z}) = \mathbf{H}_k(\mathcal{G}_{n,n+p-1}, \mathbb{Z}), \tag{2.36}$$

$$\text{(2.37)} \qquad \mathbf{H}_k(I_n^p, \mathbb{Q}) = \sum_{l_1+l_2=k} \mathbf{H}_{l_1}(\mathcal{G}_{n,n+p-1}, \mathbb{Q}) \otimes \mathbf{H}_{l_2}(\mathcal{U}_p, \mathbb{Q}),$$

$$\text{(2.38)} \qquad \mathbf{H}_k(RI_n^p/\mathcal{O}_p, \mathbb{Z}/2\mathbb{Z}) = \mathbf{H}_k(R\mathcal{G}_{n,n+p-1}, \mathbb{Z}/2\mathbb{Z}),$$

$$\text{(2.39)} \qquad \mathbf{H}_k(RI_n^p, \mathbb{Z}/2\mathbb{Z}) = \sum_{l_1+l_2=k} \mathbf{H}_{l_1}(R\mathcal{G}_{n,n+p-1}, \mathbb{Z}/2\mathbb{Z}) \otimes \mathbf{H}_{l_2}(\mathcal{O}_p, \mathbb{Z}/2\mathbb{Z}).$$

Proof: we may replace $Obs_p^-(n)$ by $Obs_p(n)$ and $RObs_p^-(n)$ by $RObs_p(n)$ in corollary 1 because it is shown in [35] that these two spaces are diffeomorphic (the theorem in [35] is for $\Sigma_{p,m}(n)$ and $\Sigma_{p,m}^-(n)$ but its proof would yield the same conclusion for observable pairs). Then (2.36) and (2.37) are implied by the fact that $Obsp(n)$ and $\mathcal{G}_{n,n+p-1}$ (resp. $RObs_p(n)$ and $R\mathcal{G}_{n,n+p-1}$) have the same integral (resp. mod 2) homology by the results of [38] also reported in [39]. Now, the Künneth theorem (see e.g. [46]) implies (2.37) and (2.39). Since $\mathbf{H}_0(E,G)$ is a direct sum on the number of connected components, the first statement follows from the connectivity properties of $\mathcal{U}_p$, $\mathcal{O}_p$, and the Grassmann manifolds. ■

The connectivity statement can of course be proved without resorting to homology from known properties of minimal pairs. For the computation of the homology of $\mathcal{G}_{k,l}$, $\mathcal{U}_p$, and $\mathcal{O}_p$ we refer the reader to [43].

3 Charts using the Schur algorithm

In this section, we study I_n^p and related manifolds using Schur analysis. We begin with a review of finite–dimensional reproducing kernel spaces, focusing on two special cases, namely $H(Q)$ and $H(\Theta)$ spaces. Relationships between these spaces lead to the tangential Schur algorithm (theorem 6). Spaces $H(Q)$ are the orthogonals of Beurling-Lax invariant spaces (see e.g. [29]) while $H(\Theta)$ spaces were first introduced and studied by L. de Branges in [18]. Analogues of $H(Q)$ spaces for general Schur functions B were defined and studied by L. de Branges and J. Rovnyak in [19], [20]. Some of the material appears in a more general context in [4].

3.1 Preliminaries and H(Q) spaces

A Hilbert space H of $\mathbb{C}^p$-valued functions defined on some set $\mathcal{E}$ is called a reproducing kernel Hilbert space if there exists a $\mathbb{C}^{p\times p}$–valued function $K(z,w)$ such that:

(1) For every choice of $c \in \mathbb{C}^p$ and $w \in \mathcal{E}$, the function $z \rightarrow K(z,w)c$ belongs to H

(2) For every $f \in H$ and w, c as above,

$$(3.1) \qquad < f \ , \ K(. \ , \ w)c >= c^* f(w)$$

where $< , >$ denotes the inner product in H.

Note that, by the Riesz representation theorem, it is equivalent to assume that the functionals

$$f \longrightarrow c^* f(w)$$

are all continuous. The function $K(z,w)$ is uniquely defined and called the reproducing kernel of H. It is positive in the sense that for every $r \in \mathbb{N}$, every $w_1, ..., w_r \in \mathcal{E}$ and $c_1, ..., c_r \in \mathbb{C}^p$, the $r \times r$ matrix with ij entry $c_j^* K(w_j, w_i) c_i$ is nonnegative. Moreover, there is a one–to–one correspondence between positive functions and reproducing kernel Hilbert spaces: whenever $K(z,w)$ is positive for z, w in $\mathcal{E}$, the completion of the linear space generated by the functions $z \rightarrow K(z,w)c$ with $w \in \mathcal{E}$ and $c \in \mathbb{C}^p$ endowed with the scalar product

$$(3.2) \qquad < K(.,w)c, K(.,v)d >= d^* K(v,w)c$$

is the reproducing kernel space with kernel $K(z,w)$; see [6], [50].

A classical example is the Hardy space H_2^p. Its reproducing kernel is $I_p/\rho_w(z)$ with z and w in $\mathbb{D}$ and ρ_w is defined in (1.10). In this case, equation (3.1) is just the Cauchy formula.

When H is finite dimensional, the kernel can be expressed explicitly as follows: let $f_1, ..., f_N$ be a basis of H over $\mathbb{C}$ and P be the matrix with $i-j$ entry $< f_j \ , \ f_i >$; then

$$(3.3) \qquad K(z,w) = \ [f_1(z), ..., f_N(z)]P^{-1}[f_1(w), ..., f_N(w)]^*.$$

Let now Q be an element of I_n^p. Left multiplication by Q is an isometry M_Q from H_2^p into itself and for $w \in \mathbb{D}$ and $c \in \mathbb{C}^p$

$$M_Q^*(c/\rho_w) = Q(w)^* c/\rho_w.$$

It follows that the function

$$(3.4) \qquad K_Q(z,w) = (I_p - Q(z)Q(w)^*)/\rho_w(z)$$

is positive for $z, w \in \mathbb{D}$. The main properties of the associated reproducing kernel space, denoted by $H(Q)$, are gathered in the next theorem.

Theorem 3 *Let Q be in I_n^p. Then $H(Q) = H_2^p \ominus QH_2^p$ and its complex dimension is equal to the McMillan degree n of Q. Furthermore, $H(Q)$ is R_0-invariant so that there exists an observable pair of matrices $C \in \mathbb{C}^{p\times n}$ and $A \in \mathbb{C}^{n\times n}$ with its spectrum in $\mathbb{D}$ such that $H(Q)$ is spanned by the columns of*

$$C(I_n - zA)^{-1}. \tag{3.5}$$

The space $H(Q)$ determines Q uniquely up to right multiplication by some unitary factor. A possible choice for Q is

$$Q(z) = I_p - (1-z)C(I_n - zA)^{-1}P^{-1}(I_n - A)^{-*}C^*, \tag{3.6}$$

where P is the solution to the Stein equation:

$$P - A^*PA = C^*C. \tag{3.7}$$

The space $H(Q)$ is real if and only if there exists a unitary matrix U such that QU is real. In this case, C and A in (3.5) may also be chosen real.

All the assertions in the theorem are widely known, at least in the complex case, and we refer the reader to [4] or [26] for a proof. Formula (3.6) deserves perhaps a word of explanation: if we take as a basis of $H(Q)$ the columns of (3.5), the corresponding Gram matrix in the H_2^p inner product is precisely the solution P of (3.7). Equating (3.4) and (3.3) yields then

$$(I_p - Q(z)Q(w)^*)/\rho_w(z) = C(I_n - zA)^{-1}P^{-1}(I_n - wA)^{-*}C^*.$$

Specializing to $w = 1$ and normalizing Q so that $Q \in NI_n^p$, we obtain (3.6). From this, the real case is easy, because A and C can be chosen real as mentionned in section 1.2 and P is also real.

3.2 $H(\Theta)$ spaces and the tangential Schur algorithm

We shall deal in the sequel with a special instance of what is called the class of J-contractive functions which were first studied by Potapov in [47]. We content ourselves with defining rational J-inner functions as follows. Put

$$J = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix} \quad J_+ = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix} \quad J_- = \begin{pmatrix} 0 & 0 \\ 0 & I_p \end{pmatrix}.$$

A $\mathbb{C}^{2p\times 2p}$-valued rational function Θ will be called J-inner if at every point of analyticity of Θ in $\mathbb{D}$,

$$\Theta(z)J\Theta(z)^* \leq J$$

and equality holds for z point of analyticity on $\mathbb{T}$.
If Θ is a rational J-inner function, the function

$$\Sigma(z) = (J_+ - \Theta(z)J_-)^{-1}(\Theta(z)J_+ - J_-)$$

is inner and the function

$$K_\Theta(z,w) = (J - \Theta(z)J\Theta(w)^*)/\rho_w(z) \tag{3.8}$$

is positive for z and w varying in the domain of analyticity of Θ, as follows from the equality

$$\frac{I_{2p} - \Sigma(z)\Sigma(w)^*)}{\rho_w(z)} = (J_+ - \Theta(z)J_-)^{-1}\, K_\Theta(z,w)\, (J_+ - \Theta(w)J_-)^{-*}.$$

We will denote by $H(\Theta)$ the associated reproducing kernel Hilbert space of $\mathbb{C}^{2p}$ valued functions. The space $H(\Theta)$ has properties similar to that of $H(Q)$ (see [4] and [26]). In particular, it is finite–dimensional, and the complex dimension is still the McMillan degree of Θ. A noteworthy difference with $H(Q)$ is that functions in $H(\Theta)$ need not lie in H_2^p, but here we shall only be concerned with the case where $H(\Theta) \subset H_2^p$. Then, 0 belongs to the domain of analyticity of Θ, and $H(\Theta)$ is again R_0-invariant. It can thus be represented as the span of the columns of a matrix of the form (3.5). Conversely we have:

Theorem 4 *Let (C, A) be an observable pair in $C \in \mathbb{C}^{2p\times n} \times \mathbb{C}^{n\times n}$ such that the spectrum of A lies in $\mathbb{D}$. Let $f_1, ..., f_n$ be the vector-valued functions defined by*

$$[f_1, ..., f_n](z) = C(I_n - zA)^{-1},$$

and $\mathcal{M} = l.s.\{f_i\}$ be endowed with the inner product

$$< f_j, f_i >_{\mathcal{M}} = < f_j, Jf_i >_{H_2^{2p}}.$$

If this inner product is positive definite, then $\mathcal{M}$ is a reproducing kernel Hilbert space with reproducing kernel of the form (3.8). In this case the function Θ is unique up to a J-unitary right multiplicative factor and, defining the matrix P by $P_{ij} = < f_j, f_i >_{\mathcal{M}}$, a possible choice for Θ is given by

$$\Theta(z) = I_{2p} - (1 - z\zeta_0^*)C(I_n - zA)^{-1}P^{-1}(I_n - \zeta_0 A)^{-*}C^*J \tag{3.9}$$

where ζ_0 is arbitrary in $\mathbb{T}$.

This result is essentially the finite–dimensional version of the Beurling-Lax theorem in H_2^p endowed with the above metric, see [8]. For a proof in the setting of reproducing kernel spaces see e.g. [26] and [5]. A direct way of seeing it is to start from (3.9) and to plug it in (3.8). Then it is easy to check that this expression for K_Θ satisfies (3.3) using the fact that P satisfies the equation:

$$P - A^*PA = C^*JC. \tag{3.10}$$

■

Corollary 3 *Let $w_0, ...w_{m-1}$ be m points in $\mathbb{D}$ and $x_0, ..., x_{m-1}$ be m elements of $\mathbb{C}^{2p}$. Suppose that the $m \times m$ matrix P with ij entry $P_{ij} = \frac{x_i^* J x_j}{\rho_{w_j}(w_i)}$ is strictly positive. Then the space*

$$\mathcal{M} = l.s.\{x_0/\rho_{w_0}, ..., x_{m-1}/\rho_{w_{m-1}}\}$$

endowed with the inner product defined by P is a finite dimensional $H(\Theta)$ space.
The function Θ can be chosen real when the w_i's are real and the x_i's are in $\mathbb{R}^{2p}$.

The next theorem describes the deep link between the spaces $H(\Theta)$ and $H(Q)$ on which the tangential Schur algorithm ultimately rests. It is a consequence of a general result due to de Branges and Rovnyak (see [19] and [3]).
For $f \in H_2^{2p}$ and $Q \in I_n^p$ define $\tau f \in H_2^p$ as

$$\tau f = [I_p, \ -Q]f. \tag{3.11}$$

Theorem 5 *Let Q be in I_n^p and let Θ be a J-inner $2p \times 2p$ rational function of McMillan degree d.*
Let $\begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix}$ be the decomposition of Θ into four $\mathbb{C}^{p\times p}$ valued blocks. Then τ defines an isometry from $H(\Theta)$ into $H(Q)$ if and only if there exists an inner function $Q^{(1)} \in I_{n-d}^p$ such that

$$Q = T_\Theta(Q^{(1)}) = (\Theta_{11}Q^{(1)} + \Theta_{12})(\Theta_{21}Q^{(1)} + \Theta_{22})^{-1}. \tag{3.12}$$

Finally, if Q and Θ are real, so is $Q^{(1)}$.

Proof: let us suppose that τ is an isometry; in particular, it is a contraction and, by the result of de Branges and Rovnyak, we can write Q as in (3.12) for some function $Q^{(1)}$ analytic and contractive in $\mathbb{D}$. The inverse of the linear fractional transformation T_Θ is $T_{\Theta^{-1}}$, so that $Q^{(1)}$ is rational. By the maximum modulus principle, $Q^{(1)}$ is in fact unitary on $\mathbb{T}$. We now show that $Q^{(1)} \in I^p_{(n-d)}$.

The operators $\tau\tau^*$ and $I - \tau\tau^*$ are orthogonal projections and thus

$$H(Q) = \mathrm{Im}(I - \tau\tau^*) \oplus \mathrm{Im}(\tau\tau^*).$$

For $u \in \mathbb{C}^p$ and $w \in \mathbb{D}$ it is easily verified (see e.g. [19] or [3]) that

$$\tau^*(K_Q(.,w)u) = K_\Theta(.,w)\begin{pmatrix} u \\ -Q(w)^*u \end{pmatrix}. \tag{3.13}$$

Thus,

$$\begin{aligned} (I - \tau\tau^*)(K_Q(.,w)u)(z) &= \\ (\Theta_{11} - Q\Theta_{21})(z)K_{Q^{(1)}}(z,w)((\Theta_{11} - Q\Theta_{21})(w))^*u, \end{aligned} \tag{3.14}$$

and since the reproducing kernel of the projection is the projection of the reproducing kernel we find that

$$(\Theta_{11} - Q\Theta_{12})(z)K_{Q^{(1)}}(z,w)((\Theta_{11} - Q\Theta_{12})(w))^*$$

is the reproducing kernel of $\mathrm{Im}(I - \tau\tau^*)$. Because τ is an isometry, this space has complex dimension $n-d$. Moreover, the function $\Theta_{11} - Q\Theta_{12}$ has nonzero determinant, so that the above complex dimension is also equal to the complex dimension of $H(Q^{(1)})$. Thus the McMillan degree of $Q^{(1)}$ is $n-d$.

Conversely, suppose that $Q = T_\Theta(Q^{(1)})$ where $Q^{(1)}$ has degree $n-d$; using again the theorem of de Branges and Rovnyak, we can assert that the map τ is a contraction. Now, a general theorem [27] states that, if Γ_1 and Γ_2 are two positive operators in a Hilbert space, then

$$\mathbf{Im}(\Gamma_1 + \Gamma_2)^{1/2} = \mathbf{Im}\Gamma_1^{1/2} + \mathbf{Im}\Gamma_2^{1/2} \tag{3.15}$$

and this decomposition is orthogonal if and only if it is direct. Since our spaces are finite–dimensional, we can dispense with the square roots:

$$H(Q) = \mathbf{Im}(I - \tau\tau^*) + \mathbf{Im}(\tau\tau^*). \tag{3.16}$$

The space $\mathbf{Im}(\tau)$ has complex dimension at most d and by (3.14) $\mathbf{Im}(I - \tau\tau^*)$ has complex dimension $n - d$. It follows that the decomposition (3.16) is direct hence orthogonal by the above mentioned theorem, and that τ is injective. Consequently, τ is an isometry.

We shall draw two consequences of Theorem 5 and Corollary 3. The first one describes the solution to the Nevanlinna–Pick problem.

Proposition 6 *Let n and m be integers with $n > 0$ and $0 \leq m \leq n$. For $Q \in I_n^p$, there exists unit vectors $u_0, ..., u_{m-1}$ in $\mathbb{C}^p$ and points $w_0, \ldots, w_{m-1}$ in $\mathbb{D}$ such that the $m \times m$ matrix P with ij entry*

$$P_{ij} = u_i^* K_Q(w_i, w_j) u_j \tag{3.17}$$

is strictly positive. The space $\mathcal{M}$ defined in Corollary 3 with

$$x_i = \begin{pmatrix} u_i \\ Q(w_i)^* u_i \end{pmatrix},$$

endowed with the inner product P, is then a $H(\Theta)$ space for some J-inner rational function Θ of degree m. There exists an element $\Sigma \in I_{n-m}^p$ such that $Q = T_\Theta(\Sigma)$. Finally, when Q is real, the w_i can be chosen real and the u_i in $\mathbb{R}^p$.

Proof: we first prove by induction on m that one can find vectors u_i as prescribed. The claim is true for $m = 1$; in fact, even more is true in this case since w_0 can be assigned arbitrarily in $\mathbb{D}$. Suppose indeed that $Q(w)^* u$ is of unit norm for all unit vectors $u \in \mathbb{C}^p$. Then,

$$u^* K_Q(w, w) u = < K_Q(., w)u, K_Q(., w)u >_{H(Q)}$$

is zero for all $u \in \mathbb{C}^p$, so that $K(., w)$ is zero. But this is impossible for (3.4) would imply that Q is a constant.
Suppose now that the claim holds for some $m < n$ and let $u_0, \ldots, u_m$ and $w_0, \ldots, w_m$ satisfy our requirements. If the claim does not hold for $m + 1$, then every unit vector $u \in \mathbb{C}^p$ and every $w \in \mathbb{D}$, is such that the space spanned by the $K_Q(., w_i)u_i$'s and $K_Q(., w)u$ is degenerate in $H(Q)$. Since

the $K_Q(., w_i)u_i$'s are linearly independent, this forces $\dim_{\mathbb{C}} H(Q) = m < n$ contradicting the assumption.
By corollary 3, $\mathcal{M}$ is a $H(\Theta)$ space and the map τ defined in (3.11) an isometry from $H(\Theta)$ onto $H(Q)$. By theorem 5, $Q = T_\Theta(\Sigma)$ for some $\Sigma \in I^p_{n-m}$. The case where Q is real is similar. ■

The special case where $m = p$, all the w_i are equal, and the u_i form a basis of $\mathbb{C}^p$ correspond to the Schur algorithm of [23]. The case where $m = 1$ will be of special interest to us and is singled out in the next corollary.

Corollary 4 *Let Q be in I^p_n , $n > 0$, and let $w \in \mathbb{D}$. Then, there exists $u \in \mathbb{C}^p$ of unit norm such that $\|Q(w)^*u\| < 1$. The complex one–dimensional space spanned by*

$$f = \begin{pmatrix} u \\ Q(w)^*u \end{pmatrix} / \rho_w, \tag{3.18}$$

endowed with the inner product

$$< f, f >= u^*(I_p - Q(w)Q(w)^*)u/(1 - |w|^2),$$

is a $H(\Theta)$ space where Θ is given by formula (3.9) with $\zeta_0 = 1$, $A = w$, and

$$C = \begin{pmatrix} u \\ Q(w)^*u \end{pmatrix}.$$

When Q is in RI^p_n, w and u may be chosen real. The function Θ is then real.

Combining Theorem 5 and Corollary 4, we obtain the tangential version of the Schur algorithm. The latter will be a tool to obtain charts of I^p_n, as explained in the next section. We first need to mention some relationships between $H(\Theta)$ spaces and interpolation (see [26],[2]).

Proposition 7 *Let $w_i, i = 0, \ldots, m-1$ be m points in $\mathbb{D}$, $u_i, i = 0, \ldots, m-1$ and $v_i, i = 0, \ldots, m-1$ be vectors in $\mathbb{C}^p$. Let*

$$f_i(z) = \frac{\begin{pmatrix} u_i \\ v_i \end{pmatrix}}{1 - w_i^* z} \tag{3.19}$$

and let $\mathcal{M}$ denote the linear span of the functions f_i. Let us suppose that

$$(3.20) \qquad < f_i, f_j >_{\mathcal{M}} = \frac{u_i^* u_j - v_i^* v_j}{1 - w_i w_j^*}$$

defines a positive quadratic form. Then, there exists a rational J–inner function Θ such that $\mathcal{M} = H(\Theta)$ and there is a one–to–one correspondance between elements Q in I_n^p such that

$$(3.21) \qquad Q(w_i)^* u_i = v_i \qquad i = 1, \ldots, m$$

and the set of $T_\Theta(\Sigma)$, where $\Sigma \in I_{n-m}^p$.

Proof: let Σ range over the set of $p \times p$ Schur functions. ¿From [26], it follows that the set of $Q = T_\Theta(\Sigma)$ describes all Schur functions which satisfy the interpolation condition (i). On another hand, the interpolation conditions (3.21) are to the effect that the map τ (defined in (3.11) is an isometry from $H(\Theta)$ onto $H(Q)$.The conclusion now follows from theorem 5. ■

Theorem 6 *(The tangential Schur algorithm) Let $Q \in I_n^p$ and let $w_0, ..., w_{n-1}$ be n (possibly non distinct points) in the open unit disk. Then, for $0 \leq i \leq n-1$, there exist $B^{(i)} \in I_{n-i}^p$, unit vectors u_i, and J-inner rational functions of degree one Θ_i given by formula (3.9) where $\xi_0 = 1$, $A = w_i^*$, and*

$$C = \begin{pmatrix} u_i \\ Q(w_i)^{(i)*} u_i \end{pmatrix},$$

such that $Q^{(0)} = Q$ and

$$(3.22) \qquad Q^{(i)} = T_{\Theta_{i+1}}(Q^{(i+1)}) \quad i = 0, ...n-1.$$

In particular, setting $\Theta = \Theta_0 .... \Theta_{n-1}$, we get $Q = T_\Theta(U)$ where $U = Q^{(n)} = Q(1)$ is a constant unitary matrix.

Finally, when Q is real, the u_i's and the w_i's can also be chosen real, and so are the Θ_i's and the $Q^{(i)}$'s.

Proof: the theorem is a recursive application of Theorem 5 and Corollary 4. Let w_0 be in $\mathbb{D}$. Applying Corollary 4, we build Θ_0 which takes the value I_{2p} at $z = 1$ and is such that the map τ is an isometry from $H(\Theta_0)$

into $H(Q)$. From Theorem 5, we obtain a linear fractional transformation $Q = T_{\Theta_0}(Q^{(1)})$ where $Q^{(1)}$ is in I^p_{n-1}. Iterating this procedure $n-2$ times, we obtain a constant $Q^{(n)} \in \mathcal{U}_p$. Since all the functions Θ_i take the value I_{2p} at $z = 1$, we have that $Q^{(n)} = Q(1)$. One proceeds similarly in the real case. ■

As explained in [4], Theorem 6 reduces to the classical Schur algorithm when $p = 1$ and at each stage the point w is taken to be the origin. It is recursive: at each step, one chooses a point w and a direction u, in order to compute the Blaschke factor. Alternatively, one may proceed in one shot from n points and n directions using Proposition 7.

3.3 Constructing the charts

In this section, we construct new charts on I^p_n in terms of transfer functions rather than realizations. We develop two (equivalent) atlases, one based on Proposition 7 and another one on the tangential Schur algorithm.

For $u_0, ..., , u_{n-1} \in \mathbb{C}$ of unit length, $w_0, ..., w_{n-1}$ in the open unit disk and $(\mathcal{V}, \vartheta)$ a chart on $\mathcal{U}_p$, we define a chart (W, ψ) by its domain:

$$\begin{aligned} &W(u_0, ..., u_{n-1}, w_0, ..., w_{n-1}, \mathcal{V}) \\ &= \{Q \in I^p_n |\ P > 0\ , Q(1) \in \mathcal{V}\} \end{aligned} \tag{3.23}$$

where P is the matrix defined in proposition 6, and its coordinate map

$$\psi(Q) = (Q(w_0)^* u_0, ..., Q(w_{n-1})^* u_{n-1}, \vartheta(Q(1))). \tag{3.24}$$

Theorem 7 *The family (W, ψ) defines a C^∞ atlas on I^p_n which is compatible with its natural structure of embedded submanifold of $H^{p\times p}_q$ fo any $1 \leq q \leq \infty$. If we choose real w_i's and and u_i's and if we restict ourselves to real coordinates and orthogonal matrices, we obtain an atlas for RI^p_n.*

Proof: that $W(u_0, ..., u_{n-1}, w_0, ..., w_{n-1}, \mathcal{V})$ is open in I^p_n is easily checked from the definition. It is equally clear that ψ is defined and smooth on some open subset of $H^{p\times p}_q$. Thanks to Proposition 7, the range of ψ is $\mathcal{B} \times \vartheta(\mathcal{V})$ where $\mathcal{B}$ denotes the set of $(v_0, \ldots, v_{n-1})$ such that the matrix defined in (3.20) is strictly positive, and is therefore open in $\mathbb{R}^{2np+p^2}$. Finally, ψ^{-1} is given by $Q = T_\Theta(Q(1))$, where Θ is rationally computed from $Q(w_i)^* u_i$ and $Q(1)$, hence is smooth. The real case is obvious. ■

The preceding theorem gives explicit charts, one drawback being that the ranges of the charts are rather involved. One may alternatively use the tangential Schur algorithm to obtain charts whose range is the product of n copies of the unit ball in $\mathbb{C}^p$ with an open subset of $\mathcal{U}_p$–but the coordinate functions, this time, are more involved. This is our last result.

We define this new atlas as follows: the chart (V, ϕ) will have domain

$$\begin{aligned} &V(u_0, ..., u_{n-1}, w_0, ..., w_{n-1}, \mathcal{V}) \\ &= \{Q \in I_n^p | \ \|Q(w_i)^{(i)*}u_i\| < 1, \ i = 0, ..., n-1, Q(1) \in \mathcal{V}\} \end{aligned} \tag{3.25}$$

and the coordinate map will be given by

$$\phi(Q) = (Q^{(0)*}(w_0)u_0, ..., Q^{(n-1)*}(w_{n-1})u_{n-1}, Q(1)), \tag{3.26}$$

where the functions $Q^{(i)}$ are defined recursively as in theorem 6. Namely, we first construct Θ_0 by setting $\xi_0 = 1$, $A = w_0^*$, and $C = \begin{pmatrix} u_0 \\ Q(w_0)^* u_0 \end{pmatrix}$ in formula (3.9). Then, we define $Q^{(1)}$ by inverting the formula $Q = T_{\Theta_0}(Q^{(1)})$ and iterate the same procedure on $Q^{(1)}$. The function Q is uniquely determined by $\phi(Q)$ and if

$$\xi = (\xi_0, ...\xi_{n-1}, U)$$

where $U \in \mathcal{V}$ and each $\xi_i \in \mathbb{C}^p$ is of norm strictly less than one, we have from Proposition 7 that

$$\phi^{-1}(\xi) = T_{\Theta_0 ... \Theta_{n-1}}(U) \tag{3.27}$$

where Θ_i is given by (3.9) with $A = w_i^*$, $\zeta_0 = 1$, and $C = \begin{pmatrix} u_i \\ \xi_i \end{pmatrix}$. Hence, the range of ϕ consists of the announced product.

Theorem 8 *The family (V, ϕ) defines a C^∞ atlas on I_n^p which is compatible with that of Theorem 7. Restricting to real parameters in the charts as in the cited theorem, we get an atlas on RI_n^p.*

The proof of theorem 8 is analogous to the proof of theorem 7 and will be omitted.

4 Conclusion

Having shown that the set of $p \times p$ inner functions of degree n is a submanifold of $H_q^{p\times p}$ in the real and complex case, we produced two different parametrization for it, one based on the set of observable pairs, and the other on Schur coefficients. Both are well–known tools in system–theory and interpolation theory respectively, stressing here a link of a topological nature between two domains which are already known to interfere strongly from the analytic viewpoint. Along the same lines, further extensions to $J - inner$ and $J - unitary$ functions are to be expected.
Perhaps the main practical contribution of the paper is to provide a mean of applying differential calculus to the set I_n^p. Such a need arises, for instance, in rational approximation and this was part of the author's motivation for studying these questions: in fact, it is easily shown (see [13], [21]) that obtaining the best L^2 approximant of degree n of a function in $H_2^{p\times m}$ is equivalent to minimize a nonlinear function on the set I_n^p. To justify a differential approach and to use gradient algorithms for the minimization, a differential structure together with an explicit parametrization are needed.

In the scalar case where $m = p = 1$, a numerical algorithm has been derived in [14] to generate local minima of the criterion, and a uniqueness theorem has been obtained in [15] for sufficiently stable Stieltjes functions (i.e. transfer-functions of relaxation systems). Both references use in an essential way the topological structure of the closure of $RI_n^1/\mathcal{O}_1$ in H_2 (which is completely different from its closure in H_∞ so that here the exponent does matter). This closure turns out to be a projective space and the global step of the uniqueness proof in [15] drops out from the corresponding Morse inequalities [12].

In the case where $p = 2$, this problem was considered in [21] and our results allow for such a study when p is arbitrary. A full generalization, however, would again require a detailed knowledge of the closure of $RI_n^p/\mathcal{O}_p$ in $H_2^{p\times p}$. For $p > 1$, this is by no means well-understood.

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