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"Hilbert 90" for Polynomial Matrices

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Abstract

A generalization of the "Hilbert 90" theorem is proved for unimodular polynomial matrices.

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1. Introduction and statement of main result

Let K/k be a finite Galois extension with Galois group $\Gamma = Gal(K/k)$. Let $GL_n(K[z_1, \dots, z_m])$ be the group of polynomial unimodular $n \times n$ matrices over K, i.e. the group of $n \times n$ matrices M with entries that are polynomials in m variables over K and such that $\det(M) \in K \setminus \{0\}$. The group Γ acts on $GL_n(K[z_1, \dots, z_m])$ by acting on the coefficients of the matrix elements.

The main theoren of this note says that the corresponding first cohomology group is zero.

THEOREM 1.1. Let K/k be a finite Galois extension with Galois group Γ. Then

$$H^1(\Gamma, GL_n(K[z_1, \dots, z_m]) = 0$$

In case $m = 0$ this reduces to the "Hilbert 90" theorem

$$H^1(\Gamma, K^*) = 0 \tag{1.2}$$

Another simple but nontrivial cases is $k = \mathbb{R}$, $K = \mathbb{C}$. In that case the theorem says that if A is a polynomial matrix over $\mathbb{C}$ such that $A\bar{A} = I_n$ then there exists a $B \in GL_n(\mathbb{C}[z_1, \dots, z_m])$ such that $A = \bar{B}B^{-1}$

Part of the interest in "Hilbert 90"-type theorems comes from the philosophy of forms, [4,5]. Two objects, e.g. algebras, over k, are K/k-forms of each other if they become isomorphic over K. Let $\text{Forms}_{K/k}(T)$ be the set with distinguished element of k-isomorphism classes of K/k-forms of the object T. Then there is a natural map

$$V : \text{Forms}_{K/k}(T) \longrightarrow H^1(\Gamma,\ \text{Aut}_K(T_K)) \tag{1.3}$$

which in a number of cases can be proved to be an isomorphism. Moreover the proof sometimes also uses a "Hilbert 90"-type result.

Let k be a perfect field and $\bar{k}$ the algebraic closure of k. Let $\Gamma = \text{Gal}(\bar{k}/k)$ be the corresponding Galois group.

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Because a continuous 1-cocycle $\mathrm{Gal}(\bar{k}/k) \to GL_n(\bar{k}[z_1, \ldots z_m])$ factors through a finite quotient $\mathrm{Gal}(K/k)$ one has the immediate corollary.

COROLLARY 1.4. $H^1(\mathrm{Gal}(\bar{k}/k), GL_n(\bar{k}[z_1, \ldots, z_m]) = 0$.

2. PROOF OF THEOREM 1.1

Below I shall usually with z instead of $z_1, \ldots, z_m$. Let $s \mapsto A_s \in GL_n(k[z])$, $s \in \Gamma = \mathrm{Gal}(K/k)$, be a 1−cocycle. This means that

$$t(A_s) = A_t^{-1} A_{ts},\ s,\ t \in \Gamma \tag{2.1}$$

For each vector $v \in K[z]^n$, let $b \in K^n$ be the vector

$$b = \sum_{s \in \Gamma} A_s s(v) \tag{2.2}$$

LEMMA 2.3. *There exists a finite set of vectors $v_1(z), \ldots, v_r(z)$ in $K[z]^n$ such that the corresponding vectors $b_1(z), \ldots, b_r(z)$ for each value $\lambda = (\lambda_1, \ldots, \lambda_m)$ of z span the vector space K_n.*

PROOF. For each fixed λ there are n vectors $v_1'(\lambda), \ldots, v_n'(\lambda)$ such that the corresponding b's span K^n at that λ. This uses a procedure of Cartier and the (Dedekind) theorem on the algebraic independence of automorphisms in Γ, [1, Ch. V, §6, Thm. 2; 4, page 159]. Given any set of vectors $V = \{v_i(z)\}_{i \in I}$ the set of values of z for which the $\{b_i(z)\}_{i \in I}$ fail to span K^n is given by an ideal $I_V \subset K[z_1, \ldots, z_m]$. By the remark just made $\cap_V I_V = \{0\}$. Because $K[z]$ is noetherian it follows that there is also a finite V for which the corresponding b satisfy the conclusion of the lemma. QED

Now take such a finite set $v_1(z), \ldots, v_r(z)$ and let B be the polynomial matrix which has the corresponding $b_1(z), \ldots, b_r(z)$ as columns. The rows of B define an algebraic vector bundle E over $\mathrm{Spec}(K[z])$.

LEMMA 2.4. *The vector bundle E is defined over k.*

PROOF. Let $u \in K^n$ and suppose that $Bu = 0$, i.e. $\Sigma_j u_j b_j = 0$. Then

$$\sum_{j,s} u_j A_s\, s(v_j) = 0 \tag{2.5}$$

Let $t \in \Gamma$ and apply t to (2.5) to find

$$\sum_{j,s} t(u_j)\, t(A_s) ts(v_j) = 0$$

Now $t(A_s) = A_t^{-1} A_{ts}$ (cf. (2.1)) and hence

$$0 = \sum_{j,s} t(u_j)\, t(A_s)\, ts(v_j) = \sum_{j,s} t(u_j) A_t^{-1}\, A_{ts} ts(v_j) = A_t^{-1} \sum_j t(u_j) b_j$$

So that also $\Sigma t(u_j) b_j = 0$. QED

2.6 We can now finish the proof of the theorem. By the Quillen-Suslin theorem, [2,3], there is a unimodular matrix $U \in GL_r(k[z])$ such that the first n columns of BU form a unimodular $n \times n$ matrix, B'. Let C be the matrix formed by the vectors $v_1, \ldots, v_r$ and C' the matrix formed by the first n columns of CU. Then

$$B' = \sum_s A_s s(C') \tag{2.7}$$

Indeed, if M_i denotes the i-th column of a matrix M, we have

$$\begin{aligned}\sum A_s s(C') &= \sum_s A_s(s(C'_1), \ldots, s(C'_n)) \\ &= \sum_s A_s(s(CU)_1, \ldots, s(CU)_n) \\ &= \sum_s A_s((s(C)U)_1, \ldots, (s(C)U)_n) \\ &= \sum_s ((A_s s(C)U)_1, \ldots, (A_s s(C)U)_n) \\ &= \text{matrix formed by the first } n \text{ columns of } \sum_s A_s s(C)U = BU\end{aligned}$$

Now B' is unimodular, and by (2.7) and (2.1)

$$t(B') = \sum_s t(A_s)\, ts(C') = \sum_s A_t^{-1} A_{ts} ts(C') = A_t^{-1} B'$$

showing that the cocycle $s \mapsto A_s$ is cohomologous to zero. QED

REFERENCES

1. N. BOURBAKI, Algébre, Ch. IV-Ch. VII, Masson, 1981.
2. T.J. LAM, Serre's conjecture, Springer, 1978.
3. D. QUILLEN, Projective modules over polynomial rings, Inv. Math. *36* (1976), 167-171.
4. J.-P. SERRE, Corps locaux, Hermann, 1962.
5. J.-P. SERRE, Cohomologie galoisienne, Springer, 1964.