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# On the Ergodicity Conditions of the Hitting Point Process of the Semi-Homogeneous, Zero-Drift Random Walk on the First Quadrant

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**Abstract**

This note corrects a flaw in a part of the proof of the ergodicity conditions for the hitting point process as presented in [1]; the theorem II.2.7.1 in [1] formulating these conditions is correct.

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## 1. Introduction

In section II.2.5 of [1] the hitting pont process $\{\mathbf{k}_m, m = 1, 2, \ldots\}$ of the semi-homogeneous random walk on the first quadrant has been introduced. Here

$$\mathbf{k}_m = (\mathbf{k}_m^{(1)}, \mathbf{k}_m^{(2)}), \quad \mathbf{k}_m^{(1)}\mathbf{k}_m^{(2)} = 0, \tag{1.1}$$

is the $m$th hitting point of the random walk with its boundary, which is formed by the lattice points of the coordinate axes of the first quadrant.

In section II.2.7 the ergodicity conditions for the $\mathbf{k}_m$-process of the random walk with *zero drift* have been investigated and formulated in theorem II.2.7.1. As it has been pointed out to the author by Professor Stam the proof of that theorem is not complete. In the present note we shall complete the proof of theorem II.2.7.1; the formulation of this theorem is correct.

We first introduce some notation and remark that for symbols used here but not defined here the reader is referred to [1], similarly for the various assumptions. In this introduction we shall point out where the flow in the proof in [1] occurs.

Put for $m = 1, 2, \ldots,$

$$\begin{aligned} K_m^{(0)} &:= \Pr\{\mathbf{k}_m^{(1)} = \ \mathbf{k}_m^{(2)} = 0\}, \\ K_m^{(1)} &:= \Pr\{\mathbf{k}_m^{(1)} > 0, \ \mathbf{k}_m^{(2)} = 0\}, \\ K_m^{(2)} &:= \Pr\{\mathbf{k}_m^{(1)} = 0, \ \mathbf{k}_m^{(2)} > 0\}; \end{aligned} \tag{1.2}$$

so $K_m^{(0)}$ is the probability that the $m$th hitting point occurs at $(0,0)$, $K_m^{(1)}(K_m^{(2)})$ that it occurs at the positive horizontal (vertical) axis. Whenever the $\mathbf{k}_m$-process is positive recurrent then since it has been assumed in [1] that the random walk is aperiodic it is readily shown that the following limits exist and are positive.

$$\begin{aligned} K_j(1) \ &:= \lim_{m\to\infty} K_m^{(j)}, \quad j = 1, 2, \\ K_0 \ &:= \lim_{m\to\infty} K_m^{(0)}, \end{aligned} \tag{1.3}$$

and

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$$K_0 + K_1(1) + K_2(1) = 1. \tag{1.4}$$

In [1] the relations, cf. (II.2.7.9),

$$\begin{aligned}&\nu_0\, K_0 + \nu_1\, K_1(1) + (\nu_2 - 1)\, K_2(1) = 0, \\ &\mu_0\, K_0 + (\mu_1 - 1)\, K_1(1) + \mu_2\, K_2(1) = 0,\end{aligned} \tag{1.5}$$

have been derived from the functional relation (II.2.7.3). Actually this derivation only leads to (1.5) if the $\mathbf{k}_m$-process is positive recurrent. If it is not positive recurrent then $K_0 = 0$ and $K_1(\hat{p}_1), K_2(\hat{p}_2)$ both tend to zero for $\hat{p}_1 \to 1$, and so (II.2.7.3) provides in the limiting case no information, a fact which has unfortunately been overseen in the derivation of (II.2.7.9). Hence, no conclusions can be drawn from (1.4) and (1.5) to distinguish the null and nonrecurrent case.

To obtain the correct relations for the case that the $\mathbf{k}_m$-process is null or nonrecurrent, see the relations (2.9) below, one has to start from (II.2.5.17), by letting $\hat{p}_1 \to 1$ and then by equating to zero the coefficients of $q^m$ in the series expansion of (II.2.5.17) into powers of $q$. However these relations (2.9) can be obtained more directly by starting from theorem II.2.4.4, as it will be shown in the next section.

## 2. Om the proof of the ergodicity conditions

If the $\mathbf{k}_m$-process is positive recurrent then the limits in (II.2.7.2) are nonzero and it follows, as in section II.2.7, that

$$\nu_0\, K_0 + \nu_1\, K_1(1) + (\nu_2 - 1)\, K_2(1) = 0, \tag{2.6}$$

$$\mu_0\, K_0 + (\mu_1 - 1)\, K_1(1) + \mu_2\, K_2(1) = 0,$$

$$K_0 + K_1(1) + K_2(1) = 1, \tag{2.7}$$

with

$$K_0 > 0,\ K_1(1) > 0,\ K_2(1) > 0. \tag{2.8}$$

The relations (2.1) and (2.3) are not compatible if $\mu_1 \geq 1$ or $\nu_2 \geq 1$ or $D \leq 0$, where $D$ is defined by

$$D := (1 - \mu_1)(1 - \nu_2) - \mu_2 \nu_1. \tag{2.9}$$

Hence

$$\mathbf{k}_m = \text{“}+\text{”} \ \Rightarrow\ \mu_1 < 1,\ \nu_2 < 1,\ D > 0, \tag{2.10}$$

where $\mathbf{k}_m$ = "+" indicates that the $\mathbf{k}_m$-process is positive recurrent, analogously $\mathbf{k}_m$ = "0" and $\mathbf{k}_m$ = "—" stand for null- and nonrecurrent, see [1].

Next we derive a set of relations for the probabilities defined in (1.2). Suppose, cf. (II.2.1.19), that

$$\mathbf{k}_m^{(1)} > 0\ \mathbf{k}_m^{(2)} = 0, \quad \text{at hitting time } \mathbf{t}_m, \tag{2.11}$$

so that $\mathbf{k}_{m+1} = (\mathbf{k}_{m+1}^{(1)}, \mathbf{k}_{m+1}^{(2)})$ is the $(m+1)$th hitting point of the boundary in the zero drift random walk starting at the point

$$(\mathbf{k}_m^{(1)} + \boldsymbol{\xi}_{\mathbf{t}_m+1} - 1, \mathbf{k}_m^{(2)} + \boldsymbol{\eta}^{(1)}_{\mathbf{t}_m+1}).$$

By using theorem II.2.4.4.ii it follows that

$$\begin{aligned}&\mathrm{E}\{\mathbf{k}_{m+1}^{(1)} - \mathbf{k}_m^{(1)} | \mathbf{k}_m^{(1)} > 0,\ \mathbf{k}_m^{(2)} = 0\} = \mu_1 - 1, \\ &\mathrm{E}\{\mathbf{k}_{m+1}^{(2)} - \mathbf{k}_m^{(2)} | \mathbf{k}_m^{(1)} > 0,\ \mathbf{k}_m^{(2)} = 0\} = \nu_1.\end{aligned} \tag{2.12}$$

Analogously

$$\mathrm{E}\{\mathbf{k}_{m+1}^{(1)} - \mathbf{k}_m^{(1)} | \mathbf{k}_m^{(1)} = 0,\ \mathbf{k}_m^{(2)} > 0\} = \mu_2, \tag{2.13}$$

$$\mathrm{E}\{\mathbf{k}_{m+1}^{(2)} - \mathbf{k}_m^{(2)} | \mathbf{k}_m^{(1)} = 0,\ \mathbf{k}_m^{(2)} > 0\} = \nu_2 - 1,$$

$$\mathrm{E}\{\mathbf{k}_{m+1}^{(1)} - \mathbf{k}_m^{(1)} | \mathbf{k}_m^{(1)} = 0,\ \mathbf{k}_m^{(2)} = 0\} = \mu_0,$$

$$\mathrm{E}\{\mathbf{k}_{m+1}^{(2)} - \mathbf{k}_m^{(2)} | \mathbf{k}_m^{(1)} = 0,\ \mathbf{k}_m^{(2)} = 0\} = \nu_0.$$

Hence it follows from (1.2), (2.7) and (2.8) that for $m = 1, 2, \ldots$,

$$\nu_0 K_m^{(0)} + \nu_1 K_m^{(1)} + (\nu_2 - 1) K_m^{(2)} = \mathrm{E}\{\mathbf{k}_{m+1}^{(2)} - \mathbf{k}_m^{(2)}\}, \tag{2.14}$$

$$\mu_0 K_m^{(0)} + (\mu_1 - 1) K_m^{(1)} + \mu_2 K_m^{(2)} = \mathrm{E}\{\mathbf{k}_{m+1}^{(1)} - \mathbf{k}_m^{(1)}\},$$

$$K_m^{(0)} + K_m^{(1)} + K_m^{(2)} = 1. \tag{2.15}$$

Put

$$\tilde{D} := D + \mu_0(1 + \nu_1 - \nu_2) + \nu_0(1 + \mu_2 - \mu_1). \tag{2.16}$$

Obviously, the ergodicity conditions for the $\mathbf{k}_m$-process are independent of $\mu_0$ and $\nu_0$ if at least $\mu_0 > 0$ or $\nu_0 > 0$; if $\mu_0 = \nu_0 = 0$ then the state (0,0) is absorbing. Hence in investigating the ergodicity conditions it is no restriction to assume that

$$\tilde{D} \neq 0. \tag{2.17}$$

It is also not a restriction to assume that

$$\mathbf{k}_1^{(1)} = 0, \quad \mathbf{k}_2^{(2)} = 0, \tag{2.18}$$

as it will be done in the following analysis, it implies that (0,0) is the starting point of the $\mathbf{k}_m$-process.
From (2.9),..., (2.12) it follows for $m = 1, 2, \ldots$,

$$\tilde{D} K_m^{(0)} = (1 + \nu_1 - \nu_2)[\mathrm{E}\{\mathbf{k}_{m+1}^{(1)} - \mathbf{k}_m^{(1)}\}] + (1 + \mu_2 - \mu_1)[\mathrm{E}\{\mathbf{k}_{m+1}^{(2)} - \mathbf{k}_m^{(2)}\}] + D. \tag{2.19}$$

Hence from (2.13) and (2.14); for $m = 1, 2, \ldots$,

$$\tilde{D} \sum_{h=1}^{m} K_h^{(0)} = (1 + \nu_1 - \nu_2)\mathrm{E}\{\mathbf{k}_{m+1}^{(1)}\} + (1 + \mu_2 - \mu_1)\mathrm{E}\{\mathbf{k}_{m+1}^{(2)}\} + m\,D, \tag{2.20}$$

or equivalently, cf. (2.11), for $m = 1, 2, \ldots$,

$$D[-m + \sum_{h=1}^{m} K_h^{(0)}] \quad = (1 + \nu_1 - \nu_2)[\mathrm{E}\{\mathbf{k}_{m+1}^{(2)}\} - \mu_0 \sum_{h=1}^{m} K_h^{(0)}] + \tag{2.21}$$

$$+(1 + \mu_2 - \mu_1)[\mathrm{E}\{\mathbf{k}_{m+1}^{(2)}\} - \nu_0 \sum_{h=1}^{m} K_h^{(0)}]. \tag{2.22}$$

Because $1 > K_m^{(0)} \geq 0$, $m = 1, 2, \ldots$, it follows from

$$\frac{1}{m+1} \sum_{h=1}^{m+1} K_h^{(0)} - \frac{1}{m} \sum_{h=1}^{m} K_h^{(0)} = \frac{-1}{m+1} \frac{1}{m} \sum_{h=1}^{m} K_h^{(0)} + \frac{1}{m+1} K_{m+1}^{(0)},$$

that the following limit exists and

$$1 \geq \lim_{m\to\infty} \frac{1}{m} \sum_{h=1}^{m} K_h^{(0)} \geq 0. \tag{2.23}$$

The random walk has been assumed to be aperiodic, cf. section II.2.1, and so the $\mathbf{k}_m$-process is aperiodic, so that $K_m^{(0)}$ has a limit for $m \to \infty$ and it is wellknown that, cf. [2], p. 155,

$$\lim_{m\to\infty} K_m^{(0)} = \lim_{m\to\infty} \frac{1}{m} \sum_{h=1}^{m} K_h^{(0)}, \tag{2.24}$$

and

$$\mathbf{k}_m = \text{“—”} \vee \text{“0”} \Longleftrightarrow \lim_{m\to\infty} K_m^{(0)} = 0. \tag{2.25}$$

From (2.9), (2.13) and (2.18) it follows that

$$v_0 \lim_{m\leftarrow\infty} K_m^{(0)} + v_1 \lim_{m\to\infty} \frac{1}{m} \sum_{h=1}^{m} K_h^{(1)} + (v_2 - 1) \lim_{m\to\infty} \frac{1}{m} \sum_{m\to\infty} K_h^{(2)} = \lim_{m\to\infty} \frac{1}{m} \mathrm{E}\{\mathbf{k}_m^{(2)}\}, \tag{2.26}$$

$$\mu_0 \lim_{m\to\infty} K_m^{(0)} + (\mu_1 - 1) \lim_{m\to\infty} \frac{1}{m} \sum_{h=1}^{m} K_h^{(1)} + \mu_2 \lim_{m\to\infty} \frac{1}{m} \sum_{h=1}^{m} K_h^{(2)} = \lim_{m\to\infty} \frac{1}{m} \mathrm{E}\{\mathbf{k}_m^{(1)}\}.$$

$$\lim_{m\to\infty} K_m^{(0)} + \lim_{m\to\infty} \frac{1}{m} \sum_{h=1}^{m} K_h^{(1)} + \lim_{m\to\infty} \frac{1}{m} \sum_{h=1}^{m} K_h^{(2)} = 1,$$

the existence of the limits in the lefthand side of (2.20) follows as before, and retails the existence of the limits in the righthand side of (2.20).

From (2.15) and (2.18) it follows that

$$\tilde{D} \lim_{m\to\infty} K_m^{(0)} = D + (1 + v_1 - v_2) \lim_{m\to\infty} \frac{1}{m} \mathrm{E}\{\mathbf{k}_{m+1}^{(1)}\} + (1 + \mu_2 - \mu_1) \lim_{m\to\infty} \frac{1}{m} \mathrm{E}\{\mathbf{k}_{m+1}^{(2)}\}. \tag{2.27}$$

Because, cf. (2.11),

$$D > 0, \quad \mu_1 \leq 1, \quad v_2 \leq 1 \Rightarrow \tilde{D} > 0, \quad \mu_1 < 1, \quad v_2 < 1, \tag{2.28}$$

it follows from (2.21) that

$$D > 0, \quad \mu_1 < 1, \quad v_2 < 1 \Rightarrow \lim_{m\to\infty} K_m(0) > 0 \Longleftrightarrow \mathbf{k}_m = \text{“+”}. \tag{2.29}$$

Consequently we have from (2.5) and (2.23),

$$\mathbf{k}_m = \text{“+”} \Longleftrightarrow D > 0, \quad \mu_1 < 1, \quad v_2 < 1. \tag{2.30}$$

Next we consider the case with

$$D = 0, \quad \mu_1 < 1, \quad v_2 < 1. \tag{2.31}$$

It then follows from (2.16) that: for $m = 1, 2, \ldots$,

$$(1 + v_1 - v_2)\Big[\frac{\mathrm{E}\{\mathbf{k}_{m+1}^{(1)}\}}{\sum_{h=1}^{m} K_h^{(0)}} - \mu_0\Big] + (1 + \mu_2 - \mu_1)\Big[\frac{\mathrm{E}\{\mathbf{k}_{m+1}^{(2)}\}}{\sum_{h=1}^{m} K_h^{(0)}} - v_0\Big] = 0, \tag{2.32}$$

and since the quotients are not negative

$$0 < \lim_{m\to\infty} \frac{\mathrm{E}\{\mathbf{k}_{m+1}^{(j)}\}}{\sum_{h=1}^{m} K_h^{(0)}} < \infty. \tag{2.33}$$

Put

$$\tilde{\mathbf{k}}_m := \mathbf{k}_m^{(1)} + \mathbf{k}_m^{(2)}, \qquad m = 1, 2, \ldots, \tag{2.34}$$

so that, cf. (1.1), for $n = 1, 2, \ldots$,

$$\Pr\{\tilde{\mathbf{k}}_m = n\} = \Pr\{\mathbf{k}_m^{(1)} = n, \mathbf{k}_m^{(2)} = 0\} + \Pr\{\mathbf{k}_m^{(1)} = 0, \mathbf{k}_m^{(2)} = n\}. \tag{2.35}$$

From (2.24) it follows that

$$D = 0, \quad \mu_1 < 1, \quad v_2 < 1 \Rightarrow \mathbf{k}_m = \text{“0”} \vee \text{“—”}. \tag{2.36}$$

Suppose that

$$\mathbf{k}_m = \text{“—”}, \tag{2.37}$$

i.e. the $\mathbf{k}_m$-process is nonrecurrent, so, cf. (2.13),

$$\sum_{m=1}^{\infty} \Pr\{\tilde{\mathbf{k}}_m = n | \tilde{\mathbf{k}}_1 = 0\} < \infty, \qquad n = 1, 2, \ldots .$$

Hence, since $\tilde{\mathbf{k}}_m \geq 0$, it follows that $\tilde{\mathbf{k}}_m \to \infty$ with probability one, and so

$$\mathrm{E}\{\liminf \tilde{\mathbf{k}}_m\} = \infty. \tag{2.38}$$

Application of the Fatou-Lebesgue theorem leads to, note that $\tilde{\mathbf{k}}_m \geq 0$,

$$\mathrm{E}\{\liminf \tilde{\mathbf{k}}_m\} \leq \liminf \mathrm{E}\{\tilde{\mathbf{k}}_m\}; \tag{2.39}$$

so from (2.32)

$$\liminf \mathrm{E}\{\tilde{\mathbf{k}}_m\} = \infty. \tag{2.40}$$

Consequently, (2.27), (2.28) and (2.34) imply that

$$\sum_{h=1}^{m} K_h^{(0)} \to \infty \qquad \text{for} \qquad m \to \infty. \tag{2.41}$$

i.e. the $\mathbf{k}_m$-process is recurrent. However, this conclusion contradicts (2.31). Hence from (2.30) it follows that

$$D = 0, \quad \mu_1 < 1, \quad \nu_2 < 1 \Longrightarrow \mathbf{k}_m = \text{“0”}. \tag{2.42}$$

Note that it follows from (2.20) for the case described in (2.25), since (2.36) implies that $K_m^{(0)} \to 0$ for $n \to \infty$, that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{h=1}^{m} K_h^{(1)} = \frac{1-\nu_2}{1+\nu_1-\nu_2}, \tag{2.43}$$

$$\lim_{m\to\infty} \frac{1}{m} \sum_{h=1}^{m} K_h^{(2)} = \frac{1-\mu_1}{1+\mu_2-\mu_1},$$

$$\lim_{m\to\infty} \frac{1}{m} \mathrm{E}\{\mathbf{k}_m^{(j)}\} = 0, \qquad j = 1, 2. \tag{2.44}$$

Next we consider the case

$$\mathbf{k}_m = \text{“—”}, \quad \mu_1 < 1, \quad \nu_2 < 1. \tag{2.45}$$

So in this case the $\mathbf{k}_m$-process is nonrecurrent i.e.

$$\sum_{h=1}^{\infty} K_m^{(0)} < \infty, \tag{2.46}$$

and as above it follows by using the Fatou-Lebesgue theorem that (2.32) also applies here. Hence from (2.16) it is seen by letting $m \to \infty$ that we should have $D < 0$, if (2.38) holds, i.e.

$$\mathbf{k}_m = \text{“—”}, \quad \mu_1 < 1, \quad v_2 < 1 \Longrightarrow D < 0. \tag{2.47}$$

Hence it is readily seen from (2.24), (2.36) and (2.40) that

$$\begin{array}{llllll} \text{if} & \mu_1 \le 1 \text{ or } v_2 \ge 1 & & \text{then} & \mathbf{k}_m & = \text{“—”}, \\ \text{if} & \mu_1 < 1,\ \nu_2 < 1 & & & & \\ & \quad \text{and} & D < 0 & ,, & \mathbf{k}_m & = \text{“—”}, \\ & & D = 0 & ,, & & = \text{“0”}, \\ & & \ \ > 0 & ,, & & = \text{“+”}, \end{array} \tag{2.48}$$

and actually (2.41) is the content of theorem II.2.7.1 concerning the $\mathbf{k}_m$-process.

Finally, it is noted that if the $\mathbf{k}_m$-process is nonrecurrent then from (2.20) it follows: for $D < 0$, $\mu_1 < 1$, $v_2 < 1$,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{h=1}^{m} K_h^{(1)} = \frac{1 - v_2}{-D} \lim_{m\to\infty} \frac{1}{m} \mathrm{E}\{\mathbf{k}_{m+1}^{(2)}\} + \frac{\mu_2}{-D} \lim_{m\to\infty} \mathrm{E}\{\mathbf{k}_{m+1}^{(1)}\}, \tag{2.49}$$

$$\lim_{m\to\infty} \frac{1}{m} \sum_{h=1}^{m} K_h^{(2)} = \frac{v_1}{-D} \lim_{m\to\infty} \frac{1}{m} \mathrm{E}\{\mathbf{k}_{m+1}^{(2)}\} + \frac{1 - \mu_1}{-D} \lim_{m\to\infty} \mathrm{E}\{\mathbf{k}_{m+1}^{(1)}\},$$

$$\frac{1 + v_1 - v_2}{-D} \lim_{m\to\infty} \frac{1}{m} \mathrm{E}\{\mathbf{k}_{m+1}^{(2)}\} + \frac{1 + \mu_2 - \mu_1}{-D} \lim_{m\to\infty} \frac{1}{m} \mathrm{E}\{\mathbf{k}_{m+1}^{(1)}\} = 1.$$

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