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Integral Solution to Systems $Ax \leq b$

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Abstract. Let A be an integral $m \times n$-matrix satisfying $\sum_{j=1}^{n} |a_{ij}| \leq 2$ for all $i = 1, \ldots, m$ and b an integral m-vector. We show that if $Ax \leq b$ has an integral solution then every half-integral solution of $Ax \leq b$ can be rounded to an integral solution. Moreover, an integral solution can be found with the same complexity as the single source problem.

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1. Introduction

Consider the following problem:

given : an integral m-vector b and an integral $m \times n$-matrix $A = (a_{ij})$ satisfying

$$\sum_{j=1}^{n} |a_{ij}| \leq 2 \qquad \text{for all } i = 1, \ldots, m; \tag{1}$$

find : an integral solution of $Ax \leq b$, if there exists any.

Using Fourier-Motzkin elimination of variables Schrijver [1991] proposed an algorithm which can be seen to solve (1) in $O(n^3)$ time and space. Moreover, Schrijver characterized the existence of a solution.

We describe here a algorithm for solving (1) in $O(mn)$ time and $O(m+n)$ space. Actually, we show that (1) can be solved with the same time and space complexity as the single source problem. (The *single source problem* is: *given:* a digraph $D = (V, A)$ with length on its arcs and a source $s \in V$; *find:* for any vertex v a shortest path, i.e. a path of minimum length, from s to v.) The algorithm we present is based on (and a proof of) the fact that problem (1) has a solution if and only if:

(i) $Ax \leq b$ has a half-integral solution and
(ii) every half-integral solution of $Ax \leq b$ can be rounded to an integral solution.

We associate with a system $Ax \leq b$, given in (1), a digraph D_A with lengths on its arcs. It turns out that a (half-integral) solution of $Ax \leq b$, if there exists any, can be found with one application of a single source shortest path algorithm and that rounding the solution found is equivalent to *2-SAT* (satisfiability with at most 2 literals per clause). Finding a half-integral solution will be the time-dominating step while the rounding step is solvable in linear time (*cf.* Even, Itai & Shamir [1976]).

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2. Graph associated with $Ax \leq b$

We associate with a matrix A satisfying (1) a digraph D_A as follows. For each column index j of A, D_A has two corresponding vertices, j^+ and j^-. If row i has nonzeros in positions j and k with $j \neq k$, then D_A has arcs of length b_i connecting j^+, j^-, k^+, k^-, as shown in Figure 1, depending on whether $(a_{ij}, a_{ik}) = (1,1), (1,-1), (-1,1)$ or $(-1,-1)$.

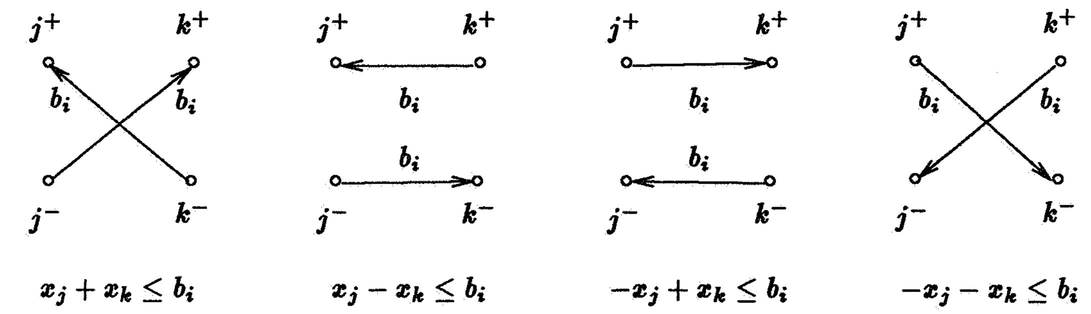

Figure 1:

If row i has only one nonzero $a_{ij} = 2$ (-2 respectively), it is represented in D_A by an arc (j^-, j^+) ((j^+, j^-) respectively) of length b_i as in Figure 2a. Finally, if row i has exactly one nonzero a_{ij} being $+1$ (-1 respectively), D_A has arcs (j^-, j^+) ((j^+, j^-) respectively) of lenght $2b_i$, as shown in Figure 2b. We shall denote the length of an arc (u, v) by length(u, v).

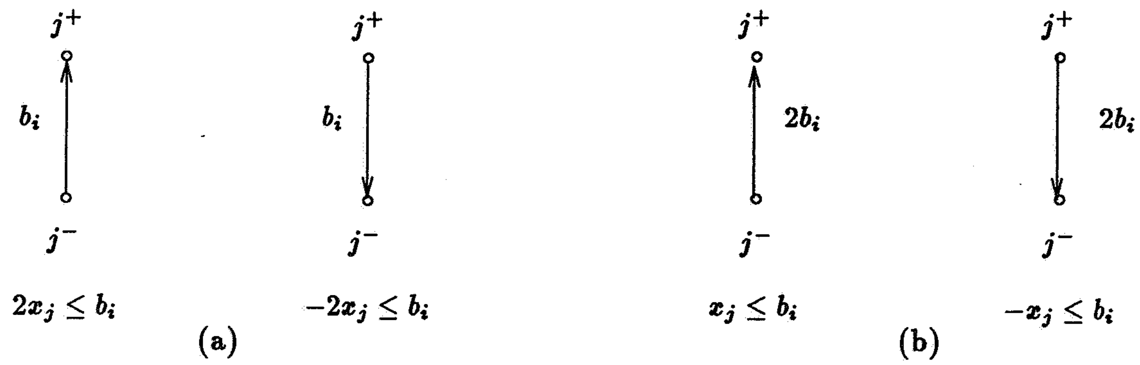

Figure 2:

3. Finding a half-integral solution

We call a function p on the vertices of a graph D satisfying

$$p(v) - p(u) \leq \text{length}(u, v) \qquad \text{for each arc } (u, v) \text{ of } D$$

a *potential function on D*. If x' is a solution of $Ax \leq b$ and p is the function defined as

$$p(j^+) := x'_j, \quad p(j^-) := -x'_j \qquad \text{for each column index } j \text{ of } A, \tag{2}$$

then p is a potential function on D_A.

Thus, any solution of $Ax \leq b$ gives rise to a potential function on D_A. Conversely, if p is a potential function on D_A then x' defined as

$$x'_j := (p(j^+) - p(j^-))/2 \qquad \text{for each column index } j \text{ of } A$$

is a solution of $Ax \leq b$. Therefore, there exists a solution of $Ax \leq b$ if and only if there exists a potential function on D_A. Moreover, if a potential function is integral then the corresponding solution of $Ax \leq b$ is half-integral. In order to find a (half-integral) solution of $Ax \leq b$ we find an (integral) potential function on D_A as it is described below.

A graph D has a potential function if and only if D has no negative cycle. (A *negative cycle* is a cycle the sum of whose arc lengths is negative.) As it is well known, to find a potential function on D we can proceed as follows. First we add to D a new vertex s and zero-length arcs (s, v) for every vertex v in D. Then for each vertex v we compute the length distance(v) of a shortest path from s to v in the augmented graph. There exists a shortest path from s to each vertex if and only if the augmented graph (and consequently D) has no negative cycle. If distance(v) is well-defined for every vertex v, then $p(v) :=$ distance(v) for each vertex v of D, is a potential function on D.

Hence, with one application of a single source shortest path algorithm we either find a potential function on D or a negative cycle proving that D has no potential function at all. Moreover, if all the arc lengths are integer then the potential function found is integral.

3. Rounding a half-integral solution

Let x^* be a half-integral solution of $Ax \leq b$. We transform the problem of rounding x^* to an integral solution of $Ax \leq b$ to 2-SAT. For each $j = 1, \ldots, n$ we introduce a variable u_j. We shall construct a collection F_{x^*} of two-literal clauses on the set $U = \{u_1, \ldots, u_n\}$ of variables such that x^* can be rounded into a solution of $Ax \leq b$ if and only if F_{x^*} is satisfiable. If a truth assignment $t : U \rightarrow \{\mathit{true, false}\}$ has $t(u_j) = \mathit{true}$ (*false* respectively) then we interpret this as "x_j^* should be rounded up (down respectively)".

The collection F_{x^*} will merely have from each inequality $a_{ij}x_j + a_{ik}x_k \leq b_i$ of $Ax \leq b$, satisfied with equality by nonintegral components x_j^* and x_k^* of x^* (allowing $j = k$), a two-literal clause, depending on whether $(a_{ij}, a_{ik}) = (1,1), (1,-1)$ or $(-1,-1)$, constructed as follows:

$$\begin{array}{rcrcr} x_j^* + x_k^* = b_i & \longrightarrow & \neg u_j & \vee & \neg u_k \\ x_j^* - x_k^* = b_i & \longrightarrow & \neg u_j & \vee & u_k \\ -x_j^* - x_k^* = b_i & \longrightarrow & u_j & \vee & u_k \end{array}$$

One easily checks that this indeed transforms the rounding step to 2-SAT. The transformation takes linear time.

Let $U = \{u_1, \ldots, u_n\}$ be a set of variables. It can be shown (see Schrijver[1978]) that a collection of two-literal clauses on U is satisfiable if and only if it contains no "doubly-odd" formula. A *doubly-odd formula* is a collection of clauses

$$(u \vee l_1), (l_2 \vee l_3), \ldots, (l_p \vee u), (\neg u \vee l_{p+1}), (l_{p+2} \vee l_{p+3}), \ldots, (l_q \vee \neg u)$$

satisfying:

(*i*) $u \in U$;

(*ii*) $l_1, \ldots, l_q$ are literals such that for $k = 1, \ldots, q/2$
either $l_{2k-1} = u_j$ and $l_{2k} = \neg u_j$
or $l_{2k-1} = \neg u_j$ and $l_{2k} = u_j$
for some $u_j \in U$.

Applying the linear-time algorithm of Even, Itai & Shamir [1976] we can either find a truth assignment satisfying simultaneously all clauses in F_{x^*} or a doubly-odd formula in F_{x^*}. The correctness of the algorithm can be derived from the following propositions. We shall denote by distance(u, v) the length of a shortest path from u to v in D_A.

Proposition 1 *If there exists $j \in \{1, \ldots, n\}$ such that*

$$-\,distance(j^+, j^-) = distance(j^-, j^+) \quad \text{and it is odd} \tag{3}$$

then $Ax \leq b$ has no integral solution.

Proof. If x' is an integral solution of $Ax \leq b$ and p is the potential function on D_A defined from x' as in (2) then $p(j^+) = p(j^-) \pmod 2$. But, the condition (3) implies that every integral potential function p on D_A must satisfy $p(j^+) = p(j^-) + \text{distance}(j^-, j^+)$. ■

Proposition 2 *Let x^* be a half-integral solution of $Ax \leq b$ and let F_{x^*} be the corresponding collection of two-literal clauses. If F_{x^*} has a doubly-odd formula then there exists $j \in \{1, \ldots, n\}$ satisfying (3).*

Proof. Let $F := (u_j \vee l_1), \ldots, (l_p \vee u_j), (\neg u_j \vee l_{p+1}), \ldots, (l_q \vee \neg u_j)$ a doubly-odd formula contained in F_{x^*}. We show that j fulfill conditions (3). Indeed, the set of arcs in D_A corresponding to clauses in F contains paths $P_{j^-j^+}$ and $P_{j^+j^-}$, from j^- to j^+ and from j^+ to j^- respectively. By definition of F_{x^*}, x_j^* is half-integer but not integer. The potential function p on D_A defined from x^* as in (2) satisfies $\text{length}(u, v) = p(v) - p(u)$ for each arc (u, v) in $P_{j^-j^+}$ or $P_{j^+j^-}$. Hence, $2x_j^* = p(j^+) - p(j^-) \leq \text{distance}(j^-, j^+) \leq \text{length}(P_{j^-j^+}) = 2x_j^*$. Analogously, $\text{distance}(j^+, j^-) = -2x_j^*$. ■

This proves the correctness of the algorithm: either a half-integral solution x^* of $Ax \leq b$ can be rounded into an integral solution or there exists a $j \in \{1, \ldots, n\}$ satisfying (3).

Theorem 1 *If $Ax \leq b$ satisfying (1) has an integral solution then every half-integral solution of $Ax \leq b$ can be rounded to an integral solution. Moreover, an integral solution can be found with the same complexity as the single source problem.* ■

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