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A correction: orthogonal representations and connectivity of graphs

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Abstract

This note corrects an error in the proof of the main result of the authors' paper "Orthogonal Representations and Connectivity of Graphs", which appeared in Linear Algebra and its Applications 114/115 (1989) 439–454. © 2000 Elsevier Science Inc. All rights reserved.

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In this note, we correct an error in the proof of the main theorem of [1]. Let $G = (V, E)$ be an undirected graph. A *d-dimensional orthogonal representation* of G is a map $f : V \longrightarrow \mathbb{R}^d$, such that $\langle f(u), f(v)\rangle = 0$ for all pairs u, v of nonadjacent nodes, where $\langle x, y\rangle$ denotes the usual inner product. An *orthonormal representation* is an orthogonal representation in which $\|f(v)\| = 1$ for all $v \in V$. The representation is in *general position* if for any $W \subseteq V$ with $|W| = d$, the set $\{f(v) : v \in W\}$ is linearly independent. The main theorem of [1] was the following.

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Theorem 1 [1, Theorem 1.1]. *If G is a graph with n nodes and $d \geqslant 1$ is an integer, then the following are equivalent*:

(i) *G is (vertex) $(n-d)$-connected*;

(ii) *G has a general-position orthogonal representation in $\mathbb{R}^d$*;

(iii) *G has an orthonormal representation in $\mathbb{R}^d$ such that for each node v, the vectors representing the nodes nonadjacent to v are linearly independent.*

The easy proof that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) was given correctly in the original paper, but the harder proof that (i) $\Rightarrow$ (ii) was incorrectly given. We review that proof, indicate the error, and correct it.

In what follows, if A is a subset of $\mathbb{R}^d$, $A^{\perp} = \{v \in \mathbb{R}^d : \langle v, a\rangle = 0\ \forall a \in A\}$ is the subspace orthogonal to A and $U(A)$ is the set of unit vectors of A. We will need the standard fact that, if A is a subspace, then there is a unique probability measure defined on $U(A)$ which is invariant under any unitary transformation of A, which we call the *uniform distribution* on $U(A)$, denoted u_A.

If G is $(n-d)$-connected, then G has minimum degree at least $n-d$. The following randomized procedure constructs a d-dimensional orthonormal representation for any graph G of minimum degree $n-d$. Fix an ordering $(v_1, v_2, \ldots, v_n)$ of V and choose $f(v_1), f(v_2), \ldots$ sequentially as follows. Select $f(v_1)$ according to the distribution $u_{\mathbb{R}^d}$. For $j \in \{2, \ldots, n\}$, having chosen $f(v_1), \ldots, f(v_{j-1})$, let $W_j = \{v_i : i < j, (v_i, v_j) \notin E\}$ and let $M_j = \{f(v_i) : v_i \in W_j\}^{\perp}$. Since v_j has at most $d-1$ non-neighbors in G, $\dim(M_j) \geqslant 1$. Choose $f(v_j)$ according to u_{M_j}. This process clearly produces an orthonormal representation of G. Theorem 1 follows from:

Theorem 2 [1, Theorem 1.2]. *If G is $(n-d)$-connected, the representation produced by the algorithm is in general position with probability* 1.

For any vertex subset W of size d, let D_W be the set of orthogonal representations f such that $\{f(w) : w \in W\}$ is linearly dependent. It is enough to show that $\mathrm{Prob}[D_W] = 0$ for all W of size d. Let us first note that this is easy for $W_0 = \{v_1, \ldots, v_d\}$. $\mathrm{Prob}[D_{W_0}] \leqslant \sum_{j=2}^{d} \mathrm{Prob}[f(v_j) \in \mathrm{span}(\{f(v_i) : i < j\})]$, and each of the terms in the sum is 0. To see this, observe first that $f(v_j)$ is chosen according to u_{M_j} and $\dim(M_j) = d - |\{v_i : i < j, (v_i, v_j) \notin E\}| \geqslant 1 + |\{v_i : i < j, (v_i, v_j) \in E\}|$. Letting $g_j(v_i)$ denote the orthogonal projection of $f(v_i)$ onto M_j, the space $\mathrm{span}(\{f(v_i) : i < j\}) \cap M_j$ is contained in (in fact, equal to) $\mathrm{span}(\{g_j(v_i) : i < j, (v_i, v_j) \in E\})$, whose dimension is strictly smaller than that of M_j.

For a permutation σ of $\{1, \ldots, n\}$, let μ_σ denote the probability distribution on orthonormal representations obtained by running the above algorithm with the vertices considered in the order $v_{\sigma(1)}, \ldots, v_{\sigma(n)}$. When σ is the identity, we write μ for μ_σ. Lemma 1.3 in [1] asserted that the distributions μ_σ are the same for all σ. This is enough to complete the proof of Theorem 2 since for any W of size d, we can choose σ such that $W = \{v_{\sigma(1)}, \ldots, v_{\sigma(d)}\}$ and then we have $\mu[D_W] = \mu_\sigma[D_W] = 0$.

Unfortunately, Lemma 1.3 is false; for example, let G be the path on v_1, v_2, v_3, v_4, and $d = 3$. When the vertices are processed by the algorithm in the natural order, $f(v_1)$ and $f(v_2)$ are independent as random variables, but when processed in the order v_4, v_1, v_2, v_3 they are not.

We replace Lemma 1.3 by a statement that is weaker, but is still strong enough to use in the argument of the previous paragraph to complete the proof of Theorem 2. Two probability measures μ and ν on the same probability space S are *mutually absolutely continuous* (mac) if for any measurable subset A of S, $\mu(A) = 0$ if and only if $\nu(A) = 0$. We show the following.

Lemma 3. *For any two vertex orderings σ and τ, μ_σ and μ_τ are* mac.

The proof of Lemma 3 is similar to the false proof of Lemma 1.3, diverging only at the end (although we have modified some of the notation from the original paper for precision and clarity). If σ is a permutation and v, w are vertices with $v = v_{\sigma(r)}$ and $w = v_{\sigma(s)}$, then *swapping v and w in σ* produces the permutation τ that is the same as σ except that $\tau(r) = \sigma(s)$ and $\tau(s) = \sigma(r)$.

It suffices to prove that for all j between 1 and $n - 1$, if τ is obtained from σ by swapping $v_{\sigma(j)}$ and $v_{\sigma(j+1)}$, then σ and τ are mac. We prove this by induction on j, with the base case and the induction step proved together.

Fix $j \geqslant 1$. For ease of notation we assume, without loss of generality, that σ is the identity permutation. For $1 \leqslant i \leqslant n$, let $V_i = \{v_1, \ldots, v_i\}$. We consider two cases depending on whether v_j and v_{j+1} are joined by a path that lies entirely in V_{j+1}.

Suppose first that there is such a path. Let P be a shortest such path and t be its length (number of edges). So $t \leqslant j$. For fixed j, we argue by induction on t. If $t = 1$, then $(v_j, v_{j+1}) \in E$. When conditioned on $\{f(v_1), \ldots, f(v_{j-1})\}$, $f(v_j)$ and $f(v_{j+1})$ are independent for both distributions μ_σ and μ_τ. Thus $\mu_\sigma = \mu_\tau$. Suppose that $t > 1$ and let v_i be any internal node of P. Now transform σ to τ by the following steps:

1. Obtain σ^1 by swapping v_i and v_j in σ. Since this can be obtained by successive adjacent swaps among the first j elements, μ_σ and μ_{σ^1} are mac by the induction hypothesis on j.
2. Obtain σ^2 from σ^1 by swapping v_i and v_{j+1}. By the induction hypothesis on t, μ_{σ^2} and μ_{σ^1} are mac.
3. Obtain σ^3 from σ^2 by swapping v_{j+1} and v_j. As in (1), μ_{σ^3} and μ_{σ^2} are mac.
4. Obtain σ^4 from σ^3 by swapping v_j and v_i. As in (2), μ_{σ^4} and μ_{σ^3} are mac.
5. Obtain τ from σ^4 by swapping v_{j+1} and v_i. As in (1), μ_τ and μ_{σ^4} are mac.

Thus μ_σ and μ_τ are mac, to complete the case that V_{j+1} contains a path from v_j to v_{j+1}.

Now assume that there is no path connecting v_j to v_{j+1} in V_{j+1}. This means that $C = V - V_{j+1}$ is a cut set, and thus $j + 1 = |V_{j+1}| \leqslant d$. Thus we can partition V_{j-1} into two sets A_j and A_{j+1} so that for $i \in \{j, j+1\}$, A_i contains all neighbors of v_i in V_{j-1}, and there are no edges from A_j to A_{j+1}.

We want to compare the distributions of μ_σ and μ_τ. For $1 \leqslant i \leqslant n$, let μ_σ^i (resp. μ_τ^i) denote the marginal distribution function induced on $f(v_1), \ldots, f(v_i)$. Note that it suffices to prove that μ_σ^{j+1} and v_τ^{j+1} are mac, since conditioned on any given assignment $f(v_1), \ldots, f(v_{j+1})$ the distributions μ_σ and μ_τ are identical.

Also, note that the marginal distributions μ_σ^{j-1} and μ_τ^{j-1} are identical. Let $x_1, \ldots, x_{j-1}$ be an arbitrary selection of vectors for the first $j-1$ vertices. Condition the two distributions μ_σ^{j+1} and μ_τ^{j+1} on $f(v_1) = x_1, \ldots, f(v_{j-1}) = x_{j-1}$. This yields two distributions ν_σ and ν_τ over pairs $(f(v_j), f(v_{j+1}))$ of vectors. It suffices to show that ν_σ and ν_τ are mac.

For $i \in \{j, j+1\}$, let L_i be the subspace spanned by $f(A_i)$. Then L_j and L_{j+1} are orthogonal (since there are no edges between A_j and A_{j+1}). Let M be the orthogonal complement of $L_j \oplus L_{j+1}$ in $\mathbb{R}^d$, so that $\dim(M) \geqslant 2$ and $L_j \oplus L_{j+1} \oplus M$ is an orthogonal decomposition of $\mathbb{R}^d$. We refine this decomposition further. For $i \in \{j, j+1\}$, let B_i be the set of vertices of A_i that are not adjacent to v_i. Let K_i be the subspace spanned by $f(B_i)$ and let H_i be the orthogonal complement of K_i in L_i. Then $M \oplus K_j \oplus H_j \oplus K_{j+1} \oplus H_{j+1}$ is an orthogonal decomposition of $\mathbb{R}^d$.

With this notation, we can describe the distribution ν_σ as follows: $f(v_j)$ is selected according to the distribution $u_{M \oplus H_j}$ and $f(v_{j+1})$ is selected according to the distribution $u_{(M \oplus H_{j+1}) \cap f(v_j)^\perp} = u_{(M \cap f(v_j)^\perp) \oplus H_{j+1})}$. Similarly, ν_τ can be described as follows: $f(v_{j+1})$ is selected according to the distribution $u_{M \oplus H_{j+1}}$ and $f(v_j)$ is selected according to the distribution $u_{(M \oplus H_j) \cap f(v_{j+1})^\perp} = u_{(M \cap f(v_{j+1})^\perp) \oplus H_j)}$.

Simplifying the notation, (letting $X_0 = M$, $X_1 = H_j$ and $X_2 = H_{j+1}$ and letting $k = \dim(M \oplus H_j \oplus H_{j+1})$) we are left to prove the following.

Lemma 4. *Let $X_0 \oplus X_1 \oplus X_2$ be an orthogonal decomposition of $\mathbb{R}^k$ for some k, with $\dim(X_i) = c_i$, and $c_0 \geqslant 2$. Let A be the subset of $\mathbb{R}^k \times \mathbb{R}^k$ consisting of pairs (x_1, x_2) such that $x_1 \in U(X_0 \oplus X_1)$ and $x_2 \in U(X_0 \oplus X_2)$ and $\langle x_1, x_2 \rangle = 0$. Let λ_1 be the distribution on A which first selects x_1 according to $u_{X_0 \oplus X_1}$ and then selects x_2 according to $u_{(X_0 \cap \{x_1\}^\perp) \oplus X_2}$. Let λ_2 be the distribution which first selects x_2 according to $u_{X_0 \oplus X_2}$ and then selects x_1 according to $u_{(X_0 \cap \{x_2\}^\perp) \oplus X_1}$. Then λ_1 and λ_2 are* mac.

Proof. We first consider the special case that X_1 and X_2 are both the $\mathbf{0}$ subspace. In that case, A is the set of pairs (x_1, x_2), where $x_1, x_2 \in U(X_0)$ are perpendicular. The invariance of the uniform distribution under unitary transformations implies that λ_1 is invariant under unitary transformations. Thus the marginal distribution of λ_1 induced on x_2 is u_{X_0} and the conditional distribution on x_1 given x_2 is uniform on $U(X_0 \cap \{x_2\}^\perp)$. Thus $\lambda_1 = \lambda_2$. Let us denote the common distribution on $U(X_0) \times U(X_0)$ in this case by κ.

Next we consider the general case. Observe that if Y and Z are orthogonal spaces, a vector $U(Y \oplus Z)$ can be written uniquelyin the form $y \cos\theta + z \sin\theta$, where

$y \in U(Y)$, $z \in U(Z)$ and $\theta \in [0, \pi/2]$. Uniform selection from $U(Y \oplus Z)$ can be described by the following process for choosing (y, z, θ): independently select y according to u_Y, z according to u_Z and select θ according to a distribution that depends only on $a = \dim(Y)$ and $b = \dim(Z)$ and will be denoted by $\zeta_{a,b}$. If $\dim(Z) = 0$ then $\theta = 0$ with probability 1. If $a, b \geqslant 1$, the only thing we need about $\zeta_{a,b}$ is that it is mac with respect to the uniform distribution on the interval $[0, \pi/2]$.

Similarly, a point $(x_1, x_2) \in A$ can be described as $(y_1 \sin\theta_1 + z_1 \cos\theta_1, y_2 \sin\theta_2 + z_2 \cos\theta_2)$, where $\theta_1, \theta_2 \in [0, \pi/2]$, $y_1, y_2 \in U(X_0)$ with y_1 orthogonal to y_2 and $z_1 \in U(X_1)$ and $z_2 \in U(X_2)$.

The distribution λ_1 can be described as the product of five independent distributions: z_1 is chosen according to u_{X_1}, z_2 is chosen according to u_{X_2}, (y_1, y_2) is selected according to κ, θ_1 is selected according to ζ_{c_0,c_1} and θ_2 is selected according to ζ_{c_0-1,c_2}. The distribution λ_2 is described similarly except that θ_2 is selected according to ζ_{c_0,c_2} and θ_1 is selected according to ζ_{c_0-1,c_1}.

Since $c_0 \geqslant 2$, we have that ζ_{c_0,c_2} and ζ_{c_0-1,c_2} are mac and ζ_{c_0,c_1} and ζ_{c_0-1,c_1} are mac, from which we deduce that λ_1 and λ_2 are mac. This completes the proof of Lemma 4, which in turn completes the proofs of Lemma 3 and the theorem. $\square$

Remarks.

1. The conclusion of Lemma 4 fails if $c_0 = 1$. In this case, if x_1 is secected first that $x_1 \notin X_1$ (which happens with probability 1), we have $x_2 \in X_2$, which has probability 0 if x_1 and x_2 are chosen in the reverse order.
2. The error in the original paper was not to take into account that the distributions $\zeta_{a,b}$ are different for different values of a and b.

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Reference

[1] L. Lovász, M. Saks, A. Schrijver, Orthogonal representations and connectivity of graphs, Linear Algebra Appl. 114/115 (1989) 439–454.