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Odd Paths and Odd Circuits in Planar Graphs with Two Odd Faces

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Abstract: We prove the following result: let G be a planar graph with at most two odd faces and let S be the collection of nodes in G with degree 1: if all the nodes not in S have even degree, then the maximum number pairwise disjoint odd circuits and odd S-paths is equal to the minimum size of a collection of edges meeting each odd circuit and each odd S-path.

The result implies that if a graph G contains a node v such that if we delete v we obtain a planar graph with at most two odd faces, then G is weakly bipartite; that is: the odd circuits in G have the weak max-flow min-cut property.

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Key Words and Phrases: graph, planar, odd paths, odd circuits.

1 Introduction

We prove the following result:

Theorem: *Let G be graph embedded in the plane such that at most two of its faces are bounded by odd circuits. Let $S \subseteq V(G)$ be the set of degree-1 nodes in G and assume that nodes not in S have even degree. Then the maximum number of pairwise edge-disjoint odd circuits and odd S-paths is equal to the minimum size of a set of edges meeting each odd circuit and each odd S-path.*

Here an S-path is a path with both its endpoints in S. The theorem has the following corollary:

Corollary: *Let G be a graph, and $v \in V(G)$. If $G \setminus v$ is planar with at most two odd faces, then G is weakly bipartite.*

A graph G is *weakly bipartite* — this name is from Grötschel and Pulleyblank [1981] — if the collection $\mathcal{C}$ of edge sets of odd circuits in G has the following *weak max-flow min-cut property*: the polyhedron $\{x \in \mathbb{R}^E | x_e \geq 0 (e \in E); \sum_{e \in C} x_e \geq 1 (C \in \mathcal{C})\}$ has integral vertices only.

The proof of the theorem is in Section 2. The proof of the corollary as well as a short overview of related results is in Section 3.

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2 Proof of the theorem

We prove the theorem for 'signed' graphs; not really an extension, but providing just a little bit more freedom.

A *signed graph* is a pair (G,σ) where G is an undirected graph and σ a function from $E(G)$ to $GF(2)$. We set $X_\sigma := \{e \in E(G) | \sigma(e) = 1\}$. A collection F of edges, like a single edge, a path or a circuit, is called *odd* (*even*) if $\sigma(F) := \sum_{e \in F} \sigma(e) = 1$ ($= 0$ respectively). (G,σ) is called *bipartite* if there exists a partition of $V(G)$ into two sets U_1 and U_2, called *the bipartition of* σ, such that $\delta(U_1) = X_\sigma$. Obviously, (G,σ) is bipartite if and only if G contains no odd circuits. If $F \subseteq E(G)$, then χ_F denotes the characteristic vector of F as a subset of $E(G)$. We consider it also as a function from $E(G)$ to $GF(2)$.

Let $\mathcal{C}(G,\sigma)$ the collection of the edge sets of the odd circuits and of the odd S-paths. Moreover we define:

$$\begin{aligned}
&\mathcal{B}(G,\sigma) := \{F \subseteq E(G) | K \in \mathcal{C}(G,\sigma) \Longrightarrow K \cap F \neq \emptyset\};\\
&\tau(G,\sigma) := \min\{\,|F|\,|F \in \mathcal{B}(G,\sigma)\};\\
&\mathcal{B}_\tau(G,\sigma) := \{F \in \mathcal{B}(G,\sigma)|\,|F| = \tau(G,\sigma)\};\\
&\mathcal{B}_{min}(G,\sigma) := \{F \in \mathcal{B}(G,\sigma) | F \text{ is inclusion-wise minimal in } \mathcal{B}(G,\sigma)\};\\
&\mathcal{B}_{odd}(G,\sigma) := \{F \subseteq E(G) | \mathcal{C}(G,\chi_F) = \mathcal{C}(G,\sigma)\}.
\end{aligned}$$

The properties (1), (2), (3), (4), (5), and (6) below, are essentially a consequence of the fact that $\mathcal{C}(G,\sigma)$ is a *binary clutter*: no member of $\mathcal{C}(G,\sigma)$ is contained in another one; and the symmetric difference of any three members of $\mathcal{C}(G,\sigma)$ contains a member of $\mathcal{C}(G,\sigma)$. We present their proofs for sake of completeness.

(1) $\quad F_1 \in \mathcal{B}_{odd}(G,\sigma) \Longleftrightarrow F_1 = \delta(U) \triangle X_\sigma$ *for some* $U \subseteq V(G) \setminus S$.

($\triangle$ denotes 'symmetrical difference'.) (1) follows from the following series of mutually equivalent assertions: $\mathcal{C}(G,\chi_F) = \mathcal{C}(G,\sigma)$; $\mathcal{C}(G,\chi_F + \sigma) = \emptyset$; $(G,\chi_{(F \triangle X_\sigma)})$ is bipartite and contains no odd S-path; $F_1 \triangle X_\sigma = \delta(U)$ for some $U \subseteq V(G) \setminus S$.

A direct consequence of (1) is (as, always, $\delta(U_1) \triangle \delta(U_2) = \delta(U_1 \triangle U_2)$):

(2) $\quad F_1, F_2 \in \mathcal{B}_{odd}(G,\sigma) \Longleftrightarrow F_1 \triangle F_2 = \delta(U)$ *for some* $U \subseteq V(G) \setminus S$.

Combining this with the fact that the nodes in a set U as in (1) have even degree, we get:

(3) $\quad F_1, F_2 \in \mathcal{B}_{odd}(G,\sigma) \Longrightarrow |F_1| \equiv |F_2| \pmod 2$.

Of the following sequence of inclusions only the middle one is not completely trivial.

(4) $\quad \mathcal{B}_\tau(G,\sigma) \subseteq \mathcal{B}_{min}(G,\sigma) \subseteq \mathcal{B}_{odd}(G,\sigma) \subseteq \mathcal{B}(G,\sigma)$.

To prove the second inclusion: assume it is wrong. Let $F \in \mathcal{B}_{min}(G,\sigma)$ and $K \in \mathcal{C}(G,\sigma) \triangle \mathcal{C}(G,\chi_F) = \mathcal{C}(G,\sigma + \chi_F)$. For each $e \in F \cap K$, let $K_e \in \mathcal{C}(G,\sigma)$ such that $F \cap K_e = \{e\}$ (K_e exists as $F \in \mathcal{B}_{min}(G,\sigma)$). Let $\widetilde{K}$ be the symmetrical difference of K and all the sets K_e with $e \in F \cap K$. Then $\widetilde{K}$ is the disjoint union of a collection $K_1, \ldots, K_\ell$ of circuits and S-paths (nodes not in S meet an even number of edges in $\widetilde{K}$). For each $i = 1, \ldots, \ell$: $K_1 \cap F \subseteq \widetilde{K} \cap F = \emptyset$; hence $\sigma(K_i) = 0$. Now the following equalities contain a contradiction: $0 = \sum_{i=1}^{\ell} \sigma(K_i) = \sigma(\widetilde{K}) = \sigma(K) + \sum_{e \in K \cap F} \sigma(K_e) = \sigma(K) + \sum_{e \in K \cap F} 1 = \sigma(K) + \chi_F(K) = (\sigma + \chi_F)(K) = 1$. So (4) must be true.

(1) and (4) immediately imply:

(5) *$F \in \mathcal{B}_\tau(G,\sigma) \iff$ for each $U \subseteq V(G) \setminus S$: $|\delta(U) \setminus F| \geq |\delta(U) \cap F|$; with equality if and only if $F \triangle \delta(U) \in \mathcal{B}_\tau(G,\sigma)$.*

Hence:

(6) $F_1, F_2 \in \mathcal{B}_\tau(G,\sigma), U \subseteq V(G) \setminus S, \delta(U) \subseteq F_1 \cup F_2 \implies F_1 \triangle \delta(U), F_2 \triangle \delta(U) \in \mathcal{B}_\tau(G,\sigma)$.

Before we start with the actual proof of the theorem, we make a small observation:

(7) *It suffices to prove the theorem for graphs with maximum degree equal to 4.*

Indeed, any node u with degree more than 4 (like in Figure 1a) can be replaced by a configuration as in Figure 1b. It is easy to see that this — standard — construction does not change the value of $\tau(G,\sigma)$ and that any collection of k edge disjoint odd circuits and odd S-paths in the new graph yields such a collection of the same size in G. Hence, from now on, we only consider graphs with maximum degree 4. We prove the theorem by contradiction.

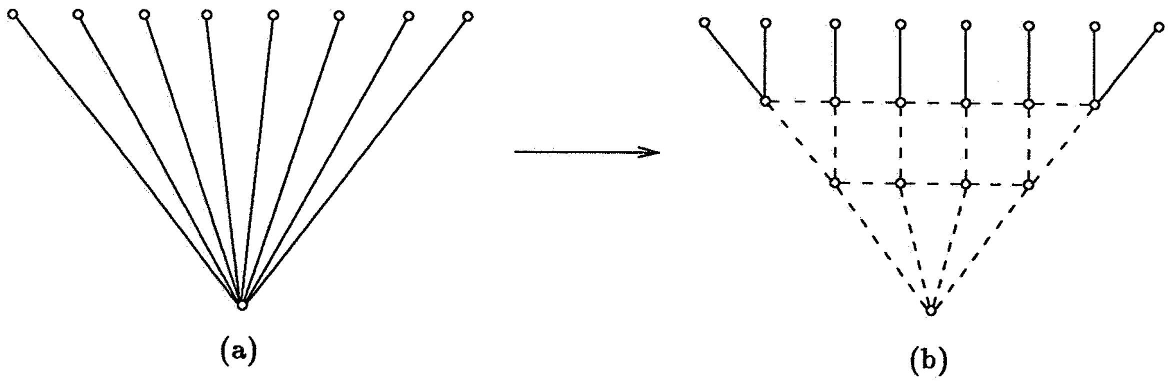

Figure 1: non-dashed edges in (b) correspond to the edges in (a), with same value in σ; the new, dashed, edges get their σ-value equal to 0.

(8) *Assume (G,σ) is a counterexample to the theorem, with $|E(G)|$ plus the number of degree-4 nodes as small as possible.*

Let G^* be the planar dual of G, and let α and β be faces of G, i.e. nodes of G^*, such that all other faces of G are bounded by even circuits.

Claim 1 *G is connected and has no nodes of degree 2, (G,σ) is not bipartite, and $S \neq \emptyset$.*
Proof of Claim 1: By (8), connectivity of G is obvious. And so is the absence of degree 2 nodes. [If u is only adjacent to v and w, replace vu and uw by vw, with $\sigma_{vw} := \sigma_{vu} + \sigma_{uw}$.] If (G,σ) is bipartite with bipartition U_1, U_2, then there are no odd circuits and the odd paths with endpoints in S are exactly the paths from $S \cap U_1$ to $S \cap U_2$. So Menger's Edge Disjoint Path Theorem, or equivalently Ford and Fulkerson's Max-Flow Min-Cut Theorem, shows that (G,σ) cannot be a counterexample to the theorem. If $S = \emptyset$, then the odd circuits are exactly the $\alpha\beta$-cuts in G^*. So again (G,σ) cannot be a counterexample, as in any graph — so also in G^* — the maximum number of pairwise disjoint $\alpha\beta$-cuts equals the length of the shortest $\alpha\beta$-path. *End of Proof of Claim 1.*

Claim 2 *Let $F \in \mathcal{B}_\tau(G,\sigma)$ and let W_1, W_2 partition $V(G)$ such that $\delta(W_1) \subseteq F$, then W_1 or W_2 is contained in S.*
Proof of Claim 2: Let F, W_1, and W_2 contradict the claim. By (4), we may assume that $\sigma = \chi_F$. Construct a new graph G^+ as follows: replace each edge u_1u_2 in $\delta(W_1)$ by two edges u_1u, uu_2 in series. Let S^+ be the set of all the nodes u thus added to the graph. For $i = 1, 2$, let G_i be the subgraph of G^+ induced by $W_i \cup S^+$. The set of degree-1 nodes in G_i is $S_i := (S \cap W_i) \cup S^+$.

Applying the '$\Leftarrow$–direction' of (5) to G_i and then the '$\Rightarrow$–direction' of (5) to G, we get that for $i = 1, 2$ that $F_i := (F \cap E(G_i)) \cup \delta(S^+) \in \mathcal{B}_\tau(G_i, \chi_{F_i})$. As neither W_1 nor W_2 are contained in S, both G_1 and G_2 have fewer edges than G. Hence, for $i = 1, 2$, we have a collection $\mathcal{C}_i$ of $\tau(G_i, \chi_{F_i})$ pairwise disjoint odd circuits and odd S_i-paths. Among these paths, there is for $i = 1, 2$ and for each $u \in S^+$ exactly 1 path, called P_i^u, with endpoint in u; the other endpoint of P_i^u lies in $S_i \setminus S_+$ ('complementary slackness'). For each $u \in S^+$, glue path P_1^u to path P_2^u. By taking these glued paths together with all other members of the collections $\mathcal{C}_1$ and $\mathcal{C}_2$, we get a collection of disjoint odd circuits and odd S-paths in (G, σ). The cardinality of that collection is $\tau(G_1, \chi_{F_1}) + \tau(G_2, \chi_{F_2}) - |S^+| = \tau(G, \sigma)$. This is a contradiction with our assumption that (G, σ) is a counterexample to the theorem.
End of Proof of Claim 2.

Let $v^* \in S$ such that its neighbour, u^*, has degree 4. Let u_1, u_2, and u_3 be the neighbours of u^* other than v^*.

Claim 3 *For each $i = 1, 2, 3$ there exists an $F_i \in \mathcal{B}_\tau(G, \sigma)$ containing both v^*u^* and u^*u_i.*
Proof of Claim 3: We prove the claim for $i = 1$. Construct a new signed graph (G^+, σ^+) as follows: split u^* into two degree-2 nodes u^+ and u^-; u^+ is adjacent to v^* and to u_1, whereas u^- is adjacent to u_2 and to u_3; σ^+ is the same as σ on the understanding that $\sigma^+(v^*u^+) := \sigma(v^*u^*)$, $\sigma^+(u^+u_1) := \sigma(u^*u_1)$, $\sigma^+(u^-u_2) := \sigma(u^*u_2)$, and $\sigma^+(u^-u_3) := \sigma(u^*u_3)$. Since G^+ has fewer degree-4 nodes than G has, G^+ satisfies the theorem. Each collection of odd circuits and odd S-paths in G^+ yields such a collection in G with the same cardinality. Hence $\tau(G^+, \sigma^+) < \tau(G, \sigma)$. A consequence of this is:

(9) *For each $F \in \mathcal{B}_\tau(G^+, \sigma^+)$, the u^+u^--paths in $(G^+, \sigma^+ + \chi_F)$ are odd.*

[If not, all u^+u^--paths in $(G^+, \sigma^+ + \chi_F)$ are even ($(G^+, \sigma^+ + \chi_F)$ is bipartite). So F — or better: its counterpart in G — meets every odd circuit and odd S-path in (G, σ). This contradicts $\tau(G^+, \sigma^+) < \tau(G, \sigma)$.]

Let $F \in \mathcal{B}_\tau(G^+, \sigma^+)$. By (9), it contains none of the four edges adjacent to u^+ or u^-. [If it would, one could, by application of (1) to $U = \{u^+\}$ or $\{u^-\}$, easily construct a counterexample to (9).] From (9) and because $F \in \mathcal{B}_{odd}(G, \sigma)$, it follows that $F_1 := F \cup \{v^*u^*, u^*u_1\} \in \mathcal{B}_{odd}(G, \sigma)$. Next, combining (3) and (4) with $|F_1| \leq \tau(G, \sigma) + 1$, we see that $F_1 \in \mathcal{B}_\tau(G, \sigma)$. *End of Proof of Claim 3.*

Claim 4 *Nodes in S have no common neighbour.*
Proof of Claim 4: In the notation of Claim 3: let, besides v^*, also $u_1 \in S$. Let F_1 be as in Claim 3. By (5), $u^*u_2, u^*u_3 \notin F_1$. Hence $F' := F_1 \triangle \delta(u^*) \in \mathcal{B}_\tau(G, \sigma)$ and $u^*u_2, u^*u_3 \in F'$, which violates Claim 2. *End of Proof of Claim 4.*

A *dual-path* is a collection of edges in G that forms in G^* an $\alpha\beta$-path. If P is a dual path and $v \in S$, then $v(P)^+ := S \cap U_1$ and $v(P)^- := S \cap U_2$, where U_1, U_2 is the bipartition of $(G, \sigma + \chi_P)$ with $v \in U_1$. Note that if P is a dual path, then $P \cup \delta(v(P)^+) \in \mathcal{B}_{odd}(G, \sigma)$.

Claim 5 *Each member of $\mathcal{B}_\tau(G,\sigma)$ is a dual path or of the form $P \cup \delta(v(P)^+)$, where P is a dual path and $v \in S$.*
Proof of Claim 5: Let $F \in \mathcal{B}_\tau(G,\sigma)$. Let P be a dual path contained in F. (P should exist as F meets every odd circuit.) If all S-paths in $G \setminus P$ are even, then $F = P$. If not, there exists a set $U \subseteq V(G)$ such that $U \cap S = v(P)^+$ (for some $v \in V(G)$) and $\delta(U) = F \setminus P$. From this and Claim 2 the claim easily follows. *End of Proof of Claim 5.*

Claim 6 *For each $i = 1, 2, 3$, there exists a dual path P_i containing edge u^*u_i, such that $P_i \cup \delta(v^*(P)^+) \in \mathcal{B}_\tau(G,\sigma)$.*
Proof of Claim 6: This is an immediate consequence of the Claims 3, 4, and 5.
End of Proof of Claim 6.

Let γ be the face of G meeting u^*, which has u^*u_1 and u^*u_2 on its boundary. We may assume (if necessary by renumbering the nodes u_1, u_2 and u_3) that the dual paths P_1 and P_2 meant in Claim 5 have the property that, going from α to β, P_1 enters γ through u^*u_1, whereas P_2 leaves γ through u^*u_2 (cf. Figure 2). Let H^* be the subgraph of G^* formed by the union of

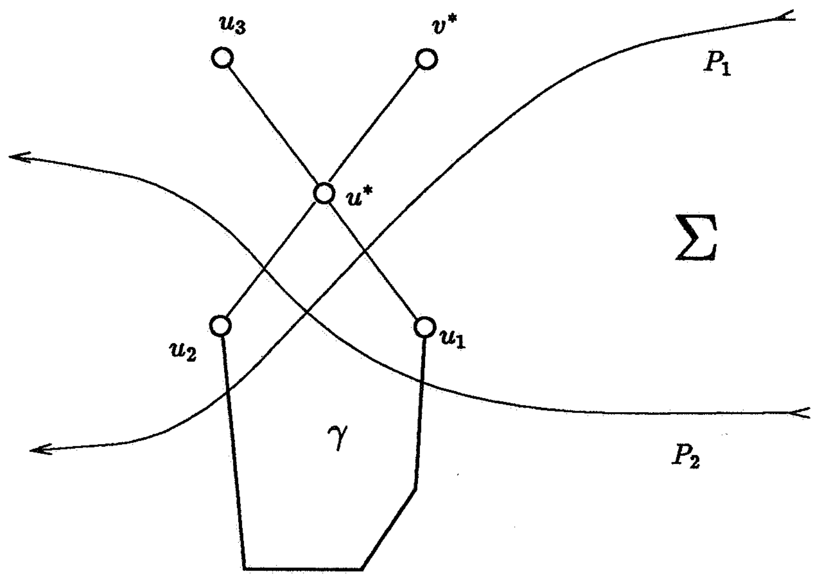

Figure 2:

P_1 and P_2 (now considered as paths in G^*, in which they are ordinary paths). Let Σ be the face of H^* containing u_1.

Claim 7 *u_2 lies in Σ.*

Proof of Claim 7: The path $v^*u^*u_1$ in G does not meet P_2 (cf. (5)). Hence it meets P_2 an even number of times and P_1 an odd number of times (namely once). So $S \cap \Sigma$ is partioned into $v^*(P_1)^+ \cap \Sigma$ and $v^*(P_2)^+ \cap \Sigma$. This implies that $U := (V(G) \cap \Sigma) \setminus S$ and $F_i := P_i \cup \delta(v^*(P_i)^+)$ $(i = 1, 2)$ satisfy the condition of (6). Hence $F_2' := F_2 \triangle \delta(U) \in \mathcal{B}_T(G, \sigma)$. Now, combining $\{v^*u^*, u^*u_1\} \subseteq F_2'$ with (5) yields $u^*u_2 \notin F_2'$; so, because $u^*u_2 \in F_2$, we have: $u^*u_2 \in \delta(U)$. As, clearly, $u^* \notin \Sigma$, this proves the claim. *End of Proof of Claim 7.*

Let K be a curve from u_1 to u_2 contained in Σ. It closes with the edges u_2u^*, u^*u_1 a closed curve $\widetilde{K}$. Assume that α lies in the inner region determined by $\widetilde{K}$. P_1 meets $\widetilde{K}$ exactly once; so β lies in the outer region. According to the definition of Σ, P_2 enters the inner region going from α to β. As also P_2 meets $\widetilde{K}$ exactly once, this is absurd (cf. Figure 3). So the theorem follows. □

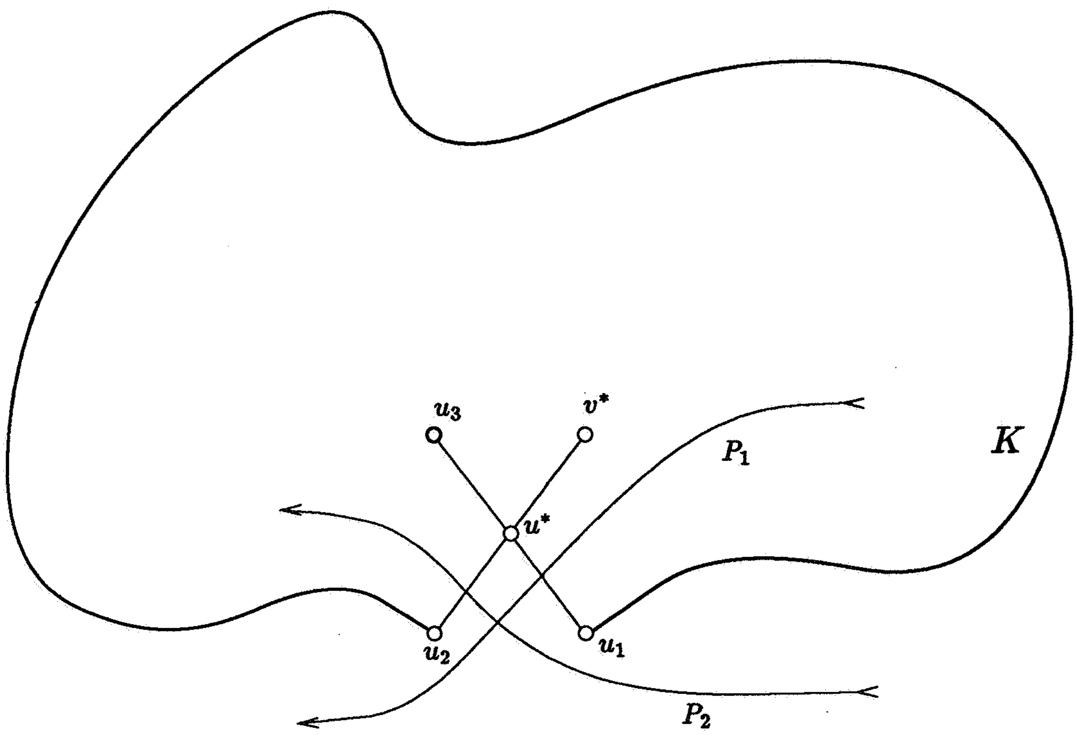

Figure 3:

3 Odd circuits with the weak max-flow min cut property

Next we prove the corollary.

Let G be an undirected graph and let $s \in V(G)$ such that deleting s from G yields a planar graph with at most two odd faces. To prove that $\{x \in \mathbb{R}^E | x_e \geq 0 (e \in E); \sum_{e \in C} x_e \geq 1 (C \in \mathcal{C})\}$ is an integral polyhedron, we have to prove that for each $w \in \mathbb{Z}^E$ the minimum weight $\sum_{e \in F} w_e$ of a set F meeting each odd circuit is equal to the maximum of $\sum_{C \in \mathcal{C}} \lambda_C$ such that: $\lambda_C \geq 0$ when $C \in \mathcal{C}$, $\sum_{C \in \mathcal{C}, C \ni e} \lambda_C \leq w_e$ when $e \in E$. Clearly, it suffices to prove the min-max relation for the case that w_e is even for each $e \in E$; in fact, it suffices to restrict ourselves to the case that G has even degree nodes only and $w_e = 1, e \in E$ — because the class of graphs under consideration is closed under adding edges parallel to already existing ones. However in that case the min-max relation which has to be proved is exactly the content of the theorem, when applied to the graph $\tilde{G}$ constructed as follows: replace node s by k new nodes $s_1, \ldots, s_k$, where k is the degree of s and replace the edges $su_1, \ldots, su_k$ by the new edges $s_1u_1, \ldots, s_ku_k$. $\tilde{G}$ is planar with at most two odd faces. □

Other classes of weakly bipartite graphs are:

Graphs embeddable on the projective plane (Lins [1981]) or on the Klein bottle (Schrijver [1989]) such that the odd circuits are exactly the circuits in G that are orientation reversing on that surface.

Extensions of these classes are: graphs embeddable on the projective plane such that at most two faces are bounded by odd circuits (Gerards [1992b]); graphs embeddable on the Klein bottle with all faces bounded by even circuits (Gerards [1992a]); graphs obtained from a member of Lins' class by identifying two of its nodes (Gerards and Schrijver [1992]). All these are classes of weakly bipartite graphs.

Graphs containing two nodes such that each odd circuit contains at least one of them (Barahona [1983]; Hu [1963]).

A common generalization of this result and of the theorem proved in the present article is that G is weakly bipartite if it contains a node v such that $G \setminus v$ contains no *odd*-K_4; that is a subdivision of K_4 in which all triangles of K_4 have become odd circuits (Gerards [1992c]).

Planar graphs or, more generally, graphs not contractable to K_5 (Seymour [1981ab], cf. Barahona [1983]).

In his seminal paper on binary clutters with the max-flow min-cut property Seymour [1977] has proved that G contains no odd-K_4 if and only if the system $x_e \geq 0 (e \in E), \sum_{e \in C} x_e \geq 1 (C \in \mathcal{C}(G, \sigma))$ is *totaly dual integral*; that means that when optimizing an integer objective function over it the corresponding dual linear programming problem admits an integer optimal solution. In fact the result in that paper is more general, but this is what it means for the clutter of odd circuits. In that same paper Seymour has made a conjecture which — when true — would imply a characterization for the class of all weakly bipartite graphs. It is easiest explained in terms of signed graphs:

Conjecture (Seymour [1977]): *A signed graph (G, σ) is* weakly bipartite, *that is $\{x \in \mathbb{R}^E | x_e \geq 0 (e \in E); \sum_{e \in C} x_e \geq 1 (C \in \mathcal{C}(G, \sigma))\}$ is integral, if and only if (G, σ) cannot* *be reduced to* $(K_5, \mathbf{1})$ *by a series of the following operations: deletion of edges, contraction of even edges, and* re-signing*: i.e. replacing* σ *by* $\sigma + \chi_{\delta(U)}$ *for some set* $U \in V(G)$.

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