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On Constructing Multi-Variate Orthonormal Polynomials Associated with Spectral Distribution Functions

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Abstract

With a spectral distribution function of a random field, we associate orthonormal polynomials in several variables. Recurrence relations between these polynomials are presented, which yield an algorithm for construction. In the special case of polynomials in a single variable, these relations were already known as opposed to the multi-dimensional case treated in this report.

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1 Introduction

In [Grenander and Szegö], Chapter 2, recurrence relations are presented between orthonormal polynomials in a single variable. These relations yield an algorithm for constructing orthonormal polynomials: each orthonormal polynomial $\phi_n(z)$ can be expressed in terms of the orthonormal polynomial $\phi_{n-1}(z)$; see Theorem 10. In time series literature where orthonormal polynomials are used for prediction, this algorithm is known as the Durbin-Levinson algorithm (see [Brockwell and Davis]; see also [Grenander and Szegö] where further applications of these orthonormal polynomials can be found).

Unlike [Grenander and Szegö], in this report orthonormal polynomials in several variables are treated and analogous recurrence relations between them are derived. This extension to the d-dimensional case requires a certain ordering, which will be described by sequences $\omega = (\omega_0, \omega_1, \ldots)$ with $\omega_i \in \mathbb{Z}^d$. It will be shown that for a special class of sequences, which we call 'periodic sequences', the recurrence relations obtained, yield analogously to the one dimensional case, an algorithm for constructing orthonormal polynomials.

In order to clarify the mentioned relationship between orthonormal polynomials and predictors, we restrict our discussion to the one dimensional case: let $\{X_t\}_{t\in\mathbb{Z}}$ be a zero mean time series with given covariances $\mathrm{Cov}(X_s, X_t) = \mathbb{E}(X_s\overline{X_t})$. Suppose that X_{N+1} has to be predicted by observation on $X_0, \ldots, X_N$. Let $\hat{X}_{N+1}$ denote the best linear unbiased estimator (BLUE) for X_{N+1}, i.e.

$$\hat{X}_{N+1} = \sum_{m=0}^{N} c_m X_m,$$

where the complex numbers c_m are chosen so that the mean squared error (MSE)

$$\mathbb{E}|X_{N+1} - \hat{X}_{N+1}|^2$$

is minimized. As is easily seen, the numbers c_m can be found by solving the system

$$(c_0, \ldots, c_N)\Gamma_N = (\mathbb{E}(X_{N+1}\overline{X}_0), \ldots, \mathbb{E}(X_{N+1}\overline{X}_N)),$$

where Γ_N is the covariance matrix of $X_0, \ldots, X_N$, i.e.

$$\Gamma_N \ := \ \mathbb{E}\begin{pmatrix} X_0 \\ \vdots \\ X_N \end{pmatrix} \overline{(X_0, \ldots, X_N)}.$$

Assume now that the time series $\{X_t\}_{t\in\mathbb{Z}}$ is weakly stationary, i.e. $\mathbb{E}(X_s\overline{X_t})$ depends only on $s-t$. Then the (Hermitian-)matrix Γ_N is Toeplitz. This additional assumption simplifies considerably the construction of $\hat{X}_{N+1}$. Indeed, due to stationarity, Bochner's theorem yields that there exists a spectral distribution function $F : (-\pi, \pi] \to \mathbb{R}$ such that for all $s, t \in \mathbb{Z}$

$$\mathbb{E}(X_s\overline{X_t}) = \int_{(-\pi,\pi]} (e^{i\lambda})^{s-t}\, dF(\lambda).$$

Hence

$$\mathbb{E}|X_{N+1} - \hat{X}_{N+1}|^2 = \int_{(-\pi,\pi]} |(e^{i\lambda})^{N+1} - \sum_{m=0}^{N} c_m (e^{i\lambda})^m|^2 \, dF(\lambda).$$

The least squares criterion can therefore be interpreted as follows: let P denote the complex vector space consisting of polynomials (in the variable z) of maximal degree $N+1$. *Assume* that the mapping $< \cdot ; \cdot >$: $\mathrm{P} \times \mathrm{P} \to \mathbb{C}$ given by

$$< f; g > := \int_{(-\pi,\pi]} f(e^{i\lambda}) \overline{g(e^{i\lambda})} \, dF(\lambda)$$

is an inner product. If $f(z) = z^{N+1} - \sum_{m=0}^{N} c_m z^m$, then

$$\mathbb{E}|X_{N+1} - \hat{X}_{N+1}|^2 = \|f\|^2,$$

where $\| \cdot \| = < \cdot ; \cdot >^{\frac{1}{2}}$ is the induced norm. So minimizing the mean squared prediction error is equivalent to finding the polynomial $f \in \mathrm{P}$ that minimizes $\| \cdot \|$ and has a leading coefficient (i.e. the coefficient of z^{N+1}) equal to one. The minimizer f can easily be expressed in terms of orthonormal polynomials: let e_n denote the monomial $z \longmapsto z^n$ and let ϕ_n denote the orthonormal polynomial constructed by a Gramm-Schmidt orthogonalization of the monomials $e_0, \ldots, e_n$, in which the coefficient of e_n (denoted by k_n) is a positive real number, i.e.

$$\frac{\phi_n}{k_n} = e_n - \sum_{k=0}^{n-1} < e_n; \phi_k > \phi_k.$$

Then the polynomial $f \in \mathrm{P}$ that minimizes $\| \cdot \|$ and has a leading coefficient equal to one, is $\frac{\phi_{N+1}}{k_{N+1}}$ (see subsection 2.5). So if

$$\frac{\phi_{N+1}}{k_{N+1}} = e_{N+1} + \sum_{m=0}^{N} a_m e_m,$$

then the coefficients c_m in $\hat{X}_{N+1}$ are given by

$$c_m = -a_m.$$

Surely, the Gramm-Schmidt orthogonalization procedure yields relations which enable one to construct the orthonormal polynomials recurrently: the polynomial ϕ_n is constructed once the polynomials $\phi_0, \ldots, \phi_{n-1}$ are known. This procedure of constructing orthonormal polynomials is quite elaborate. However, due to stationarity *each orthonormal polynomial ϕ_n can be expressed in terms of the orthonormal polynomial ϕ_{n-1}* (see [Grenander and Szegö] or subsection 3.1).

As was mentioned earlier, the desired extensions to the d-dimensional case require ordering described by a sequence $\omega = (\omega_0, \omega_1, \ldots)$ with $\omega_i \in \mathbb{Z}^d$. For a fixed ω we establish in

Theorem 11, the relationships (47) and (48) between orthonormal polynomials, which for periodic sequences ω can be written down in algorithmic form (see Theorem 15). Finally for 'block sequences' (an important special case of periodic sequences), we will introduce orthonormal matrix polynomials. The relation between these matrix polynomials and the 'ordinary' orthonormal polynomials associated with a block sequence ω, are expressed in Corollary 2.

2 Orthonormal Polynomials

2.1 Basic Notions

In this subsection polynomials in the variable $z = (z_1, z_2, \ldots, z_d) \in \mathbb{C}^d$ will be considered. On the complex vector space of these polynomials, an inner product $< \cdot; \cdot >$ will be defined with respect to a spectral distribution function F (see formula (2)).

Since polynomials require powers of the variable z, such powers will be defined. By using the usual multi-index notations, z to the power η for $z = (z_1, z_2, \ldots, z_d) \in \mathbb{C}^d$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_d) \in \mathbb{Z}^d$ is defined by

$$z^\eta := z_1^{\eta_1} \cdot z_2^{\eta_2} \cdots z_d^{\eta_d} \in \mathbb{C},$$

so that

$$z^\eta \cdot z^\nu = z^{\eta+\nu}$$

for all $\eta, \nu \in \mathbb{Z}^d$. Once powers of the variable z are known, monomials can be defined. A monomial is then a function $\mathbb{C}^d \to \mathbb{C}$ such that

$$z \longmapsto z^\eta$$

for some $\eta \in \mathbb{Z}^d$. This mapping will be denoted by e_η. Notice that

$$e_\eta \cdot e_\nu = e_{\eta+\nu} \tag{1}$$

for all $\eta, \nu \in \mathbb{Z}^d$. Finally a polynomial is a finite linear combination of monomials.

Let $(\mathrm{P}^d, \mathbb{C})$ denote the complex vector space consisting of the polynomials. On this vector space, an inner product will be defined. Let $F : \Lambda \to \mathbb{R}$ be a spectral distribution function, where $\Lambda = (-\pi, \pi]^d$. Then *it is assumed that* the mapping $< \cdot; \cdot >: \mathrm{P}^d \times \mathrm{P}^d \to \mathbb{C}$ given by

$$< f; g >:= \int_\Lambda f(e^{i\lambda})\overline{g(e^{i\lambda})}\ dF(\lambda) \tag{2}$$

is an inner product, where the numbers $e^{i\lambda}$ for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{R}^d$ are defined by

$$e^{i\lambda} := (e^{i\lambda_1}, e^{i\lambda_2}, \ldots, e^{i\lambda_d}) \in \mathbb{C}^d.$$

So the set

$$\{e^{i\lambda} | \lambda \in \Lambda\}$$

is just the d-torus. Notice that

$$\overline{(e^{i\lambda})^\eta} = (e^{i\lambda})^{-\eta}$$

for all $\eta \in \mathbb{Z}^d$, which yields

$$< e_{-\eta}; e_{-\nu} >=< e_\nu; e_\eta > \tag{3}$$

for all $\eta, \nu \in \mathbb{Z}^d$.

The norm $\|\cdot\| : \mathrm{P}^d \to \mathbb{R}$ induced by the inner product is given by

$$\|\cdot\| = <\cdot;\cdot>^{\frac{1}{2}} .$$

Recall that $F : \Lambda \to \mathbb{R}$ is a spectral distribution function if

- $\Delta_A F \geq 0$ for all $A \subset \Lambda$ of the form $A = (\mu_1, \nu_1] \times \cdots \times (\mu_d, \nu_d]$, in which $-\pi < \mu_j < \nu_j \leq \pi$ for all j. Here $\Delta_A F := \sum \mathrm{sgn}_A x \cdot F(x)$, the sum extending over the 2^d vertices $x = (x_1, \ldots, x_d)$ of A ($x_j = \mu_j$ or $x_j = \nu_j$) and $\mathrm{sgn}_A x$, the signum of the vertex, be $+1$ or -1, according as the number of j ($1 \leq j \leq d$) satisfying $x_j = \mu_j$ is even or odd.

- $F : \Lambda \to \mathbb{R}$ is bounded.

- F is continuous from above, i.e. suppose $\{\lambda^{(n)}\}_{n\geq 0}$ is a sequence in Λ with $\lambda^{(n)} = (\lambda_1^{(n)}, \ldots, \lambda_d^{(n)})$, $\lambda = (\lambda_1, \ldots, \lambda_d) \in \Lambda$ and $\lambda^{(n)} \downarrow \lambda$ in the sense that $\lambda_j^{(n)} \downarrow \lambda_j$ as $n \to \infty$ for each $j \in \{1, \ldots, d\}$. Then $F(\lambda^{(n)}) \to F(\lambda)$.

(see e.g. [Billingsley]).

By assuming that (2) defines an inner product, we have actually restricted the class of spectral distribution functions F, which however includes all spectral distribution functions F with derivatives f whose geometric mean is positive, i.e.

$$\int_{(-\pi,\pi]^d} \log f(\lambda_1, \ldots, \lambda_d)\, d\lambda_1 \ldots d\lambda_d > -\infty$$

(see e.g. [Rosenblatt]).

2.2 Orthonormal Polynomials

In the previous subsection, the inner product space $(\mathrm{P}^d, \mathbb{C}, <\cdot;\cdot>)$ was constructed. In this subsection orthonormal polynomials will be introduced, which belong to this space. The orthonormal polynomials will be given explicitly in (8), (9) and (15).

Theorem 1 *Let*

$$e_{\omega_0}, e_{\omega_1}, e_{\omega_2}, e_{\omega_3}, \ldots$$

be an ordered system of monomials, with $\omega = (\omega_0, \omega_1, \ldots) \in \mathbb{Z}^d \times \mathbb{Z}^d \times \cdots$ *and* $\omega_i \neq \omega_j$ *for all* $i \neq j$. *Then a unique system of polynomials* $\{\phi_{\omega,n}\}_{n\in\mathbb{N}}$ *exists such that*

- $\phi_{\omega,n}$ *is a linear combination of the monomials* $e_{\omega_0}, \ldots, e_{\omega_n}$.

- *the coefficient of* e_{ω_n} *in* $\phi_{\omega,n}$ *is a positive real number.*

- *the polynomials* $\{\phi_{\omega,n}\}_{n\in\mathbb{N}}$ *are orthonormal, i.e.*

$$<\phi_{\omega,n};\phi_{\omega,m}>=\begin{cases}1 & \text{if } n=m\\ 0 & \text{if } n\neq m\end{cases}$$

Proof: See e.g. [Grenander and Szegö]. □

Notice that ω may have a finite length, i.e. $\omega=(\omega_0,\ldots,\omega_N)$ for some $N\in\mathbb{N}$. In this case we have a unique finite system $\{\phi_{\omega,n}\}_{n=0}^N$ of orthonormal polynomials.

Corollary 1 *Let* $\omega=(\omega_0,\omega_1,\ldots)\in\mathbb{Z}^d\times\mathbb{Z}^d\times\cdots$ *and* $\omega_i\neq\omega_j$ *for all* $i\neq j$. *Fix* $\eta\in\mathbb{Z}^d$. *Then the system of polynomials* $\{e_\eta\phi_{\omega,n}\}_{n\in\mathbb{N}}$ *is orthonormal.*

Proof: This is a consequence of the special form of the functions e_{ω_i}. □

For further reference let $k_{\omega,n}$ and $l_{\omega,n}$ denote the coefficient in $\phi_{\omega,n}$ of e_{ω_n} and e_{ω_0} respectively. Now the orthonormal polynomials $\phi_{\omega,n}$ will be given explicitly.

Fix an arbitrary $\omega=(\omega_0,\omega_1,\ldots)\in\mathbb{Z}^d\times\mathbb{Z}^d\times\cdots$ with $\omega_i\neq\omega_j$ for all $i\neq j$. Then for each $n\in\mathbb{N}$ define the Hermitian (or self-adjoint) matrices $\mathrm{H}_{\omega,n}$ by

$$\begin{aligned}\mathrm{H}_{\omega,n} &:= \int_\Lambda \begin{pmatrix}(e^{i\lambda})^{\omega_0}\\ \vdots\\ (e^{i\lambda})^{\omega_n}\end{pmatrix}\overline{((e^{i\lambda})^{\omega_0},\ldots,(e^{i\lambda})^{\omega_n})}\,dF(\lambda)\\ &= \begin{pmatrix}<e_{\omega_0};e_{\omega_0}> & <e_{\omega_0};e_{\omega_1}> & \ldots & <e_{\omega_0};e_{\omega_n}>\\ <e_{\omega_1};e_{\omega_0}> & <e_{\omega_1};e_{\omega_1}> & \ldots & <e_{\omega_1};e_{\omega_n}>\\ \ldots & \ldots & \ldots & \ldots\\ <e_{\omega_n};e_{\omega_0}> & <e_{\omega_n};e_{\omega_1}> & \ldots & <e_{\omega_n};e_{\omega_n}>\end{pmatrix}\end{aligned} \tag{4}$$

and let $\mathrm{D}_{\omega,n}$ denote the determinant of $\mathrm{H}_{\omega,n}$. Notice that by (3)

$$\mathrm{H}_{-\omega,n}=\overline{\mathrm{H}}_{\omega,n} \tag{5}$$

and

$$\mathrm{D}_{-\omega,n}=\mathrm{D}_{\omega,n}. \tag{6}$$

Define the Hermitian forms $\mathrm{T}_{\omega,n}:\mathbb{C}^{n+1}\to\mathbb{R}$ by

$$\begin{aligned}\mathrm{T}_{\omega,n}(u_0,\ldots,u_n) &:= \sum_{\mu,\nu=0}^{n} u_\mu\,(\mathrm{H}_{\omega,n})_{\mu,\nu}\overline{u_\nu}\\ &= (u_0,\ldots,u_n)\,\mathrm{H}_{\omega,n}\overline{\begin{pmatrix}u_0\\ \vdots\\ u_n\end{pmatrix}}\\ &= \int_\Lambda |u_0(e^{i\lambda})^{\omega_0}+\cdots+u_n(e^{i\lambda})^{\omega_n}|^2\,dF(\lambda)\\ &= \|u_0\,e_{\omega_0}+\ldots+u_n\,e_{\omega_n}\|^2.\end{aligned} \tag{7}$$

Since the monomials $e_{\omega_0}, \ldots, e_{\omega_n}$ are linearly independent and $\| \cdot \|$ is a norm (by assumption), it follows from (7) that the matrices $\mathrm{H}_{\omega,n}$ are positive definite and so the determinants $\mathrm{D}_{\omega,n}$ are strictly positive. Then the unique orthonormal polynomials $\phi_{\omega,n}$ are given by

$$\phi_{\omega,n} = \frac{1}{\sqrt{\mathrm{D}_{\omega,n-1}\mathrm{D}_{\omega,n}}} \begin{vmatrix} < e_{\omega_0}; e_{\omega_0} > & < e_{\omega_0}; e_{\omega_1} > & \ldots & < e_{\omega_0}; e_{\omega_n} > \\ \ldots & \ldots & \ldots & \ldots \\ < e_{\omega_{n-1}}; e_{\omega_0} > & < e_{\omega_{n-1}}; e_{\omega_1} > & \ldots & < e_{\omega_{n-1}}; e_{\omega_n} > \\ e_{\omega_0} & e_{\omega_1} & \ldots & e_{\omega_n} \end{vmatrix} \tag{8}$$

if $n \in \mathbb{N}^+$ and

$$\phi_{\omega,0} = \frac{1}{\sqrt{\mathrm{D}_{\omega,0}}} e_{\omega_0}. \tag{9}$$

This can be verified by expanding the determinant in (8) in the cofactors of its last row and then calculating $< e_{\omega_k}; \phi_{\omega,n} >$ for $k \in \{0, \ldots, n\}$. Observe that for $n \in \mathbb{N}^+$

$$k_{\omega,n} = \frac{\sqrt{\mathrm{D}_{\omega,n-1}}}{\sqrt{\mathrm{D}_{\omega,n}}} \in \mathbb{R}^+.$$

So for all $n \in \mathbb{N}$ we have by (6)

$$k_{-\omega,n} = k_{\omega,n}. \tag{10}$$

Moreover for $n \in \mathbb{N}^+$

$$l_{\omega,n} = \frac{(-1)^n}{\sqrt{\mathrm{D}_{\omega,n-1}\mathrm{D}_{\omega,n}}} \begin{vmatrix} < e_{\omega_0}; e_{\omega_1} > & \ldots & < e_{\omega_0}; e_{\omega_n} > \\ \ldots & \ldots & \ldots \\ < e_{\omega_{n-1}}; e_{\omega_1} > & \ldots & < e_{\omega_{n-1}}; e_{\omega_n} > \end{vmatrix}$$

which yields that for all $n \in \mathbb{N}$

$$l_{-\omega,n} = \bar{l}_{\omega,n}. \tag{11}$$

Example: Let $F : \Lambda \to \mathbb{R}$ be given by

$$F(\lambda_1, \ldots, \lambda_d) = \frac{\sigma^2 \prod_{m=1}^{d} \lambda_m}{(2\pi)^d},$$

with $\sigma \in \mathbb{R}^+$. Then $\mathrm{D}_{\omega,0} = \sigma^2$ and

$$\phi_{\omega,n} = \frac{e_{\omega_n}}{\sigma},$$

whatever ω may be. In terms of spectral distribution functions, this function F is associated with so called 'White Noise'.

Throughout this report formula (8) for a fixed n is abbreviated to

$$e_{\omega_0}, \ldots, e_{\omega_n} \stackrel{GS}{\longrightarrow} \phi_{\omega,n}, \tag{12}$$

which refers to the fact that the unique system $\{\phi_{\omega,n}\}_{n\in\mathbb{N}}$ is also obtained by a Gramm-Schmidt orthogonalization of the system of monomials $\{e_{\omega_k}\}_{k\in\mathbb{N}}$, i.e. for $n \in \mathbb{N}$

$$\frac{\phi_{\omega,n}}{k_{\omega,n}} = e_{\omega_n} - \sum_{k=0}^{n-1} < e_{\omega_n}; \phi_{\omega,k} > \phi_{\omega,k}. \tag{13}$$

We will also need the construction of $\frac{\phi_{\omega,n}}{k_{\omega,n}}$, which will be abbreviated to

$$e_{\omega_0}, \ldots, e_{\omega_n} \stackrel{GS\bullet}{\longrightarrow} \frac{\phi_{\omega,n}}{k_{\omega,n}}. \tag{14}$$

Observe that the orthonormal polynomials $\phi_{\omega,n}$ can be represented alternatively to (8). Since $\mathrm{H}_{\omega,n}$ is invertible, for $n \in \mathbb{N}^+$ the orthonormal polynomials $\phi_{\omega,n}$ are given by

$$\phi_{\omega,n} = \frac{\sqrt{\mathrm{D}_{\omega,n}}}{\sqrt{\mathrm{D}_{\omega,n-1}}} (0, \cdots, 0, 1) \, (\mathrm{H}_{\omega,n})^{-1} \begin{pmatrix} e_{\omega_0} \\ \vdots \\ e_{\omega_n} \end{pmatrix}. \tag{15}$$

This can be verified by calculating $< e_{\omega_k}; \phi_{\omega,n} >$ for $k \in \{0, \ldots, n\}$.

2.3 Sequence Sets and their Transformations

We will need in the sequel certain transformations, which transform sequences ω. These transformations are defined on particular sequence sets.

Denote

$$0_d := (0, 0, \ldots, 0) \in \mathbb{Z}^d.$$

The basic sequence sets are defined as follows: for $n \in \mathbb{N}^+$

$$\Omega^{\neq}_{n,n-1} := \{(\omega_0, \omega_1, \ldots, \omega_n) \;\mid\; \omega_i \in \mathbb{Z}^d, \omega_0 = 0_d \;,\; \forall i,j \in \{0, \ldots, n-1\} : (i \neq j) \Rightarrow (\omega_i \neq \omega_j)\}, \tag{16}$$

and for $n \in \mathbb{N}$

$$\Omega^{\neq}_{n,n} := \{(\omega_0, \omega_1, \ldots, \omega_n) \;\mid\; \omega_i \in \mathbb{Z}^d, \omega_0 = 0_d \;,\; \forall i,j \in \{0, \ldots, n\} : (i \neq j) \Rightarrow (\omega_i \neq \omega_j)\}. \tag{17}$$

On these sets several mappings will be defined:

- If $n \in \mathbb{N}$, then the identity mapping $I_n : \Omega^{\neq}_{n,n} \to \Omega^{\neq}_{n,n}$ is defined by

$$I_n((\omega_0, \omega_1, \ldots, \omega_n)) := (\omega_0, \omega_1, \ldots, \omega_n). \tag{18}$$

- The mapping T_n 'shifts' coordinates of $\omega \in \Omega^{\neq}_{n,n}$. For $n \in \mathbb{N}^+$ define this mapping $T_n : \Omega^{\neq}_{n,n} \to \Omega^{\neq}_{n,n-1}$ by
$$T_n((\omega_0, \omega_1, \ldots, \omega_n)) := (\omega_1 - \omega_1, \omega_2 - \omega_1, \ldots, \omega_n - \omega_1, \omega_n). \tag{19}$$
Notice that this mapping is bijective.

- The mapping V_n will be used in the next subsection to define the multi-dimensional version of a reciprocal polynomial. For $n \in \mathbb{N}$ this mapping $V_n : \Omega^{\neq}_{n,n} \to \Omega^{\neq}_{n,n}$ is defined by
$$V_n((\omega_0, \omega_1, \ldots, \omega_n)) := (\omega_n - \omega_n, \omega_n - \omega_{n-1}, \ldots, \omega_n - \omega_0). \tag{20}$$
Notice that for each $n \in \mathbb{N}$ the mapping $V_n : \Omega^{\neq}_{n,n} \to \Omega^{\neq}_{n,n}$ is its own inverse, i.e.
$$V_n \circ V_n = I_n. \tag{21}$$

Fix now an arbitrary $n \in \mathbb{N}$ and consider the mappings:
$$\begin{aligned} T_{n+1} &: \Omega^{\neq}_{n+1,n+1} \to \Omega^{\neq}_{n+1,n} \\ V_{n+1} &: \Omega^{\neq}_{n+1,n+1} \to \Omega^{\neq}_{n+1,n+1} \\ V^{\diamond}_n &: \Omega^{\neq}_{n+1,n+1} \to \Omega^{\neq}_{n+1,n} \end{aligned}$$
where the mapping $V^{\diamond}_n : \Omega^{\neq}_{n+1,n+1} \to \Omega^{\neq}_{n+1,n}$ is defined by
$$V^{\diamond}_n((\omega_0, \omega_1, \ldots, \omega_{n+1})) := (\omega_n - \omega_n, \omega_n - \omega_{n-1}, \ldots, \omega_n - \omega_0, \omega_{n+1}). \tag{22}$$
Observe that
$$V^{\diamond}_n \circ V^{\diamond}_n = I_{n+1}. \tag{23}$$
Then it follows that
$$T_{n+1} = V^{\diamond}_n \circ V_{n+1} \tag{24}$$
and
$$T_{n+1} \circ V_{n+1} = V^{\diamond}_n. \tag{25}$$

2.4 Reciprocal Polynomials

In this subsection reciprocal polynomials will be introduced. These polynomials will be used intensively throughout the remaining part of this report.

Let $n \in \mathbb{N}$ and fix an arbitrary $\omega = (\omega_0, \omega_1, \ldots, \omega_n) \in \Omega^{\neq}_{n,n}$. The Gramm-Schmidt orthogonalization procedure yields
$$e_{\omega_0}, e_{\omega_1}, \ldots, e_{\omega_k} \xrightarrow{GS} \phi_{\omega,k}$$
where $k \in \{0, \ldots, n\}$ is arbitrary. The polynomial reciprocal to $\phi_{\omega,k}$ is defined by
$$\phi^*_{\omega,k} := e_{\omega_k} \phi_{-\omega,k}. \tag{26}$$
The next theorem shows the relation between the coefficients of $\phi_{\omega,k}$ and $\phi^*_{\omega,k}$:

Theorem 2 *Let $n \in \mathbb{N}$ and fix $\omega = (\omega_0, \omega_1, \ldots, \omega_n) \in \Omega^{\neq}_{n,n}$. Choose $k \in \{0, \ldots, n\}$. Suppose that*

$$\phi_{\omega,k} = \sum_{m=0}^{k} \varphi_{\omega,m} e_{\omega_m},$$

where $\varphi_{\omega,m} \in \mathbb{C}$. Then

$$\phi^*_{\omega,k} = \sum_{m=0}^{k} \overline{\varphi}_{\omega,m} e_{\omega_k - \omega_m}.$$

Proof: For $k = 0$ the statement is obviously true. Take $k \in \mathbb{N}^+$. Then according to (1), (5), (6) and (15)

$$e_{\omega_k} \phi_{-\omega,k} = \frac{\sqrt{\mathrm{D}_{\omega,k}}}{\sqrt{\mathrm{D}_{\omega,k-1}}} (0, \cdots, 0, 1) (\overline{\mathrm{H}}_{\omega,k})^{-1} \begin{pmatrix} e_{\omega_k - \omega_0} \\ \vdots \\ e_{\omega_k - \omega_k} \end{pmatrix},$$

which yields the statement of this theorem. □

The next theorem shows how $\phi^*_{\omega,n}$ is constructed by the Gramm-Schmidt orthogonalization procedure.

Theorem 3 *Let $n \in \mathbb{N}$ and fix $\omega = (\omega_0, \omega_1, \ldots, \omega_n) \in \Omega^{\neq}_{n,n}$. Then*

$$e_{\omega_k - \omega_0}, e_{\omega_k - \omega_1}, \ldots, e_{\omega_k - \omega_k} \xrightarrow{GS} \phi^*_{\omega,k} \tag{27}$$

for $k \in \{0, \ldots, n\}$.

Proof: We have

$$e_{-\omega_0}, e_{-\omega_1}, \ldots, e_{-\omega_k} \xrightarrow{GS} \phi_{-\omega,k}.$$

Now using the assertion of Corollary 1 with $\eta = \omega_k$ and formula (26) we get

$$e_{\omega_k - \omega_0}, e_{\omega_k - \omega_1}, \ldots, e_{\omega_k - \omega_k} \xrightarrow{GS} \phi^*_{\omega,k}.$$

□

Remark: For $k \in \{0, \ldots, n\}$ we have by definition (20) that

$$e_{\omega_n - \omega_n}, e_{\omega_n - \omega_{n-1}}, \ldots, e_{\omega_n - \omega_1}, e_{\omega_n - \omega_{n-k}} \xrightarrow{GS} \phi_{V_n(\omega),k}. \tag{28}$$

Hence by Theorem 3

$$e_{\omega_n - \omega_{n-k}}, e_{\omega_{n-1} - \omega_{n-k}}, \ldots, e_{\omega_{n-k} - \omega_{n-k}} \xrightarrow{GS} \phi^*_{V_n(\omega),k}.$$

2.5 Extremum Properties

Similarly to the one dimensional case (see [Grenander and Szegö]), the orthonormal polynomials $\phi_{\omega,n}$ can be used for constrained minimization of the Hermitian forms as defined in (7).

Theorem 4 *Let $n \in \mathbb{N}$ and fix $\omega = (\omega_0, \ldots, \omega_n) \in \Omega^{\neq}_{n,n}$.*

Then the polynomial $\frac{\phi_{\omega,n}}{k_{\omega,n}}$ minimizes $\|g_{\omega,n}\|^2$ where $g_{\omega,n} = a_0\, e_{\omega_0} + \ldots + a_n\, e_{\omega_n}$ is an arbitrary polynomial in which $a_n = 1$. The minimizing polynomial is unique and the minimum itself is $\frac{1}{k^2_{\omega,n}}$.

Proof: Represent $g_{\omega,n}$ as

$$g_{\omega,n} = v_0\, \phi_{\omega,0} + \cdots + v_n\, \phi_{\omega,n}$$

where $v_j \in \mathbb{C}$ for all j. Since $a_n = 1$, it follows that $v_n\, k_{\omega,n} = 1$. Then

$$\|g_{\omega,n}\|^2 = |v_0|^2 + \cdots + |v_n|^2 \geq |v_n|^2 = \frac{1}{k^2_{\omega,n}}$$

□

Theorem 5 *Let $n \in \mathbb{N}$ and fix $\omega = (\omega_0, \ldots, \omega_n) \in \Omega^{\neq}_{n,n}$.*

Denote by $u_{\omega,n}$ the unique polynomial that minimizes $\|g_{\omega,n}\|^2$ where $g_{\omega,n} = a_0\, e_{\omega_0} + \ldots + a_n\, e_{\omega_n}$ is an arbitrary polynomial in which $a_0 = 1$. Then

$$e_{\omega_n}, \ldots, e_{\omega_0} \xrightarrow{GS\bullet} u_{\omega,n}.$$

Proof: Analogous to the proof of Theorem 4. □

2.6 Relations between Minimizing Polynomials

In this subsection some relations between polynomials $\phi_{\omega,n}$, $u_{\omega,n}$ and their reciprocals are presented. Recall that the minimizing polynomials $u_{\omega,n}$ were defined in Theorem 5.

Let $n \in \mathbb{N}$ and fix an arbitrary $\omega = (\omega_0, \omega_1, \ldots, \omega_n) \in \Omega^{\neq}_{n,n}$. As was seen in the previous subsection

$$e_{\omega_0}, e_{\omega_1}, \ldots, e_{\omega_k} \xrightarrow{GS} \phi_{\omega,k}$$

and

$$e_{\omega_k - \omega_0}, e_{\omega_k - \omega_1}, \ldots, e_{\omega_k - \omega_k} \xrightarrow{GS} \phi^*_{\omega,k}$$

where $k \in \{0, \ldots, n\}$ is arbitrary. For $k \in \{0, \ldots, n\}$, the polynomial $u_{\omega,k}$ is defined in Theorem 5 by

$$e_{\omega_k}, \ldots, e_{\omega_1}, e_{\omega_0} \xrightarrow{GS\bullet} u_{\omega,k}, \tag{29}$$

which is obviously equivalent (for $k \in \{1, \ldots, n\}$) to

$$e_{\omega_1}, \ldots, e_{\omega_k}, e_{\omega_0} \xrightarrow{GS\bullet} u_{\omega,k}. \tag{30}$$

By definition the polynomial reciprocal to $u_{\omega,k}$ is given by

$$u^*_{\omega,k} := e_{\omega_k} u_{-\omega,k}. \tag{31}$$

By arguments similar to those used in the course of proving Theorem 3, formulas (29) and (31) yield

$$e_{\omega_k-\omega_k}, \ldots, e_{\omega_k-\omega_1}, e_{\omega_k-\omega_0} \xrightarrow{GS\bullet} u^*_{\omega,k}. \tag{32}$$

Theorem 6 *Let $n \in \mathbb{N}$ and fix an arbitrary $\omega = (\omega_0, \omega_1, \ldots, \omega_n) \in \Omega^{\neq}_{n,n}$. Then*

$$u^*_{\omega,n} = \frac{\phi_{V_n(\omega),n}}{k_{V_n(\omega),n}} \tag{33}$$

and

$$u_{\omega,n} = \frac{\phi^*_{V_n(\omega),n}}{k_{V_n(\omega),n}}. \tag{34}$$

Proof: By (28) and (32) with $k = n$ we have (33). Hence

$$\begin{aligned} u_{\omega,n} &= e_{\omega_n} u^*_{-\omega,n} \\ &= e_{\omega_n} \frac{\phi_{V_n(-\omega),n}}{k_{V_n(-\omega),n}} \\ &= e_{\omega_n-\omega_0} \frac{\phi_{-V_n(\omega),n}}{k_{-V_n(\omega),n}} \\ &= \frac{\phi^*_{V_n(\omega),n}}{k_{V_n(\omega),n}}. \end{aligned}$$

□

Theorem 7 *Let $n \in \mathbb{N}^+$ and fix an arbitrary $\omega = (\omega_0, \omega_1, \ldots, \omega_n) \in \Omega^{\neq}_{n,n}$. Then*

$$u^*_{\omega,n-1} = \frac{\phi_{V^\diamond_{n-1}(\omega),n-1}}{k_{V^\diamond_{n-1}(\omega),n-1}} \tag{35}$$

and

$$u_{\omega,n-1} = \frac{\phi^*_{V^\diamond_{n-1}(\omega),n-1}}{k_{V^\diamond_{n-1}(\omega),n-1}}. \tag{36}$$

Proof: Formula (32) for $k = n-1$ reduces to

$$e_{\omega_{n-1}-\omega_{n-1}}, \ldots, e_{\omega_{n-1}-\omega_1}, e_{\omega_{n-1}-\omega_0} \xrightarrow{GS\bullet} u^*_{\omega,n-1}.$$

On the other hand, by definition (22) we have

$$e_{\omega_{n-1}-\omega_{n-1}}, \ldots, e_{\omega_{n-1}-\omega_1}, e_{\omega_{n-1}-\omega_0} \xrightarrow{GS} \phi_{V^\diamond_{n-1}(\omega),n-1}.$$

Thus (35) holds as well as (36), since

$$\begin{aligned}
u_{\omega,n-1} &= e_{\omega_{n-1}} u^*_{-\omega,n-1} \\
&= e_{\omega_{n-1}} \frac{\phi_{V^\circ_{n-1}(-\omega),n-1}}{k_{V^\circ_{n-1}(-\omega),n-1}} \\
&= e_{\omega_{n-1}-\omega_0} \frac{\phi_{-V^\circ_{n-1}(\omega),n-1}}{k_{-V^\circ_{n-1}(\omega),n-1}} \\
&= \frac{\phi^*_{V^\circ_{n-1}(\omega),n-1}}{k_{V^\circ_{n-1}(\omega),n-1}}.
\end{aligned}$$

□

2.7 Applications to Random Fields

Let $\{X_t\}_{t\in\mathbb{Z}^d}$ be a complex valued zero mean weakly stationary random field with given covariances, i.e.

- $\forall t \in \mathbb{Z}^d \;\; \mathbb{E}(X_t) = 0$ and $\mathbb{E}(|X_t|^2) < \infty$.
- $\forall s, t \in \mathbb{Z}^d$

  $$\mathrm{Cov}(X_s, X_t) = \mathbb{E}(X_s \overline{X_t})$$

  depends only on $s - t$.

Due to stationarity, there exists a spectral distribution function $F : \Lambda \to \mathbb{R}$ with $\Lambda = (-\pi, \pi]^d$ such that for all $s, t \in \mathbb{Z}^d$

$$\mathbb{E}(X_s \overline{X_t}) = \int_\Lambda (e^{i\lambda})^{s-t}\, dF(\lambda) =< e_s; e_t >$$

Here the multi-dimensional Bochner theorem is used (see [Yaglom]).

The covariance matrix of $X_{\omega_0}, \ldots, X_{\omega_n}$ with $\omega = (\omega_0, \ldots, \omega_n) \in \Omega^{\neq}_{n,n}$ is

$$\begin{aligned}
\Gamma_{\omega,n} &:= \mathbb{E}\begin{pmatrix} X_{\omega_0} \\ \vdots \\ X_{\omega_n} \end{pmatrix} \overline{(X_{\omega_0}, \ldots, X_{\omega_n})} \\
&= \begin{pmatrix} < e_{\omega_0}; e_{\omega_0} > & < e_{\omega_0}; e_{\omega_1} > & \ldots & < e_{\omega_0}; e_{\omega_n} > \\ < e_{\omega_1}; e_{\omega_0} > & < e_{\omega_1}; e_{\omega_1} > & \ldots & < e_{\omega_1}; e_{\omega_n} > \\ \ldots & \ldots & \ldots & \ldots \\ < e_{\omega_n}; e_{\omega_0} > & < e_{\omega_n}; e_{\omega_1} > & \ldots & < e_{\omega_n}; e_{\omega_n} > \end{pmatrix} \\
&= H_{\omega,n}.
\end{aligned}$$

Now let $n \in \mathbb{N}$ and $\omega = (\omega_0, \ldots, \omega_{n+1}) \in \Omega^{\neq}_{n+1,n+1}$. Then the best *linear* predictor for $X_{\omega_{n+1}}$, based on $X_{\omega_0}, \ldots, X_{\omega_n}$, is given by

$$\hat{X}_{\omega_{n+1}} = \sum_{m=0}^{n} c_{\omega,m} X_{\omega_m},$$

where the coefficients $c_{\omega,m} \in \mathbb{C}$ are chosen so that the mean squared error

$$\mathbb{E}|X_{\omega_{n+1}} - \hat{X}_{\omega_{n+1}}|^2$$

is minimized. This means that the random variable $X_{\omega_{n+1}}$ is *approximated* by a random variable $\hat{X}_{\omega_{n+1}}$ belonging to the linear space spanned by the random variables $X_{\omega_0}, \ldots, X_{\omega_n}$. The coefficients $c_{\omega,m}$ can be found by solving the system

$$(c_{\omega,0}, \ldots, c_{\omega,n})\Gamma_{\omega,n} = (< e_{\omega_{n+1}}; e_{\omega_0} >, \ldots, < e_{\omega_{n+1}}; e_{\omega_n} >).$$

As was already noted in the introduction, the stationarity assumption simplifies considerably the construction of $\hat{X}_{\omega_{n+1}}$ by using orthonormal polynomials. Indeed, let

$$d_{\omega,m} = \begin{cases} 1 & \text{if } m = n+1 \\ -c_{\omega,m} & \text{if } m \in \{0, \ldots, n\}. \end{cases}$$

Then

$$\begin{aligned}
\mathbb{E}|X_{\omega_{n+1}} - \hat{X}_{\omega_{n+1}}|^2 &= \mathbb{E}|X_{\omega_{n+1}} - \sum_{m=0}^{n} c_{\omega,m} X_{\omega_m}|^2 \\
&= \mathbb{E}|\sum_{m=0}^{n+1} d_{\omega,m} X_{\omega_m}|^2 \\
&= \sum_{m=0}^{n+1}\sum_{p=0}^{n+1} d_{\omega,m}\bar{d}_{\omega,p}\mathbb{E}(X_{\omega_m}\overline{X}_{\omega_p}) \\
&= \sum_{m=0}^{n+1}\sum_{p=0}^{n+1} d_{\omega,m}\bar{d}_{\omega,p} < e_{\omega_m}; e_{\omega_p} > \\
&= \|\sum_{m=0}^{n+1} d_{\omega,m} e_{\omega_m}\|^2 \\
&= \int_{\Lambda} |(e^{i\lambda})^{\omega_{n+1}} - \sum_{m=0}^{n} c_{\omega,m}(e^{i\lambda})^{\omega_m}|^2 \, dF(\lambda).
\end{aligned}$$

If

$$\phi_{\omega,n+1} = k_{\omega,n+1} e_{\omega_{n+1}} + \sum_{m=0}^{n} \varphi_{\omega,n+1,m}\, e_{\omega_m},$$

then according to Theorem 4, the coefficients $c_{\omega,m}$ minimizing the mean squared error are identified with the coefficients of $\frac{\phi_{\omega,n+1}}{k_{\omega,n+1}}$, i.e.

$$c_{\omega,m} = -\frac{\varphi_{\omega,n+1,m}}{k_{\omega,n+1}}$$

for all $m \in \{0, \ldots, n\}$. With this choice of the numbers $c_{\omega,m}$, the mean squared error reduces to

$$\mathbb{E}|X_{\omega_{n+1}} - \hat{X}_{\omega_{n+1}}|^2 = \frac{1}{k^2_{\omega,n+1}}.$$

Notice that $\hat{X}_{\omega_{n+1}}$ can be expressed as a linear combination of the innovations $[X_{\omega_m} - \hat{X}_{\omega_m}]$: according to (13)

$$\begin{aligned} \frac{\phi_{\omega,n+1}}{k_{\omega,n+1}} &= e_{\omega_{n+1}} - \sum_{m=0}^{n} < e_{\omega_{n+1}}; \phi_{\omega,m} > \phi_{\omega,m} \\ &= e_{\omega_{n+1}} - \sum_{m=0}^{n} k_{\omega,m} < e_{\omega_{n+1}}; \phi_{\omega,m} > \frac{\phi_{\omega,m}}{k_{\omega,m}}. \end{aligned}$$

Hence

$$\sum_{m=0}^{n} c_{\omega,m} e_{\omega_m} = \sum_{m=0}^{n} k_{\omega,m} < e_{\omega_{n+1}}; \phi_{\omega,m} > \frac{\phi_{\omega,m}}{k_{\omega,m}},$$

i.e.

$$\hat{X}_{\omega_{n+1}} = \sum_{m=0}^{n} k_{\omega,m} < e_{\omega_{n+1}}; \phi_{\omega,m} > [X_{\omega_m} - \hat{X}_{\omega_m}], \tag{37}$$

where $\hat{X}_{\omega_0} = \mathbb{E}(X_{\omega_0}) = 0$.

2.8 Kernel Polynomials

In this subsection kernel polynomials will be introduced. These polynomials fully describe the inverse of $\mathrm{H}_{\omega,n}$ (see formula (40)). We will use these polynomials here to proof that

$$(\mathrm{H}_{\omega,n})^{-1} = \mathrm{L}^*_{\omega,n}\, \mathrm{L}_{\omega,n},$$

where $\mathrm{L}_{\omega,n}$ is the lower triangular matrix with real positive diagonal elements given by (41) and $\mathrm{L}^*_{\omega,n}$ is the conjugate transpose of $\mathrm{L}_{\omega,n}$.

Let $n \in \mathbb{N}$ and fix $\omega = (\omega_0, \ldots, \omega_n) \in \Omega^{\neq}_{n,n}$. For fixed $a \in \mathbb{C}^d$, define the kernel polynomial $s_{\omega,n}(a) : \mathbb{C}^d \to \mathbb{C}$ by

$$s_{\omega,n}(a) = \sum_{k=0}^{n} \overline{\phi_{\omega,k}(a)} \phi_{\omega,k} \tag{38}$$

The term 'kernel polynomial' is made clear by the next theorem.

Theorem 8 *Let $n \in \mathbb{N}$ and fix $\omega = (\omega_0, \ldots, \omega_n) \in \Omega^{\neq}_{n,n}$. Let $g_{\omega,n} = b_0\, e_{\omega_0} + \ldots + b_n\, e_{\omega_n}$ be an arbitrary polynomial. Fix $a \in \mathbb{C}^d$. Then*

$$< g_{\omega,n}; s_{\omega,n}(a) >= g_{\omega,n}(a). \tag{39}$$

The kernel polynomial $s_{\omega,n}(a)$ is uniquely determined by (39).

Proof: Represent $g_{\omega,n}$ as

$$g_{\omega,n} = v_0\,\phi_{\omega,0} + \cdots + v_n\,\phi_{\omega,n}$$

where $v_j \in \mathbb{C}$ for all j. Then

$$\begin{aligned}
< g_{\omega,n}; s_{\omega,n}(a) > &= \sum_{k=0}^{n} v_k < \phi_{\omega,k}; s_{\omega,n}(a) > \\
&= \sum_{k=0}^{n} v_k\,\phi_{\omega,k}(a) \\
&= g_{\omega,n}(a).
\end{aligned}$$

Now suppose that along with $s_{\omega,n}(a)$ there is another polynomial $s^{\sim}_{\omega,n}(a)$ which has the kernel property (39). Then

$$< \phi_{\omega,k}; s_{\omega,n}(a) - s^{\sim}_{\omega,n}(a) >= 0$$

for all $k \in \{0,\dots,n\}$. So $s_{\omega,n}(a) = s^{\sim}_{\omega,n}(a)$, which yields the uniqueness of the kernel polynomial $s_{\omega,n}(a)$. □

The kernel polynomial $s_{\omega,n}(a)$ was defined by (38). The next theorem shows that $s_{\omega,n}(a)$ has an alternative representation.

Theorem 9 *Let $n \in \mathbb{N}$ and fix $\omega = (\omega_0,\dots,\omega_n) \in \Omega^{\neq}_{n,n}$. Then for fixed $a \in \mathbb{C}^d$*

$$s_{\omega,n}(a) = \overline{(e_{\omega_0}(a),\dots,e_{\omega_n}(a))}\,(\mathrm{H}_{\omega,n})^{-1} \begin{pmatrix} e_{\omega_0} \\ \vdots \\ e_{\omega_n} \end{pmatrix}. \tag{40}$$

Proof: Fix $m \in \{0,\dots,n\}$ and $a \in \mathbb{C}^d$. Then formula (40) gives

$$\begin{aligned}
< e_{\omega_m}; s_{\omega,n}(a) > &= (e_{\omega_0}(a),\dots,e_{\omega_n}(a))\,(\overline{\mathrm{H}}_{\omega,n})^{-1} \begin{pmatrix} < e_{\omega_m}; e_{\omega_0} > \\ \vdots \\ < e_{\omega_m}; e_{\omega_n} > \end{pmatrix} \\
&= (e_{\omega_0}(a),\dots,e_{\omega_n}(a))\,(0,\cdots,0,1,0,\cdots,0)^* \\
&= e_{\omega_m}(a),
\end{aligned}$$

where the 1 in the unit vector is at the mth place. Due to linearity it follows that (39) holds. Then uniqueness of the kernel polynomial yields (40). □

Let now $\mathrm{L}_{\omega,n}$ be the lower triangular matrix consisting of the coefficients of the orthonormal polynomials $\phi_{\omega,0},\dots,\phi_{\omega,n}$, i.e.

$$\mathrm{L}_{\omega,n} = \begin{pmatrix}
k_{\omega,0} & 0 & \dots & \dots\dots & 0 \\
\varphi_{\omega,1,0} & k_{\omega,1} & 0 & \dots\ \dots & 0 \\
\varphi_{\omega,2,0} & \varphi_{\omega,2,1} & k_{\omega,2} & 0\ \dots & 0 \\
\dots & \dots & \dots & \dots\ \dots & \ddots \\
\dots & \dots & \dots & \dots\ \dots & 0 \\
\varphi_{\omega,n,0} & \varphi_{\omega,n,1} & \varphi_{\omega,n,2} & \dots\ \dots & k_{\omega,n}
\end{pmatrix}, \tag{41}$$

with the non-zero elements defined by the identity

$$\phi_{\omega,n} = k_{\omega,n}\, e_{\omega_n} + \sum_{m=0}^{n-1} \varphi_{\omega,n,m}\, e_{\omega_m}.$$

Then (38) gives

$$\begin{aligned} s_{\omega,n}(a) &= \sum_{k=0}^{n} \overline{\phi_{\omega,k}(a)}\phi_{\omega,k} \\ &= \left(\mathrm{L}_{\omega,n}\begin{pmatrix} e_{\omega_0}(a) \\ \vdots \\ e_{\omega_n}(a)\end{pmatrix}\right)^{*}\left(\mathrm{L}_{\omega,n}\begin{pmatrix} e_{\omega_0} \\ \vdots \\ e_{\omega_n}\end{pmatrix}\right) \\ &= \overline{(e_{\omega_0}(a),\ldots,e_{\omega_n}(a))}\,\mathrm{L}^{*}_{\omega,n}\,\mathrm{L}_{\omega,n}\begin{pmatrix} e_{\omega_0} \\ \vdots \\ e_{\omega_n}\end{pmatrix}. \end{aligned} \tag{42}$$

Now formulas (40) and (42) yield

$$(\mathrm{H}_{\omega,n})^{-1} = \mathrm{L}^{*}_{\omega,n}\,\mathrm{L}_{\omega,n}. \tag{43}$$

This decomposition of $(\mathrm{H}_{\omega,n})^{-1}$ is unique, since the diagonal elements of $\mathrm{L}_{\omega,n}$ are positive real numbers.

3 Constructing Orthonormal Polynomials

As noted in the introduction, the Gramm-Schmidt orthogonalization procedure yields relations for constructing the orthonormal polynomials $\{\phi_{\omega,n}\}_{n\in\mathbb{N}}$. However the special structure (4) of $\mathrm{H}_{\omega,n}$ allows one to establish alternative relations between these orthonormal polynomials (see Theorem 11). As applied to the particular case of 'periodic sequences ω', these relations yield a useful scheme for constructing the orthonormal polynomials; see Theorem 15.

3.1 The One Dimensional Case

In this subsection a scheme is given for recurrently constructing the orthonormal polynomials $\{\phi_{\omega,n}\}_{n\in\mathbb{N}}$ in the one dimensional case, with $\omega = (0, 1, 2, 3, \ldots)$ kept fixed, i.e. *each polynomial* $\phi_{\omega,n+1}$ *is expressed in terms of the polynomial* $\phi_{\omega,n}$ (see Theorem 10). For the basic formula (44) we refer to [Grenander and Szegö] or to the next subsection, where (44) is proved at once in the d-dimensional setting as well as the formulas (45) and (46), which are only implicitly presented in [Grenander and Szegö].

Since the sequence ω is fixed, the subscript ω is redundant and therefore omitted in the notations of this subsection. The orthonormal polynomials $\{\phi_n\}_{n\in\mathbb{N}}$ are recurrently related by the relation

$$\frac{\phi_{n+1}}{k_{n+1}} = e_1\frac{\phi_n}{k_n} + \frac{l_{n+1}}{k_{n+1}k_n}\phi_n^*, \tag{44}$$

where ϕ_n^* is the polynomial reciprocal to ϕ_n, i.e.

$$\phi_n^*(z) := z^n\overline{\phi}_n(z^{-1}).$$

In the next subsection we provide for the proof of the general Theorem 13, which as applied to the present special case, can be formulated as follows:

Theorem 10 *Assume that* ϕ_n *is known. Then the following three steps enable one to construct the polynomial* ϕ_{n+1}*:*

- *Determine*

$$\frac{l_{n+1}}{k_{n+1}k_n} = -< e_1\phi_n; e_0 > . \tag{45}$$

- *Determine* $k_{n+1} \in \mathbb{R}^+$ *according to* (45) *and the relation*

$$\frac{1}{k_{n+1}^2} = \frac{1}{k_n^2} - \frac{|l_{n+1}|^2}{k_{n+1}^2k_n^2}. \tag{46}$$

- *Determine* ϕ_{n+1} *according to* (44).

Proof: See the remarks at the beginning of subsection 3.5. □

3.2 Recurrence Relations

The assertion of Theorem 10 can be extended to the d-dimensional case as it is formulated in Theorem 13, which is an obvious consequence of Theorems 11 and 12.

Theorem 11 *Let* $n \in \mathbb{N}$ *and fix* $\omega = (\omega_0, \omega_1, \ldots, \omega_{n+1}) \in \Omega^{\neq}_{n+1,n+1}$. *Define* $V_{n+1}(\omega)$ *and* $V_n^\diamond(\omega)$ *as in subsection 2.3. Then the orthonormal polynomials and their reciprocals are related as follows:*

$$\frac{\phi^*_{\omega,n+1}}{k_{\omega,n+1}} = \frac{\phi^*_{T_{n+1}(\omega),n}}{k_{T_{n+1}(\omega),n}} + \frac{l_{\omega,n+1}}{k_{\omega,n+1}k_{V_n^\diamond(\omega),n}} e_{\omega_{n+1}-\omega_n}\phi_{V_n^\diamond(\omega),n} \tag{47}$$

and

$$\frac{\phi_{\omega,n+1}}{k_{\omega,n+1}} = e_{\omega_1}\frac{\phi_{T_{n+1}(\omega),n}}{k_{T_{n+1}(\omega),n}} + \frac{l_{\omega,n+1}}{k_{\omega,n+1}k_{V_n^\diamond(\omega),n}}\phi^*_{V_n^\diamond(\omega),n}. \tag{48}$$

Moreover

$$\frac{l_{\omega,n+1}}{k_{\omega,n+1}k_{V_n^\diamond(\omega),n}} = -< e_{\omega_{n+1}-\omega_n}\phi_{V_n^\diamond(\omega),n}; e_{\omega_0} > . \tag{49}$$

Proof: Since for all $k \in \{0, \ldots, n\}$

$$e_{\omega_1-\omega_1}, e_{\omega_2-\omega_1}, \ldots, e_{\omega_{k+1}-\omega_1} \xrightarrow{GS} \phi_{T_{n+1}(\omega),k},$$

by definition (19), Corollary 1 yields

$$e_{\omega_1}, z^{\omega_2}, \ldots, e_{\omega_{k+1}} \xrightarrow{GS} e_{\omega_1}\phi_{T_{n+1}(\omega),k}. \tag{50}$$

According to (30)

$$e_{\omega_1}, e_{\omega_2}, \ldots, e_{\omega_{n+1}}, e_{\omega_0} \xrightarrow{GS\bullet} u_{\omega,n+1},$$

which means that by (50)

$$u_{\omega,n+1} = e_{\omega_0} - \sum_{k=0}^{n} < e_{\omega_0}; e_{\omega_1}\phi_{T_{n+1}(\omega),k} > e_{\omega_1}\phi_{T_{n+1}(\omega),k}. \tag{51}$$

Applying similar arguments to $u_{\omega,n}$, we get

$$u_{\omega,n} = e_{\omega_0} - \sum_{k=0}^{n-1} < e_{\omega_0}; e_{\omega_1}\phi_{T_{n+1}(\omega),k} > e_{\omega_1}\phi_{T_{n+1}(\omega),k}. \tag{52}$$

So (51) and (52) give

$$u_{\omega,n+1} = u_{\omega,n} - < e_{\omega_0}; e_{\omega_1}\phi_{T_{n+1}(\omega),n} > e_{\omega_1}\phi_{T_{n+1}(\omega),n}. \tag{53}$$

Next $< e_{\omega_0}; e_{\omega_1}\phi_{T_{n+1}(\omega),n} >$ will be expressed in terms of coefficients of orthonormal polynomials. By formula (34)

$$u_{\omega,n+1} = e_{\omega_{n+1}}\frac{\phi_{V_{n+1}(-\omega),n+1}}{k_{V_{n+1}(\omega),n+1}}.$$

Thus by (11) the coefficient of $e_{\omega_{n+1}}$ in $u_{\omega,n+1}$ is

$$\frac{\bar{l}_{V_{n+1}(\omega),n+1}}{k_{V_{n+1}(\omega),n+1}}.$$

Observe that the polynomial

$$u_{\omega,n+1} - e_{\omega_0} - \frac{\bar{l}_{V_{n+1}(\omega),n+1}}{k_{V_{n+1}(\omega),n+1}} e_{\omega_{n+1}}$$

is an element of span$\{e_{\omega_1}, \ldots, e_{\omega_n}\}$ if $n \in \mathbb{N}^+$ and it is identically zero if $n = 0$. Due to (50) with $k = n$ this yields

$$< u_{\omega,n+1} - e_{\omega_0} - \frac{\bar{l}_{V_{n+1}(\omega),n+1}}{k_{V_{n+1}(\omega),n+1}} e_{\omega_{n+1}}; e_{\omega_1}\phi_{T_{n+1}(\omega),n} >= 0 \tag{54}$$

for all $n \in \mathbb{N}$. Since the function $e_{\omega_1}\phi_{T_{n+1}(\omega),n}$ is an element of span$\{e_{\omega_1}, \ldots, e_{\omega_{n+1}}\}$, formula (29) yields

$$< u_{\omega,n+1}; e_{\omega_1}\phi_{T_{n+1}(\omega),n} >= 0. \tag{55}$$

By (54) and (55)

$$\begin{aligned} < e_{\omega_0}; e_{\omega_1}\phi_{T_{n+1}(\omega),n} > &= -\frac{\bar{l}_{V_{n+1}(\omega),n+1}}{k_{V_{n+1}(\omega),n+1}} < e_{\omega_{n+1}}; e_{\omega_1}\phi_{T_{n+1}(\omega),n} > && (56) \\ &= -\frac{\bar{l}_{V_{n+1}(\omega),n+1}}{k_{V_{n+1}(\omega),n+1} k_{T_{n+1}(\omega),n}}, && (57) \end{aligned}$$

since

$$\begin{aligned} < e_{\omega_{n+1}}; e_{\omega_1}\phi_{T_{n+1}(\omega),n} > &= \frac{1}{k_{T_{n+1}(\omega),n}} < k_{T_{n+1}(\omega),n}\, e_{\omega_{n+1}}; e_{\omega_1}\phi_{T_{n+1}(\omega),n} > \\ &= \frac{1}{k_{T_{n+1}(\omega),n}} < e_{\omega_1}\phi_{T_{n+1}(\omega),n}; e_{\omega_1}\phi_{T_{n+1}(\omega),n} > \\ &= \frac{1}{k_{T_{n+1}(\omega),n}}. \end{aligned}$$

By (53) and (57) we have

$$u_{\omega,n+1} = u_{\omega,n} + \frac{\bar{l}_{V_{n+1}(\omega),n+1}}{k_{V_{n+1}(\omega),n+1} k_{T_{n+1}(\omega),n}} e_{\omega_1}\phi_{T_{n+1}(\omega),n}. \tag{58}$$

Now (58) will be expressed in terms of orthonormal polynomials. According to formula (34) and (36)

$$u_{\omega,n+1} = \frac{\phi^*_{V_{n+1}(\omega),n+1}}{k_{V_{n+1}(\omega),n+1}}$$

and

$$u_{\omega,n} = \frac{\phi^*_{V_n^\diamond(\omega),n}}{k_{V_n^\diamond(\omega),n}}.$$

So (58) becomes

$$\frac{\phi^*_{V_{n+1}(\omega),n+1}}{k_{V_{n+1}(\omega),n+1}} = \frac{\phi^*_{V_n^\diamond(\omega),n}}{k_{V_n^\diamond(\omega),n}} + \frac{\bar{l}_{V_{n+1}(\omega),n+1}}{k_{V_{n+1}(\omega),n+1}k_{T_{n+1}(\omega),n}} e_{\omega_1}\phi_{T_{n+1}(\omega),n}. \tag{59}$$

Substituting $\omega = (\omega_0, \ldots, \omega_{n+1})$ in (57) and (59) by $V_{n+1}(\omega) = (\omega_{n+1} - \omega_{n+1}, \omega_{n+1} - \omega_n, \ldots, \omega_{n+1} - \omega_0)$ (this is allowed since $V_{n+1}(\omega) \in \Omega^{\neq}_{n+1,n+1}$), we get by (21), (24) and (25)

$$< e_{\omega_0}; e_{\omega_{n+1}-\omega_n}\phi_{V_n^\diamond(\omega),n} >= -\frac{\bar{l}_{\omega,n+1}}{k_{\omega,n+1}k_{V_n^\diamond(\omega),n}} \tag{60}$$

and

$$\frac{\phi^*_{\omega,n+1}}{k_{\omega,n+1}} = \frac{\phi^*_{T_{n+1}(\omega),n}}{k_{T_{n+1}(\omega),n}} + \frac{\bar{l}_{\omega,n+1}}{k_{\omega,n+1}k_{V_n^\diamond(\omega),n}} e_{\omega_{n+1}-\omega_n}\phi_{V_n^\diamond(\omega),n}.$$

Thus formula (47) is proved as well as (49), since this last formula follows by conjugating (60). It remains to prove (48).

By the definition of reciprocals, formula (47) is equivalent to

$$\frac{e_{\omega_{n+1}}\phi_{-\omega,n+1}}{k_{\omega,n+1}} = \frac{e_{\omega_{n+1}-\omega_1}\phi_{T_{n+1}(-\omega),n}}{k_{T_{n+1}(\omega),n}} + \frac{\bar{l}_{\omega,n+1}}{k_{\omega,n+1}k_{V_n^\diamond(\omega),n}} e_{\omega_{n+1}-\omega_n}\phi_{V_n^\diamond(\omega),n}.$$

Due to (11) and the fact that

$$e_{\omega_n}\phi_{V_n^\diamond(-\omega),n} = \phi^*_{V_n^\diamond(\omega),n}$$

this yields

$$\frac{\phi_{\omega,n+1}}{k_{\omega,n+1}} = e_{\omega_1}\frac{\phi_{T_{n+1}(\omega),n}}{k_{T_{n+1}(\omega),n}} + \frac{l_{\omega,n+1}}{k_{\omega,n+1}k_{V_n^\diamond(\omega),n}}\phi^*_{V_n^\diamond(\omega),n},$$

cf. (48). □

Formula (48) and (49) show that $\frac{\phi_{\omega,n+1}}{k_{\omega,n+1}}$ can be determined once $\phi_{T_{n+1}(\omega),n}$ and $\phi_{V_n^\diamond(\omega),n}$ are known. The next theorem shows how $k_{\omega,n+1}$ can be determined once $\phi_{T_{n+1}(\omega),n}$ and $\phi_{V_n^\diamond(\omega),n}$ are known.

Theorem 12 *Let $n \in \mathbb{N}$ and fix $\omega = (\omega_0, \omega_1, \ldots, \omega_{n+1}) \in \Omega^{\neq}_{n+1,n+1}$. Then*

$$\frac{1}{k^2_{\omega,n+1}} = \frac{1}{k^2_{T_{n+1}(\omega),n}} - \frac{|l_{\omega,n+1}|^2}{k^2_{\omega,n+1}k^2_{V_n^\diamond(\omega),n}}. \tag{61}$$

<u>Proof:</u> Formula (53) gives

$$< u_{\omega,n+1}; e_{\omega_0} >=< u_{\omega,n}; e_{\omega_0} > -| < e_{\omega_0}; e_{\omega_1}\phi_{T_{n+1}(\omega),n} > |^2. \tag{62}$$

This combined with (57) yields

$$\frac{1}{k^2_{V_{n+1}(\omega),n+1}} = \frac{1}{k^2_{V_n^\diamond(\omega),n}} - \frac{|l_{V_{n+1}(\omega),n+1}|^2}{k^2_{V_{n+1}(\omega),n+1} k^2_{T_{n+1}(\omega),n}}, \tag{63}$$

since

$$\begin{aligned} < u_{\omega,n+1}; e_{\omega_0} > &= < e_{\omega_{n+1}}; e_{\omega_0} \frac{\phi_{V_{n+1}(\omega),n+1}}{k_{V_{n+1}(\omega),n+1}} > \\ &= \frac{1}{k_{V_{n+1}(\omega),n+1}} < e_{\omega_{n+1}}; \phi_{V_{n+1}(\omega),n+1} > \\ &= \frac{1}{k^2_{V_{n+1}(\omega),n+1}} \end{aligned} \tag{64}$$

and

$$\begin{aligned} < u_{\omega,n}; e_{\omega_0} > &= < e_{\omega_n}; e_{\omega_0} \frac{\phi_{V_n^\diamond(\omega),n}}{k_{V_n^\diamond(\omega),n}} > \\ &= \frac{1}{k_{V_n^\diamond(\omega),n}} < e_{\omega_n}; \phi_{V_n^\diamond(\omega),n} > \\ &= \frac{1}{k^2_{V_n^\diamond(\omega),n}}. \end{aligned} \tag{65}$$

Substituting $\omega = (\omega_0, \ldots, \omega_{n+1})$ in (63) by $V_{n+1}(\omega) = (\omega_{n+1} - \omega_{n+1}, \omega_{n+1} - \omega_n, \ldots, \omega_{n+1} - \omega_0)$ (this is allowed since $V_{n+1}(\omega) \in \Omega^{\neq}_{n+1,n+1}$) and using (21), (24) and (25), we get

$$\frac{1}{k^2_{\omega,n+1}} = \frac{1}{k^2_{T_{n+1}(\omega),n}} - \frac{|l_{\omega,n+1}|^2}{k^2_{\omega,n+1} k^2_{V_n^\diamond(\omega),n}}.$$

□

The next theorem shows how $\phi_{\omega,n+1}$ is determined once $\phi_{T_{n+1}(\omega),n}$ and $\phi_{V_n^\diamond(\omega),n}$ are known.

Theorem 13 *Let $n \in \mathbb{N}$ and fix $\omega = (\omega_0, \omega_1, \ldots, \omega_{n+1}) \in \Omega^{\neq}_{n+1,n+1}$. Assume that $\phi_{T_{n+1}(\omega),n}$ and $\phi_{V_n^\diamond(\omega),n}$ are known. Then the following three steps enable one to construct the polynomial $\phi_{\omega,n+1}$:*

- *Determine $\frac{l_{\omega,n+1}}{k_{\omega,n+1} k_{V_n^\diamond(\omega),n}}$ according to (49).*
- *Determine $k_{\omega,n+1} \in \mathbb{R}^+$ according to (61).*
- *Determine $\phi_{\omega,n+1}$ according to (48).*

Proof: Use Theorems 11 and 12. □

Observe that one can derive the basic relationship (48) alternatively by using geometric arguments as follows. Notice first that (48) is equivalent to

$$\frac{\phi_{\omega,n+1}}{k_{\omega,n+1}} = e_{\omega_1}\frac{\phi_{T_{n+1}(\omega),n}}{k_{T_{n+1}(\omega),n}} - < e_{\omega_1}\frac{\phi_{T_{n+1}(\omega),n}}{k_{T_{n+1}(\omega),n}}; \phi^*_{V_n^\diamond(\omega),n} > \phi^*_{V_n^\diamond(\omega),n}, \tag{66}$$

since by (26), (36), (49) and (50)

$$\begin{aligned} < e_{\omega_1}\frac{\phi_{T_{n+1}(\omega),n}}{k_{T_{n+1}(\omega),n}}; \phi^*_{V_n^\diamond(\omega),n} > &= < e_{\omega_{n+1}}; \phi^*_{V_n^\diamond(\omega),n} > \\ &= < e_{\omega_{n+1}-\omega_n}\phi_{V_n^\diamond(\omega),n}; e_{\omega_0} > \\ &= -\frac{l_{\omega,n+1}}{k_{\omega,n+1}k_{V_n^\diamond(\omega),n}}. \end{aligned} \tag{67}$$

In order to verify formula (66), we use the following notations:

- $\mathrm{L}(\omega_1; \ldots; \omega_n)$ denotes $\mathrm{span}(e_{\omega_1}; \ldots; e_{\omega_n})$.
- $\mathrm{L}(\omega_0; \ldots; \omega_{n+1})$ denotes $\mathrm{span}(e_{\omega_0}; \ldots; e_{\omega_{n+1}})$.
- $\mathrm{L}^\perp(\omega_1; \ldots; \omega_n)$ is defined by

$$\mathrm{L}(\omega_0; \ldots; \omega_{n+1}) = \mathrm{L}(\omega_1; \ldots; \omega_n) \oplus \mathrm{L}^\perp(\omega_1; \ldots; \omega_n).$$

Notice that $\mathrm{L}^\perp(\omega_1; \ldots; \omega_n)$ has dimension two.

Since $\phi^*_{V_n^\diamond(\omega),n}$ and $e_{\omega_1}\phi_{T_{n+1}(\omega),n}$ are linearly independent polynomials belonging to $\mathrm{L}^\perp(\omega_1; \ldots; \omega_n)$, a basis of $\mathrm{L}^\perp(\omega_1; \ldots; \omega_n)$ is

$$\{\phi^*_{V_n^\diamond(\omega),n}; e_{\omega_1}\phi_{T_{n+1}(\omega),n}\}.$$

An orthogonal basis of $\mathrm{L}^\perp(\omega_1; \ldots; \omega_n)$ is

$$\{\phi^*_{V_n^\diamond(\omega),n}; e_{\omega_1}\phi_{T_{n+1}(\omega),n} - < e_{\omega_1}\phi_{T_{n+1}(\omega),n}; \phi^*_{V_n^\diamond(\omega),n} > \phi^*_{V_n^\diamond(\omega),n}\}.$$

Since $\phi_{\omega,n+1} \in \mathrm{L}^\perp(\omega_1; \ldots; \omega_n)$ and $< \phi_{\omega,n+1}; \phi^*_{V_n^\diamond(\omega),n} >= 0$, it follows that $\phi_{\omega,n+1} \in \mathrm{span}(e_{\omega_1}\phi_{T_{n+1}(\omega),n} - < e_{\omega_1}\phi_{T_{n+1}(\omega),n}; \phi^*_{V_n^\diamond(\omega),n} > \phi^*_{V_n^\diamond(\omega),n})$. By comparing the coefficients of $e_{\omega_{n+1}}$ we get (66).

3.3 Applications to Random Fields (continuation)

In this subsection it will be shown that formula (48) yields a relationship between random variables in a random field and their predictors. Moreover the partial autocorrelation between random variables will be expressed in terms of coefficients of orthonormal polynomials.

Consider a random field as defined in subsection 2.7. Let $n \in \mathbb{N}$ and fix $\omega \in \Omega^{\neq}_{n+1,n+1}$. Let $\hat{X}_{\omega_{n+1}}$ be the best linear predictor for $X_{\omega_{n+1}}$ that belongs to $\mathrm{span}(X_{\omega_0}, \ldots, X_{\omega_n})$, i.e. in view of subsection 2.7

$$\hat{X}_{\omega_{n+1}} = -\sum_{m=0}^{n} \frac{\varphi_{\omega,n+1,m}}{k_{\omega,n+1}} X_{\omega_m},$$

where the coefficients are defined by the identity

$$\phi_{\omega,n+1} = k_{\omega,n+1} e_{\omega_{n+1}} + \sum_{m=0}^{n} \varphi_{\omega,n+1,m}\, e_{\omega_m}.$$

Let $\hat{X}^{\sim}_{\omega_{n+1}}$ be the best linear predictor for $X_{\omega_{n+1}}$ that belongs to $\mathrm{span}(X_{\omega_1}, \ldots, X_{\omega_n})$ and let $\hat{X}^{\sim}_{\omega_0}$ be the best linear predictor for X_{ω_0} that belongs to $\mathrm{span}(X_{\omega_1}, \ldots, X_{\omega_n})$. By the considerations similar to that of subsection 2.7, we get

$$\hat{X}^{\sim}_{\omega_{n+1}} = -\sum_{m=1}^{n} \frac{\varphi_{T_{n+1}(\omega),n+1,m}}{k_{T_{n+1}(\omega),n}} X_{\omega_m}$$

and

$$\hat{X}^{\sim}_{\omega_0} = -\sum_{m=1}^{n} \frac{\varphi^*_{V_n^\circ(\omega),n+1,m}}{k_{V_n^\circ(\omega),n}} X_{\omega_m},$$

where the coefficients are defined by the identities

$$e_{\omega_1} \phi_{T_{n+1}(\omega),n} = k_{T_{n+1}(\omega),n} e_{\omega_{n+1}} + \sum_{m=1}^{n} \varphi_{T_{n+1}(\omega),n+1,m} e_{\omega_m}$$

and

$$\phi^*_{V_n^\circ(\omega),n} = k_{V_n^\circ(\omega),n} e_{\omega_0} + \sum_{m=1}^{n} \varphi^*_{V_n^\circ(\omega),n+1,m} e_{\omega_m}.$$

Hence (48) yields the following relationship between the innovations $[X_{\omega_{n+1}} - \hat{X}_{\omega_{n+1}}]$, $[X_{\omega_{n+1}} - \hat{X}^{\sim}_{\omega_{n+1}}]$ and $[X_{\omega_0} - \hat{X}^{\sim}_{\omega_0}]$:

$$[X_{\omega_{n+1}} - \hat{X}_{\omega_{n+1}}] = [X_{\omega_{n+1}} - \hat{X}^{\sim}_{\omega_{n+1}}] + \frac{l_{\omega,n+1}}{k_{\omega,n+1} k_{V_n^\circ(\omega),n}} [X_{\omega_0} - \hat{X}^{\sim}_{\omega_0}].$$

Moreover formula (67) yields

$$\begin{aligned} \mathrm{Cov}(X_{\omega_{n+1}} - \hat{X}^{\sim}_{\omega_{n+1}}, X_{\omega_0} - \hat{X}^{\sim}_{\omega_0}) &= < e_{\omega_1} \frac{\phi_{T_{n+1}(\omega),n}}{k_{T_{n+1}(\omega),n}}; \frac{\phi^*_{V_n^\circ(\omega),n}}{k_{V_n^\circ(\omega),n}} > \\ &= -\frac{l_{\omega,n+1}}{k_{\omega,n+1} k^2_{V_n^\circ(\omega),n}}. \end{aligned}$$

Therefore the partial autocorrelation between $X_{\omega_{n+1}}$ and X_{ω_0}, defined as the correlation between $[X_{\omega_{n+1}} - \hat{X}^{\sim}_{\omega_{n+1}}]$ and $[X_{\omega_0} - \hat{X}^{\sim}_{\omega_0}]$ (see [Brockwell and Davis]), is

$$\mathrm{Corr}(X_{\omega_{n+1}} - \hat{X}^{\sim}_{\omega_{n+1}}, X_{\omega_0} - \hat{X}^{\sim}_{\omega_0}) = -\frac{l_{\omega,n+1} k_{T_{n+1}(\omega),n}}{k_{\omega,n+1} k_{V_n^\circ(\omega),n}}.$$

3.4 'Periodic' Sequence Sets and their Transformations

Unlike subsection 2.3, we will define first certain sequence sets consisting of sequences ω with infinitely many entries and then look at their truncations.

Define

$$\Omega^{\neq} := \{(\omega_0, \omega_1, \ldots) \mid \omega_i \in \mathbb{Z}^d, \omega_0 = 0_d, \forall i, j : (i \neq j) \Rightarrow (\omega_i \neq \omega_j)\} \tag{68}$$

and

$$\Omega^{\text{per}} := \{(\omega_0, \omega_1, \ldots) \mid (\omega_0, \omega_1, \ldots) \in \Omega^{\neq}, \exists p \in \mathbb{N}^+ : \forall i \in \mathbb{N}\, \omega_{i+p} = \omega_i + \omega_p\} \tag{69}$$

If $\omega \in \Omega^{\text{per}}$, then ω is called periodic. The period of ω, denoted by p, is defined as the smallest number in $\mathbb{N}^+$ such that $\forall i \in \mathbb{N} : \omega_{i+p} = \omega_i + \omega_p$.

On $\Omega^{\neq}$ and Ω^{per} several mappings will be defined:

- The identity mapping $I : \Omega^{\neq} \to \Omega^{\neq}$ is defined by

$$I(\omega) = \omega. \tag{70}$$

- For each $n \in \mathbb{N}$, the mapping $S_n : \Omega^{\neq}(\Omega^{\text{per}}) \to \Omega^{\neq}(\Omega^{\text{per}})$ is defined by

$$S_n((\omega_0, \omega_1, \ldots)) := (\omega_n - \omega_n, \omega_{n+1} - \omega_n, \omega_{n+2} - \omega_n, \ldots). \tag{71}$$

Fix now an $\omega \in \Omega^{\text{per}}$ with period p. Then $\omega_R \in \Omega^{\text{per}}$ associated with this ω is defined by

$$\omega_R := ([\omega_R]_0, [\omega_R]_1, \ldots) \text{ with } [\omega_R]_{pk+q} = \omega_{(k+1)p} - \omega_{p-q}, \tag{72}$$

where $k \in \mathbb{N}$ and $q \in \{0, \ldots, p-1\}$. Consequently by (69), for all $l \in \mathbb{N}$

$$[\omega_R]_{pk+q} = \omega_{(k+l+1)p} - \omega_{(l+1)p-q}.$$

Notice that

$$(\omega_R)_R = \omega. \tag{73}$$

For $\omega \in \Omega^{\neq}$ we want to make use of formulas from the previous subsections. Since these formulas are only valid for sequences ω of finite length, we define for $n \in \mathbb{N}$ the 'truncation' of $\omega = (\omega_0, \omega_1, \ldots) \in \Omega^{\neq}$ as follows:

$$\omega^{(n)} := (\omega_0, \ldots, \omega_n) \in \Omega^{\neq}_{n,n}. \tag{74}$$

3.5 Recurrence Schemes

We show that a scheme for constructing orthonormal polynomials $\{\phi_{\omega,n}\}_{n\in\mathbb{N}}$ is considerably simplified if ω is periodic. Notice that in this subsection ω has infinitely many entries.

As the simpliest example of a periodic sequence, consider a sequence ω with period one, i.e. fix $\omega_1 \neq 0_d$ and let $\omega = (0\cdot\omega_1, 1\cdot\omega_1, 2\cdot\omega_1, 3\cdot\omega_1, \ldots)$. For such a sequence ω, (48),(49) and (61) become

$$\begin{aligned}
\frac{\phi_{\omega,n+1}}{k_{\omega,n+1}} &= e_{\omega_1}\frac{\phi_{\omega,n}}{k_{\omega,n}} + \frac{l_{\omega,n+1}}{k_{\omega,n+1}k_{\omega,n}}\phi^*_{\omega,n},\\
\frac{l_{\omega,n+1}}{k_{\omega,n+1}k_{\omega,n}} &= -< e_{\omega_1}\phi_{\omega,n}; e_{\omega_0} > \text{ and}\\
\frac{1}{k^2_{\omega,n+1}} &= \frac{1}{k^2_{\omega,n}} - \frac{|l_{\omega,n+1}|^2}{k^2_{\omega,n+1}k^2_{\omega,n}}.
\end{aligned}$$

cf. Theorem 13. The orthonormal polynomials $\{\phi_{\omega,n}\}_{n\in\mathbb{N}}$ can now recurrently be constructed in the same way as in Theorem 10. We extend now the above scheme to an arbitrary periodic sequence ω.

Fix an $\omega \in \Omega^{\text{per}}$ with period p. The scheme for building the orthonormal polynomials $\{\phi_{\omega,n}\}_{n\in\mathbb{N}}$ will involve collections of orthonormal polynomials $C_{\omega,n}$ defined by

$$C_{\omega,n} := \{\phi_{S_0(\omega)^{(n)},n}, \ldots, \phi_{S_{p-1}(\omega)^{(n)},n}\} \cup \{\phi_{S_0(\omega_R)^{(n)},n}, \ldots, \phi_{S_{p-1}(\omega_R)^{(n)},n}\}. \tag{75}$$

Notice that if $\omega = \omega_R$, i.e.

$$\forall k \in \{0, \ldots, p-1\} \;:\; \omega_p = \omega_k + \omega_{p-k},$$

(this is of course always true if $p = 1$), then the sets $C_{\omega,n}$ are given by

$$C_{\omega,n} = \{\phi_{S_0(\omega)^{(n)},n}; \ldots; \phi_{S_{p-1}(\omega)^{(n)},n}\}.$$

An example of a periodic sequence in $\mathbb{Z}^2$ with period $p = 3$ and $\omega = \omega_R$ is given by

$$\omega = (\omega_0, \omega_1, \ldots)$$

with $\omega_0 = (0,0)$, $\omega_1 = (2,1)$, $\omega_2 = (3,2)$ and $\omega_3 = (5,3)$.

Theorem 14 tells us that once the set $C_{\omega,n}$ is known, the set $C_{\omega,n+1}$ can recurrently be constructed by using the orthonormal polynomials from $C_{\omega,n}$, i.e. each polynomial from $C_{\omega,n+1}$ can be expressed in terms of polynomials from $C_{\omega,n}$. Obviously if the sets $\{C_{\omega,n}\}_{n\in\mathbb{N}}$ are known, then the orthonormal polynomials $\{\phi_{\omega,n}\}_{n\in\mathbb{N}}$ are known as well, since for all $n \in \mathbb{N}$

$$\phi_{\omega,n} = \phi_{S_0(\omega)^{(n)},n} \in C_{\omega,n}. \tag{76}$$

Theorem 14 *Fix* $\omega \in \Omega^{\text{per}}$ *with period* p. *For* $n \in \mathbb{N}$ *let the sets* $C_{\omega,n}$ *of orthonormal polynomials be defined by* (75). *Suppose that* $C_{\omega,0}$ *and* $C_{\omega,1}$ *are known. Then for each* $n \in \mathbb{N}^+$ *the set* $C_{\omega,n+1}$ *can recurrently be constructed according to the scheme in Theorem 13, by using the orthonormal polynomials from* $C_{\omega,n}$.

Proof: Fix $n \in \mathbb{N}^+$ and suppose that $C_{\omega,n}$ is known. It suffices to show that for all $m \in \{0,\dots,p-1\}$, the polynomials $\phi_{T_{n+1}(S_m(\omega)^{\langle n+1\rangle}),n}$, $\phi^*_{V_n^\diamond(S_m(\omega)^{\langle n+1\rangle}),n}$, $\phi_{T_{n+1}(S_m(\omega_R)^{\langle n+1\rangle}),n}$ and $\phi^*_{V_n^\diamond(S_m(\omega_R)^{\langle n+1\rangle}),n}$ belong to $C_{\omega,n}$, because if so, then the polynomials $\phi_{S_m(\omega)^{\langle n+1\rangle},n+1}$ and $\phi_{S_m(\omega_R)^{\langle n+1\rangle},n+1}$ can be constructed by applying Theorem 13.

The first question is whether $\phi_{T_{n+1}(S_m(\omega)^{\langle n+1\rangle}),n}$ and $\phi^*_{V_n^\diamond(S_m(\omega)^{\langle n+1\rangle}),n}$ belong to $C_{\omega,n}$ for an arbitrary $m \in \{0,\dots,p-1\}$. So fix an $m \in \{0,\dots,p-1\}$. It is easily verified that

$$T_{n+1}(S_m(\omega)^{\langle n+1\rangle}) = (\omega_{m+1}-\omega_{m+1}, \omega_{m+2}-\omega_{m+1}, \dots, \omega_{m+n+1}-\omega_{m+1}).$$

Therefore

$$e_{\omega_{m+1}-\omega_{m+1}}, e_{\omega_{m+2}-\omega_{m+1}}, \dots, e_{\omega_{m+n+1}-\omega_{m+1}} \xrightarrow{GS} \phi_{T_{n+1}(S_m(\omega)^{\langle n+1\rangle}),n}. \tag{77}$$

Define now

$$m_p := (m+1) \bmod p \in \{0,\dots,p-1\}. \tag{78}$$

Then

$$e_{\omega_{m_p}-\omega_{m_p}}, e_{\omega_{m_p+1}-\omega_{m_p}}, \dots, e_{\omega_{m_p+n}-\omega_{m_p}} \xrightarrow{GS} \phi_{S_{m_p}(\omega)^{\langle n\rangle},n}. \tag{79}$$

Due to periodicity of ω, formulas (77) and (79) imply

$$\phi_{T_{n+1}(S_m(\omega)^{\langle n+1\rangle}),n} = \phi_{S_{m_p}(\omega)^{\langle n\rangle},n}. \tag{80}$$

So (80) yields $\phi_{T_{n+1}(S_m(\omega)^{\langle n+1\rangle}),n} \in C_{\omega,n}$. It remains to show that also

$$\phi^*_{V_n^\diamond(S_m(\omega)^{\langle n+1\rangle}),n} \in C_{\omega,n}.$$

To this end, we likewise verify first that

$$V_n^\diamond(S_m(\omega)^{\langle n+1\rangle}) = (\omega_{m+n}-\omega_{m+n}, \omega_{m+n}-\omega_{m+n-1}, \dots, \omega_{m+n}-\omega_m, \omega_{m+n+1}-\omega_m)$$

and conclude then that

$$e_{\omega_{m+n}-\omega_{m+n}}, e_{\omega_{m+n}-\omega_{m+n-1}}, \dots, e_{\omega_{m+n}-\omega_m} \xrightarrow{GS} \phi_{V_n^\diamond(S_m(\omega)^{\langle n+1\rangle}),n} \tag{81}$$

$$e_{\omega_{m+n}-\omega_m}, e_{\omega_{m+n-1}-\omega_m}, \dots, e_{\omega_m-\omega_m} \xrightarrow{GS} \phi^*_{V_n^\diamond(S_m(\omega)^{\langle n+1\rangle}),n}. \tag{82}$$

We now need the numbers $\theta \in \mathbb{N}$ and $\kappa \in \{0,\dots,p-1\}$, uniquely defined by

$$m+n = \theta p + (p-\kappa) \Leftrightarrow m+n+\kappa = (\theta+1)p, \tag{83}$$

in order to write

$$S_\kappa(\omega_R)^{\langle n\rangle} = (\omega_{(\theta+1)p-\kappa}-\omega_{(\theta+1)p-\kappa}, \dots, \omega_{(\theta+1)p-\kappa}-\omega_{(\theta+1)p-\kappa-n})$$

and consequently

$$\begin{aligned}&e_{\omega_{(\theta+1)p-\kappa}-\omega_{(\theta+1)p-\kappa}}, e_{\omega_{(\theta+1)p-\kappa}-\omega_{(\theta+1)p-\kappa-1}},\\ &\quad \ldots, e_{\omega_{(\theta+1)p-\kappa}-\omega_{(\theta+1)p-\kappa-n}} \xrightarrow{GS} \phi_{S_\kappa(\omega_R)^{\langle n\rangle},n} \qquad (84)\\ &e_{\omega_{(\theta+1)p-\kappa}-\omega_{(\theta+1)p-\kappa-n}}, e_{\omega_{(\theta+1)p-\kappa-1}-\omega_{(\theta+1)p-\kappa-n}},\\ &\quad \ldots, e_{\omega_{(\theta+1)p-\kappa-n}-\omega_{(\theta+1)p-\kappa-n}} \xrightarrow{GS} \phi^*_{S_\kappa(\omega_R)^{\langle n\rangle},n}. \qquad (85)\end{aligned}$$

Due to the periodicity of ω, it follows from (81) and (84) that

$$\phi_{V_n^\diamond(S_m(\omega)^{\langle n+1\rangle}),n} = \phi_{S_\kappa(\omega_R)^{\langle n\rangle},n} \qquad (86)$$

and from (82) and (85) that

$$\phi^*_{V_n^\diamond(S_m(\omega)^{\langle n+1\rangle}),n} = \phi^*_{S_\kappa(\omega_R)^{\langle n\rangle},n}. \qquad (87)$$

Thus $\phi^*_{V_n^\diamond(S_m(\omega)^{\langle n+1\rangle}),n} \in C_{\omega,n}$.

The second question is whether $\phi_{T_{n+1}(S_m(\omega_R)^{\langle n+1\rangle}),n}$ and $\phi^*_{V_n^\diamond(S_m(\omega_R)^{\langle n+1\rangle}),n}$ belong to $C_{\omega,n}$, for if so, then $\phi_{S_m(\omega_R)^{\langle n+1\rangle},n+1}$ can be constructed. Since $\omega_R \in \Omega^{\text{per}}$ with period p, formulas (80), (86) and (87) with ω substituted by ω_R yield

$$\begin{aligned}\phi_{T_{n+1}(S_m(\omega_R)^{\langle n+1\rangle}),n} &= \phi_{S_{m_p}(\omega_R)^{\langle n\rangle},n} &(88)\\ \phi_{V_n^\diamond(S_m(\omega_R)^{\langle n+1\rangle}),n} &= \phi_{S_\kappa(\omega)^{\langle n\rangle},n} &(89)\\ \phi^*_{V_n^\diamond(S_m(\omega_R)^{\langle n+1\rangle}),n} &= \phi^*_{S_\kappa(\omega_R)^{\langle n\rangle},n}, &(90)\end{aligned}$$

where (73) is used. So $\phi_{T_{n+1}(S_m(\omega_R)^{\langle n+1\rangle}),n}$ and $\phi^*_{V_n^\diamond(S_m(\omega_R)^{\langle n+1\rangle}),n}$ belong to $C_{\omega,n}$. □

If $\omega \in \Omega^{\text{per}}$ with period p, $m \in \{0,\ldots,p-1\}$ and m_p and κ are defined by resp. (78) and (83), then formulas (48),(49),(61) and Theorem 14 yield:

- $$\begin{aligned}\frac{\phi_{S_m(\omega)^{\langle n+1\rangle},n+1}}{k_{S_m(\omega)^{\langle n+1\rangle},n+1}} &= e_{\omega_{m+1}-\omega_m}\frac{\phi_{S_{m_p}(\omega)^{\langle n\rangle},n}}{k_{S_{m_p}(\omega)^{\langle n\rangle},n}}\\ &+ \frac{l_{S_m(\omega)^{\langle n+1\rangle},n+1}}{k_{S_m(\omega)^{\langle n+1\rangle},n+1}k_{V_n^\diamond(S_m(\omega)^{\langle n+1\rangle}),n}}\phi^*_{S_\kappa(\omega_R)^{\langle n\rangle},n}. \qquad (91)\end{aligned}$$

- $$\begin{aligned}\frac{l_{S_m(\omega)^{\langle n+1\rangle},n+1}}{k_{S_m(\omega)^{\langle n+1\rangle},n+1}k_{V_n^\diamond(S_m(\omega)^{\langle n+1\rangle}),n}} &= -< e_{\omega_{m+n+1}-\omega_{m+n}}\phi_{V_n^\diamond(S_m(\omega)^{\langle n+1\rangle}),n}; e_{\omega_0} >\\ &= -< e_{\omega_{p-\kappa+1}-\omega_{p-\kappa}}\phi_{S_\kappa(\omega_R)^{\langle n\rangle},n}; e_{\omega_0} >. \qquad (92)\end{aligned}$$

- $$\frac{1}{k^2_{S_m(\omega)^{\langle n+1\rangle},n+1}} = \frac{1}{k^2_{S_{m_p}(\omega)^{\langle n\rangle},n}} - \frac{|l_{S_m(\omega)^{\langle n+1\rangle},n+1}|^2}{k^2_{S_m(\omega)^{\langle n+1\rangle},n+1}k^2_{V_n^\diamond(S_m(\omega)^{\langle n+1\rangle}),n}}. \qquad (93)$$

-
$$\frac{\phi_{S_m(\omega_R)^{\langle n+1\rangle},n+1}}{k_{S_m(\omega_R)^{\langle n+1\rangle},n+1}} = e_{\omega_{p-m}-\omega_{p-m-1}}\frac{\phi_{S_{m_p}(\omega_R)^{\langle n\rangle},n}}{k_{S_{m_p}(\omega_R)^{\langle n\rangle},n}} + \frac{l_{S_m(\omega_R)^{\langle n+1\rangle},n+1}}{k_{S_m(\omega_R)^{\langle n+1\rangle},n+1}k_{V_n^\circ(S_m(\omega_R)^{\langle n+1\rangle}),n}}\phi^*_{S_\kappa(\omega)^{\langle n\rangle},n}. \quad (94)$$

-
$$\frac{l_{S_m(\omega_R)^{\langle n+1\rangle},n+1}}{k_{S_m(\omega_R)^{\langle n+1\rangle},n+1}k_{V_n^\circ(S_m(\omega_R)^{\langle n+1\rangle}),n}} = -< e_{\omega_{p+\kappa}-\omega_{p+\kappa-1}}\phi_{V_n^\circ(S_m(\omega_R)^{\langle n+1\rangle}),n}; e_{\omega_0} > = -< e_{\omega_{p+\kappa}-\omega_{p+\kappa-1}}\phi_{S_\kappa(\omega)^{\langle n\rangle},n}; e_{\omega_0} > . \quad (95)$$

-
$$\frac{1}{k^2_{S_m(\omega_R)^{\langle n+1\rangle},n+1}} = \frac{1}{k^2_{S_{m_p}(\omega_R)^{\langle n\rangle},n}} - \frac{|l_{S_m(\omega_R)^{\langle n+1\rangle},n+1}|^2}{k^2_{S_m(\omega_R)^{\langle n+1\rangle},n+1}k^2_{V_n^\circ(S_m(\omega_R)^{\langle n+1\rangle}),n}}. \quad (96)$$

Theorem 14 and formulas (91) - (96) yield the following explicit scheme for constructing the sets $C_{\omega,n}$:

Theorem 15 *Let $n \in \mathbb{N}$ and fix $\omega \in \Omega^{\text{per}}$ with period p. Assume that the set $C_{\omega,n}$ is known. For a fixed $m \in \{0,\ldots,p-1\}$, let m_p and κ be defined by resp. (78) and (83). Then the following steps have to be carried out for all $m \in \{0,\ldots,p-1\}$ in order to construct the set $C_{\omega,n+1}$:*

- *Determine* $\frac{l_{S_m(\omega)^{\langle n+1\rangle},n+1}}{k_{S_m(\omega)^{\langle n+1\rangle},n+1}k_{V_n^\circ(S_m(\omega)^{\langle n+1\rangle}),n}}$ *according to* (92).
- *Determine* $k_{S_m(\omega)^{\langle n+1\rangle},n+1} \in \mathbb{R}^+$ *according to* (93).
- *Determine* $\phi_{S_m(\omega)^{\langle n+1\rangle},n+1}$ *according to* (91).
- *Determine* $\frac{l_{S_m(\omega_R)^{\langle n+1\rangle},n+1}}{k_{S_m(\omega_R)^{\langle n+1\rangle},n+1}k_{V_n^\circ(S_m(\omega_R)^{\langle n+1\rangle}),n}}$ *according to* (95).
- *Determine* $k_{S_m(\omega_R)^{\langle n+1\rangle},n+1} \in \mathbb{R}^+$ *according to* (96).
- *Determine* $\phi_{S_m(\omega_R)^{\langle n+1\rangle},n+1}$ *according to* (94).

□

We have seen in the beginning of this subsection that in case of $p = 1$ the fact that $\omega = \omega_R$ simplifies the above scheme considerably. For $\omega \in \Omega^{\text{per}}$ with period $p = 2$ the situation is different. The next theorem however shows that the sets $\{C_{\omega,n}\}_{n\in\mathbb{N}}$ in Theorem 14 can be replaced by the sets $\{\hat{C}_{\omega,n}\}_{n\in\mathbb{N}}$ defined by

$$\hat{C}_{\omega,n} := \{\phi_{S_0(\omega)^{\langle n\rangle},n}, \phi_{S_1(\omega)^{\langle n\rangle},n}\}. \quad (97)$$

Theorem 16 *Fix $\omega \in \Omega^{\text{per}}$ with period $p = 2$. For $n \in \mathbb{N}$ let the sets $\hat{C}_{\omega,n}$ of orthonormal polynomials be defined by (97). Suppose that $\hat{C}_{\omega,0}$ and $\hat{C}_{\omega,1}$ are known. Then for each $n \in \mathbb{N}^+$ the set $\hat{C}_{\omega,n+1}$ can recurrently be constructed according to the scheme in Theorem 13, by using the orthonormal polynomials from $\hat{C}_{\omega,n}$.*

Proof: Fix $n \in \mathbb{N}^+$ and suppose that $\hat{C}_{\omega,n}$ is known. It suffices to show that for all $m \in \{0,1\}$, the polynomials $\phi_{T_{n+1}(S_m(\omega)^{\langle n+1\rangle}),n}$ and $\phi^*_{V_n^\diamond(S_m(\omega)^{\langle n+1\rangle}),n}$ belong to $\hat{C}_{\omega,n}$ (see the proof of Theorem 14). According to formula (80), the polynomials $\phi_{T_{n+1}(S_0(\omega)^{\langle n+1\rangle}),n}$ and $\phi_{T_{n+1}(S_1(\omega)^{\langle n+1\rangle}),n}$ belong to $\hat{C}_{\omega,n}$, so it remains to show that $\phi^*_{V_n^\diamond(S_0(\omega)^{\langle n+1\rangle}),n}$ and $\phi^*_{V_n^\diamond(S_1(\omega)^{\langle n+1\rangle}),n}$ belong to $\hat{C}_{\omega,n}$.

As we have already seen

$$\begin{aligned}
e_{\omega_n}, e_{\omega_{n-1}}, \dots, e_{\omega_0} &\xrightarrow{GS} \phi^*_{V_n^\diamond(S_0(\omega)^{\langle n+1\rangle}),n} \\
e_{\omega_{n+1}-\omega_1}, e_{\omega_n-\omega_1}, \dots, e_{\omega_1-\omega_1} &\xrightarrow{GS} \phi^*_{V_n^\diamond(S_1(\omega)^{\langle n+1\rangle}),n} \\
e_{\omega_n-\omega_0}, e_{\omega_n-\omega_1}, \dots, e_{\omega_n-\omega_n} &\xrightarrow{GS} \phi^*_{S_0(\omega)^{\langle n\rangle},n} \\
e_{\omega_{n+1}-\omega_1}, e_{\omega_{n+1}-\omega_2}, \dots, e_{\omega_{n+1}-\omega_{n+1}} &\xrightarrow{GS} \phi^*_{S_1(\omega)^{\langle n\rangle},n}.
\end{aligned}$$

Depending whether n is even or odd we proceed as follows:

- If n is even ($n = pN, N \in \mathbb{N}^+$), then

$$\phi^*_{V_n^\diamond(S_0(\omega)^{\langle n+1\rangle}),n} = \phi^*_{S_1(\omega)^{\langle n\rangle},n}$$

  and

$$\phi^*_{V_n^\diamond(S_1(\omega)^{\langle n+1\rangle}),n} = \phi^*_{S_0(\omega)^{\langle n\rangle},n}.$$

- If n is odd ($n = pN + 1, N \in \mathbb{N}$), then

$$\phi^*_{V_n^\diamond(S_0(\omega)^{\langle n+1\rangle}),n} = \phi^*_{S_0(\omega)^{\langle n\rangle},n}$$

  and

$$\phi^*_{V_n^\diamond(S_1(\omega)^{\langle n+1\rangle}),n} = \phi^*_{S_1(\omega)^{\langle n\rangle},n}.$$

In both cases $\phi^*_{V_n^\diamond(S_0(\omega)^{\langle n+1\rangle}),n}$ and $\phi^*_{V_n^\diamond(S_1(\omega)^{\langle n+1\rangle}),n}$ belong to $\hat{C}_{\omega,n}$. □

4 Orthonormal Matrix Polynomials

In this section orthonormal matrix polynomials will be defined. These matrix polynomials $\{\Phi_n\}_{n\in\mathbb{N}}$ are functions in the variable Z, which is a $p\times p$ matrix. They are associated with particular periodic sequences ω, the so called 'block sequences', which will be constructed in subsection 4.1. In subsection 4.2 uniqueness of the system of orthonormal matrix polynomials will be proved and moreover it will be shown that if $\{\Phi_n\}_{n\in\mathbb{N}}$ are known, then $\{\phi_{\omega,n}\}_{n\in\mathbb{N}}$ are known, and vice versa. More details about matrix polynomials in the variable $Z = zI$ (with $z \in \mathbb{C}$) can be found in [Gohberg].

4.1 Block Sequences

In this subsection block sequences ω will be introduced.

By definition, a block sequence $\omega = (\omega_0, \omega_1, \ldots)$ is a periodic sequence ($\omega \in \Omega^{\text{per}}$) with period p such that

- for all $k \in \{0, \ldots, p-1\}$, $(e^{i\lambda})^{\omega_k}$ depends only on $\lambda_1, \ldots, \lambda_{d-1}$

and

- $(e^{i\lambda})^{\omega_p}$ depends only on λ_d.

The following vector which depends on $z \in \mathbb{C}$, will be used in the remaining part of this report: for $z \in \mathbb{C}$ define

$$V(z) = \begin{pmatrix} z^{\omega_0} \\ \vdots \\ z^{\omega_{p-1}} \end{pmatrix}. \tag{98}$$

Observe that for block sequences ω, the vector $V(e^{i\lambda})$ does not depend on λ_d.

Finally we give an example of a block sequence. Fix $(b_0, \ldots, b_{d-1}) \in \mathbb{N}^+ \times \ldots \times \mathbb{N}^+$ with $b_0 = 1$ and define

$$p = \prod_{k=0}^{d-1} b_k.$$

Every number $q \in \mathbb{N}$ can be decomposed uniquely as

$$\begin{aligned} q &= \sum_{k=0}^{d-1} \left(\prod_{j=0}^{k} b_j \right) q_k \\ &= \sum_{k=0}^{d-2} \left(\prod_{j=0}^{k} b_j \right) q_k + p\, q_{d-1}, \end{aligned}$$

where

- $q_k \in \{0, \ldots, (b_{k+1} - 1)\}$ when $0 \le k \le (d-2)$

and

- $q_{d-1} \in \mathbb{N}$.

For each $q \in \mathbb{N}$ decomposed in this way, define

$$\omega_q = (q_0, \ldots, q_{d-1}).$$

Then

$$\omega_p = (0, \ldots, 0, 1)$$

and $\omega = (\omega_0, \omega_1, \ldots) \in \Omega^{\text{per}}$ with period p.

4.2 Orthonormal Matrix Polynomials

In this subsection matrix polynomials are introduced. On the 'space' of these polynomials a 'bilinear form' (which maps into the space of $p \times p$ complex valued matrices), will be introduced that will play the 'role of an inner product' (see formula (99)). Then for a fixed block sequence ω, orthonormal matrix polynomials will be introduced, which turn out to be directly related to orthonormal polynomials $\{\phi_{\omega,n}\}_{n\in\mathbb{N}}$.

Let $\mathrm{C}_{p\times p}$ denote the set of all $p \times p$ complex valued matrices. A matrix polynomial of degree n is a mapping $\mathrm{C}_{p\times p} \to \mathrm{C}_{p\times p}$ such that

$$Z \longmapsto \sum_{k=0}^{n} A_k Z^k,$$

where $A_k \in \mathrm{C}_{p\times p}$ for $k \in \{0, \ldots, n\}$. Let $\mathbb{P}^d$ denote the set of all matrix polynomials. Then the bilinear form $< \cdot ; \cdot >_{\mathbb{F}}$: $\mathbb{P}^d \times \mathbb{P}^d \to \mathrm{C}_{p\times p}$ is given by

$$< P; Q >_{\mathbb{F}} = \int_{(-\pi,\pi]} P(e^{i\lambda_d} I)\, d\mathbb{F}(\lambda_d)[Q(e^{i\lambda_d} I)]^*, \tag{99}$$

where

$$d\mathbb{F}(\lambda_d) = \int_{(-\pi,\pi]^{d-1}} V(e^{i\lambda})\, dF(\lambda_1, \ldots, \lambda_d)[V(e^{i\lambda})]^*. \tag{100}$$

Theorem 17 *There exists a unique system of matrix polynomials $\{\Phi_n\}_{n\in\mathbb{N}}$ such that*

- *Φ_n is a matrix polynomial of degree n.*
- *the coefficient of Z^n in $\Phi_n(Z)$ is a lower triangular matrix with positive real diagonal elements.*
- *the matrix polynomials $\{\Phi_n\}_{n\in\mathbb{N}}$ are orthonormal, i.e.*

$$< \Phi_n; \Phi_m >_{\mathbb{F}} = \begin{cases} I & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Proof: First such a system will be constructed explicitly. Let

$$\phi_{\omega,m} = \sum_{k=0}^{m} \varphi_{\omega,m,k}\, e_{\omega_k},$$

where $\phi_{\omega,m,k} = 0$ if $k > m$. Define

$$\Phi_{n,k} = \begin{pmatrix} \varphi_{\omega,np,kp} & \cdots & \varphi_{\omega,np,kp+p-1} \\ \vdots & & \vdots \\ \varphi_{\omega,np+p-1,kp} & \cdots & \varphi_{\omega,np+p-1,kp+p-1} \end{pmatrix} \tag{101}$$

and

$$\Phi_n(Z) = \sum_{k=0}^{n} \Phi_{n,k}\, Z^k. \tag{102}$$

Obviously Φ_n is a matrix polynomial of degree n and $\Phi_{n,n}$ is a lower triangular matrix with positive real diagonal elements. Moreover

$$\begin{aligned} < \Phi_n; \Phi_m >_{\mathbb{F}} &= \int_{(-\pi,\pi]} \Phi_n(e^{i\lambda_d} I)\, d\mathbb{F}(\lambda_d)[\Phi_m(e^{i\lambda_d} I)]^* \\ &= \int_{\Lambda} \Phi_n(e^{i\lambda_d} I) V(e^{i\lambda})\, dF(\lambda)[\Phi_m(e^{i\lambda_d} I) V(e^{i\lambda})]^* \\ &= \int_{\Lambda} \begin{pmatrix} \phi_{\omega,np}(e^{i\lambda}) \\ \vdots \\ \phi_{\omega,np+p-1}(e^{i\lambda}) \end{pmatrix} \overline{(\phi_{\omega,mp}(e^{i\lambda}), \ldots, \phi_{\omega,mp+p-1}(e^{i\lambda}))}\, dF(\lambda) \\ &= \begin{cases} I & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \end{aligned}$$

Now uniqueness of the system $\{\phi_{\omega,n}\}_{n\in\mathbb{N}}$ guarantees uniqueness of the system $\{\Phi_n\}_{n\in\mathbb{N}}$: if $\{\Phi_n^{\sim}\}_{n\in\mathbb{N}}$ is another system having the same properties as $\{\Phi_n\}_{n\in\mathbb{N}}$, we can construct a system $\{\phi_{\omega,n}^{\sim}\}_{n\in\mathbb{N}}$ by using the coefficients of the matrix polynomials $\Phi_n^{\sim}$. The polynomials $\{\phi_{\omega,n}^{\sim}\}_{n\in\mathbb{N}}$ are related to the matrix polynomials $\{\Phi_n^{\sim}\}_{n\in\mathbb{N}}$ in the same way as the polynomials $\{\phi_{\omega,n}\}_{n\in\mathbb{N}}$ to $\{\Phi_n\}_{n\in\mathbb{N}}$. The polynomials $\{\phi_{\omega,n}^{\sim}\}_{n\in\mathbb{N}}$ are orthonormal. Hence Theorem 1 gives $\phi_{\omega,n} = \phi_{\omega,n}^{\sim}$. So $\Phi_n = \Phi_n^{\sim}$. □

Theorem 17 yields the following corollary:

Corollary 2 *If $\{\Phi_n\}_{n\in\mathbb{N}}$ are known, then $\{\phi_{\omega,n}\}_{n\in\mathbb{N}}$ are known, and vice versa.*

Surely, recurrent relations between the orthonormal polynomials $\{\phi_{\omega,n}\}_{n\in\mathbb{N}}$ yield in view of Corollary 2 recurrent relations between the orthonormal matrix polynomials $\{\Phi_n\}_{n\in\mathbb{N}}$ so that explicit schemes for constructing the matrix polynomials can be written down, but we do not enter in details here.

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