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On Semisimple Cocommutative Bialgebras

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Abstract

It is well known that a semisimple finite dimensional commutative Hopf-algebra over an algebraically closed field k is a group algebra. The standard proofs, cf. e.g. [1,2] are rather deeply imbedded in the theory. It is a purpose of this note to give a simple direct computational proof that also works in the infinite dimensional case and also gives results if there is no co-unit, or if the co-multiplication is not associative.

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1. Introduction

Let N be a set with a partial multiplication. I.e. there is given a partial mapping $N \times N \to N$. Let K be a field (or more generally a ring). Consider the ring of finite support functions $F(N) = \{f : N \to k : \text{Supp}(f) = \{n \in N : f(n) \neq 0\}$ is finite$\}$ with pointwise multiplication and addition. As a vector space $F(N)$ has the basis a_n, $n \in N$, $a_n(n') = \delta_{n,n'}$ where δ is the Kronecker delta. Define

$$\mu : F(N) \to F(N) \otimes F(N), \quad a_n \mapsto \sum_{n'n''=n} a_{n'} \otimes a_{n''} \tag{1.1}$$

where the sum is equal to zero if there are no n', n'' such that $n'n'' = n$. This defines an algebra homomorphism μ. If the partial map is associative i.e. $n(n'n'') = (nn')n''$ in the sense that if either of these two expressions is defined that so is the other and then both are equal, then μ is coassociative:

$$(\mu \otimes 1)\mu = (1 \otimes \mu)\mu : F(N) \to F(N) \otimes F(N) \otimes F(N) \tag{1.2}$$

If there is a unit in N, i.e. an element e such that $en = ne = n$ for all n, then $\epsilon : F(N) \to k$, $\epsilon(a_e) = 1$, $\epsilon(a_n) = 0$ for $n \neq e$ defines a co-unit for μ, i.e. an algebra homomorphism such that

$$(\epsilon \otimes 1)\mu = \text{id}, \quad (1 \otimes \epsilon)\mu = \text{id} \tag{1.3}$$

Thus $F(N)$ is a bialgebra if N is an associative partial semigroup with unit. If N is a finite group, then $r(a_n) = a_{n^{-1}}$ defines an antipode $\iota : F(N) \to F(N)$, i.e. an anti-algebra homomorphism such that

$$m(\iota \otimes \text{id})\mu = e\epsilon, \quad m(\text{id} \otimes \iota)\mu = e\epsilon \tag{1.4}$$

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where $m: F(N) \otimes F(N) \to F(N)$ is the multiplication on $F(N)$ and $e: k \to F(N)$ is the unit, $1 \mapsto \Sigma_n a_n$. In this case $F(N)$ is a Hopf algebra.

1.5. THEOREM. *Let A be an algebra over k which as an algebra is isomorphic to a direct sum of copies of k. Let $\mu: A \to A \otimes A$ be a comultiplication, i.e. an algebra morphism. Then $A \simeq F(N)$ for some N with a partial multiplication. If A has a counit, i.e. an algebra map $\epsilon: A \to k$ such that (1.3) holds then N has a unit element. If μ is co-associative, then the partial multiplication on N is associative. If A is a bialgebra then N is a partial monoid. If A is a finite dimensional Hopf algebra then N is a finite group.*

2. PROOF OF THE THEOREM. Choose a basis a_i, $i \in N$ for A such that $A = \oplus_{i \in N} k a_i$, $a_i a_j = \delta_{ij} a_i$. Let

$$\mu(a_i) = \sum_{j,k} c_i^{jk} a_j \otimes a_k \tag{2.1}$$

Now μ is a homomorphism of algebras. Therefore

$$\mu(a_i) = \mu(a_i^2) = \mu(a_i)^2, \quad \mu(a_i a_j) = \mu(a_i)\mu(a_j) = 0 \quad \text{if} \quad i \neq j \tag{2.2}$$

The second condition gives for $i \neq j$

$$0 = (\sum_{r,s} c_i^{rs} a_r \otimes a_s)(\sum_{t,u} c_j^{tu} a_t \otimes a_u) = \sum_{t,s,t,u} c_i^{rs} c_j^{tu} \delta_{rt} \delta_{su} (a_r \otimes a_s) = \sum_{r,s} c_i^{rs} c_j^{rs} (a_r \otimes a_s) \tag{2.3}$$

Thus

$$c_i^{rs} c_j^{rs} = 0 \quad \text{all} \quad i \neq j, \quad i, r, s. \tag{2.4}$$

Similarly, from $\mu(a_i) = \mu(a_i)^2$ it follows that

$$(c_i^{rs})^2 = c_i^{rs} \quad \text{all} \quad i, r, s. \tag{2.5}$$

Thus for all r, s, there is at most one i such that $c_i^{rs} \neq 0$ and then $c_i^{rs} = 1$. This defines a partial multiplication on N, viz.

$$(t, s) \mapsto i \quad \text{iff} \quad c_i^{rs} \neq 0$$

Now let $\epsilon: A \to k$ be a co-unit. Then $(\epsilon \otimes 1)\mu = \text{id}$ which translates as

$$\sum_{j,k} c_i^{jk} \epsilon(a_j) a_k = a_i \tag{2.6}$$

so that

$$\sum_{j} c_i^{jk} \epsilon(a_j) = \delta_{ki} \tag{2.7}$$

Now $\epsilon(a_i)\epsilon(a_j) = 0$ for $i \neq j$. So there is at most one i such that $\epsilon(a_i) \neq 0$. And because of (2.7) there is then precisely one such i. Let this be the index e. Then $\epsilon(a_e^2) = \epsilon(a_e)$, so that $\epsilon(a_e) = 1$. Thus (2.7) implies

$$c_i^{ek} = \delta_{ki} \tag{2.8}$$

and similarly, using $(1 \otimes \epsilon)\mu = \text{id}$,

$$c_i^{ke} = \delta_{ki} \tag{2.9}$$

Thus $c_k^{ke} = 1 = c_k^{ke}$ for all k, proving that e acts as a unit for the partial multiplication on N.
Let $r.s \in N$ be, if it exists, the unique index such that $c_{r.s}^{rs} \neq 0$. Now

$$\mu(a_i) = \sum_{i=r\cdot s} a_r \otimes a_s \tag{2.10}$$

If μ is co-associative then

$$(1 \otimes \mu)\mu(a_i) = \sum_{i=r\cdot s} a_r \otimes \mu(a_s) = \sum_{\substack{i=r\cdot s\\ s=u\cdot t}} a_r \otimes a_u \otimes a_t$$

$$= (\mu \otimes 1)\mu(a_i) = \sum_{i=s\cdot t} \mu(a_s) \otimes a_t = \sum_{\substack{i=s\cdot t\\ s=r\cdot u}} a_r \otimes a_u \otimes a_t \tag{2.11}$$

Thus $r \cdot (u \cdot t)$ exists iff $(r \cdot u) \cdot t$ is does and then both are equal. So that the partial group structure on N is associative.
Finally let ι be an antipode. Then ι

$$\iota(a_i) = \sum b_i^j a_j \tag{2.12}$$

Because $\iota(a_i a_k) = \delta_{ik}\iota(a_i)$ it follows that $(b_i^j)^2 = b_i^j$ and that for each j at most one i exists for which $b_i^j \neq 0$, (and then $b_i^j = 1$).

Now

$$m(\text{id} \otimes \iota)\mu(a_e) = m(\text{id} \otimes \iota)(\sum_{r\cdot s=e} a_r \otimes a_s) = m(\sum_{r\cdot s=e} a_r \otimes \iota(a_s))$$

$$= \sum_{j,r\cdot s=e} a_r b_s^j a_j = \sum_{j,r\cdot s=e} \delta_{rj} b_s^j a_r = \sum_{r\cdot s=e} b_s^r a_r$$

Thus, by the antipode property (1.4), if A is finite dimensional, so that $\Sigma_n a_n$ is a unit,

$$\sum_{r\cdot s=e} b_s^r a_r = \sum_n a_n \tag{2.13}$$

Now we have seen that for all r there is at most one s for which $b_s^r \neq 0$. From (2.13) it now follows that there is precisely one such s. Denote this one by $\bar{r}$. Then $r \mapsto \bar{r}$ defines a right inverse on N, and $r(a_r) = a_{\bar{r}}$. Now by the second of the two antipode properties (1.4)

$$m(\iota \otimes \text{id})\mu(a_e) = \sum_{r\cdot s=e} \iota(a_r)a_s = \sum_{r\cdot s=e} a_{\bar{r}} a_s = \sum_{r\cdot s=e} \delta_{\bar{r},s} a_s = \sum_n a_n$$

so that for all s there is an r with $r \cdot s = e$ (and $\bar{r} = s$), so that left inverses exist also (and are equal to right inverses).
Finally the existence of inverses and a unit also shows that the partial multiplication on N is everywhere defined making N a group. Indeed for all u, t, $(u^{-1} \cdot u) \cdot t = e \cdot t = t$ exists and hence so does $u^{-1} \cdot (u \cdot t)$ by the coassociativity and hence in particular $u \cdot t$. This concludes the proof.

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