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# Stochastic Scheduling Games with Markov Decision Arrival Processes

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## Abstract

The dependent arrival processes MDAP were introduced by Hordijk & Koole [1,2] to model special arrival processes into controlled queueing networks. These were applied to solve control problems with several controllers having a common objective, where the output from one controlled network is fed into a second one. In case that objectives of the controllers are different, one may choose a min-max (worst case) approach where typically a controller tries to obtain the best performance under the worst possible (unknown) strategies of the other controllers. We use MDAP to model such situations or situations of control in an unknown environment. We apply this approach to several scheduling problems, including scheduling of customers and scheduling of servers. We consider different information patterns including delayed information. For all these models we obtain several structural results of the optimal policies.

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## 1. INTRODUCTION

Recently Hordijk & Koole [1,2] introduced the *Markov Decision Arrival Process*, a Markovian arrival process by which not only independent arrivals can be modeled, but also arrival processes which depend in a certain way on the system into which the customers arrive. A typical example is the following tandem model. Customers arrive according to a Poisson process at  $m$  parallel  $M|M|1$  queues. Dynamically the customers have to be assigned to one of the queues. After being served the customers arrive at a second station, where we have again  $m$  parallel queues to choose from. The question is how to assign the customers at the second center, when all service parameters are equal. Here we cannot use the standard results on the optimality of shortest queue routing for independent arrivals, as the actions in the first center are allowed to depend on the state of the whole system. Now the arrivals from the first center can be modeled as an MDAP, and for this type of arrival process it was shown in Hordijk & Koole [1] that shortest queue routing at the second center is still optimal, for various objective functions. This is done using dynamic programming, where the action consists of two components, viz. the action in the MDAP and the assignment to one of the parallel queues (of the second center). This can also be seen as two controllers, one at the MDAP and one at the queues, who are cooperating.

In this paper we consider the situation where the controllers do not cooperate, and might have different objectives. If the second controller does not know the objective of the controller of the MDAP, then he may still try to use a min-max approach, i.e. to design a control strategy that guarantees the best performance under the worst possible strategy of the MDAP controller. This naturally leads to a zero-sum stochastic game, where the MDAP controller plays against the second controller.

Another possible motivation for this model is when there is only one controller of a queueing system; the arrival process to that system is characterized by some parameters that may change in time in a way unpredictable by the controller. The controller may wish to design a control strategy that guarantees the best performance under the worst possible (time dependent) parameters of the arrival process. Here again, we end up with a zero-sum game between the MDAP (player 1), that models the arrival process, and the controller (player 2).

The use of the MDAP in the setting of a zero-sum game allows to obtain structural results for the optimal strategy of player 2. We illustrate this by two examples where the optimal min-max strategies of player 2 are in fact explicitly obtained. In the next section we will show the optimality result for the asymmetric model of [1] for scheduling of customers, of which the optimality of shortest queue routing (known as SQP) is a special case. We use a model known as a “stochastic game with switching controller”, for which deterministic policies exist for both players.

In Section 3 we extend this result to the case of control with delayed information. More specifically, we first consider the problem where the state of the MDAP reaches the controller (player 2) after some delay. This results in a model known as a “stochastic game with complete information”, for which deterministic policies exist for both players. We then study the case where the information on both the state and the action of the MDAP is delayed. This results in a standard stochastic game for which the optimal policy for both players may need randomization.

In section 4 we consider the model of [2] where one or more servers are to be assigned to customers of different classes. Here the results for player two are different than in the case of cooperation between the players, but in the same spirit.

## 2. RESULTS ON THE CUSTOMER ASSIGNMENT MODEL

We start by formulating the model of the stochastic game for the uniformized model.

We consider a state space given by a product of two spaces: the state of the MDAP  $\mathbf{X}$ , assumed to be finite, times the state of the queueing system  $\mathbf{L} = \prod_{j=1}^M \mathbf{L}_j$  where  $\mathbf{L}_j = \{0, 1, \dots, L_j\}$ , and  $L_j > 0$  is the size of queue  $j$  (which may be either finite or infinite). Let  $L = (L_1, \dots, L_j)$  be the vector of queue lengths. A typical element of the state space is denoted by  $(x, i)$ , with  $x \in \mathbf{X}$  and  $i = (i_1, \dots, i_m) \in \mathbf{L}$  the number of customers in the  $m$  queues *including* the ones in service. Let  $e_i$  denote a  $m$  dimensional vector with all entries zero except for the  $i$ th entry, which is one.

The probability of a successful service in queue  $j$  is  $\mu_j$ . Without loss of generality we assume that  $\mu_1 \geq \dots \geq \mu_m$ .

The finite space of actions of the MDAP (player 1) is  $\mathbf{A}$  (different actions may be

available in different states).  $\lambda_{xay}$  is the probability that the MDAP moves from  $x$  to  $y$  if action  $a$  was chosen, and  $q_{xay}$  is the probability that a customer arrives if the arrival process moves from  $x$  to  $y$  using action  $a$ . We assume without loss of generality that for any  $(x, i)$ ,  $\sum_y \lambda_{xay} + \sum_{j=1}^m \mu_j = 1$ .

The finite space of actions available to the second player (controller) is  $\mathbf{B} = \{1, \dots, m\}$ . An action  $b \in \mathbf{B}$  has the meaning of assigning a customer to queue  $b$ . However, we assume that if a queue is full then a customer cannot be assigned to it. If all queues are full then the customer is lost. We assume in this section that player 2 takes an action immediately after an arrival occurs, (hence after a transition in the MDAP occurs) already knowing the new state of the MDAP.

A precise description of the decision process (transition probabilities, and the state space) is given in [3] Section 5.1. (The state should in fact include a mark of whether an arrival just occurred. Only at such states can player 2 take actions, whereas player 1 can take actions in the remaining states. The resulting game is then seen to be in fact a “switching controller” game. It is known that for these games there exist optimal policies which do not require randomizations). In our model it will however suffice to construct the dynamic programming equation.

Let  $U$  be the set of policies for player 1 and  $W$  the set of policies for player 2. Let  $(X(s), I(s)), A(s), B(s), s = 0, \dots, n$  denote the state and action processes.

The cost for a horizon of length  $n$ , for an initial state  $(x, i)$  and policies  $u, w$  is denoted by  $v^n(x, i, u, w)$ . We assume that there is a terminal cost  $v^0(x, i) = v^0(x, i, u, w)$  that satisfies the following equations (where  $n=0$ ):

$$v_{(x, i+e_{j_1})}^n \leq v_{(x, i+e_{j_2})}^n \text{ if } i_{j_1} \leq i_{j_2}, j_1 \leq j_2, i + e_{j_1} + e_{j_2} \leq L \quad (1)$$

$$v_{(x, i)}^n \leq v_{(x, i+e_j)}^n \text{ if } i + e_j \leq L \quad (2)$$

$$v_{(x, i)}^n \leq v_{(x, i^*)}^n \text{ if } i_j^* = \begin{cases} i_j & \text{if } j \neq j_1, j_2, \\ i_{j_2} & \text{if } j = j_1, \\ i_{j_1} & \text{if } j = j_2, \end{cases} \quad \begin{cases} \max(i_{j_1}, i_{j_2}) \leq \max(L_{j_1}, L_{j_2}), \\ i_{j_1} > i_{j_2} \text{ and } j_1 \leq j_2. \end{cases} \quad (3)$$

In particular, we may choose  $v^0(x, i) = v^0(x, i, u, w) = \sum_{j=1}^m i_j$ .  $v^n(x, i, u, w)$  is given by  $v^n(x, i, u, w) = E_{(x, i)}^{u, w} v^0(X(n), I(n))$ .

The objective of the controller (player 2) is (P0): find a policy  $w^*$  that achieves

$$\sup_{u \in U} v^n(x, i, u, w) \geq \sup_{u \in U} v^n(x, i, u, w^*) =: v_{(x,i)}^n, \quad \forall w \in W.$$

It is well known that  $v_{(x,i)}^n = \sup_{u \in U} \inf_{w \in W} v^n(x, i, u, w)$ , and moreover, there is a pair of policies  $(u^*, w^*)$  for the two players such that

$$\inf_{w \in W} v^n(x, i, u^*, w) = v^n(x, i, u^*, w^*) = v_{(x,i)}^n.$$

A policy for player 2 is called Shorter Queue Faster Server Policy (SQFSP) if it satisfies the following property: if  $i_{j_1} < i_{j_2}$  and  $j_1 < j_2$  (and  $i + e_{j_1} + e_{j_2} \leq L$ ) then an arriving customer will not be assigned to queue  $j_2$ . In particular, if the fastest queue has the smallest number of customers then  $w^*$  sends an arriving customer to it. This gives the optimality of the SQP in the symmetric case.

**Theorem 1:** The optimal policy  $w^*$  for player 2 is SQFSP.

**Proof:** We formulate the dynamic programming equation. If the system is not full then

$$v_{(x,i)}^{n+1} = \max_a \left\{ \sum_y \lambda_{xay} \left( q_{xay} \min_b \{v_{(y,i+e_b)}^n\} + (1 - q_{xay}) v_{(y,i)}^n \right) \right\} + \sum_{j=1}^m \mu_j v_{(x,(i-e_j)+}^n \quad (4)$$

When the system is full then we have for all three problems:

$$v_{(x,i)}^{n+1} = \max_a \left\{ \sum_y \lambda_{xay} v_{(y,i)}^n \right\} + \sum_{b=1}^m \mu_b v_{(x,(i-e_b))}^n.$$

We show by induction on  $n$  that  $v^n$  satisfies (1), (2) and (3). The optimality of SQFSP follows from (1) and is seen to be independent of the action in the MDAP.

The terms in (4) corresponding to the departures,  $\sum_{j=1}^m \mu_j v_{(x,(i-e_j)+}^n$  satisfy the properties following the same arguments as in the proof of theorem 3.1 [1]. We show how the proof for the other term, corresponding to the arrivals, deviates from the the proof of theorem 3.1 [1]. We begin with (1). Introduce the following notation:

$$f(i, a) = \sum_y \lambda_{xay} \left( q_{xay} \min_j \{v_{(y,i+e_j)}^n\} + (1 - q_{xay}) v_{(y,i)}^n \right)$$

It is shown in [1] that  $f(i + e_{j_1}, a) \leq f(i + e_{j_2}, a)$  for all  $a$  and suitable  $j_1$  and  $j_2$ . If  $a^*$  is the minimizing action for the MDAP in state  $(x, i + e_{j_2})$  (note that in [1] both the MDAP and the controller minimize) then the proof in [1] proceeds as follows:

$$\min_a f(i + e_{j_1}, a) \leq f(i + e_{j_1}, a^*) \leq f(i + e_{j_2}, a^*) = \min_a f(i + e_{j_2}, a)$$

This can easily be adapted to the current setting. Let  $a^*$  be maximizing action of the MDAP (player 1) in  $(x, i + e_{j_1})$ . Then

$$\max_a f(i + e_{j_1}, a) = f(i + e_{j_1}, a^*) \leq f(i + e_{j_2}, a^*) \leq \max_a f(i + e_{j_2}, a),$$

which establishes (1). The remaining proofs of (2) and (3) are similar. ■

### 3. CUSTOMER ASSIGNMENT MODEL WITH DELAYED INFORMATION

We consider the same model as in the previous section with one exception. Player 2 takes an action immediately after an arrival occurs; however, due to information delay, it does not have the knowledge of the new state of the MDAP. As a result, we may consider this action to have been taken already prior to the arrival (since no new information is obtained by player 2 in the arrival epoch).

We shall thus assume that the decision instants for the players are the same; each time a transition occurs (departure, or a transition in the MDAP), both players take a decision. The decision of player 2 should be interpreted however as the action to be taken when there will be a future arrival.

We further consider two versions of that game, depending on whether or not the information on the action of player 1 is delayed too.

(P1) When a customer arrives then player 2 already has the information on the last action of player 1. Hence, at each decision epoch, player 1 takes a decision first and only then player 2 takes a decision, knowing the decision of player 1.

(P2) When a customer arrives then player 2 does not yet have the information on the last action of player 1. Hence, at each decision epoch, the players take their actions independently.



To summarize, the information available to each player at a given decision epoch consists of all previous states and actions of both players, as well as the current state of the system. Moreover, in (P1), at any time  $n$ , player 2 has the information on the decision of player 1 at time  $n$ . Problem (P1) is known as a stochastic game with full information. It is known that for these games there exist optimal policies which do not require randomizations (for both players), whereas in (P2) randomized policies are usually needed to obtain optimality. Since the action of player 2 is interpreted as the decision to be taken when a future arrival occurs, the knowledge of the current state indeed grasps the fact that information is delayed, and thus when that arrival will occur and the MDAP will change its state, the new state will not be available to player 2.

Since the amount of information that player 2 possesses in (P1) when making a decision is less than in (P2), and that is less of the information he has in (P0) (of the previous section), the value  $v^n$  will satisfy

$$v_{(P0)}^n \leq v_{(P1)}^n \leq v_{(P2)}^n.$$

The transition probabilities (for all three scenarios) are given by:

$$\mathcal{P}_{(x,i),a,b,(y,k)} = \begin{cases} \lambda_{xay}q_{xay} & \text{if } k = i + e_b, \\ \lambda_{xay}(1 - q_{xay}) + \sum_{j=1}^m \mu_j 1_{\{i_j = 0, y = x\}} & \text{if } k = i, \\ \mu_j & \text{if } y = x, k = i - e_j, i_j > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(We assume that the rates are already normalised so that for any  $(x, i)$  we have  $\sum_{a,b} \mathcal{P}_{(x,i),a,b,(y,k)} = 1$ .) If all queues are full then

$$\mathcal{P}_{(x,i),a,b,(y,k)} = \begin{cases} \lambda_{xay} & \text{if } k = i, \\ \mu_j & \text{if } y = x, k = i - e_j, \\ 0 & \text{otherwise.} \end{cases}$$

For a set  $\Pi$ , let  $M(\Pi)$  be the set of probability measure over  $\Pi$ . For a function (matrix)  $f : \mathbf{A} \times \mathbf{B}$  and  $\alpha \in M(\mathbf{A}), \beta \in M(\mathbf{B})$ , we denote

$$f(\alpha, \beta) := \int_{\mathbf{A}} f(a, \beta) \alpha(da)$$

$$f(a, \beta) := \int_{\mathbf{B}} f(a, b) \beta(db)$$

$$f(\alpha, \beta) := \int_{\mathbf{B}} \int_{\mathbf{A}} f(a, b) \alpha(da) \beta(db)$$

Let  $\text{val} f$  denote the value of the matrix game  $f$ , which is given by

$$\text{val} f = \sup_{\alpha \in M(\mathbf{A})} \inf_{\beta \in M(\mathbf{B})} f(\alpha, \beta).$$

$\text{val} f$  is known to satisfy

$$\text{val} f = \inf_{\beta \in M(\mathbf{B})} \sup_{\alpha \in M(\mathbf{A})} f(\alpha, \beta).$$

Moreover, there exists a pair  $(\alpha^*, \beta^*)$ ,  $\alpha^* \in M(\mathbf{A})$ ,  $\beta^* \in M(\mathbf{B})$ , such that

$$\text{val} f = \inf_{\beta \in M(\mathbf{B})} f(\alpha^*, \beta) = \sup_{\alpha \in M(\mathbf{A})} f(\alpha, \beta^*) = f(\alpha^*, \beta^*).$$

$(\alpha^*, \beta^*)$  are said to be optimal for the matrix game  $f$ .

**Theorem 2:** Consider problems (P1) and (P2). The optimal policy  $w^*$  for player 2 is SQFSP.

**Proof:** We formulate the dynamic programming equation for the three problems. If the system is not full then

$$(P1) : v_{(x,i)}^{n+1} = \max_a \min_b \left\{ \sum_y \lambda_{xay} \left( q_{xay} v_{(y,i+e_b)}^n + (1 - q_{xay}) v_{(y,i)}^n \right) \right\} + \sum_{j=1}^m \mu_j v_{(x,(i-e_j)+)}^n,$$

$$(P2) : v_{(x,i)}^{n+1} = \text{val} \left\{ \sum_y \lambda_{xay} \left( q_{xay} v_{(y,i+e_b)}^n + (1 - q_{xay}) v_{(y,i)}^n \right) \right\} + \sum_{j=1}^m \mu_j v_{(x,(i-e_j)+)}^n.$$

When the system is full then we have for all three problems:

$$v_{(x,i)}^{n+1} = \max_a \left\{ \sum_y \lambda_{xay} v_{(y,i)}^n \right\} + \sum_{b=1}^m \mu_b v_{(x,(i-e_b))}^n.$$

We show by induction on  $n$  that  $v^n$  satisfies (1), (2) and (3). Property (1) will then enable us to obtain the optimality of SQFSP.

The terms for (P1) and (P2) corresponding to the departures,  $\sum_{j=1}^m \mu_j v_{(x,(i-e_j)+)}^n$ , satisfy the properties following the same arguments as in the proof of theorem 3.1 [1].

We show that the arrival terms in the expression for  $v^{n+1}$  of (P1) satisfy (1). Let

$$f(i, a) = \min_b \left\{ \sum_y \lambda_{xay} \left( q_{xay} v_{(y, i+e_b)}^n + (1 - q_{xay}) v_{(y, i)}^n \right) \right\}.$$

Let  $b^*$  be the minimizing action for  $f(i + e_{j_2}, a)$ . We show that  $f(i + e_{j_1}, a) \leq f(i + e_{j_2}, a)$  for all  $a$  and suitable  $j_1$  and  $j_2$  (as specified by equation (1)). First consider the case  $b^* = j_1$ . Then

$$f(i + e_{j_1}, a) \leq \sum_y \lambda_{xay} \left( q_{xay} v_{(y, i+e_{j_1}+e_{j_2})}^n + (1 - q_{xay}) v_{(y, i+e_{j_1})}^n \right) \leq f(i + e_{j_2}, a),$$

the last inequality follows by (1). If  $b^* \neq j_1, j_2$  then

$$f(i + e_{j_1}, a) \leq \sum_y \lambda_{xay} \left( q_{xay} v_{(y, i+e_{j_1}+e_{b^*})}^n + (1 - q_{xay}) v_{(y, i+e_{j_1})}^n \right) \leq f(i + e_{j_2}, a),$$

the last inequality follows again by (1). By (1) we know that  $b^* \neq j_2$ . This establishes that  $f(i + e_{j_1}, a) \leq f(i + e_{j_2}, a)$  for all  $a$  and suitable  $j_1$  and  $j_2$ . Using the same arguments as in the proof of Theorem 1 we conclude that (1) holds for the arrival term in the expression for  $v^{n+1}$ .

Now consider problem (P2). Define, for  $\alpha \in M(\mathbf{A})$ ,

$$\begin{aligned} \tilde{f}(i, \alpha) &= \min_{\beta} \left\{ \sum_b \beta(b) \sum_a \alpha(a) \sum_y \lambda_{xay} \left( q_{xay} v_{(y, i+e_b)}^n + (1 - q_{xay}) v_{(y, i)}^n \right) \right\} = \\ &= \min_b \left\{ \sum_a \alpha(a) \sum_y \lambda_{xay} \left( q_{xay} v_{(y, i+e_b)}^n + (1 - q_{xay}) v_{(y, i)}^n \right) \right\}. \end{aligned}$$

Similarly to the previous case, we can show that  $\tilde{f}(i + e_{j_1}, \alpha) \leq \tilde{f}(i + e_{j_2}, \alpha)$ , and therefore (1) holds for the arrival term in the expression for  $v^{n+1}$ .

It can be shown similarly that (2) and (3) also hold for both models (P1) and (P2).

For problem (P1), the optimality of SQFSP follows immediately from (1) and is seen to be independent of the action in the MDAP. Consider (P2) and choose  $\beta \in M(\mathbf{B})$ . Suppose  $\beta(j_2) > 0$ , while there is a  $j_1$  such that  $j_1$  and  $j_2$  satisfy the conditions of equation (1). Then the value can be decreased, for each policy  $\alpha$  of player 1, by using  $\tilde{\beta}$ , with  $\tilde{\beta}(j) = \beta(j)$  if  $j \neq j_1, j_2$ ,  $\tilde{\beta}(j_1) = \beta(j_1) + \beta(j_2)$  and  $\tilde{\beta}(j_2) = 0$ . ■

**Remark 1:** In general in (P2), both players need to randomize, but player 2 does not use (w.p.1) actions which are not optimal according to the SQFSP policy. However, in some states (in some models like the symmetric, in all states) player 2 uses one action with probability 1. In these states player 1 does not randomize either.

**Remark 2:** In [1] model (P0) was analyzed in an MDP framework, i.e. both players cooperate in order to minimize the (common) cost. (P1) and (P2) could also be considered in such a framework and would yield the optimality of the SQFSP policy by the same technique as we used in this section. Note however that in the MDP case, (P1) and (P2) have the same optimal values and optimal (deterministic) policies. Indeed, since in the MDP case an optimal deterministic policy is known to exist, the controllers need not be told the information on the actions of each other in order to know them accurately, since these actions can be deduced directly from the state of the system.

#### 4. RESULTS ON THE SERVER ASSIGNMENT MODEL

In [2] both multiple and single server systems are studied. We start with the single server model. Customers arrive according to an MDAP (again with transition probabilities  $\lambda_{xay}$ ) in  $m$  queues, where  $q_{xay}^j$  is the probability of an arrival in queue  $j$ . Customers in queue  $j$  have an exponential service time distribution with parameter  $\mu_j$ . In this model, there is a single server, and player 2 has to decide to which queue the server will be assigned.

We shall use the same notation as in Sections 2 and 3; the only difference is that this time the meaning of an action  $b \in \mathbf{B}$  of player 2 is to assign the server to queue  $b$ . Each player takes decisions based on the history of previous states and actions as well as the current state. We get the following dynamic programming equation:

$$v_{(x,i)}^{n+1} = \max_a \left\{ \sum_y \lambda_{xay} \left( \sum_{j=1}^m q_{xay}^j v_{(y,i+e_j)}^n + \left( 1 - \sum_{j=1}^m q_{xay}^j \right) v_{(y,i)}^n \right) \right\} + \min_b \left\{ \mu_b v_{(x,i-e_b)}^n + (\mu - \mu_b) v_{(x,i)}^n \right\}.$$

Note that the dynamic programming equation resembles the one for (P1) or (P2) in the previous section rather than (P0). This is related to the fact that in (P0) in Section 2

player 2 could take an action only immediately after an arrival occurred (and a transition in the state of the MDAP).

For independent arrivals and linear costs, i.e.  $v_{(x,i)}^0 = \sum_j c_j i_j$ , the  $\mu c$ -rule is known to be optimal. Reorder the queues such that  $\mu_1 c_1 \geq \dots \geq \mu_m c_m$ . For arrivals according to an MDAP the extra condition  $\mu_1 \leq \dots \leq \mu_m$  was needed in [2]. In the present setting, where maximizing actions are chosen in the MDAP, we have to assume  $\mu_1 \geq \dots \geq \mu_m$  instead. Indeed, under this assumption we can rewrite the following basic inequality for the proof of optimality

$$\mu_{j_1} v_{(x,i-e_{j_1})}^n + (\mu - \mu_{j_1}) v_{(x,i)}^n \leq \mu_{j_2} v_{(x,i-e_{j_2})}^n + (\mu - \mu_{j_2}) v_{(x,i)}^n \quad \text{for } j_1 < j_2$$

as

$$\mu_{j_1} v_{(x,i-e_{j_1})}^n \leq \mu_{j_2} v_{(x,i-e_{j_2})}^n + (\mu_{j_1} - \mu_{j_2}) v_{(x,i)}^n \quad \text{for } j_1 < j_2.$$

Similarly as in the previous section, if we write

$$f(i, a) = \min_a \left\{ \sum_y \lambda_{xay} \left( \sum_{j=1}^m q_{xay}^j v_{(y,i+e_j)}^n + (1 - \sum_{j=1}^m q_{xay}^j) v_{(y,i)}^n \right) \right\}$$

and if  $a^*$  is the maximizing action in  $(x, i - e_{j_1})$ , then

$$\begin{aligned} \mu_{j_1} \max_a f(i - e_{j_1}, a) &= \mu_{j_1} f(i - e_{j_1}, a^*) \leq \mu_{j_2} f(i - e_{j_2}, a^*) + (\mu_{j_1} - \mu_{j_2}) f(i, a^*) \leq \\ &\mu_{j_2} \max_a f(i - e_{j_2}, a) + (\mu_{j_1} - \mu_{j_2}) \max_a f(i, a), \end{aligned}$$

proving the optimality of the  $\mu c$ -rule if  $\mu_1 \geq \dots \geq \mu_m$ . For the multiple server case,  $\mu_1 \leq \dots \leq \mu_m$  is also required for proving the inequality for the terms concerning service of customers. Thus in this case the  $\mu c$ -rule is only optimal if  $\mu_1 = \dots = \mu_m$ .

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