A strange application of Kolmogorov complexity

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A Strange Application of Kolmogorov Complexity

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Abstract

The paper deals with two similar inequalities:

(1) $2K(\langle A,B,C\rangle) \le K(\langle A,B\rangle) + K(\langle A,C\rangle) + K(\langle B,C\rangle) + O(\log n)$,

(2) $2KP(\langle A,B,C\rangle) \le KP(\langle A,B\rangle) + KP(\langle A,C\rangle) + KP(\langle B,C\rangle) + O(1)$,

where K denotes simple Kolmogorov entropy (i.e. the very first version of Kolmogorov complexity having been introduced by Kolmogorov himself) and KP denotes prefix entropy (self-delimiting complexity by the terminology of M. Li and P. M. B. Vitanyi [1]). It turns out that from (1) one can infer the following well-known geometric fact:

$$|V|^2 \le |S_{xy}| \cdot |S_{yz}| \cdot |S_{xz}| ,$$

where V is a set in three-dimensional space, S_{xy}, S_{yz}, S_{xz} are its three two-dimensional projections, and $|W|$ is the volume (or the area) of W. And (2), in its turn, is a corollary of the well-known Cauchy-Schwarz inequality. So the connection between geometry and Kolmogorov complexity works in both directions.

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1. Inequality

For any binary words A, B and C whose length does not exceed n we have

$$2K(\langle A,B,C\rangle) \le K(\langle A,B\rangle) + K(\langle A,C\rangle) + K(\langle B,C\rangle) + O(\log n)$$

(By K we denote Kolmogorov complexity as defined in original Kolmogorov article, see, e.g., [1], p. 198; it is called sometimes "simple Kolmogorov entropy" and denoted by KS. Because of the term $O(\log n)$ other versions may be used, see above.)

2. Proof

We rewrite the inequality as

$$(K(\langle A,B,C\rangle) - K(\langle A,B\rangle)) + (K(\langle A,B,C\rangle) - K(\langle A,C\rangle)) \le K(\langle B,C\rangle) + O(\log n)\ .$$

Now we use the equality $K(\langle X,Y\rangle) = K(X) + K(Y|X) + O(\log n)$ where $K(Y|X)$ is conditional complexity of Y when X is known and get

$$K(C|\langle A,B\rangle) + K(B|\langle A,C\rangle) \le K(\langle B,C\rangle) + O(\log n).$$

Now it remains to use that $K(C|\langle A,B\rangle) \le K(C|B)$, $K(B|\langle A,C\rangle) \le K(B)$ and refer to the equality mentioned above.

3. Application

We use this inequality to prove a (well-known) geometric fact which seems to have nothing in common with Kolmogorov complexity. Assume that we have a set V in a three-dimensional space with coordinates x, y, z. Consider three projections S_{xy}, S_{yz} and S_{xz}. By $|Z|$ we mean volume (or area) of Z.

Theorem (A continuous version.)

$$|V|^2 \le |S_{xy}| \cdot |S_{yz}| \cdot |S_{xz}|.$$

To avoid difficulties (non-measurable sets etc.) we shall consider a finite version:

Theorem (A discrete version.) Let X, Y, Z be finite sets, $V \subset X \times Y \times Z$ and $S_{xy} \subset X \times Y$, $S_{xz} \subset X \times Z$, $S_{yz} \subset Y \times Z$ are projections of V. Then

$$(\#V)^2 \le (\#S_{xy}) \cdot (\#S_{yz}) \cdot (\#S_{xz}).$$

Proof. Choose a random sequence $v = \langle v_1 \dots v_n\rangle \in V^n$. Its complexity is $n \cdot (\log \#V) + O(\log n)$. Now remember that v_i is a triple $\langle x_i, y_i, z_i\rangle$, therefore v can be considered as a triple $\langle x, y, z\rangle$ where $x = \langle x_1 \dots x_n\rangle$, $y = \langle y_1 \dots y_n\rangle$, $z = \langle z_1 \dots z_n\rangle$. Now use our inequality:

$$2K(v) \le K(\langle x,y\rangle) + K(\langle y,z\rangle) + K(\langle x,z\rangle) + O(\log n).$$

Remember that $\langle x, y\rangle$ can be considered also as a sequence of n elements of S_{xy}, therefore $K(\langle x, y\rangle) \le n \cdot \log \#S_{xy} + O(\log n)$. Using similar inequalities for $\langle y, z\rangle$ and $\langle x, z\rangle$ we get

$$2n \log \#V = 2K(v) \le n \cdot (\log \#S_{xy} + \log \#S_{yz} + \log \#S_{xz}) + O(\log n).$$

Dividing by n and using that $\frac{\log n}{n} \to 0$ we get

$$2 \log \#V \le \log \#S_{xy} + \log \#S_{yz} + \log \#S_{xz}.$$

4. Prefix entropy and L^2-inequality

Because of the $O(\log n)$ term the inequality of section 1 is valid for all usual variants of Kolmogorov complexity (entropy) such as decision entropy, monotone entropy and prefix entropy. But the case of prefix entropy (called also self-delimiting complexity, see [1], p. 209) is somehow special because logarithmic term may be omitted:

$$2KP(\langle A, B, C\rangle) \le KP(\langle A, B\rangle) + KP(\langle A, C\rangle) + KP(\langle B, C\rangle) + O(1)\,.$$

It turns out that this inequality can be proved analytically, using Cauchy–Schwarz inequality. Indeed, recall that KP may be defined as a $-\log_2 P$ where P is an *a priori* probability on the set of natural numbers. Therefore, we should prove that

$$P^2(\langle A, B, C\rangle) \ge c \cdot P(\langle A, B\rangle) \cdot P(\langle A, C\rangle) \cdot P(\langle B, C\rangle)$$

for some positive constant c. The *a priori* probability is defined as a maximum enumerable from below function on natural numbers with a finite sum; therefore it is enough to show that

$$\sum_{A,B,C} \sqrt{P(\langle A, B\rangle) \cdot P(\langle A, C\rangle) \cdot P(\langle B, C\rangle)} < +\infty.$$

This fact is a consequence of the following inequality which we prefer to write using integrals instead of sums:

$$\iiint f(x,y)g(x,z)h(y,z)\,dx\,dy\,dz \le$$
$$\le \left(\iint f^2(x,y)\,dx\,dy\right)^{1/2} \left(\iint g^2(x,z)\,dx\,dz\right)^{1/2} \left(\iint h^2(x,y)\,dy\,dy\right)^{1/2}.$$

This is a version of Cauchy–Schwarz inequality containing three L^2-norm in the right-hand side instead of two; it can be easily reduced to an ordinary Cauchy–Schwarz inequality:

$$\iiint f(x,y)g(x,z)h(y,z)\,dx\,dy\,dz =$$
$$= \iint h(y,z) \left(\int f(x,y)g(x,z)\,dx\right) dy\,dz \le$$
$$\le \iint h(y,z) \left(\int f^2(x,y)\,dx\right)^{1/2} \left(\int g^2(x,z)\,dx\right)^{1/2} dy\,dz \le$$
$$\le \left(\iint h^2(y,z)\,dy\,dz\right)^{1/2} \left(\iint \underbrace{\left(\int f^2(x,y)\,dx\right)}_{\varphi(y)} \underbrace{\left(\int g^2(x,z)\,dx\right)}_{\psi(z)} dy\,dz\right)^{1/2} =$$

$$= \left(\iint h^2(y,z)\,dy\,dz\right)^{1/2}\left(\iint \varphi(y)\psi(z)\,dy\,dz\right)^{1/2} =$$

$$= \left(\iint h^2(y,z)\,dy\,dz\right)^{1/2}\left(\int \varphi(y)\,dy \int \psi(z)\,dz\right)^{1/2} =$$

$$= \left(\iint h^2(y,z)\,dy\,dz\right)^{1/2}\left(\iint f^2(x,y)\,dx\,dy\right)^{1/2}\left(\iint g^2(x,z)\,dx\,dz\right)^{1/2}.$$

Thus, the connection between geometrical facts and Kolmogorov complexity works in both directions.

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