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Understanding the world: from facts to concepts, from concepts to propositions

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Understanding the World: from Facts to Concepts, from Concepts to Propositions

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Abstract

This paper gives an introduction to uses of lattice theory in formal concept formation, and the theory of reasoning with formal concepts.

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1 Introduction

We start out from the basic observation that no machine or human can process all signals from the real world that it or (s)he receives. Machines and men focus on a small subset of the available signals. These are the facts in focus. Facts are used to form concepts, and concepts are used to form a picture of the world, in the shape of some kind of relational database, say.

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Call this the recognized world. We assume that for purposes of reasoning and learning, the relational database is somehow transformed into a kind of deductive database. Call this the realized world. In this paper we produce a picture of what this transformation from recognized world to realized world may look like.

2 Concept Analysis with Lattice Theory

It is generally agreed that concepts are crucial for thinking. A formal analysis of the notion of a concept was proposed by R. Wille (see [18, 19, 20, 21]). For a general introduction for the uses of lattice theory in concept analysis, see [3, Chapter 11]. Further applications of concept analysis are data analysis (see [13, 16, 4, 23, 22, 5], [12]) and the analysis of inference in conceptual knowledge systems [24],[7].

3 The Basics of Concept Theory

We start with some definitions from Wille [18], namely the definitions of the extent and the intent of a concept, and the definitions of the operations $\vee, \wedge$ on concepts. Let O be a set of objects and A a set of attributes. Then any relation $W \subseteq O \times A$ is a set of facts, where $(o, a) \in W$ is read as: the fact that object o has attribute a is a fact of world W. We say that W is the recognized world, that O is the recognized object set and that A is the recognized attribute set. Each $(o, a) \in W$ is a fact. Now consider two functions ρ and λ, defined as follows.

$$\text{for } M \subseteq A \quad \rho(M) \stackrel{def}{=} \{o \mid \forall a \in M \ \ (o,a) \in W\} \tag{1}$$

$$\text{for } X \subseteq O \quad \lambda(X) \stackrel{def}{=} \{a \mid \forall x \in X \ \ (x,a) \in W\} \tag{2}$$

Obviously,

$$M \subseteq \lambda\rho(M) \tag{3}$$

$$\text{if} \quad M \subseteq N \ \ then \ \lambda\rho(N) \subseteq \lambda\rho(M) \tag{4}$$

$$\lambda\rho(\lambda\rho(M)) = \lambda\rho(M) \tag{5}$$

There is a Galois connection between the power sets of O and A, that mean there are two functions ρ and λ from the power set of A to power set of O and from the power set of O to the power set of A such that:

$$\text{if } M \subseteq N \text{ then } \rho(M) \supseteq \rho(N)$$

$$\text{if } X \subseteq Y \text{ then } \lambda(X) \supseteq \lambda(Y)$$

$$\begin{aligned} M &\subseteq \lambda\rho(M) \\ X &\subseteq \rho\lambda(X) \\ \rho\lambda\rho &= \rho \\ \lambda\rho\lambda &= \lambda \end{aligned}$$

$$\rho(\bigcup_j M_j) = \bigcap_j \rho(M_j)$$

$$\lambda(\bigcup_j X_j) = \bigcap_j \lambda(X_j)$$

A subset M of A (X of O) is called **closed** if $\lambda\rho(M) = M$ ($\rho\lambda(X) = X$). For all $M \subseteq A$, all $X \subseteq O$ the subset $\rho(M)$ and $\lambda(X)$ is closed and is called **generated by** M (**generated by** X).If M is closed and $\rho(M) = X$, then the cartesian product $C = X \times M$ is called a **concept** (a W-**concept**), and M is called the **intent** of C, denoted by $int(C)$, X is called the **extent** of C, denoted by $ext(C)$. In other words, a W-concept is a maximal rectangle inside the recognized world W. If $C = X \times M$ and $C' = X' \times M'$ and $X \subseteq X'$, then we say C is a subconcept of C' and denote it by $C \leq C'$, so we have a lattice of W-concepts with the operations $\vee$ and $\wedge$ on it.

$$(X \times M) \vee (Y \times N) = (X \cup Y) \times (M \cap N) \quad (6)$$

$$(X \times M) \wedge (Y \times N) = (X \cap Y) \times (M \cup N) \quad (7)$$

We denote the minimum and the maximum of the concept lattice by $W -$ *zero* and $W -$ *unit* or *zero* and *unit* (see [3]).

Example 1 *The recognized world consists of 7 objects with 11 possible attributes. We use the rows for the objects and the columns for the attributes.*

O

		♡	♡	♡						
		♡	♡	♡	♡					
		♡	♡	♡	♡	♡	♡	♡		
		♡	♡	♡	♡	♡			♡	♡
♡	♡	♡	♡	♡	♡	♡	♡		♡	
♡	♡	♡	♡	♡	♡	♡	♡	♡		♡

A

In Example 1 we can distinguish a concept (call it "clubsuit") as follows.

O

		♡	♡	♡						
		♡	♡	♡	♡					
		♣	♣	♣	♣	♣	♡	♡		
		♣	♣	♣	♣	♣			♡	♡
♡	♡	♣	♣	♣	♣	♣		♡		♡
♡	♡	♣	♣	♣	♣	♣	♡		♡	

A

W contains all places filled by ♡ or ♣. The concept consists of all places filled by ♣, which is a maximal rectangle in the object/attribute matrix. Just the names of the concepts and the relation $\leq$ are not enough to reconstruct the recognized world W. For this reconstruction we need some new relations. Suppose a is an element of A and C a concept. Then we stipulate that $axi(C, a)$ holds if $int(C)$ is a closed subset of $\{a\}$. (Here $axi(C, a)$ stands for: '$\{a\}$ is an axiom of C.) $axi(C,a)$ holds iff it holds that if C is a maximum concept with intent containing a (that mean $int(C)$ is minimum containing a). Suppose x is an element of O and C is a concept, then we stipulate that $gen(x,C)$ holds if $ext(C)$ is closed subset of $\{x\}$ ($gen(x,C)$ stands for: ' $\{x\}$ is generator of $ext(C)$'). Thus, $gen(x,C)$ holds iff C is a minimum concept with extent containing x, so x is in $ext(C)$. See [17] for the operations γ and μ. We define $imp(C,C')$ as the relation, which holds between C and C' iff $C \leq C'$ and there is not another C" such that $C \leq C$" and C" $\leq C'$. Thus, the relation $\leq$ is the transitive closure of the relation imp. We find that if $axi(C,a)$ and $imp(C,C')$ and $gen(C',x)$ then $(a,x) \in C \cap C'$. In this way we can reconstruct every concept from the three relations axi, gen, imp. Notice that C in the three relations is only a symbol, not a concept (as a rectangle in the W), whence it is called a **concept symbol**. Instead of concept lattice the machine has to put into its memory three relations, but the first two are functions.

Def. The three relations together are called the **triple** of W or W-**triple**.

Notice that the diagram of a concept lattice (see [18]) is the map of "*imp*", and that there are two different worlds, with the same diagram of concept lattice, but with different triples.

Example 2 *[Triple $\mathbf{T}_C$ of the world in Example 1]*

Here are the concepts with its extents:

C_1	with extent	$\{x_1\}$
C_2	with extent	$\{x_2\}$
C_3	with extent	$\{x_3\}$
C_4	with extent	$\{x_4\}$
C_5	with extent	$\{x_1,x_2\}$
C_6	with extent	$\{x_1,x_3\}$
C_7	with extent	$\{x_1,x_4\}$
C_8	with extent	$\{x_2,x_3\}$
C_9	with extent	$\{x_2,x_4\}$
C_{10}	with extent	$\{x_1,x_2,x_3,x_4\}$
C_{11}	with extent	$\{x_1,x_2,x_3,x_4,x_5\}$
C_{12}	with extent	$\{x_1,x_2,x_3,x_4,x_5,x_6\}$

Here are the triple of the concepts (the *gen*, *axi*, *imp* relations for the concepts):

$gen(x_1,C_1)$, $gen(x_2,C_2)$, $gen(x_3,C_3)$, $gen(x_4,C_4)$, $gen(x_5,C_{11})$, $gen(x_6,C_{12})$,

$axi(C_5,a_1)$, $axi(C_5,a_2)$, $axi(C_{12},a_3)$, $axi(C_{12},a_4)$, $axi(C_{12},a_5)$, $axi(C_{11},a_6)$, $axi(C_{10},a_7)$, $axi(C_7,a_8)$, $axi(C_9,a_9)$, $axi(C_6,a_{10})$, $axi(C_8,a_{11})$,

$imp(C_1,C_5)$, $imp(C_1,C_6)$, $imp(C_1,C_7)$, $imp(C_2,C_5)$, $imp(C_2,C_8)$,

$imp(C_2,C_9)$, $imp(C_3,C_6)$, $imp(C_3,C_8)$, $imp(C_4,C_7)$, $imp(C_4,C_9)$,

$imp(C_5,C_{10})$, $imp(C_6,C_{10})$, $imp(C_7,C_{10})$, $imp(C_8,C_{10})$, $imp(C_9,C_{10})$,

$imp(C_{10}, C_{11})$, $imp(C_{11}, C_{12})$.

The three relations can be represented in a schema as follows (where the rectangles represent the diagram of concept lattice, the arrows between the rectangles represent the *imp* relation, the arrows from the circles to the rectangles represent the *gen* relation, and the arrows from the rectangles to the circles represent the *axi* relation).

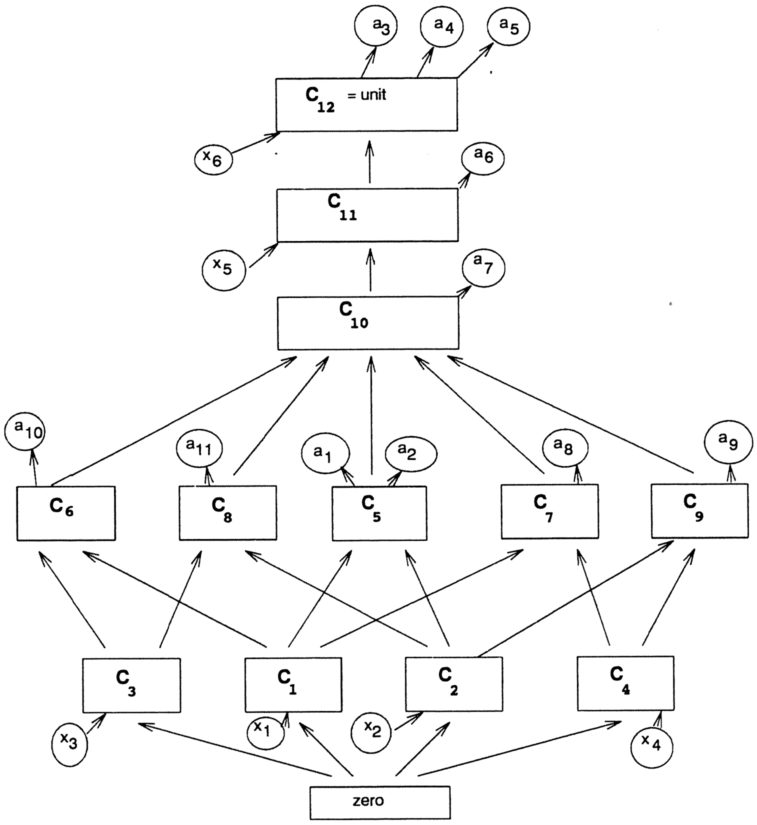


Diagram of the Triple T_C

Example 3 *[Triple of diagonal]* Δ *is the diagonal in the* $B \times B$*.Then the* Δ*-concept is the rectangle* $\{b\} \times \{b\}$ *for all* b *belonging to* B*. Thus, the* Δ*-triple consists of* $gen(b, \{b\} \times \{b\})$, $axi(\{b\} \times \{b\}, b)$ *and nothing for "imp", except for* $imp(zero, .)$ *and* $imp(., unit)$.

Example 4 *[Triple of partition]*

If W *is an equivalence relation on* B*, then a* W*-concept is a rectangle* $M \times M$*,where* M *is an equivalence class, and the* W*-triple consists of* $gen(b, M \times M)$, $axi(M \times M, b)$ *for all* $b \in M, M$ *is an equivalence class and nothing for "imp", except for* $imp(zero, .)$ *and* $imp(., unit)$*. We see that the triple is "isomorphic" to triple of the diagonal of* B/W.

Example 5 *[Triple of function] Suppose* W *is the map of a function* f *from* O *to* A*.Then the* W*-triple consists of* $gen(x, X \times \{a\})$, $axi(X \times \{a\}, a)$*, where* X *is* $f^{-1}(a)$*, and nothing for "imp", except for* $imp(zero, .)$ *and* $imp(., unit)$.

Having constructed the three relations *axi*, *gen*, *imp* and put them in memory, the machine can reconstruct all the concepts and therefore reconstruct the recognized world W. The three relations together form **realized world**.

DUALITY.

Notice that A and O play the same role, so we can exchange them and arrive at the dual situation. This show that *axi* and *gen* are dual concepts.

ANTICONCEPT.

Analogously, considering the relation $O \times A - W$, we have the definition of **anticoncept**, **antiintent**,**antiextent**, **antiaxi**, **antigen**, **antiimp**. And from the last three relations we can reconstruct the relation $O \times A - W$ and then W itself.

4 Building Concept Towers

From the world W we constructed the three relations *gen*, *axi*, *imp* and now we can construct the triple for these relations, and so on. Notice that since *gen* and *axi* are fuctions, so we can temporally ignore them. After having constructed the "tower" of concepts, we attach them again to the tower. First of all, let us look at the following example:

Example 6 *[the triples of triples]. (see Example 2).*

Constructing the triple $\mathbf{T}_D$ of the diagram of $\mathbf{T}_C$, we have 12 concepts as follows:

D_1	is	$\{C_1\}$	$\times$	$\{C_5, C_6, C_7\}$
D_2	is	$\{C_2\}$	$\times$	$\{C_5, C_8, C_9\}$
D_3	is	$\{C_3\}$	$\times$	$\{C_6, C_8\}$
D_4	is	$\{C_4\}$	$\times$	$\{C_7, C_9\}$
D_5	is	$\{C_5, C_6, C_7, C_8, C_9\}$	$\times$	$\{C_{10}\}$
D_6	is	$\{C_{10}\}$	$\times$	$\{C_{11}\}$
D_7	is	$\{C_{11}\}$	$\times$	$\{C_{12}\}$
D_8	is	$\{C_1, C_2\}$	$\times$	$\{C_5\}$
D_9	is	$\{C_1, C_3\}$	$\times$	$\{C_6\}$
D_{10}	is	$\{C_1, C_4\}$	$\times$	$\{C_7\}$
D_{11}	is	$\{C_2, C_3\}$	$\times$	$\{C_8\}$
D_{12}	is	$\{C_2, C_4\}$	$\times$	$\{C_9\}$

and the triple $\mathbf{T}_D$ of these concepts is :
$gen(C_1, D_1)$, $gen(C_2, D_2)$, $gen(C_3, D_3)$, $gen(C_4, D_4)$, $gen(C_5, D_5)$, $gen(C_6, D_5)$, $gen(C_7, D_5)$, $gen(C_8, D_5)$, $gen(C_9, D_5)$, $gen(C_{10}, D_6)$, $gen(C_{11}, D_7)$, $gen(C_{12}, unit)$
$axi(D_1, zero)$, $axi(D_2, zero)$, $axi(D_3, zero)$, $axi(D_4, zero)$, $axi(D_5, C_{10})$, $axi(D_6, C_{11})$, $axi(D_7, C_{12})$, $axi(D_8, C_5)$, $axi(D_9, C_6)$, $axi(D_{10}, C_7)$, $axi(D_{11}, C_8)$, $axi(D_{12}, C_9)$ $imp(D_1, D_8)$, $imp(D_1, D_9)$, $imp(D_1, D_{10})$, $imp(D_2, D_8)$, $imp(D_2, D_{11})$, $imp(D_2, D_{12})$, $imp(D_3, D_9)$, $imp(D_3, D_{11})$, $imp(D_4, D_{10})$, $imp(D_4, D_{12})$

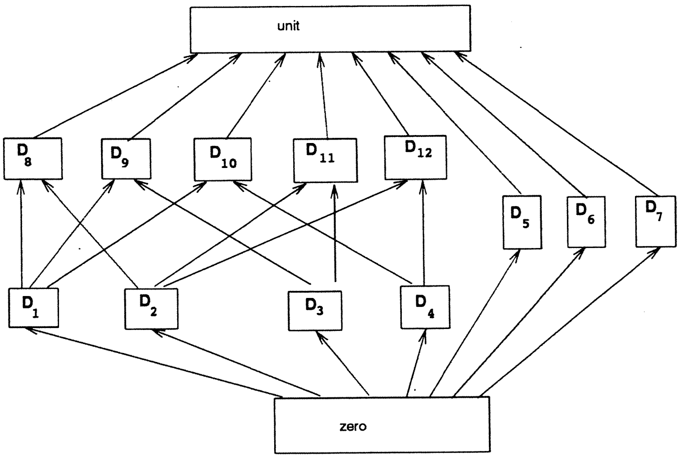


Diagram of the concept lattice in the triple T_D

Constructing the triple $\mathbf{T}_E$ of the diagram of the triple $\mathbf{T}_D$ above, we have 9 concepts:

E_1 is $\{D_1\} \times \{D_8, D_9, D_{10}\}$ E_2 is $\{D_2\} \times \{D_8, D_{11}, D_{12}\}$
E_3 is $\{D_3\} \times \{D_9, D_{11}\}$ E_4 is $\{D_4\} \times \{D_{10}, D_{12}\}$
E_5 is $\{D_1, D_3\} \times \{D_9\}$ E_6 is $\{D_1, D_4\} \times \{D_{10}\}$
E_7 is $\{D_2, D_3\} \times \{D_{11}\}$ E_8 is $\{D_2, D_4\} \times \{D_{12}\}$
E_9 is $\{D_1, D_2\} \times \{D_8\}$

and the triple $\mathbf{T}_E$ of these concepts is:
$gen(D_1, E_1)$, $gen(D_2, E_2)$, $gen(D_3, E_3)$, $gen(D_4, E_4)$, $gen(D_5, unit)$, $gen(D_6, unit)$, $gen(D_7, unit)$, $gen(D_8, unit)$, $gen(D_9, unit)$, $gen(D_{10}, unit)$, $gen(D_{11}, unit)$,
$axi(E_9, D_8)$, $axi(E_5, D_9)$, $axi(E_6, D_{10})$, $axi(E_7, D_{11})$, $axi(E_8, D_{12})$, $axi(zero, D_1)$, $axi(zero, D_2)$, $axi(zero, D_3)$, $axi(zero, D_4)$, $axi(zero, D_5)$, $axi(zero, D_6)$, $axi(zero, D_7)$,
$imp(E_1, E_5)$, $imp(E_1, E_6)$, $imp(E_2, E_7)$, $imp(E_2, E_8)$, $imp(E_3, E_5)$, $imp(E_3, E_7)$, $imp(E_4, E_6)$, $imp(E_4, E_8)$

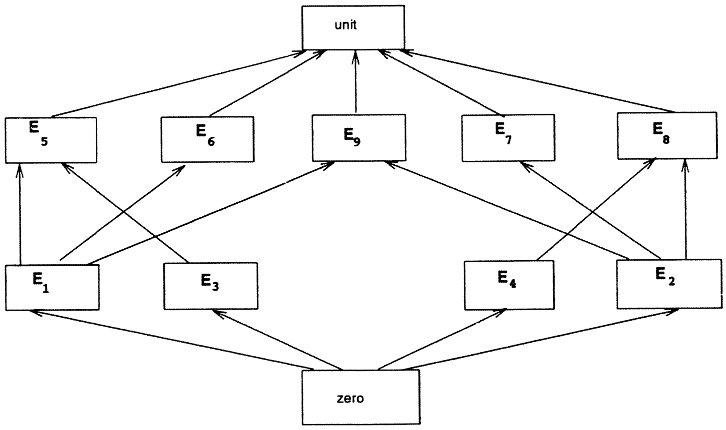


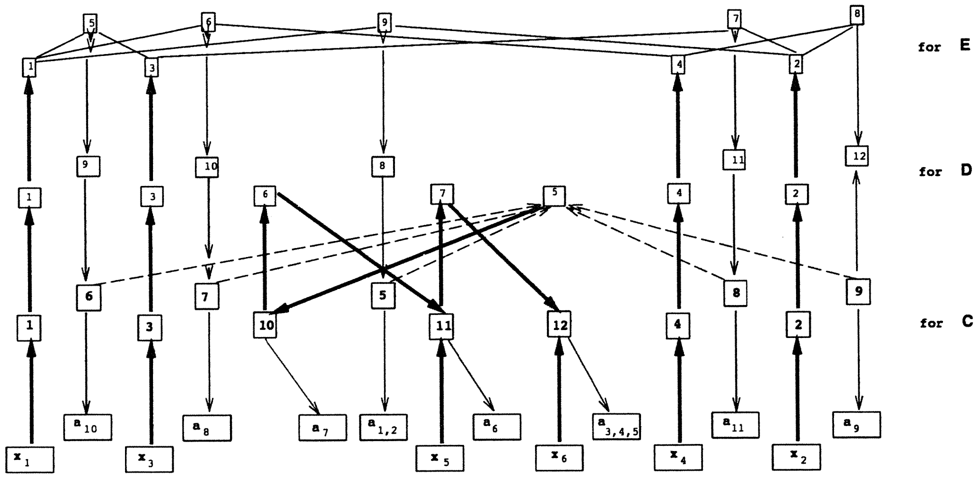
Diagram of the concept lattice of the triple T_E

If we perform the construction once more, then we have 9 concepts, with the same arrow pattern (the same function *imp*).

Question: for which triple, does it hold that it and its triple are the same ?.

Def. The diagram of all the *axi*-relations, all the *gen*-relations, all the *imp*-relations is called the **Concept Tower of the world** W. And the diagram of all the *axi*-relations, *gen*-relations and the last *imp*-relation is called the **Skeleton of the Concept Tower.**

Let us look at the Skeleton of the Concept Tower of W from example 1, in the following figure:



The Skeleton of Concept Tower related with the world W

(The scheme without true's and false's)

The Skeleton of the Concept Tower of W gives us "complete" information about W.

Proposition 1 **From the *axi*-functions,*gen*-functions,and the last *imp*-relation we can reconstruct the world** W.

For example,

$x_1 \stackrel{gen}{\rightarrow} C_1 \stackrel{gen}{\rightarrow} D_1 \stackrel{gen}{\rightarrow} E_1 \stackrel{imp}{\rightarrow} E_5 \stackrel{axi}{\rightarrow} D_9 \stackrel{axi}{\rightarrow} C_6 \stackrel{gen}{\rightarrow} D_5 \stackrel{axi}{\rightarrow} C_{10} \stackrel{gen}{\rightarrow} D_6 \stackrel{axi}{\rightarrow} C_{11} \stackrel{axi}{\rightarrow} a_6$

Notice that $imp(C, C')$ holds iff there are D, D' such that $gen(C, D)$, $imp(D, D')$, and $axi(D', C')$.

Def. In a diagram of concept lattice the pair of sets of concept symbols $C_1, ..., C_k$ and $C'_1, ..., C'_n$ is called a **section** of the diagram iff:

(a) if $imp(C, C')$ is in the diagram, and C is in the first set then C' is in the second set

(b) if $imp(C, C')$ is in the diagram, and C' is in the second set then C is in the first set

(c) considering the relation *imp* in the union of the two sets as a graph, then the graph is completely connected (that is, for any "note" C and C' in the graph there is a path from C to C').

Lemma 2 **For two concept symbols C and C' in one set of a section there is not more than one concept symbol C" in the other set connected with both of C and C' by *imp*.**

Proof. If C and C' are in the first set and $imp(C, C")$ and $imp(C', C")$ then $C" = C \vee C'$, if C and C' are in the second set and $imp(C", C)$ and $imp(C", C')$ then $C" = C \wedge C'$.

Proposition 3 **If in a section each part has more than one concept symbol then the diagram of the section is isomorphic to itself.**

Proof. By the lemma each new concept D is defined by one concept symbol C in the first set (so we can refer to D by C_*), or by one concept symbol C in the second set (so we can refer to D by C^*). And we have $imp(C'_*, C^*)$ iff $imp(C', C)$. Thus the section is isomorphic to its diagram of concept lattice.

Proposition 4 **The diagram of a section has only one element iff one of its parts has only one element**

Notice that in the triple of a diagram of concept lattice, the concept symbols in two different sections are not "related" by *imp*. Thus we have:

Proposition 5 **In the triple of a diagram of concept lattice, a path from *zero* to *unit* goes through at most two concepts.**

Because of Proposition 5 no tower has more than 3 stages.

Notice that a *Ferrers relation* W (satisfying the condition: if $(x, a), (y, b) \in W$ then (x, b) or (y, a) is in W) has a chain as its concept lattice (see [23]), the concept lattice of this chain is a set – no relation *imp*, except for concections with *unit* or *zero*), and the concept lattice of that set has only one concept,except *zero* and *unit*.

5 Accepting a New Fact

Now we consider the situation, in which a machine receives a new fact (x, a) and changes its memory to "incorporate the new fact".

First of all, the machine looks for any concept C such that $gen(x, C)$.

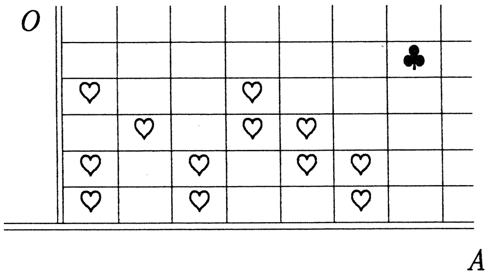
(2.A).

If it fail, then x is not in the recognized object set A, so the machine looks for any $axi(C', a)$.

(2.A.a).

If it fails, too, then a is not in the recognized attribute set O, (that is a is new, too, and (x, a) stands apart in the recognized world). Inthis situation the machine have only two news items: $axi(C^*, a)$, $gen(x, C^*)$, and nothing for *imp*.

Example 7 *The new fact (x, a) is on the upper right corner.*



The new concept is the rectangle $\{x\} \times \{a\}$

(2.A.b).

If there is $axi(C, a)$, we have only one new concept C', and the machine put the following into its memory :

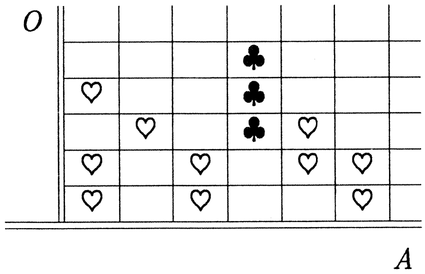
$axi(C', a)$, (and erases $axi(C, a)$),

$gen(x, C')$,

$gen(y, C')$, (iff there is $gen(y, C)$) and we erase $gen(y, C)$,

$imp(C, C')$ (iff there exists b and C' such that $b \neq a$, $axi(C', b)$ and either C' is C or $imp(C', C)$).

Example 8 *The new fact (x, a) is in the first row.*



C is the 'clubsuit' rectangle.

(2.B).

If there is $gen(x, C)$. The machine looks for $axi(C', a)$, too.

(2.B.a).

If the search fails we have the dual of case (2.A.a).

(2.B.b).

If, on the other hand $axi(C', a)$ is found, we have the following subcases.

(2.B.b.i).

If there is $axi(C, a)$ and $gen(x, C')$ and $Imp(C, C')$ then (x, a) is in C, and is in C',too.This mean that (x, a) is already in W. In this case nothing has to be changed.

(2.B.b.ii).

In the other case, there is something to be changed

Example 9 *The new fact (x, a) is the "clubsuit" in the site (3,2).*

O						
♡			♡			
	♡		♡	♡		
♡	♣	♡		♡	♡	
♡		♡			♡	

A

The machine now has to change all concepts concerned with a or x.

The algorithm is complex but the idea is simple: only a new concept C^* is created if there are two concepts C' and $C"$ such that $int(C') = \{a\} \cup int(C")$ and $ext(C') \cup \{x\} = ext(C")$ and $C^* = ext(C") \times int(C')$. Then we also have $imp(C', C^*)$ and all $imp(C^+, C^*)$ such that $imp(C^+, C")$.

In this case we have 2 new concepts

$\{x_2\} \times \{a_1, a_2, a_3, a_5, a_6\}$

$\{x_2, x_3\} \times \{a_2, a_5\}$

So,we have:

Proposition 6 **There is an algorithm for adding one more fact into the machine's triple.**

Erasing facts.

Using anticoncepts we can erase a fact.

6 Operations on Triples

We have the definition of W-triple. Now we give some definitions of operations on triples.

Def. A **Preconcept** is a rectangle in the product $O \times A$

Def. A **Preconcept System** is a set of preconcepts on $O \times A$ closed under the operations $\vee$ and $\wedge$.

Def. A **Pretriple** on $O \times A$ is a set of three relations $\{att, isa, sub\}$:

$$att \subseteq \mathbf{C} \times A$$

$$isa \subseteq O \times \mathbf{C}$$

where $\mathbf{C}$ is a new set,and it becomes an ordered set with the relation sub.

(a) For an element $C \in \mathbf{C}$ we denote by:

$Dom(C)$ the set $\{x \mid \exists C' \text{ such that} : isa(x, C') \text{ and } sub(C', C)\}$

$Att(C)$ the set $\{a \mid \exists C'$ such that : $att(C', a)$ and $sub(C, C')\}$

Then a name C define a preconcept $rec(C) = Dom(C) \times Att(C)$. Denote by W the union of all the concepts.

(b) Construct new preconcepts from the preconcepts $rec(C)$ with the name defined by the name of the preconcepts and the sign $\vee$ and $\wedge$, and construct the corresponding relation *att*, *isa*, *sub*. We have a new preconcept system.

We have the proposition:

Proposition 7 **In a preconcept system,each rectangle is maximal for the $\subseteq$ iff it is a W-concept.**

Proof. Of course, any W-concept is maximal for the $\subseteq$. Now let $X \times M$ is maximal.Suppose the W-concept $Y \times N$ contain the $X \times M$.For $(y, b) \in Y \times N$, there is a rectangle $R_{y,b}$ containing (y, b).Then

$$\text{for all } y \; R_y = \wedge_b R_{y,b} \text{ contains all } (y, b) \in Y \times N$$

$$R = \vee_y R_y \text{ contains all}(y, b) \in Y \times N$$

R is belong to the preconcept system and $X \times M$ is maximal, then $R = X \times M$

Proposition 8 **There exists an algorithm to construct a triple of the world defined by a pretriple.**

(1) Construct the rectangles defined by the preconcept.

(2) Construct the world W beeing the union of the rectangles, construct the preconcept system (the names of new concepts are constructed by the elements of **C** and the signs $\vee$ and $\wedge$)

(3) From the preconcepts (old and new), all maximal preconcept are marked.

(4) Construct three relation axi, gen, imp.

The three relation axi, gen, imp is defined as following:

For a name C of a marked rectangle we have $axi(C, a)$ iff

$Att(C)$ is smallest of all C' such that $att(C', a)$,

and in the same way we have $gen(x, C)$ iff

$Dom(C)$ is smallest of all C' such that $isa(x, C')$,

and we have $imp(C, C')$ iff

$Dom(C) \subseteq Dom(C')$ and there is not another $C"$ such that $Imp(C, C")$ and $Imp(C", C')$.

Then $\{axi, gen, imp\}$ is the triple of the world W Now, we give the definition of "union" and the "intersection" of two triples.

Defs.

(1.a) **Union** of two preconcept systems $\{R_i \mid i \in I\}$ and $\{R_k \mid k \in K\}$ is the preconcept system generated by $\{R_j \mid j \in (I \cup K)$, and denote it by the sign "∪"

(1.b) **Intersection** of two preconcept system $\{R_i \mid i \in I\}$ and $\{R_k \mid k \in K\}$ is the preconcept system generated by $\{R_{(i,k)} \mid i \in I, k \in K\}$ and denote it by the sign "∩"

(1.c) **Cartesian Product** of the two preconcept system in $O \times A$ and in $U \times B$ consist of all $(X \times Y) \times (M \times N)$ iff $X \times M$ and $Y \times N$ are the preconcepts of the two system.Denote it by the sign "×"

(1.d) **Disjoint Union** of the two preconcept system is the union of them when consider $O \cap U = \emptyset$ and $A \cap B = \emptyset$

For example, the case (2.A.a) in the example 7 of the section 5: the new preconcept system is disjoint union of the old one and the rectangle $\{x\}\times\{a\}$.

(1.e) **Join** of the two preconcept system on $O \times A$ and on $A \times B$ is generated by all rectangles $X \times H$ such that there are two concepts $X \times M$ and $N \times H$ satisfying the condition: $(M \cap N) \neq \emptyset$

Then the join of the two preconcept system on $O \times A$ and on $A \times B$ consists of all rectangles $X \times H$ such that $(x, b) \in X \times H$ iff there exists $a \in A$ satisfying the conditions:" (x, a) in a rectangle of the first system,and (a, b) in a rectangle of the second one".

(1.f) **Iterative** of the preconcept system on $A \times A$ is the result of the "join" enough times such that it become stable under the "join".

From the operations of preconcept systems we have the corresponding operations of triples.

(2.a) **A Morphism** from the triple (gen, axi, imp) on $(O,A, \mathbf{C})$ to the triple (gen', axi', imp') on $(O', A', \mathbf{C'})$ is a mapping from O to O', from A to A', from **C** to **C'** such that:

$f(unit) = unit$

$f(zero) = zero$

if $gen(x, C)$ then $gen'(fx, fC)$

if $axi(C, a)$ then $axi'(fC, fa)$

if $Imp(C, C')$ then $Imp(fC, fC')$.

(2.b) **Cartesian Product** of two triples T_1 and T_2 is a triple such that:

$gen((x,y),(C,D))$	iff	$gen(x,C)$	and	$gen(y,D)$
$axi((C,D),(a,b))$	iff	$axi(C,a)$	and	$axi(D,b)$
$imp((C,D),(C',D'))$	iff	$imp(C,C')$	and	$imp(D,D')$

We have the projections pr_1, pr_2 from the triple to T_1 and to T_2:

Proposition 9 **All triples with their morphisms form the category of triples, which has cartesian product** . *(see [2]).*

(2.c) Suppose W and V are two worlds on $O \times A$ and on $U \times B$, and $O \cap U = \emptyset$,

$A \cap B=\emptyset$. The $W \cup V$-triple is called the **disjoint union** of the W-triple and the V-triple.

For example, the case (2.A.a) in example 8 of section 5: the new triple is the disjoint union of the old one and the $\{x\} \times \{a\}$-triple.

(2.d) Suppose W and V are two worlds on $O \times A$ and on $O \times B$, and $A \cap B=\emptyset$. The $W \cup V$-triple is called the **halfdisjoint union** of the W-triple and the V-triple. (analogously, we have a halfdisjoint union of two triples on $O \times A$ and on $U \times A$). We have:

Proposition 10 **Suppose W and V are worlds on $O \times A$ and on $O \times B$, then a subset X of O is an *extent* of a $W \cup V$-concept C iff there are W-concept C' and V-concept $C"$ such that $X = ext(C') \cap ext(C")$ $(int(C) = (int(C') \cup int(C"))$**

Proof. Suppose that $C = X \times (M \cup N)$ is a $W \cup V$-concept (M, N is subset of A, B), then $X \times M \subseteq W$ and $X \times N \subseteq V$. Let denote by $C' = X' \times M'$ and $C" = X" \times N"$ the concepts generated by X in W and in V, then $X' \supset X$, $M' \supset M$, $X" \supset X$, $N" \supset N$. Denote by Y the intersection of X' and $X"$ then we have $Y \times (M' \cup N")$ is in $W \cup V$. Thus $Y = X$,and $M' = M$, $N" = N$.

Vice versa, let $C' = X' \times M'$ be a W-concept and $C" = X" \times N"$ be V-concept, and $X = X' \cap X"$. Then $X \times (M' \cup N") \subseteq W \cup V$. If $\{x\} \times (M' \cup N") \subseteq (W \cup V)$ then $\{x\} \times M' \subseteq W$, and $x \in X'$, analoguosly, $x \in X"$. We have $X = X' \cap X"$.

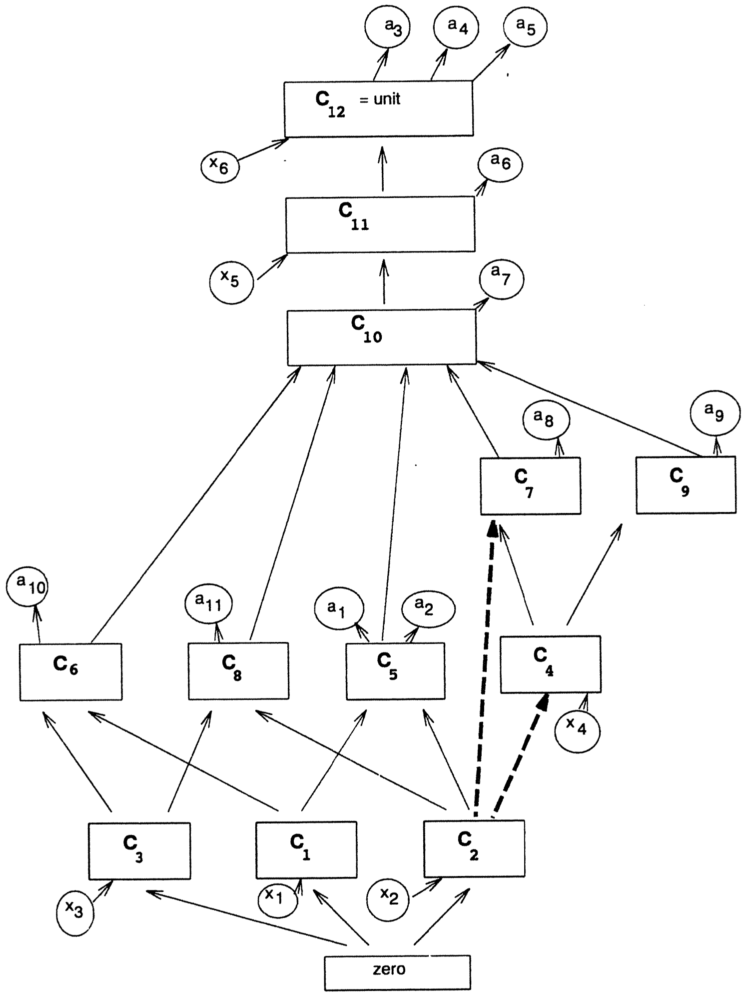
Notice that if C' is a W-concept then C' and *unit* define a $W \cup V$-concept, which has the same *int* as C', so we can denote it by C' itself.

In the section 3, the case (2.A.b) as an example for cohalfdisjoint union and the case (2.B.a) as an example for halfdisjoint union.

7 Accepting a New Rule

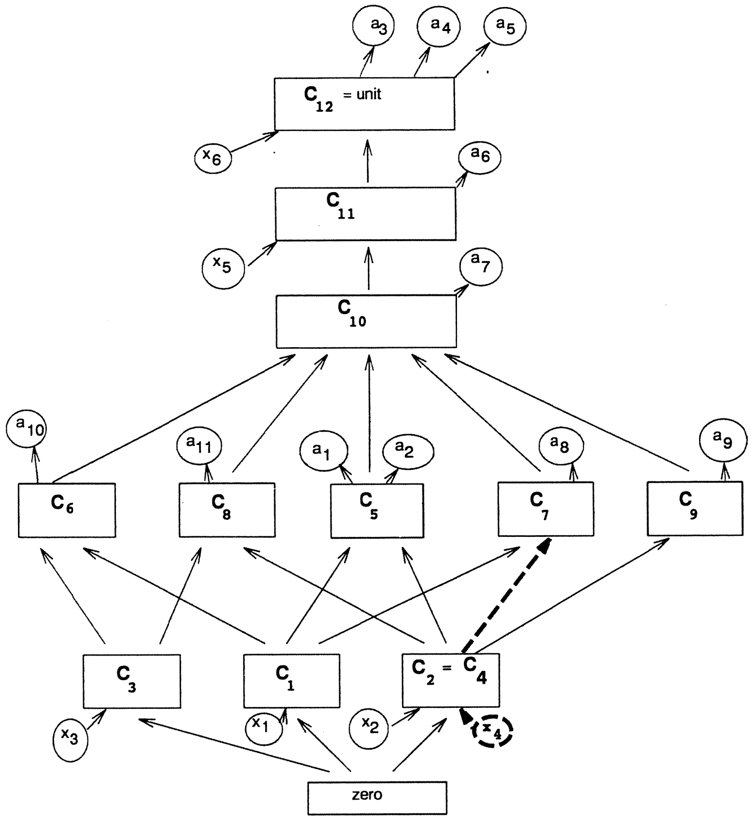
Suppose W is a relation on $O \times A$, and T_1 is its triple with the diagram of concept lattice is D_1, now we add a rule $C_0 \leftarrow C_1, ..., C_k$,that is the intersection of the concepts $C_1, ..., C_k$ has its

domain in the domain of C_0. If $ext(C_1) \cap ... \cap ext(C_k) \in ext(C_0)$ then there is nothing to be changed. See the example 2, the rule $C_9 \leftarrow C_5, C_8$. If $ext(C_1) \cap ... \cap ext(C_k)$ and $ext(C_0)$ is uncompairable in the lattice then we add the edge from $C_1 \wedge ... \wedge C_k$ to C_0. See example 2, the rule $C_4 \leftarrow C_5, C_8$. The edge from C_2 (which is the concept $C_5 \wedge C_8$) to C_4 is added, and the edges from *zero* to C_4 and from C_2 to C_9 erased.The effect of tihs is that the facts (x_2, a_8) and (x_2, a_9) are added to the world W.



Add the rule $C_4 \longleftarrow C_5$, C_8

If there is an edge from C_0 to $C_1 \wedge ... \wedge C_k$ in the concept lattice then the two concepts are merged into one. In the example situation the rule $C_2 \leftarrow C_7, C_9$ will identify two concepts C_2 and C_4.



Add one more rule $C_2 \longleftarrow C_7$, C_9

If the rule introduces a new concept then it add the new concept into the diagram the suitable edges, for example the added rule is $C_{13} \leftarrow C_6, C_{14}$,then we add three concepts C_{13}, C_{14} and $C_6 \wedge C_{14}$ with three edges : from $C_6 \wedge C_{14}$ to C_6, to C_{13}, to C_{14}.Thus, **we can get one more fact or rule at any stage in the Skeleton of concept tower.**

8 Structure on the Attribute Set

Now we introduce some extra structure on the attribute set, in order to cope with more complex cases.

8.1 Cartesian Product

Instead of A, now, we consider the product $A_1 \times A_2$. For any h of A_2 we have a world section $W_h = \{(x, a) \mid (x, a, h) \in W\}$

For this relation we have a triple $T_h = (axi_h, gen_h, imp_h)$. For the subset H of A_2, we construct from the triple (axi_h, gen_h, imp_h) the intersection T_H^* of them. The intersection triple define the world, which is the intersection of the worlds defined by the component triples. If we start out from the definition of concept on $(O \times A_1) \times A_2$, a concept is a pair of a subset of A_2 and a triple on $O \times A_1$. If we consider the product $O \times (A_1 \times A_2)$, then we have the situation that the concept is a pair consisting of a subset of O and a triple on $A_1 \times A_2$. Thus we arrive at the following four cases:

1- the product $O \times A_1 \times A_2$

2- the product $(O \times A_2) \times A_1$

3- the product $(O \times A_1) \times A_2$

4- the product $O \times (A_1 \times A_2)$

With every product we can consider two kinds of concepts depending on our choice of objects and attributes. Thus we have 7 kinds of triples for this world. For the first case, we consider the concept as a maximal 3-dimension rectangle inside W. For the second, the third and the fourth case we consider the concept as a pair of a set and a triple.

8.2 Cover

Now, with A we consider a cover of A, that is a set $A_1, ..., A_n$ of subsets of A such that A is the union of $A_1, ..., A_n$. And we have the relation W^* as follows:

$$W^* = W \cup \{(o, A_i) \mid \exists a \in A_i : (o, a) \in W\} \tag{8}$$

The second part of W^* is denoted by W'. A W'-concept (its intent) is a relation scheme, the W'-triple is the database scheme. Let $R = X \times \{A_{i_1}, ..., A_{i_k}\}$ are W'-triple and $B = \bigcup_{m=1}^{k} A_{i_m}$, the $W \cap (X \times B)$ is called a relation r over R. Then we arrive at r-relation. (see [8]).

9 Using Concept Triples in Reasoning: Logic Programs

We use the notation of Apt [1], Lloyd [9], [10], [11] with some modifications. Suppose our first-order language has an alphabet **without function symbols**. Instead of the terms we have concepts in a triple. In fact, every program will be related to a particular concept triple. The alphabet consists of

-constants (including objects and attributes of the triple)

-concept symbols of the triple that the program is about

-variables

-relation symbols

-connectives $\vee$, $\wedge$, $\leftarrow$

-punctuation symbols

If p is an n-ary relation symbol and $C_1, ..., C_n$ are variables or concept symbols then $p(C_1, ..., C_n)$ is an **atom**.

A **clause** is a list of the following form:

$$A_1, ..., A_k \leftarrow B_1, ..., B_n$$

where $A_1, ..., A_k, B_1, ..., B_n$ are atoms.

$A_1, ..., A_k$ are called **conclusions** of the clause, $B_1, ..., B_n$ are called **premises** of the clause. A disjunctive logic program is a finite nonempty set of clauses, with a nonempty conclusions (see [15]). In the general case as in normal logic programs we have to work with an infinite set of terms, in our application to triples we only have a finite set of concepts.

We consider the function *Imp* in a triple. The following program defines it:

$Imp(X, X) \leftarrow$

$Imp(X, Y) \leftarrow imp(X, Y)$

$Imp(X, Y) \leftarrow imp(X, Z), Imp(Z, Y)$

$Imp(0, X) \leftarrow$

$Imp(X, 1) \leftarrow$

$\neg Imp(1, 0) \leftarrow$

where $1, 0$ are *unit* and *zero* of the concept lattice.

Semantically, the *Imp* make concepts ordered set, but we can't prove this by mean of the program.

As another example, we consider the predicate "equality".

$equ(X, X) \leftarrow$

$equ(X, Y) \leftarrow equ(Y, X)$

$equ(X, Y) \leftarrow Imp(X, Y), Imp(Y, X)$

$equ(X, Y) \leftarrow equ(X, Z), equ(Z, Y)$

$imp(X, Y) \leftarrow imp(X, Z), equ(Z, Y)$

$imp(X, Y) \leftarrow imp(Z, Y), equ(X, Z)$

$\neg equ(0, 1)$

$\neg equ(X, Y) \leftarrow imp(X, Y)$

$\neg equ(X, Y) \leftarrow \neg Imp(X, Y)$

$\neg equ(X, Y) \leftarrow imp(X, Z), Imp(Z, Y)$

Next example: a program for the funtion $fac(x, a)$ (it gives the facts (x, a) in our triple). The program is :

$fac(x, a) \leftarrow gen(x, X), Imp(X, Y), axi(Y, a)$

The fuction $obj(x, X)$ (for: x is an object of concept X) has the following program:

$obj(x, X) \leftarrow gen(x, Y), Imp(Y, X)$

Next, we consider the operations $\vee$ and $\wedge$ in the concept lattice. The program for *vee*, which defines $\vee$:

$upv(X,Y,Z) \leftarrow Imp(X,Z), Imp(Y,Z)$

$Imp(X,Z) \leftarrow upv(X,Y,Z)$

$Imp(Y,Z) \leftarrow upv(X,Y,Z)$

$\neg upv(X,Y,Z) \leftarrow \neg Imp(X,Z)$

$\neg upv(X,Y,Z) \leftarrow \neg Imp(Y,Z)$

$upv(X,Y,Z) \leftarrow vee(X,Y,Z)$

$\neg vee(X,Y,Z) \leftarrow upt(X,Y,Z)$

$Imp(U,V) \leftarrow vee(X,Y,U), upv(X,Y,V)$

Similarly, for $\wedge$.

Now suppose we have two worlds W and V, W on $O \times A$, and V on $U \times B$. A relation between W and V is given by a program .

(i) For example, if $U = O$ and $A \cap B = \emptyset$ then we can take the halfdisjoint union of two triples, as in the following program.

$gen(x,(X \wedge Y)) \leftarrow gen(x,X), gen(x,Y)$

$axi((X \wedge 1), a) \leftarrow axi(X,a)$

$axi((1 \wedge Y, b) \leftarrow axi(Y,b)$

$Imp((X \wedge Y),(X' \wedge Y')) \leftarrow Imp(X \wedge X'), Imp(Y \wedge Y')$

$imp((X \wedge Y),(X' \wedge Y)), equ((X \wedge Y),(X' \wedge Y)) \leftarrow imp(X,X')$

$imp((X \wedge Y),(X \wedge Y')), equ((X \wedge Y),(X \wedge Y')) \leftarrow imp(Y,Y')$

$obj(x,(X \wedge Y)) \leftarrow obj(x,X), obj(x,Y)$

$\neg obj(x,(X \wedge Y) \leftarrow \neg obj(x,X)$

$\neg obj(x,(X \wedge Y) \leftarrow \neg obj(x,Y)$

(ii) If $A = B$, and $O \cap U = \emptyset$, we have the cohalfdisjoint union. See the following program.

$gen(x,(X \vee 0)) \leftarrow gen(x,X)$

$gen(u,(0 \vee Y)) \leftarrow gen(u,Y)$

$axi((X,Y),a) \leftarrow axi(X,a), axi(Y,a)$

$Imp(X \vee Y, X' \vee Y') \leftarrow Imp(X,X'), Imp(Y,Y')$

$imp(X \vee Y, X' \vee Y), equ(X \vee Y, X' \vee Y) \leftarrow imp(X,X')$

$imp(X \vee Y, X \vee Y'), equ(X \vee Y, X \vee Y') \leftarrow imp(Y,Y')$

...

(iii) The disjoint union of a W-triple and a V-triple is given by :

$gen(x,X) \leftarrow gen_W(x,X)$

$gen(x, X) \leftarrow gen_V(x, X)$

$\neg gen(x, X) \leftarrow \neg gen_W(x, X), \neg gen_V(x, X)$

and similarly for *axi* and for *imp*.

(iv) The join of a triple T_W on $O \times A$ and a triple T_V on $A \times B$ is given by:

$gen(x, X) \leftarrow gen_W(x, X)$

$\neg gen(x, X) \leftarrow \neg gen_W(x, X)$

$axi(X, b) \leftarrow axi_V(X, b)$

$\neg axi(X, b) \leftarrow \neg gen_V(X, b)$

$imp(X, Y) \leftarrow imp_W(X, Y)$

$imp(X, Y) \leftarrow imp_V(X, Y)$

$imp(X, Y) \leftarrow axi_W(X, a), gen_V(a, Y).$

(v) Consider an extreme halfdisjoint union of T_W and T_V: for any V-concept D there is a W-concept of C such that $ext(C) = ext(D)$. In this case the triple of $T_{W \cup V}$ is the result of adding axi_V to the triple T_W. We say: there is a dependency of T_C from T_D.

(vi) Consider another extreme of the halfdisjoint union notion: for any W-concept C and V-concept D $ext(C) \neq ext(D)$. In this case the triple $T_{W \cup V}$ is the cartesian product of T_W and T_V. We say that T_C and T_D are independent.

9.1 Monadic Predicate Logic

9.1.1 Atoms, Conjunctions and Universal Quantifications

Consider A as a set of "relation symbols", and $(x, a) \in W$ as a holds in x. Then a W-concept $X \times M$ is considered as a pair of of theory M and its model X: all universal propositions belonging to M are valid in X and all universal propositions valid in X are in the set M. Suppose $a_1, ..., a_n$ are in A, and M in the closed set of $\{a_1, ..., a_n\}$, and $b_1, ..., b_k \in M$, we write

$$a_1 \& ... \& a_n \quad \rightarrow \quad b_1 \& ... \& b_k \tag{9}$$

This formula has the meaning of the following formula of monadic predicate logic:

$$\forall x (a_1(x) \& ... \& a_n(x) \rightarrow b_1(x) \& ... \& b_k(x)) \tag{10}$$

We have used $\subseteq$, $\cup$, $\cap$ for sets, $\leq$, $\vee$, $\wedge$ *unit*, *zero* for latices and now we use $\rightarrow$, $|$, $\&$ for implication, disjunction and conjunction in monadic predicate logic.

The $\rightarrow$-relation has the following properties:

(1) **reflex**

$$a_1 \& ... \& a_n \rightarrow a_1 \& ... \& a_n$$

(2) **union**

$$\begin{array}{lll} if & & a_1\&...\&a_n \rightarrow b_1\&...\&b_k \\ & and & a_1\&...\&a_n \rightarrow c_1\&...\&c_l \\ then & & a_1\&...\&a_n \rightarrow b_1\&...\&b_k\&c_1\&...\&c_l \end{array}$$

(3) **cut**

$$\begin{array}{ll} if & a_1\&...\&a_n \rightarrow b_1\&...\&b_i\&...\&b_k \\ then & a_1\&...\&a_n \rightarrow b_1\&...\&b_i \end{array}$$

9.1.2 Negations

Now for any $a \in A$ we add a new sign $\neg a$. Along with the world W we consider the world $W_\neg$ defined as follows:

$$\text{for all } a \in A \ (x, \neg a) \in W_\neg \text{ iff } (x, a) \notin W \tag{11}$$

Then we can form $W \cup W_\neg$-concept $X \times (M \cup N_\neg)$ when (x, a) with $a \in M$ mean that a holds in x and (x, a) with $a \in M_\neg$ mean that a does not hold in x. By this way, the concept is considered as a "model" for the "propositions" $a_1 \wedge ... \wedge a_n \wedge \neg b_1 \wedge ... \wedge \neg b_k$ with $M = \{a_1, ..., a_n\}$ and $N = \{b_1, ..., b_k\}$. See [14].

9.1.3 Disjunctions and Existential quantifications

Suppose $M = \{a_1, ..., a_n\}$ is a subset of A then we add to A a new element M^* and add to W some new "facts" as in the part "**Cover**" of section 8.2 (8). Thus M^* has the same meaning as the disjunction $a_1|...|a_n$ See [15]. Thus we have:

Proposition 11 **If stating from $W \subseteq O \times A$ we add $W_\neg$, then consider the power set of $M \cup M_\neg$, and construct W" as above, we get the Boolean Lattice of W"-concepts.***(see [3]).*

By duality, we have the existential case: if for the subset $x_1, ..., x_n$ of O we add a new object X^*, then the fact that a holds at X^* means that $\exists i$ such that a holds at $\{x_i\}$.

9.1.4 Possibility

See [6]. In this part we consider $W \cup W_\neg$. Suppose $a_1, ..., a_n$ is a subset of A and M is its $W \cup W_\neg$-closed set. Assume $M_\neg$ is the intersection of M and $A_\neg$. If $b_1, ..., b_k$ is a subset of A and for all i: $\neg b_i \notin M_\neg$, we write:

$$a_1\&...\&a_n \quad \hookrightarrow \quad b_1\&...\&b_k \tag{12}$$

Then $\hookrightarrow$ means the "possibility", or in other words:

$$\exists x(a_1(x)\&...\&a_n(x)\&b_1(x))\& \quad ... \quad \&\exists x(a_1(x)\&...\&a_n(x)\&b_k(x)) \quad (13)$$

This $\hookrightarrow$-relation has the same three properties as $\rightarrow$ in 9.2.1 .

9.1.5 Higher-order logic

We have interpretes of first-order logic in concept lattices. Instead of concept lattices we consider concept towers have shall have intepretes of higher-order logic, because we shall work with concepts of concepts,ect...See section (4), we have, for examples:

$$D_9\&D_{10} \rightarrow D_8$$

$$E_6\&E_9 \rightarrow E_5$$

(instead of $a_1\&...\&a_n$, we write C iff $int(C)$ is closure of $\{a_1, ..., a_n\}$.)

9.1.6 Counterfactual Conditions

Consider the example of "penguins are birds, birds fly,penguins do not fly". See [6] One way to think about this situation is as follows. Think of: " If there weren't any being characterized by 'penguin', then birds fly". The counterfactual condition in this proposition is about the world, not about objects. Then we consider here conditions about the world: something is included into or excluded from the world. Now we consider the case with $\neg$.

(1) Suppose $a_1, ..., a_n \in A$ and M is its closure set, and $X \times M$ is a concept C. From W we exclude all facts of x such that $gen(x, C)$, and we arrive at W^-. If in the W^--concept we have a 'theorem' p, then we write:

$$excl(a_1\&...\&a_n) \quad \rightsquigarrow \quad p \qquad (14)$$

(If W^- is an unreal world, then $excl(a_1\&...\&a_n)$ is a counterfactual condition). Thus, we can write:

$$excl(penguins) \rightsquigarrow (birdsfly)$$

See example 1: $\{a_5, a_6\}$ defines the concept C:

$$C = \{x_1, x_2, x_3, x_4, x_5\} \times \{a_3, a_4, a_5, a_6\}$$

and $gen(x_5, C)$, and if erase x_5 from the world then $\{a_5, a_6\}$ defines the concept

$$\{x_1, x_2, x_3, x_4\} \times \{a_3, a_4, a_5, a_6, a_7\}$$

then we have:

$$excl(a_5\&a_6) \rightsquigarrow (a_3\&a_6 \rightarrow a_7)$$

(2) Suppose p is a formula (as a rule). If the world W accepts p as a new rule (see section 7) we have an unreal world W^+, and if in W^+ we have a valid formula q, then we write:

$$incl(p) \quad \leadsto q \tag{15}$$

For the examples in section 7, we can write:

$$incl(C_5 \& C_6 \to C_4) \leadsto (C_2 \to C_4)$$
$$incl(C_5 \& C_6 \to C_4, C_7 \& C_9 \to C_2) \leadsto (C_4 \to C_5)$$

10 Conclusion

This concludes our introduction to the use of lattice theory in formal concept formation. In fact, the approach outlined here has many more connections with theories of concept analysis from philosophy and artificial intelligence than we have been able to point out. It is our hope that the lattice theoretic approach to the subject outlined in this paper may one day serve as a common framework to compare and evaluate such theories.

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