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The Correlated M/G/1 Queue

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Abstract
This paper considers a single server queue in which the service time of each customer depends on the length of the preceding interarrival interval. The dependence structure corresponds to the situation where work arrives at a gate according to a process with independent increments, while at exponential intervals the gate is opened and - after the addition of an independent component - the work is delivered to the server as a single customer. This model unifies and generalizes other recently studied M/G/1 queues with dependence between interarrival and service times. The main result is the joint steady-state distribution of the waiting and service time of a customer. As a by-result, the sojourn time distribution is obtained; a decomposition for its Laplace-Stieltjes transform is presented and interpreted.

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1 Introduction

In this paper we consider the following situation. Work arrives at a single server queue according to a process with stationary non-negative independent increments. This work, however, does not immediately enter the queue of the service facility; instead it is accumulated behind a gate. At exponential intervals the gate is opened and - after the addition of an independent component - the work is collected and delivered as a single customer at the queue of the service facility. The additional component might be viewed as a set-up time.

Due to the exponentially distributed interarrival times of customers, we can view the service facility as an M/G/1 queue in which the interarrival and service time for each customer are positively correlated. Indeed, if the interval between two consecutive openings of the gate is relatively long (short), it is likely that a relatively large (small) amount of work has accumulated in that interval.

In the present paper we analyse the M/G/1 queue with the above-sketched dependence structure. Our motivation for this analysis is threefold. Firstly, single server queues with the above-sketched dependence structure arise quite naturally in many situations. An example is mail pick-up. Other examples occur in computer-communications. One might think of the collection of packets in a 'train' on a LAN with interconnected rings to be delivered at a bridge queue; see [1, 2] for an example related to the CRMA protocol.

Secondly, the present model is a unification and generalization of the so far studied M/G/1 queues with a positive correlation between interarrival and service times [1, 2, 5, 7]. Thirdly,

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it allows a detailed analysis of the main performance measures, thus giving insight into the effect of the dependence structure on those performance measures. In particular it is possible to derive a closed expression for the Laplace-Stieltjes Transform of the joint distribution of the waiting and service time of an arbitrary customer.

In [1, 2] the following special case of the model has been analysed. Customers with service time distribution $B(\cdot)$ arrive at a pick-up point according to a Poisson process with rate λ. At exponentially distributed intervals with mean $\frac{1}{\gamma}$, a bus collects these customers and delivers them as a batch at a single server facility. In this way, the service facility together with the arrival process of batch customers is an M/G/1 queue with dependence between interarrival and service times. Actually, given that the interarrival time $\boldsymbol{\sigma}_n$ between the $(n-1)st$ and the $n-th$ batch equals u, the amount of work $\boldsymbol{\tau}_n$ in that $n-th$ batch is distributed as the state of a Compound Poisson Process (CPP) $\mathbf{Y}(\cdot)$ at time u, where the intensity of the jumps is λ, the jump-sizes have distribution $B(\cdot)$ and $\mathbf{Y}(0) = 0$. The Laplace-Stieltjes Transform (LST) of $\mathbf{Y}(u)$ is given by

$$E(e^{-\omega \boldsymbol{\tau}_n} \mid \boldsymbol{\sigma}_n = u) = e^{-u\lambda(1-\beta(\omega))}, \quad \omega \geq 0, \tag{1.1}$$

in which $\beta(\cdot)$ is the LST of the service time distribution $B(\cdot)$.
Cidon et al.[5] analyse the M/G/1 queue in which the interarrival and service time are related through $\boldsymbol{\tau}_n = \alpha\boldsymbol{\sigma}_n + \tilde{\boldsymbol{\tau}}_n$, $(0 < \alpha < 1)$, with $\tilde{\boldsymbol{\tau}}_n$ an amount of work that is independent of the interarrival time. For this model

$$E(e^{-\omega \boldsymbol{\tau}_n} \mid \boldsymbol{\sigma}_n = u) = E(e^{-\omega \tilde{\boldsymbol{\tau}}_n})e^{-u\alpha\omega}, \quad \omega \geq 0. \tag{1.2}$$

As Compound Poisson Processes and the process $\mathbf{Y}(u) = \alpha u$ are special examples of processes with stationary non-negative independent increments, these models are special cases of our model.

In the remainder of this section we present a more detailed model description, a brief survey of related literature, and an overview of the paper.

Model description
The goal of the present study is to extend the analysis of [1, 2, 5, 7] to M/G/1 queues with arrival rate γ in which the LST of the service time $\boldsymbol{\tau}_n$, given that the interarrival time $\boldsymbol{\sigma}_n$ equals u, is of the following form

$$E(e^{-\omega \boldsymbol{\tau}_n} \mid \boldsymbol{\sigma}_n = u) = v(\omega)e^{-\phi(\omega)u}, \quad \omega \geq 0, \tag{1.3}$$

with $\phi(0) = 0$, and $\phi(\omega)$ having a completely monotone derivative.
The service time $\boldsymbol{\tau}_n$ consists of two parts: a component which depends on the interarrival time, represented by $e^{-\phi(\omega)u}$, and an 'ordinary' M/G/1 service time with LST $v(\omega)$, which does not depend on the interarrival time. The dependent part of the service time will be shown to represent increments during an exponential gate opening interval of an arbitrary process with stationary non-negative independent increments. In the next section this process is described in detail; here we only mention that $\phi'(0)$ and $(-\phi''(0)+(\phi'(0))^2)$ are respectively the first and second moment of the amount of work arriving at the gate per unit of time. For the independent component of the service time, the first two moments are $-v'(0)$ and $v''(0)$. The bivariate LST of $\boldsymbol{\sigma}_n$ and $\boldsymbol{\tau}_n$ follows from (1.3):

$$E(e^{-\zeta\boldsymbol{\sigma}_n-\omega\boldsymbol{\tau}_n}) = \frac{\gamma v(\omega)}{\gamma+\zeta+\phi(\omega)}, \quad \zeta \geq 0, \omega \geq 0. \tag{1.4}$$

Expression (1.4) leads to

$$E\boldsymbol{\tau}_n = -v'(0) + \frac{\phi'(0)}{\gamma}, \qquad E\boldsymbol{\tau}_n^2 = v''(0) - 2\frac{v'(0)\phi'(0)}{\gamma} - \frac{\gamma\phi''(0) - 2(\phi'(0))^2}{\gamma^2}, \tag{1.5}$$

$$Cov(\boldsymbol{\sigma}_n, \boldsymbol{\tau}_n) = \frac{\phi'(0)}{\gamma^2} \geq 0, \tag{1.6}$$

$$correl(\boldsymbol{\sigma}_n, \boldsymbol{\tau}_n) = \left[1 + \frac{-\gamma\phi''(0)}{(\phi'(0))^2} + \left(\frac{\gamma}{\phi'(0)}\right)^2 (v''(0) - (v'(0))^2)\right]^{-1/2} \in [0,1]. \tag{1.7}$$

Note that $correl(\boldsymbol{\sigma}_n, \boldsymbol{\tau}_n) \to 1$ if $\gamma \to 0$ and $correl(\boldsymbol{\sigma}_n, \boldsymbol{\tau}_n)$ decreases for increasing γ and increasing variability of the independent service time component.

Related literature

A few studies have appeared that analyse a queueing system with correlation between interarrival and service times. Borst et al.[1, 2] consider the model with dependence structure (1.1). They obtain the distributions of the queue length, waiting and sojourn times of the customers consisting of the collected work, as well as of the individual customers arriving at the pick-up point. Borst et al.[1, 2] show that the model with the pick-up or gate mechanism can be interpreted as an ordinary M/G/1 queue with server vacations, leading to interesting decomposition results. In [1, 2] it is proven that the waiting and service time of a batch of customers are negatively correlated, and that a positive correlation between interarrival and service times leads to a reduction in mean waiting times.

Takahashi[15] studies the same collection procedure of customers arriving at a gate for the special case where the service times of these customers are exponentially distributed. His service facility has multiple servers. Takahashi[15] obtains the generating function for the joint distribution of the number of customers at the gate and at the service facility.

Cidon et al.[5] study dependence structure (1.2). Their model contains the models in Conolly[6] and Conolly & Hadidi[8], in which the M/M/1 queue with linearly related interarrival and service times is studied.

Conolly & Choo[7] study an M/M/1 queue in which $\boldsymbol{\sigma}_n$ and $\boldsymbol{\tau}_n$ have a bivariate exponential distribution. For this last model, (i) Hadidi [10] shows that the waiting times are hyperexponentially distributed; (ii) Hadidi [11] examines the sensitivity of the waiting time distribution to the value of the correlation coefficient; (iii) Langaris [13] studies the busy period distribution. However, as pointed out by Borst & Combé[3], he overlooks the fact that successive busy periods are correlated.

In the next section it is shown that the above-mentioned models are all examples of the dependence structure as given by (1.3).

A model not covered by this structure is presented in Jacobs[12], who obtains heavy traffic results for the waiting time in queues with sequences of ARMA correlated negative exponentially distributed interarrival and service times.

Overview of the paper

Section 2 describes the dependence structure in more detail. We give a few examples and show how previously analysed models for M/G/1 queues with dependence fit into this structure. Section 3 is devoted to the analysis of the joint distribution of the waiting and service time of an arbitrary customer in steady state. There we also present a vacation-type workload decomposition for the M/G/1 queue with the exponential gating mechanism that was described in the beginning of this section.

2 The dependence structure

In section 1 the dependent part of the service time distribution was characterized by the LST $e^{-\phi(\omega)u}$; here $e^{-\phi(\omega)}$ is the LST of a non-negative random variable with an infinitely divisible distribution (cf. Theorem 1 on p.450 of Feller[9]):

Theorem 2.1:

The function $\psi(\omega)$ is the LST of an infinitely divisible probability distribution if and only if $\psi(\omega) = e^{-\phi(\omega)}$, where $\phi(\omega)$ has a completely monotone derivative and $\phi(0) = 0$.

Remark 2.1:

An equivalent definition of an infinitely divisible distribution is that it has an LST of the form $\psi(\omega) = e^{-\phi(\omega)}$ where

$$\phi(\omega) = \int_0^\infty \frac{1 - e^{-\omega x}}{x} dP(x), \quad \omega \geq 0, \tag{2.1}$$

and P is a measure such that

$$\int_1^\infty x^{-1} dP(x) < \infty. \tag{2.2}$$

Remark 2.2: In Feller ([9],p.303) it is shown that the following classes of probability distributions are identical:

(i) Infinitely divisible distributions.
(ii) Distributions of increments in processes with stationary independent increments.
(iii) Limits of sequences of compound Poisson distributions.

A process $\{\mathbf{Y}(u),\ u \geq 0\}$, has stationary independent increments when the distribution of $\mathbf{Y}(t+s) - \mathbf{Y}(s)$ is independent of s, for all $t, s \geq 0$. In terms of a collecting procedure, this means that the rate of increments of work at the pick-up point (gate) does not depend on the length of the elapsed collecting (gate opening) interval. An important characterization of a process $\mathbf{Y}(u)$ with stationary independent increments with $\mathbf{Y}(0) = 0$ is: $\mathbf{Y}(t+s) \stackrel{d}{=} \mathbf{Y}(t) + \mathbf{Y}(s)$, $s, t > 0$, where $\stackrel{d}{=}$ denotes equality in distribution (cf. Feller[9], p.180). Obviously

$$Ee^{-\omega \mathbf{Y}(u)} = \left[Ee^{-\omega \mathbf{Y}(1)} \right]^u = e^{-u\phi(\omega)}, \quad u \geq 0, \omega \geq 0. \tag{2.3}$$

We next present a few examples from the class of processes with stationary independent increments, in which the character of the dependence structure comes more to light. We also show how some studied dependence structures fit into this class. In example i, the ϕ function studied will be denoted by $\phi_i(\cdot)$, $i = 1, \ldots, 4$.

Example 1: The Compound Poisson Process.
This is the dependence structure as studied by Borst et al.[1]. From (1.1) and (1.3) we see that $\phi_1(\omega)$ can be expressed as $\phi_1(\omega) = \lambda(1-\beta(\omega))$, with λ the intensity of jump occurrences, and $\beta(\cdot)$ the LST of the jump-size distribution $B(\cdot)$. For this example $dP(x) = \lambda x dB(x)$.

Example 2: Linear Dependency.
This is the dependence structure as studied by Cidon et al.[5]. Here $\phi_2(\omega) = \alpha\omega$ (cf. (1.2)). It is readily verified that the underlying infinitely divisible distribution is the limit of a sequence of Compound Poisson Processes, i.e., $\phi_2(\omega) = \lim_{n\to\infty} \lambda_n(1 - \beta_n(\omega))$, for all $\omega \geq 0$, when $\lambda_n \to \infty$, $\lambda_n\beta_n \to \alpha$ and $\frac{\beta_n^{(2)}}{\beta_n} \to 0$ as $n \to \infty$. Here β_n and $\beta_n^{(2)}$ denote the first and second moment of jump-size distribution $B_n(\cdot)$ respectively. For linear dependency we find $dP(x) = \alpha\delta(x)$, where $\delta(x)$ is the Dirac delta function.

Example 3: Gamma Distributions.
In this example $\mathbf{Y}(u)$ has a Gamma(ζ^{-1}, u) distribution ($\zeta > 0$), i.e. $\psi(\omega) = \left(\frac{1}{1+\zeta\omega}\right)$. Rewriting to the standard exponential form (cf. (2.1)) we obtain: $\phi_3(\omega) = \int\limits_0^\infty \frac{1-e^{-\zeta\omega y}}{y} e^{-y} dy$, leading to $dP(x) = e^{-\frac{x}{\zeta}} dx$. This process can be obtained as the limit of a sequence of CPP's with $\lambda_n = n$, and B_n being a Gamma(ζ^{-1}, n^{-1}) distribution, $n = 1, 2, \ldots$.

Example 4: Subordination of Processes with Stationary Independent Increments.
Let $\mathbf{Z}(t)$ and $\mathbf{T}(t)$, $t \geq 0$ be processes with stationary non-negative independent increments with $\mathbf{Z}(0) = \mathbf{T}(0) = 0$. Define the process $\mathbf{Y}(t), t \geq 0$, as follows. Let $U_t(\cdot)$ and $Q_t(\cdot)$, $t \geq 0$, denote the distributions of the states of $\mathbf{Z}$ and $\mathbf{T}$ at time t respectively. Then $U_t^0(\cdot)$, the distribution of the state of $\mathbf{Y}(t)$ at time $t \geq 0$, is defined by

$$U_t^0(x) = \int\limits_{s=0}^{\infty} U_s(x) dQ_t(s), \qquad x \geq 0. \tag{2.4}$$

We can view $\mathbf{T}$ as a transformer of the time in the process $\mathbf{Z}$. A simple illustration of such a subordination is with $\mathbf{Z}$ a CPP as described in example 1 and $\mathbf{T}$ a Gamma distribution as described in example 3. This Gamma randomization yields another process $\mathbf{Y}$ with stationary non-negative independent increments, with

$$\phi_4(\omega) = \int\limits_0^\infty \frac{1 - e^{-\zeta\lambda(1-\beta(\omega))x}}{x} e^{-x} dx, \quad \omega \geq 0. \tag{2.5}$$

Comparison with example 3 shows that $\phi_4(\omega) = \phi_3(\phi_1(\omega))$.

So far in this section we only considered the dependent component of the service time of a customer. Adding an independent 'ordinary' M/G/1 component further widens the range of the class of service time distributions. For example, this class, as described by (1.3), now also includes the case where the interarrival and service time have a joint bivariate exponential distribution with density function:

$$g(s,t) = \zeta\mu(1-r)e^{-\zeta s-\mu t}I_0[2\{\zeta\mu rst\}^{1/2}], \tag{2.6}$$

where $I_0[\cdot]$ is a zero-order modified Bessel function of the first kind, and $r \in [0,1)$ is the correlation between $\boldsymbol{\sigma}_i$ and $\boldsymbol{\tau}_i$. The marginal distributions of the interarrival and service time are exponential. It is readily verified that the service time, given that the interarrival time equals u, is the sum of a CPP with jump intensity $r\zeta$ and $\exp(\mu)$ distributed jump-sizes plus an additional independent $\exp(\mu)$ distributed service time. Conolly & Choo[7] study the queue with this bivariate distribution for the interarrival and service time.

3 The waiting and sojourn time

In this section we derive the LST for the joint steady-state distribution of the waiting and service time of an arbitrary customer. First some notation. Define the vector $(\mathbf{W}_n, \boldsymbol{\tau}_n)$, $n = 1, 2, \ldots$. $\mathbf{W}_n$ denotes the amount of work at the service facility just before the $n-th$ opening of the gate, hence $\mathbf{W}_n$ is the waiting time of the $n-th$ customer.
$\boldsymbol{\tau}_n$ is the service time of the $n-th$ customer, including the independent part. One can view $\boldsymbol{\tau}_n$ also as the amount of work that is present at the gate just before the $n-th$ opening of the gate.
$\mathbf{R}_n$, the sojourn time of the $n-th$ customer, is given by $\mathbf{R}_n = \mathbf{W}_n + \boldsymbol{\tau}_n$. Note that the waiting and service times are not independent.

Our analysis starts with the following recurrence relation for the vector $(\mathbf{W}_n, \boldsymbol{\tau}_n)$:

$$(\mathbf{W}_{n+1}, \boldsymbol{\tau}_{n+1}) = (\max\{0, \mathbf{W}_n + \boldsymbol{\tau}_n - \boldsymbol{\sigma}_{n+1}\}, \boldsymbol{\tau}_{n+1}), \qquad n = 1, 2, \ldots. \tag{3.1}$$

From (1.4) if follows that the workload of the server $\rho = \frac{E\boldsymbol{\tau}}{E\boldsymbol{\sigma}} = -\gamma v'(0) + \phi'(0)$. We assume that $\rho < 1$ and without proof we claim that this is a necessary and sufficient condition for the existence of a proper limiting distribution of $(\mathbf{W}_n, \boldsymbol{\tau}_n)$ for $n \to \infty$.
Let $\mathbf{W}, \boldsymbol{\tau}$ and $\mathbf{R}$ denote the random variables with the limiting distributions of $\mathbf{W}_n, \boldsymbol{\tau}_n$ and $\mathbf{R}_n$ respectively.
Define $H(x,y) = Pr\{\mathbf{W} \le x, \boldsymbol{\tau} \le y\}$, and let $H^*(\omega_1, \omega_2)$, $\omega_1, \omega_2 \ge 0$ denote the LST of this joint distribution. Denote the LST of $\mathbf{R}$ by $r(\omega)$, $\omega \ge 0$.
By conditioning on $\boldsymbol{\sigma}$ we derive from (3.1)

$$H(x,y) = \int_{u=0}^{\infty} Pr\{\mathbf{R} \le x+u\} Pr\{\boldsymbol{\tau} \le y \mid \boldsymbol{\sigma} = u\} \gamma e^{-\gamma u} du, \qquad x, y \ge 0. \tag{3.2}$$

From (3.2) we obtain

$$H^*(\omega_1, \omega_2) = \frac{\gamma v(\omega_2)}{\omega_1 - \gamma - \phi(\omega_2)} \left[\frac{\omega_1}{\gamma + \phi(\omega_2)} r(\gamma + \phi(\omega_2)) - r(\omega_1) \right], \qquad \omega_1, \omega_2 \ge 0. \tag{3.3}$$

We see that $H^*(\omega_1, \omega_2)$ is expressed in terms of $r(\cdot)$. Also, $\mathbf{W} + \boldsymbol{\tau} = \mathbf{R}$ implies $H^*(\omega, \omega) = r(\omega)$. Hence we next solve (3.3) for $r(\omega)$.
Rewriting (3.3) for $\omega_1 = \omega_2 = \omega$, $\omega \ge 0$, we find

$$r(\omega) = \frac{\gamma\ \omega\ v(\omega)}{(\omega + \gamma v(\omega) - \gamma - \phi(\omega))(\gamma + \phi(\omega))} r(\gamma + \phi(\omega)), \qquad \omega \geq 0. \tag{3.4}$$

Remark 3.1:
Letting $\omega \downarrow 0$ in (3.4) we obtain

$$r(\gamma) = 1 + \gamma v'(0) - \phi'(0) = 1 - \rho. \tag{3.5}$$

This formula also follows from

$$r(\gamma) = \int_{u=0}^{\infty} e^{-\gamma u} dPr\{\mathbf{R} < u\} = Pr\{\mathbf{R} < \boldsymbol{\sigma}\}. \tag{3.6}$$

The latter term equals the probability that an arriving customer sees the server idle, while the steady-state probability that the server is idle obviously equals $1 - \rho$. Due to the PASTA property both probabilities are equal.

Next define for $\omega \geq 0, k = 1, 2, \dots$,

$$g(\omega) := \gamma + \phi(\omega), \quad g^{(0)}(\omega) := \omega, \quad g^{(k)}(\omega) := g(g^{(k-1)}(\omega)). \tag{3.7}$$

With (3.4) and (3.7) we find after M iterations for $\omega \geq 0,\ \ M = 0, 1, \dots$

$$r(\omega) = r(g^{(M+1)}(\omega)) \prod_{k=0}^{M} \frac{\gamma\ g^{(k)}(\omega)\ v(g^{(k)}(\omega))}{(g^{(k)}(\omega) + \gamma v(g^{(k)}(\omega)) - g^{(k+1)}(\omega))\, g^{(k+1)}(\omega)}. \tag{3.8}$$

Lemma 3.1:

(i). The equation $\omega = g(\omega)$, $\omega \geq 0$ has a unique real solution ω^*.
(ii). $\lim_{M \to \infty} g^{(M)}(\omega) = \omega^*$ for all $\omega \geq 0$.
(iii). The right hand side of (3.8) converges for $M \to \infty$, for all $\omega \geq 0$.

Proof:
(i) & (ii): Theorem 2.1 implies that $g(\omega)$ is concave and non-decreasing. Also $g(\omega) \geq 0$ for $\omega \geq 0$ and $g(0) = \gamma > 0$, thus $g(\omega) = \omega$ has at most one positive solution. Moreover, the stability condition $\rho < 1$ leads to $0 \leq \phi'(0) < 1$. Finally $\mid g^{(k+1)}(\omega) - g^{(k)}(\omega) \mid \leq \phi'(0) \mid g^{(k)}(\omega) - g^{(k-1)}(\omega) \mid$, $k = 1, 2, \dots$, hence $g(\cdot)$ is a contraction. This completes the proof of (i) and (ii).
(iii): From the theory of infinite products, cf. Titchmarsh[16] p.18,

$$\prod_{k=0}^{\infty} a_k \quad \text{converges} \Leftrightarrow \sum_{k=0}^{\infty} (1 - a_k) \quad \text{converges.} \tag{3.9}$$

We find that this condition holds for our infinite product by using the fact that $g(\omega)$ is a contraction (part (i) & (ii)) and that the terms of the infinite sum converge to a positive constant, while being strictly positive for all k. The latter follows from monotonicity properties of $\omega + \gamma v(\omega) - g(\omega)$ and $\rho < 1$. $\square$

Lemma 3.1 leads to

Theorem 3.1:

$$r(\omega) = r(\omega^*) \prod_{k=0}^{\infty} \frac{\gamma\ g^{(k)}(\omega)\ v(g^{(k)}(\omega))}{(g^{(k)}(\omega) + \gamma v(g^{(k)}(\omega)) - g^{(k+1)}(\omega))\ g^{(k+1)}(\omega)}, \quad \omega \geq 0. \tag{3.10}$$

Here $r(\omega^*)$ follows from $r(0) = 1$.

By substituting $r(\omega)$ in (3.3) we obtain a closed expression for $H^*(\cdot, \cdot)$, the LST of the joint distribution of the waiting and service time of a customer. For our purposes it suffices to rewrite (3.3), using (3.4):

$$\begin{aligned} H^*(\omega_1, \omega_2) &= \frac{\gamma \omega_1 v(\omega_2)}{\omega_1 - g(\omega_2)} \left[\frac{r(g(\omega_2))}{g(\omega_2)} - \frac{r(\omega_1)}{\omega_1} \right] \\ &= \frac{\omega_1}{\omega_1 - g(\omega_2)} \left[\omega_2 + \gamma v(\omega_2) - g(\omega_2) \right] \frac{r(\omega_2)}{\omega_2} - \frac{\gamma v(\omega_2)}{\omega_1 - g(\omega_2)} r(\omega_1), \quad \omega_1, \omega_2 \geq 0. \end{aligned} \tag{3.11}$$

The LST of the marginal distributions of the waiting time $\mathbf{W}$ and the service time $\boldsymbol{\tau}$ follow from (3.11), or more easily from (3.3). In particular,

$$H^*(\omega, 0) = Ee^{-\omega \mathbf{W}} = \frac{\omega r(\gamma) - \gamma r(\omega)}{\omega - \gamma}, \quad \omega \geq 0. \tag{3.12}$$

Remark 3.2:
Expression (3.11) also provides the covariance of $\mathbf{W}$ and $\boldsymbol{\tau}$, i.e., the covariance of the waiting and service time of a customer

$$Cov(\mathbf{W}, \boldsymbol{\tau}) = -\frac{\phi'(0)}{\gamma} \left[\frac{1 - r(\gamma)}{\gamma} + r'(\gamma) \right] < 0, \tag{3.13}$$

the inequality following from the Taylor expansion of $r(0)$ around γ. This result is intuitively clear; a customer having a relatively long (short) interarrival time is likely to have a relatively short (long) waiting time, but also a relatively long (short) service time, the latter being due to the dependence.

Remark 3.3:
The generating function of the distribution of $\mathbf{N}$, the steady-state number of customers at the service facility, can be obtained by using the distributional form of Little's law; $Ez^{\mathbf{N}} = Ee^{-\gamma(1-z)\mathbf{R}}$.

With (3.3) and (3.10) we can evaluate most of the characteristics of interest for the M/G/1 queue with the gating mechanism. Expressions (3.3) and (3.10) can be used for numerical evaluation (cf. [1, 2] for the special case (1.1)).
More insight can be obtained by looking at the model from the more general perspective of single server queues, or dam models, with arrival process a process with stationary non-negative independent increments. In [1, 2] the M/G/1 queue with dependence structure (1.1) is related to an ordinary M/G/1 queue *without* collection. Analogously, here we relate the model with an exponential gate to a single server queue in which work directly arrives at the server queue, according to a process $\mathbf{Y}(u), u \geq 0$ with stationary non-negative independent increments. For dependence structure (1.3) the corresponding process would be the superposition of the dependent arrival process with a Compound Poisson Process with arrival rate γ

and jump-size LST $v(\omega)$, the latter representing the independent part of the service time of a customer. This superposition again is a process with stationary non-negative independent increments and is characterized by

$$Ee^{-\omega \mathbf{Y}(u)} = e^{-\chi(\omega)u}, \quad u \geq 0, \omega \geq 0, \tag{3.14}$$

with $\chi(\omega) = \phi(\omega) + \gamma(1 - v(\omega))$.
For the single server queue with the arrival process characterized by (3.14) the steady-state distribution of the amount of work $\mathbf{V}_{nogate}$ in the system exists if $\chi'(0) < 1$ and in that case its LST $\tilde{V}_{nogate}(\omega)$ is given by (cf. Prahbu[14] p.249)

$$\tilde{V}_{nogate}(\omega) = \frac{\omega(1 - \chi'(0))}{\omega - \chi(\omega)}, \quad \omega \geq 0. \tag{3.15}$$

From (3.3), (3.5), (1.4) and (3.15) it follows that

$$r(\omega) = E(e^{-\omega \mathbf{V}_{nogate}}) E(e^{-\omega \boldsymbol{\tau}}) \frac{r(\gamma + \phi(\omega))}{r(\gamma)}, \quad \omega \geq 0. \tag{3.16}$$

The interpretation of the term $\frac{r(\gamma+\phi(\omega))}{r(\gamma)}$ is as follows (cf. [1, 2] for a similar interpretation of a special case).

In the model with the gating mechanism, denote by $\mathbf{H}$ the sojourn time of a customer leaving no customers behind at the server. Such a sojourn time has distribution $H(\cdot)$ with

$$dH(t) = \frac{e^{-\gamma t} dPr\{\mathbf{R} \leq t\}}{\int_{u=0}^{\infty} e^{-\gamma u} dPr\{\mathbf{R} \leq u\}}, \quad t \geq 0. \tag{3.17}$$

Denote by $\mathbf{U}$ the amount of work arriving at the gate during such a sojourn time.

$$Ee^{-\omega \mathbf{U}} = \frac{\int_{t=0}^{\infty} e^{-\phi(\omega)t} e^{-\gamma t} dR(t)}{\int_{u=0}^{\infty} e^{-\gamma u} dR(u)} = \frac{r(\gamma + \phi(\omega))}{r(\gamma)}, \quad \omega \geq 0. \tag{3.18}$$

So

$$\mathbf{R} \stackrel{d}{=} \mathbf{V}_{nogate} + \boldsymbol{\tau} + \mathbf{U} \stackrel{d}{=} \mathbf{V}_{nogate} + \boldsymbol{\tau}_{dep} + \boldsymbol{\tau}_{indep} + \mathbf{U}, \tag{3.19}$$

$\stackrel{d}{=}$ denoting equality in distribution, $\boldsymbol{\tau}_{dep}$ having LST $\frac{\gamma}{\gamma+\phi(\omega)}$ and $\boldsymbol{\tau}_{indep}$ having LST $v(\omega)$.
This decomposition can be interpreted in the following way. Because of the PASTA property, $\mathbf{R} \stackrel{d}{=} \mathbf{V} + \boldsymbol{\tau}_{indep}$, with $\mathbf{V}$ the steady-state amount of work in the system (at the gate and at the server). Denote by η a stochastic variable with distribution the stationary distribution of the amount of work at the gate at times when the server is idle. Now the following work decomposition holds, cf. Boxma[4]:

$$\mathbf{V} \stackrel{d}{=} \mathbf{V}_{nogate} + \eta, \tag{3.20}$$

$\mathbf{V}_{nogate}$ and η being independent. The interpretation of the decomposition (3.19) is completed by observing that η consists of two independent components: (i) the amount of work $\mathbf{U}$ at the gate that has arrived during the sojourn time of the last customer before the server became idle, and (ii) the amount of work at the gate that has arrived during the past part of the idle period since the departure of the last customer. The latter term has the same distribution as τ_{dep}.

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