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Isomorphisms between State and Predicate Transformers

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Abstract

We study the relation between state transformers based on directed complete partial orders and predicate transformers. Concepts like 'predicate', 'liveness', 'safety' and 'predicate transformers' are formulated in a topological setting. We treat state transformers based on the Hoare, Smyth and Plotkin powerdomains and consider continuous, monotonic and unrestricted functions. We relate the transformers by isomorphisms thereby extending and completing earlier results and giving a complete picture of all the relationships.

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Contents

1 Introduction

In this paper we give a full picture of the relationship between state transformers and predicate transformers. For the state transformers we consider the Hoare, Smyth and Plotkin power domains. We give a full picture in the sense that we consider algebraic directed complete partial orders (with a bottom element) (and not only flat domains), we consider not only continuous state transformers, but also the monotonic ones and the full function space, we do not restrict to bounded nondeterminism, and we treat all the three power domains with or without empty set. The first item is important when we want to use domains for concurrency semantics. The second and third item give more freedom in the sense that we can use these transformations also for specification purposes without constraints on computability. Having the empty set in a power domain can be important to treat deadlock. Our treatment includes the Plotkin power domain.

For state transformers we use an extension of the standard power domains. For predicate transformers we start from the (informal) classification of predicates in liveness and safety predicates of Lamport [Lam77]. Later Smyth [Smy83] followed by [AS85, Kwi91] used topology to formalize this classification. Also we use topology for defining predicates and safety and liveness predicate transformers. We consider a new kind of predicate transformers with predicates that are the intersection of safety and liveness predicates.

We prove that the Hoare state transformers are isomorphic to safety predicate transformers, that the Smyth state transformers are isomorphic to the liveness predicate transformers, and that the Plotkin state transformers are isomorphic to the "intersection" predicate transformers. So for the first time we are able to give a full picture of all the relationships filling several gaps that were present in the literature.

Next we discuss how this paper is related to previous work. Power domains for dcpo's were introduced in [Plo76], [Smy78] and [Plo81]. Our power domains are slightly more general in the sense that we do no restrict to non-empty (Scott-) compact sets. Besides the standard ways of adding the empty set to the Smyth [Smy83] and to the Plotkin [MM79, Plo81, Abr91] power domains, we also add the empty set in all the three power domains as a separate element, comparable only with itself and with the bottom.

Predicate transformers were introduced in [Dij76] with a series of healthness conditions. Back and von Wright [Bac80, vW90] use only the monotonicity restriction on predicate transformers. They use predicate transformers for refinement and provide a nice lattice theoretical framework. Nelson [Nel89] has (for the flat case) used "compatible" pairs of predicate transformers for giving semantics to a language with backtracking. Smyth [Smy83] introduced predicate transformers (with the Dijkstra healthiness conditions) for non-flat domains. Our definition of predicate transformers is parametric with respect to the collection of predicates (observable, liveness and safety). Furthermore, a new (generalization) of the multiplicativity restriction is introduced and a generalization to the non-flat case of "compatible" pairs of predicate transformers is given.

Isomorphisms between state and predicate transformers have been given for the flat case of the Smyth power domain in [Plo79] (and for countable nondeterminism in [AP86]), and for the flat case of the Hoare power domain in [Plo81]. Also De Bakker and De Roever [Bak80, Roe76] studied (from a semantical point of view) for the flat case the relation between state transformer and predicate transformer semantics. Moreover, for the flat case of the Plotkin power domain we have proposed an isomorphism in [BK92].

For the general case of the compact Smyth power domain in the paper [Smy83] an isomorphism is given for continuous state transformers. He uses a topological technique which Plotkin later used in [Plo81] for the continuous Hoare state transformers. A recent work includes an operational point of view in Van Breugel [Bre93]. In the present paper we give some new isomorphisms for the Hoare and the Smyth power domains, showing also how the previous ones can be obtained as combinations of the new isomorphisms. Our definition of multiplicativity for predicate transformers permits us to use a technique similar to that used for the flat case. Furthermore we give isomorphisms for the Plotkin power domain. As far as we know no isomorphism was known for the non-flat Plotkin power domain (as for example is remarked in [Plo81] and in [Smy83]).

2 Mathematical Preliminaries

We introduce some basic notions on domain theory and topology. For a more detailed discussions on domain theory consult for example [Plo81], and for topology we refer to [Eng77].

A *partial order* $\sqsubseteq$ on a set P is a reflexive, transitive and antisymmetric relation in $P \times P$. Let P be ordered by the partial order $\sqsubseteq$, $x \in P$ and A a subset of P. Define $x\uparrow = \{y | y \in P \ \wedge \ x \sqsubseteq y\}$ and $A\uparrow = \bigcup\{x\uparrow | x \in A\}$. A set A is called upper-closed if $A = A\uparrow$. A subset A of P is called an *ω-chain* if there is a enumeration $x_0, x_1, ...$ of the elements of A such that $x_i \sqsubseteq x_{i+1}$ for every $i \in N$. A generalization of ω-chain is a directed set; $A \subseteq P$ is said to be *directed* if it is non empty and every finite subset of A has an upper bound in A. P is a *directed complete partially order set* (dcpo) if there exists a least element $\bot$ and every directed subset A of P has least upper bound (lub) $\bigsqcup A$ in P. A directed set A is *eventually constant* if $\bigsqcup A \in A$.

A partially ordered set P has a least upper bound for every ω-chain if and only if it contains all least upper bounds of all directed countable sets [SP82]. In this case P is called a ω-complete partially ordered set (ω-cpo).

An element b of a dcpo P is *finite* if for every directed set $A \subseteq P$, $b \sqsubseteq \bigsqcup A$ implies $b \sqsubseteq x$ for some $x \in A$. The set of all finite elements of P is denoted by B_P and is called *base*. A dcpo P is *algebraic* if for every element $x \in P$ the set $\{b | b \in B_P \ \wedge \ b \sqsubseteq x\}$ is directed and has least upper bound x; it is ω*-algebraic* if it is algebraic and its base is denumerable.

Let P, Q be two partially ordered sets. A function $f : P \to Q$ is *monotone* (denoted by $f : P \to_m Q$) if for all $x, y \in P$ with $x \sqsubseteq_P y$ we have $f(x) \sqsubseteq_Q f(y)$. If P is a dcpo we say f is *continuous* (denoted by $f : P \to_c Q$) if $f(\bigsqcup A) = \bigsqcup f(A)$ for each directed set $A \subseteq P$; moreover f is *stabilizing* (denoted by $f : P \to_z Q$) if it is continuous and for every directed set $A \subseteq P$, $f(A)$ is an eventually constant directed set in Q. If $f : P \to_c Q$ is continuous then f is monotone. Given a set $B \subseteq Q$, a continuous function $f : P \to_c Q$ is said B*-algebraic* (denoted by $f : P \to_{alg(B)} Q$) if for every directed set $S \subseteq P$ and for every $q \in B$ such that $q \sqsubseteq f(\bigsqcup S)$ there exists an $x \in S$ such that $q \sqsubseteq f(x)$. Clearly, a continuous and stabilizing function $f : P \to_z Q$ is B-algebraic for every $B \subseteq Q$. Moreover we have the following:

Lemma 2.1 *Let P, Q be two dcpo's such that Q is algebraic with base B_Q. Then a function $f : P \to Q$ is continuous if and only if f is B_Q-algebraic.*

A function f is *strict* (denoted by $f : P \to_s Q$) if $f(\bot_P) = \bot_Q$; dually f is *top preserving* (denoted by $f : P \to_t Q$) if and only if $f(\top_P) = \top_Q$. If f is onto and monotone then it is also strict.

We now introduce some basic topological notions. A *topology* $\mathbf{O}(X)$ on a set X is a collection of subsets of X that is closed under finite intersections and arbitrary unions. Every topology $\mathbf{O}(X)$ is ordered by subset inclusion and forms a complete lattice with bottom element the empty set and as top element X. The pair $(X, \mathbf{O}(X))$ is called *topological space* and the elements of $\mathbf{O}(X)$ are the *open sets* of the space X. A *base* of a topology $\mathbf{O}(X)$ on X is a subset $\mathbf{B} \subseteq \mathbf{O}(X)$ such that every open set is the union of elements of $\mathbf{B}$. The topologies on a set X form a complete lattice when ordered by inclusion, with bottom element the *trivial* topology $\mathbf{O}_t(X) = \{\emptyset, X\}$ and top the *discrete* topology $\mathbf{O}_d(X) = \mathcal{P}(X)$. A set $S \subseteq X$ is *dense* if and only if $X \setminus S$ contains no non-empty open sets. A G_δ*-set* is a countable (finite or infinite) intersection of open sets.

Given a partially ordered set X, its *Alexandroff topology* $\mathbf{O}_{Al}(X)$ consists of all the upper-closed subsets of X. If X is a dcpo, a finer topology of X is the Scott topology $\mathbf{O}_{Sc}(X)$, where $o \in \mathbf{O}_{Sc}(X)$ if and only if o is upper-closed and for any directed set $S \subseteq X$ if $\bigsqcup S \in o$ then $S \cap o \neq \emptyset$.

Let A be a collection of subsets of X; the closure under arbitrary intersection of A is denoted by $A^\cap = \{Q | Q = \bigcap A' \ \wedge \ A' \subseteq A\}$.

We can describe a topology by its closed sets instead of its open sets. A subset of a set X is *closed* if and only if it is the complement of an open set of a given topology on X. The collection of closed sets of a topological space is denoted by $\mathbf{C}(X)$ and, dually to the case of open sets, is closed under finite unions and arbitrary intersections. Closed sets are ordered by superset inclusion and form a complete lattice. For every $A \subseteq X$ there exists a closed set c and a dense set d such that $A = c \cap d$.

Given a partially ordered set P, the closed sets of the Alexandroff topology are all the lower closed sets, while a set $c \subseteq P$ is closed with respect to the Scott topology if c is lower closed and for every directed set $S \subseteq P$ if $S \subseteq c$ then $\bigsqcup S \in c$.

Let X be an algebraic dcpo and let $\mathbf{O}(X)$ be a topology on X. A subset $A \subseteq X$ is *compact* in $\mathbf{O}(X)$ if and only if for every collection of open sets $o_i \in \mathbf{O}(X)$ with $i \in I$ such that $A \subseteq \bigcup_I o_i$ there exists a finite subcollection o_j such that $A \subseteq \bigcup_J o_j$. For example $A \subseteq X$ is compact in

$\mathbf{O}_d(X)$ if and only if it is a finite set. The intersection of a closed set and a compact set is compact. Sometimes we use a different characterization of compactness:

Lemma 2.2 *Let X be a topological space with topology $\mathbf{O}(X)$. A subset $A \subseteq X$ is compact in $\mathbf{O}(X)$ if and only if for every directed set $S \subseteq \mathbf{O}(X)$ such that $A \subseteq \bigcup S$ there exists an $s \in S$ with $A \subseteq s$.*

Proof: If $A \subseteq X$ is compact and $S \subseteq \mathbf{O}(X)$ is a directed set, then $A \subseteq \bigcup S$ implies by the compactness that there exists a finite $S' \subseteq S$ such that $A \subseteq \bigcup S'$. But S is directed, thus there exists $s \in S$ such that $s' \subseteq s$ for every $s' \in S'$. Therefore $A \subseteq \bigcup S' \subseteq s$.

Let now $A \subseteq X$ and assume that for every directed set $S \subseteq \mathbf{O}(X)$ if $A \subseteq \bigcup S$ then there exists an $s \in S$ with $A \subseteq s$. Then for every cover $C \subseteq \mathbf{O}(X)$ such that $A \subseteq \bigcup C$ define the set $S(C) = \{o_{C'} \in \mathbf{O}(X) | o_{C'} = \bigcup C' \wedge C' \subseteq_{fin} C\}$. The set $S(C)$ is directed because if $o_{C'}, o_{C''} \in S(C)$ with C', C'' finite subsets of C then also $C' \cup C''$ is a finite subset of C, that is $o_{C' \cup C''} \in S(C)$. Moreover, $o_{C'} \subseteq o_{C' \cup C''}$ and also $o_{C''} \subseteq o_{C' \cup C''}$, hence $S(C)$ is directed. We have also $C \subseteq S(C)$, hence $A \subseteq \bigcup C \subseteq \bigcup S(C)$. Thus there exists $s \in S(C)$ such that $A \subseteq s$, say $s = o_{\hat{C}} = \bigcup \hat{C}$ where $\hat{C} \subseteq_{fin} C$. Therefore $A \subseteq \bigcup \hat{C}$. □

3 Predicates and Predicate Transformers

A *predicate* P is a function from a set X to the boolean set $\{tt, ff\}$ or, equivalently, is a subset of X. Topology provides an elegant way of selecting classes of predicates of programs ([Smy83], [Kwi91]) in which the open sets of a topological space X are the *observable predicates*, closed sets are the *safety predicates* and arbitrary intersections of open sets are the *liveness predicates.*

In this paper we consider algebraic dcpo's together with the Scott topology. The Scott-open sets of a dcpo Y represent the observable predicates and are *finitary* in the sense that $y \in o$ if and only if there exists a $b \in B_Y$ such that $b \in o$ and $b \sqsubseteq y$. In other words, a predicate P is finitary if we can test P holds for y by testing only the finite elements smaller than y. Notice that if P is ω-algebraic then a predicate P is finitary if it can be tested to hold for y by testing only a finite number of elements smaller than y. The observable predicates are ordered by subset inclusion. The Scott-closed sets are the safety predicates and are ordered by superset inclusion. Liveness predicates are the arbitrary intersections of Scott-open sets (that is Alexandroff open sets). The liveness predicates are ordered by subset inclusion.

Consider as an example the set of sequences (finite and infinite) over an alphabet Σ ordered by the prefix ordering (for more examples see [Kwi91]). For each kind of predicate we give an example:

- Safety predicate: *always* $a = \{x | x = a^* \vee x = a^\omega\}$ (Scott closed),
- Observable predicate: *eventually* $a = \{xa | x \in \Sigma^*\}\uparrow$ (Scott open),
- Observable predicate: *start with* $x = x\uparrow$ (Scott open), where $x \in \Sigma^*$,
- Liveness predicate: *infinitely often* $a = \bigcap_{n \in N}(\{x | x \in \Sigma^* \wedge |x|_a = n\}\uparrow)$ (Alexandroff open but not Scott open),
- Neither safety nor liveness predicate: *always a but starting with* x = *always* $a \wedge$ *start with* x.

The *safety* and *liveness predicates* are introduced informally in [Lam77]. Smyth [Smy83, Smy92] introduces the topological view that closed sets represent *safety predicates* and the *liveness predicates* are intersections of open sets. Due to computability, Smyth uses closedness under countable intersection (such specifications are known in topology as G_δ sets: countable intersections of open sets) while we take arbitrary intersections of open sets as liveness predicates. In [Kwi91] liveness predicates are also G_δ-sets. We differ from [AS85] where liveness predicates are the dense sets (the complement does not contain non-empty open sets).

To provide some more intuition we show that taking different topologies corresponds to different restrictions on the function space:

Lemma 3.1 *Let* **Bool** $= \{tt, ff\}$ *with* $ff \sqsubseteq tt$. *Then* $\mathbf{O}_d(X)$ *is isomorphic to the set of all the predicates from* X *to* **Bool**, $\mathbf{O}_{Al}(X)$ *is isomorphic to the set of all the monotone predicates from* X *to* **Bool** *and* $\mathbf{O}_{Sc}(X)$ *is isomorphic to the set of all the continuous predicates from* X *to* **Bool**.

Now we come to the definition of predicate transformers. Let $P(Y)$ be a collection of predicates on the space Y and let $P(X)$ be a collection of predicates on the space X. We define *predicate transformers* as the monotone functions from $P(Y)$ to $P(X)$.

Given topologies on Y and X the collection of predicate transformers that map observable predicates to observable predicates is denoted by $P(Y) \to_O P(X)$ (Recall that an observable predicate is an open set in the topology.) All the predicate transformers in the sequel will be multiplicative in the following sense:

Definition 3.2 *A predicate transformer* $\pi : P(Y) \to P(X)$ *is called multiplicative, denoted by* $\pi : P(Y) \to_M P(X)$, *if for all collections of predicates* $P, Q \subseteq P(Y)$ *we have*

$$\bigcap P \subseteq \bigcap Q \;\Rightarrow\; \bigcap_{p \in P} \pi(p) \subseteq \bigcap_{q \in Q} \pi(q)$$

This definition of multiplicativity is a generalization of the standard definition of multiplicativity as is shown in the next lemma:

Lemma 3.3 *Let* $\pi : P(Y) \to P(X)$ *be a predicate transformer and assume that* $P(Y)$ *is closed under arbitrary intersection, that is* $P(Y)^\cap = P(Y)$. *Then* π *is multiplicative if and only if* $\pi(\bigcap P) = \bigcap_{p \in P} \pi(p)$ *for every collection of predicates* $P \subseteq P(Y)$.

Proof:

$\Leftarrow$) Let $P, Q \subseteq P(Y)$ such that $\bigcap P \subseteq \bigcap Q$. Since $\pi(\bigcap R) = \bigcap_{r \in R} \pi(r)$ for every $R \subseteq P(Y)$, the predicate transformer π is monotone, and hence $\pi(\bigcap P) \subseteq \pi(\bigcap Q)$. But then $\bigcap_{p \in P} \pi(p) = \pi(\bigcap P) \subseteq \pi(\bigcap Q) = \bigcap_{q \in Q} \pi(q)$.

$\Rightarrow$) Let $P \subseteq P(Y)$ and $q = \bigcap P \in P(Y)$. Then by multiplicativity we have $q = \bigcap P$ implies $\pi(q) = \bigcap_{p \in P} \pi(p)$. But then $\pi(\bigcap P) = \pi(q) = \bigcap_{p \in P} \pi(p)$.

□

Even if $P(Y)^\cap \neq P(Y)$ we have for a multiplicative predicate transformer π that for any $\bigcap P \in P(Y)$ for a collection of predicates $P \subseteq P(Y)$ then $\pi(\bigcap P) = \bigcap_{p \in P} \pi(p)$. Hence our notion of multiplicativity keeps all possible multiplicativity allowed by the collection of predicates $P(Y)$. Because the empty intersection of predicates in $P(Y)$ is the predicate Y, we have the following:

Lemma 3.4 *Let $\pi : P(Y) \to_M P(X)$ be a multiplicative predicate transformer. If $Y \in P(Y)$ then π is top preserving.*

Proof: Consider the collection of predicates $P = \emptyset \subseteq P(Y)$ and $Q = \{Y\}$. Then by multiplicativity we obtain $Y = \bigcap P \subseteq \bigcap Q = Y$ implies $X = \bigcap_{p \in P} \pi(p) \subseteq \bigcap_{q \in Q} \pi(q) = \pi(Y)$. But $\pi(Y) \subseteq X$, thus $\pi(Y) = X$. □

Given a predicate transformer $\pi : P(Y) \to P(X)$ its *dual* $\pi^\circ : P(Y)^\circ \to P(X)^\circ$ is given by $\pi^\circ(p) = X \setminus \pi(Y \setminus p)$, for every $p \in P(Y)^\circ$, where $P(Y)^\circ$ is defined as the set $\{Y \setminus p | p \in P(Y)\}$. A predicate transformer is *additive* (denoted by $\pi : P(Y) \to_A P(X)$) if and only if its dual is multiplicative.

Definition 3.5 *A predicate transformer $\pi : P(Y) \to P(X)$ is called intersection extensible, denoted by $\pi : P(Y) \to_I P(X)$, if $\bigcap_{p \in P} \pi(p) \in P(X)$ for every collection of predicates $P \subseteq P(Y)$. Dually, if $\bigcup_{p \in P} \pi(p) \in P(Y)$ for every collection of predicates $P \subseteq P(Y)$ then π is called union extensible (denoted by $\pi : P(Y) \to_U P(X)$).*

Intuitively, multiplicative predicate transformers $\pi : P(Y) \to_M P(X)$ preserve the logical '$\forall$' on predicates on Y, while the additive ones preserve the logical '$\exists$' even if they are not in their domain. If π is intersection (union) extensible then the logical '$\forall$' ('$\exists$') of π of an arbitrary collection of predicates on Y is always a predicate in $P(X)$.

For multiplicative predicate transformers we prove the following *stability lemma* (a generalization of a lemma due to Plotkin [Plo79, AP86])in which the collections of predicates are not necessarily closed under arbitrary intersection:

Lemma 3.6 *Let $\pi : P(Y) \to_M P(X)$ be a multiplicative predicate transformer between two collections of predicates. Then*

$$x \in \pi(\hat{p}) \;\Leftrightarrow\; \bigcap\{p | x \in \pi(p)\} \subseteq \hat{p}$$

for every $\hat{p} \in P(Y)$ and $x \in X$.

Proof:

$\Rightarrow$) If $x \in \pi(\hat{p})$ then $\hat{p} \in \{p | x \in \pi(p)\}$ and hence $\bigcap\{p | x \in \pi(p)\} \subseteq \hat{p}$.

$\Leftarrow$) Suppose $\bigcap\{p | x \in \pi(p)\} \subseteq \hat{p}$. Then by multiplicativity $x \in \bigcap_{x \in \pi(p)} \pi(p) \subseteq \pi(\hat{p})$, that is $x \in \pi(\hat{p})$.

□

Next we define a restricted version of the Cartesian product on multiplicative predicate transformers by requiring multiplicativity on the intersection.

Definition 3.7 *Let $P_1(Y)$, $P_2(Y)$ be two collections of predicates on Y and $Q_1(X)$, $Q_2(X)$ be two collections of predicates on X. Define $(P_1(Y) \to_M Q_1(X)) \bowtie (P_2(Y) \to_M Q_2(X))$ as*

$$\{(\pi, \rho) | \;\; \forall\, P, P' \subseteq P_1(Y), Q, Q' \subseteq P_2(Y) : \bigcap P \cap \bigcap Q \subseteq \bigcap P' \cap \bigcap Q' \Rightarrow \bigcap_{p \in P} \pi(p) \cap \bigcap_{q \in Q} \rho(q) \subseteq \bigcap_{p' \in P'} \pi(p') \cap \bigcap_{q' \in Q'} \rho(q')\}.$$

Tuples are ordered pointwise.

For these pairs of multiplicative predicate transformers we have a new *stability lemma*:

Lemma 3.8 *Let $(\pi, \rho) : (P_1(Y) \to_M Q_1(X)) \boxtimes (P_2(Y) \to_M Q_2(X))$. Then for every $x \in X$, $\hat{p} \in P_1(Y)$, and $\hat{q} \in P_2(Y)$ we have:*

1. $x \in \pi(\hat{p}) \Leftrightarrow \bigcap\{p | x \in \pi(p)\} \cap \bigcap\{q | x \in \rho(q)\} \subseteq \hat{p}$,
2. $x \in \rho(\hat{q}) \Leftrightarrow \bigcap\{p | x \in \pi(p)\} \cap \bigcap\{q | x \in \rho(q)\} \subseteq \hat{q}$.

Proof:

1. $\Rightarrow$) If $x \in \pi(\hat{p})$ then $\hat{p} \in \{p | x \in \pi(p)\}$ and hence

$$\bigcap\{p | x \in \pi(p)\} \cap \bigcap\{q | x \in \rho(q)\} \subseteq \bigcap\{p | x \in \pi(p)\} \subseteq \hat{p}.$$

$\Leftarrow$) Suppose $\bigcap\{p | x \in \pi(p)\} \cap \bigcap\{q | x \in \rho(q)\} \subseteq \hat{p}$, then

$$\bigcap\{p | x \in \pi(p)\} \cap \bigcap\{q | x \in \rho(q)\} \subseteq \hat{p} \cap \bigcap\{q | x \in \rho(q)\}.$$

But $(\pi, \rho) \in (P_1(X) \to_M Q_1(Y)) \boxtimes (P_2(X) \to_M Q_2(Y))$, thus we have

$$x \in \bigcap_{x \in \pi(p)} \pi(p) \cap \bigcap_{x \in \rho(p)} \rho(p) \subseteq \pi(\hat{p}) \cap \bigcap_{x \in \rho(p)} \rho(p),$$

that is $x \in \pi(\hat{p})$.

2. Similar.

□

Now we come to the definition of safety and liveness predicate transformers used in this paper. Recall that we defined safety predicates as closed sets and liveness predicates as 'arbitrary intersections of open sets.

Definition 3.9 *Let X and Y be algebraic dcpo's. The safety predicate transformers are functions in*

$$\mathbf{C}_{Sc}(Y) \to_M \mathbf{C}(X)$$

for some collection of closed sets $\mathbf{C}(X)$ on X. They are ordered pointwise by superset inclusion.

The liveness predicate transformers are functions in

$$\mathbf{O}_{Sc}(Y)^\cap \to_{OM} \mathbf{O}(X)^\cap$$

for some topology $\mathbf{O}(X)$ on X. They are ordered pointwise by subset inclusion.

Note that the domain of predicate transformers consists of the safety or the liveness predicates associated with the Scott topology. In the codomain we allow more freedom in the sense that we can choose the topology. As we will see later, taking different topologies corresponds to describing different classes of state transformers. We can define liveness and safety predicate transformers also in terms of observable predicate transformers due to the following isomorphism:

Theorem 3.10 *Let X and Y be algebraic dcpo's. The following order isomorphisms hold:*

1. $(\mathbf{O}_{Sc}(Y)^{\cap} \rightarrow_{OM} \mathbf{O}(X)^{\cap}) \cong (\mathbf{O}_{Sc}(Y) \rightarrow_M \mathbf{O}(X))$,
2. $(\mathbf{C}_{Sc}(Y) \rightarrow_M \mathbf{C}(X)) \cong (\mathbf{O}_{Sc}(Y) \rightarrow_A \mathbf{O}(X))$.

4 State Transformers

In this section we give generalizations of the three "classical" power domains on ω-algebraic dcpo's, the so-called Hoare, Smyth and Plotkin power domains ([Plo76], [Smy78] and [Plo81]). and on these power domains we base our state transformers. We need the following two closure operators for the definition of the power domains. Let A be a subset of an algebraic dcpo X:

1. $\overline{A} = \{x | \forall b \in B_X : b \sqsubseteq x \Rightarrow \exists x_b \in A : b \sqsubseteq x_b\}$,
2. $A^* = \{x | (\exists x' \in A : x' \sqsubseteq x) \;\wedge\; (\forall b \in B_X : b \sqsubseteq x \Rightarrow \exists x_b \in A : b \sqsubseteq x_b)\}$.

For every $A \subseteq X$ we have $\overline{A} = A$ if and only if $A \in \mathbf{C}_{Sc}(X)$. Hence $A \subseteq \overline{A}$ and $\overline{\overline{A}} = \overline{A}$. The intersection of an upper closed set with a Scott closed set is *-closed. Further $A \subseteq A^*$, $(A^*)^* = A^*$, and $A = A^*$ if and only if $A = A\!\uparrow \cap \overline{A}$. This means that if A is *-closed then $(A\!\uparrow \cap \overline{A})\!\uparrow = A\!\uparrow$ and $\overline{(A\!\uparrow \cap \overline{A})} = \overline{A}$. We characterize the pairs of sets (A, B) such that $(B \cap A)\!\uparrow = B$ and $\overline{(B \cap A)} = A$ (this implies clearly that A is Scott-closed and B is upper closed):

Lemma 4.1 *Let X be an algebraic dcpo and $A, B \subseteq X$.*

1. *If $A \subseteq B$ then $\overline{(B\!\uparrow \cap \overline{A})} = \overline{A}$,*
2. *if $A \supseteq B$ then $(B\!\uparrow \cap \overline{A})\!\uparrow = B\!\uparrow$,*
3. *$\overline{(B \cap A)} = A$ and $(B \cap A)\!\uparrow = B$ if and only if there exists $C \subseteq X$ such that $C\!\uparrow = B$ and $\overline{C} = A$.*

Proof:

1. Let $A \subseteq B \subseteq X$. Since $(B\!\uparrow \cap \overline{A}) \subseteq \overline{A}$ then $\overline{(B\!\uparrow \cap \overline{A})} \subseteq \overline{A}$. On the other hand, $A \subseteq B$ implies $A \subseteq B\!\uparrow$ and since $A \subseteq \overline{A}$ we obtain $A \subseteq (B\!\uparrow \cap \overline{A})$, which implies $\overline{A} \subseteq \overline{(B\!\uparrow \cap \overline{A})}$. Therefore $\overline{(B\!\uparrow \cap \overline{A})} = \overline{A}$.
2. Suppose now $A \supseteq B$. Since $(B\!\uparrow \cap \overline{A}) \subseteq B\!\uparrow$ then $(B\!\uparrow \cap \overline{A})\!\uparrow \subseteq B\!\uparrow$. Moreover $B \subseteq A$ implies $B \subseteq \overline{A}$ and since $B \subseteq B\!\uparrow$ we obtain $B \subseteq (B\!\uparrow \cap \overline{A})$, that implies $B\!\uparrow \subseteq (B\!\uparrow \cap \overline{A})\!\uparrow$. Therefore $(B\!\uparrow \cap \overline{A})\!\uparrow = B\!\uparrow$.

3. If $(B \cap A)\!\uparrow = B$ and $\overline{(B \cap A)} = A$ then take $C = B \cap A$. For the other direction, if there exists $C \subseteq X$ such that $C\!\uparrow = B$ and $\overline{C} = A$ then - by item 1 - $\overline{(B \cap A)} = \overline{(C\!\uparrow \cap \overline{C})} = \overline{C} = A$ and also - by item 2 - $(B \cap A)\!\uparrow = (C\!\uparrow \cap \overline{C})\!\uparrow = C\!\uparrow = B$.

□

Next we define the (generalizations of the) power domains:

Definition 4.2 *Let X be an algebraic dcpo. Define*

- *the Hoare power domain $\mathcal{H}(X) = \langle \{A | A \subseteq X \ \wedge \ A = \overline{A}\}, \sqsubseteq_H \rangle$, where $A \sqsubseteq_H B$ if $A \subseteq B$,*
- *the Smyth power domain $\mathcal{S}(X) = \langle \{A | A \subseteq X \ \wedge \ A = A\!\uparrow\}, \sqsubseteq_S \rangle$, where $A \sqsubseteq_S B$ if $A \supseteq B$,*
- *the Plotkin power domain $\mathcal{P}(X) = \langle \{A | A \subseteq X \ \wedge \ A = A^*\}, \sqsubseteq_P \rangle$, where $A \sqsubseteq_P B$ if $A\!\uparrow \sqsubseteq_S B\!\uparrow$ and $\overline{A} \sqsubseteq_H \overline{B}$.*

When we consider only bounded nondeterminism then we can restrict the power domains by considering only the compact sets in the Scott topology (denoted by the subscript *co*). Every Scott closed set of a dcpo contains the bottom, hence it is compact in the Scott topology. This implies $\mathcal{H}_{co}(X) = \mathcal{H}(X)$. The standard definitions of the Hoare, Smyth and Plotkin power domains are $\mathcal{H}^+(X), \mathcal{S}^+_{co}(X)$ and $\mathcal{P}^+_{co}(X)$, where the superscript $^+$ states that the power domains should be taken without the empty set.

For an algebraic dcpo X, the Hoare power domain is a complete lattice. Furthermore it is also an algebraic dcpo with finite elements the Scott closure of finite subsets of B_X [Plo81]. Also the Smyth power domain is a complete lattice, but in general it is not an algebraic dcpo. However its restriction $\mathcal{S}_{co}(X)$ is an algebraic dcpo with finite elements the upper closures of finite subsets of B_X [Plo81, Smy78, Smy83]. In general, the Plotkin power domain is neither a complete lattice nor a (pointed) dcpo (there is no bottom element because the empty set is related only with itself and $\{\bot\}$ is less than any other set different from the empty set). However if X is an ω-algebraic dcpo then $\mathcal{P}^+_{co}(X)$ is also an ω-algebraic dcpo with finite elements the *-closures of finite subsets of B_X (cf. [Plo76], [Plo81]). For a treatment of more general algebraic dcpo's we refer to [Hrb87] and [Hrb89].

Definition 4.3 *Let X, Y be two algebraic dcpo's. Define the Hoare state transformers as functions (ordered pointwise) in $X \rightarrow \mathcal{H}(Y)$, the Smyth state transformers as functions (ordered pointwise) in $X \rightarrow \mathcal{S}(Y)$ and the Plotkin state transformers as functions (ordered pointwise) in $X \rightarrow \mathcal{P}(Y)$.*

5 Predicate Transformers and State Transformers

In this section we will show some isomorphisms between the Predicate Transformers and State Transformers defined in the previous sections.

5.1 Safety Predicate Transformers and Hoare State Transformers

The following theorem shows the relation between safety predicate transformers and state transformers based on the Hoare power domain (the Hoare state transformers):

Theorem 5.1 *Let X and Y be two algebraic dcpo's. We have the following order-isomorphisms between the partial orders:*

1. $X \to \mathcal{H}(Y) \cong \mathbf{C}_{Sc}(Y) \to_M \mathbf{C}_d(X)$,
2. $X \to \mathcal{H}^+(Y) \cong \mathbf{C}_{Sc}(Y) \to_{sM} \mathbf{C}_d(X)$,
3. $X \to_m \mathcal{H}(Y) \cong \mathbf{C}_{Sc}(Y) \to_M \mathbf{C}_{Al}(X)$,
4. $X \to_c \mathcal{H}(Y) \cong \mathbf{C}_{Sc}(Y) \to_M \mathbf{C}_{Sc}(X)$,
5. $X \to_z \mathcal{H}(Y) \cong \mathbf{C}_{Sc}(Y) \to_{UM} \mathbf{C}_{Sc}(X)$.

In all cases the isomorphism is given by the function γ:

$$\gamma(m)(c) = \{x | m(x) \subseteq c\}.$$

The function γ can be seen as a generalization of the weakest liberal precondition. Its inverse is given by $\gamma^{-1}(\rho)(x) = \bigcap\{c | x \in \rho(c)\}$. Because the isomorphism is the same for every case we can combine different cases of the theorem to obtain more isomorphisms. For example by combining 2. and 4. we get as a corollary the result of [Plo81]:

$$X \to_c \mathcal{H}^+(Y) \cong \mathbf{C}_{Sc}(Y) \to_{sM} \mathbf{C}_{Sc}(X)).$$

Next we prove Theorem 5.1. We split the theorem in the five lemma's corresponding to the five parts of the theorem. For each part we also give some intuition.

Lemma 5.2 *Let X and Y be two algebraic dcpo's. Then*

$$X \to \mathcal{H}(Y) \cong \mathbf{C}_{Sc}(Y) \to_M \mathbf{C}_d(X).$$

Proof: We start by proving that for every $m \in X \to \mathcal{H}(Y)$ the function $\gamma(m)$ is multiplicative:

> Let $P, Q \in \mathbf{C}_{Sc}(Y)$ such that $\bigcap P \subseteq \bigcap Q$. If $x \in \bigcap_{p \in P} \gamma(m)(p)$ then for every $p \in P$, $x \in \gamma(m)(p)$, and hence $m(x) \subseteq p$, for every $p \in P$. Hence $m(x) \subseteq \bigcap P \subseteq \bigcap Q$. Thus $m(x) \subseteq q$, for every $q \in Q$, that is $x \in \bigcap_{q \in Q} \gamma(m)(q)$. Therefore $\bigcap_{p \in P} \gamma(m)(p) \subseteq \bigcap_{q \in Q} \gamma(m)(q)$.

Now we show that both the functions γ and γ^{-1} are monotone:

> Let $m_1 \sqsubseteq_H m_2$, that is $m_1(x) \subseteq m_2(x)$ for every $x \in X$. Thus, for every $c \in \mathbf{C}_{Sc}(Y)$ if $x \in \gamma(m_2)(c)$ then $m_1(x) \subseteq m_2(x) \subseteq c$. Therefore for every $c \in \mathbf{C}_{Sc}(Y)$
>
> $$\gamma(m_2)(c) = \{x | m_2(x) \subseteq c\} \subseteq \{x | m_1(x) \subseteq c\} = \gamma(m_1)(c).$$
>
> Let now $\rho_1, \rho_2 \in \mathbf{C}_{Sc}(Y) \to_M \mathbf{C}_d(X)$ be such that $\rho_1(c) \supseteq \rho_2(c)$ for every $c \in \mathbf{C}_{Sc}(Y)$. Then $\{c | x \in \rho_1(c)\} \supseteq \{c | x \in \rho_2(c)\}$ for every $x \in X$. Therefore
>
> $$\gamma^{-1}(\rho_1)(x) = \bigcap\{c | x \in \rho_1(c)\} \subseteq \bigcap\{c | x \in \rho_2(c)\} = \gamma^{-1}(\rho_2)(x),$$

that is $\gamma^{-1}(\rho_1) \sqsubseteq_H \gamma^{-1}(\rho_2)$.

Finally we prove γ and γ^{-1} are each other inverses.

$(\gamma^{-1} \circ \gamma = id_{(X \to \mathcal{H}(Y))})$

Let $m \in X \to \mathcal{H}(Y)$ and $x \in X$. Then

$$\gamma^{-1}(\gamma(m))(x)$$

$= \{$ definition γ^{-1} $\}$

$$\bigcap\{c | x \in (\gamma(m))(c)\}$$

$= \{$ definition γ $\}$

$$\bigcap\{c | m(x) \subseteq c\}$$

$= \{$ $m(x) \in \mathcal{H}(Y) \equiv \mathbf{C}_{Sc}(Y)$ $\}$

$$m(x).$$

$(\gamma \circ \gamma^{-1} = id_{(\mathbf{C}_{Sc}(Y) \to_M \mathbf{C}_d(X))})$

Let $\rho \in \mathbf{C}_{Sc}(Y) \to_M \mathbf{C}_d(X)$ and $\hat{c} \in \mathbf{C}_{Sc}(Y)$. Then

$$\gamma(\gamma^{-1}(\rho))(\hat{c})$$

$= \{$ definition γ $\}$

$$\{x | \gamma^{-1}(\rho)(x) \subseteq \hat{c}\}$$

$= \{$ definition γ^{-1} $\}$

$$\{x | \bigcap\{c | x \in \rho(c)\} \subseteq \hat{c}\}$$

$= \{$ stability lemma 3.6 $\}$

$$\{x | x \in \rho(\hat{c})\}$$

$=$

$$\rho(\hat{c}).$$

□

We can consider this isomorphism as a starting point for finding the other isomorphisms by considering various restrictions on the left and the right hand sides. We start with the restriction of "excluded miracles", that is, predicate transformers ρ such that $\rho(\emptyset) = \emptyset$. This restriction translates to the Hoare state transformers to consider the Hoare power domain without the empty set:

Lemma 5.3 *Let X and Y be two algebraic dcpo's. Then*

$$X \to \mathcal{H}^+(Y) \cong \mathbf{C}_{Sc}(Y) \to_{sM} \mathbf{C}_d(X).$$

Proof: Let $m \in X \to \mathcal{H}^+(Y)$. Then $\gamma(m)(\emptyset) = \{x | m(x) \subseteq \emptyset\} = \emptyset$ because $m(x) \neq \emptyset$ for every $x \in X$. Hence $\gamma(m)$ is strict.

Consider now $\rho \in \mathbf{C}_{Sc}(Y) \to_{sM} \mathbf{C}_d(X)$ and assume $\gamma^{-1}(\rho)(x) = \bigcap\{c | x \in \rho(c)\} = \emptyset$. Then we get following contradiction:

$$x \in \bigcap_{x \in \rho(c)} \rho(c) = \rho(\emptyset) = \emptyset.$$

Therefore $\gamma^{-1}(\rho)(x) \in \mathcal{H}^+(Y)$ for every $x \in X$. □

If we consider Hoare state transformers that are monotonic, then the safety predicate transformers have Alexandroff closed sets as codomain. Notice that in this case the safety predicate transformers are union extensible.

Lemma 5.4 *Let X and Y be two algebraic dcpo's. Then*

$$X \to_m \mathcal{H}(Y) \cong \mathbf{C}_{Sc}(Y) \to_M \mathbf{C}_{Al}(X).$$

Proof: Let $m \in X \to_m \mathcal{H}(Y)$ and $x_2 \in \gamma(m)(c)$ for $x_2 \in X$ and $c \in \mathbf{C}_{Sc}(Y)$. If $x_1 \in X$ is such that $x_1 \sqsubseteq x_2$, then $m(x_1) \sqsubseteq_H m(x_2)$, that is, $m(x_1) \subseteq m(x_2) \subseteq c$. Therefore $x_1 \in \gamma(m)(c)$. Therefore $\gamma(m)(c)$ is lower closed and hence $\gamma(m)(c) \in \mathbf{C}_{Al}(X)$.

Let now $\rho \in \mathbf{C}_{Sc}(Y) \to_M \mathbf{C}_{Al}(X)$ and $x_1 \sqsubseteq x_2$ for $x_1, x_2 \in X$ and $x_2 \in \rho(c)$. Then $x_1 \in x_2 \downarrow \subseteq \rho(c) \in \mathbf{C}_{Al}(X)$. Thus $\{c | x_2 \in \rho(c)\} \subseteq \{c | x_1 \in \rho(c)\}$ and hence

$$\gamma^{-1}(\rho)(x_2) = \bigcap\{c | x_2 \in \rho(c)\} \supseteq \bigcap\{c | x_1 \in \rho(c)\} = \gamma^{-1}(\rho)(x_1),$$

that is, $\gamma^{-1}(\rho)(x_1) \sqsubseteq_H \gamma^{-1}(\rho)(x_2)$. Therefore $\gamma^{-1}(\rho)$ is monotone. □

Restricting the Hoare state transformers to continuous functions corresponds to the safety predicate transformers having Scott closed sets as codomain. Since Scott closed set are not closed under arbitrary union, the safety predicate transformers are in general not union extensible.

Lemma 5.5 *Let X and Y be two algebraic dcpo's. Then*

$$X \to_c \mathcal{H}(Y) \cong \mathbf{C}_{Sc}(Y) \to_M \mathbf{C}_{Sc}(X).$$

Proof: Let $m \in X \to_c \mathcal{H}(Y)$. If m is continuous then it is also monotone, thus $\gamma(m)(c) \in \mathbf{C}_{Al}(X)$ for every $c \in \mathbf{C}_{Sc}(Y)$, and hence is lower closed. Let now $S \subseteq X$ be a directed set and suppose $x_i \in \gamma(m)(c)$ for every $x_i \in S$. Then $m(x_i) \subseteq c$ for every $x_i \in S$ and hence $\bigcup_{x_i \in S} m(x_i) \subseteq c$. Therefore, applying the Scott closure operator we obtain:

$$\overline{\bigcup_{x_i \in S} m(x_i)} = \bigsqcup_{x_i \in S} m(x_i) = m(\bigsqcup S) \subseteq c$$

because m is continuous, $\overline{(.)}$ is monotone and $c \in \mathbf{C}_{Sc}(Y)$. Thus $\bigsqcup S \in \gamma(m)(c)$ and hence $\gamma(m)(c) \in \mathbf{C}_{Sc}(X)$.

Let now $\rho \in \mathbf{C}_{Sc}(Y) \rightarrow_M \mathbf{C}_{Sc}(X)$. We have to prove $\gamma^{-1}(\rho)$ continuous. Since $\rho(c) \in \mathbf{C}_{Sc}(X) \subseteq \mathbf{C}_{Al}(X)$ we have $\gamma^{-1}(\rho)$ monotone. Thus

$$\bigsqcup_{x_i \in S} \gamma^{-1}(\rho)(x_i) \sqsubseteq_H \gamma^{-1}(\rho)(\bigsqcup S),$$

for every directed set $S \subseteq X$. Since arbitrary intersection of closed sets is closed, appling the stability lemma 3.6 we have $x_i \in \rho(\bigcap\{c | x_i \in \rho(c)\}) = \rho(\gamma^{-1}(\rho)(x_i))$ for every $x_i \in S$. But

$$\gamma^{-1}(\rho)(x_i) \subseteq \bigcup_{x_i \in S} \gamma^{-1}(\rho)(x_i) \subseteq \overline{\bigcup_{x_i \in S} \gamma^{-1}(\rho)(x_i)} = \bigsqcup_{x_i \in S} \gamma^{-1}(\rho)(x_i) \in \mathbf{C}_{Sc}(Y),$$

and thus, because ρ is multiplicative, and hence monotone, we have

$$x_i \in \rho(\gamma^{-1}(\rho)(x_i)) \subseteq \rho(\bigsqcup_{x_i \in S} \gamma^{-1}(\rho)(x_i)),$$

for every $x_i \in S$. Since $\rho(c) \in \mathbf{C}_{Sc}(X)$ for every $c \in \mathbf{C}_{Sc}(Y)$, we obtain

$$\forall x_i \in S : x_i \in \rho(\bigsqcup_{x_i \in S} \gamma^{-1}(\rho)(x_i)) \;\Rightarrow\; \bigsqcup S \in \rho(\bigsqcup_{x_i \in S} \gamma^{-1}(\rho)(x_i)).$$

Thus, by another application of the stability lemma 3.6 we obtain

$$\gamma^{-1}(\rho)(\bigsqcup S) = \bigcap\{c | \bigsqcup S \in \rho(c)\} \subseteq \bigsqcup_{x_i \in S} \gamma^{-1}(\rho)(x_i),$$

that is $\gamma^{-1}(\rho)(\bigsqcup S) \sqsubseteq_H \bigsqcup_{x_i \in S} \gamma^{-1}(\rho)(x_i)$. Therefore $\bigsqcup_{x_i \in S} \gamma^{-1}(\rho)(x_i) = \gamma^{-1}(\rho)(\bigsqcup S)$. □

Union extensible safety predicate transformers correspond to continuous and stabilizing Hoare state transformers. This is a severe restriction because for example the identity predicate transformers $\rho(c) = c$ for every $c \in \mathbf{C}_{Sc}(Y)$ is not union extensible (Scott closed sets need not to be closed under arbitrary union). One can say that in this respect the monotone (or unrestricted) Hoare state transformers behave better than the continuous Hoare state transformers.

Lemma 5.6 *Let X and Y be two algebraic dcpo's. Then*

$$X \rightarrow_z \mathcal{H}(Y) \cong \mathbf{C}_{Sc}(Y) \rightarrow_{UM} \mathbf{C}_{Sc}(X).$$

Proof: Let $m \in X \rightarrow_z \mathcal{H}(Y)$. Since m is continuous, and hence monotone, $\gamma(m)(c) \in \mathbf{C}_{Al}(Y)$ for every $c \in \mathbf{C}_{Sc}(Y)$. Thus for every $P \in \mathbf{C}_{Sc}(Y)$, $\bigcup_{p \in P} \gamma(m)(p) \in \mathbf{C}_{Al}(Y)$ because $\mathbf{C}_{Al}(Y)$ is closed under arbitrary union. Let $S \subseteq X$ be a directed set and assume $S \subseteq \bigcup_{p \in P} \gamma(m)(p)$. Then for every $x \in S$ there exists $p_x \in P$ such that $x \in \gamma(m)(p_x)$, that is, $m(x) \subseteq p_x$. But m stabilizes, thus $\bigsqcup_{x \in S} m(x) = m(\bigsqcup S) = m(\hat{x})$ for some $\hat{x} \in S$. Therefore also $\bigsqcup_{x \in S} m(x) \subseteq p_{\hat{x}}$, and hence $\bigsqcup_{x \in S} m(x) \in \bigcup_{p \in P} \gamma(m)(p)$, that is $\bigcup_{p \in P} \gamma(m)(p) \in \mathbf{C}_{Sc}(X)$.

Let now $\rho \in \mathbf{C}_{Sc}(Y) \rightarrow_{UM} \mathbf{C}_{Sc}(X)$. We have to prove $\gamma^{-1}(\rho)$ is continuous and stabilizing. Continuity is clear because $\rho \in \mathbf{C}_{Sc}(Y) \rightarrow_M \mathbf{C}_{Sc}(X)$. Let now $S \subseteq X$ be a directed set. Since $\gamma^{-1}(\rho)(x) \in \mathbf{C}_{Sc}(Y)$, we have by stability lemma 3.6 that $x \in \rho(\gamma^{-1}(\rho)(x))$. Hence, for every $x \in S$, $x \in \bigcup_{x \in S} \rho(\gamma^{-1}(\rho)(x))$. Furthermore, $\bigcup_{x \in S} \rho(\gamma^{-1}(\rho)(x)) \in \mathbf{C}_{Sc}(X)$ because ρ is union extensible, and hence $\bigsqcup S \in \bigcup_{x \in S} \rho(\gamma^{-1}(\rho)(x))$ as it is Scott closed. This means $\bigsqcup S \in \rho(\gamma^{-1}(\rho)(x_k))$ for some $x_k \in S$, and applying the stability lemma 3.6 we obtain

$$\gamma^{-1}(\rho)(\bigsqcup S) = \bigcap\{c \mid \bigsqcup S \in \rho(c)\} \subseteq \gamma^{-1}(\rho)(x_k),$$

that is $\gamma^{-1}(\rho)(\bigsqcup S) \sqsubseteq_H \gamma^{-1}(\rho)(x_k)$. But $\gamma^{-1}(\rho)$ monotone implies $\gamma^{-1}(\rho)(x_k) \sqsubseteq_H \gamma^{-1}(\rho)(\bigsqcup S)$. Therefore $\gamma^{-1}(\rho)(\bigsqcup S) = \gamma^{-1}(\rho)(x_k)$, that is $\gamma^{-1}(\rho)$ is continuous and stabilizing. □

5.2 Liveness Predicate Transformers and Smyth State Transformers

Now we relate liveness predicate transformers and Smyth state transformers:

Theorem 5.7 *Let X and Y be two algebraic dcpo's. We have the following isomorphisms between the partial orders:*

1. $X \to \mathcal{S}(Y) \cong \mathbf{O}_{Sc}(Y) \to_M \mathbf{O}_d(X)$,
2. $X \to \mathcal{S}^+(Y) \cong \mathbf{O}_{Sc}(Y) \to_{sM} \mathbf{O}_d(X)$,
3. $X \to \mathcal{S}_{co}(Y) \cong \mathbf{O}_{Sc}(Y) \to_{cM} \mathbf{O}_d(X)$,
4. $X \to_m \mathcal{S}(Y) \cong \mathbf{O}_{Sc}(Y) \to_M \mathbf{O}_{Al}(X)$,
5. $X \to_{alg(\beta)} \mathcal{S}(Y) \cong \mathbf{O}_{Sc}(Y) \to_M \mathbf{O}_{Sc}(X)$, *where* $\beta = \{B\!\uparrow \mid B \subseteq B_Y\}$,
6. $X \to_z \mathcal{S}(Y) \cong \mathbf{O}_{Sc}(Y) \to_{IM} \mathbf{O}_{Sc}(X)$.

In all cases the isomorphism is given by the function ω:

$$\omega(m)(o) = \{x \mid m(x) \subseteq o\}$$

The function ω is a generalization of the weakest precondition and its inverse is given by $\omega^{-1}(\pi)(x) = \bigcap\{o \mid x \in \pi(o)\}$. We can combine 2., 3. and 5. to obtain the result of [Smy83], because $\mathcal{S}_{co}(Y)$ is an algebraic dcpo with finite elements the upper closures of the finite subsets of B_Y, thus every continuous function from X to $\mathcal{S}_{co}(Y)$ is also β-algebraic. We start by proving the first item:

Lemma 5.8 *Let X and Y be two algebraic dcpo's. Then*

$$X \to \mathcal{S}(Y) \cong \mathbf{O}_{Sc}(Y) \to_M \mathbf{O}_d(X).$$

Proof: First we prove that for every $m \in X \to \mathcal{S}(Y)$ function $\omega(m)$ is multiplicative:

> Let $P, Q \in \mathbf{O}_{Sc}(Y)$ such that $\bigcap P \subseteq \bigcap Q$. If $x \in \bigcap_{p \in P} \omega(m)(p)$ then for every $p \in P$, $x \in \omega(m)(p)$, and hence $m(x) \subseteq p$, for every $p \in P$. Hence $m(x) \subseteq \bigcap P \subseteq \bigcap Q$. Thus $m(x) \subseteq q$, for every $q \in Q$, that is $x \in \bigcap_{q \in Q} \omega(m)(q)$. Therefore $\bigcap_{p \in P} \omega(m)(p) \subseteq \bigcap_{q \in Q} \omega(m)(q)$.

Next we show that both ω and ω^{-1} are monotone:

> Let $m_1, m_2 \in X \to \mathcal{S}(Y)$ be such that $m_1 \sqsubseteq_S m_2$, that is $m_1(x) \supseteq m_2(x)$ for every $x \in X$. Thus, for every $o \in \mathbf{O}_{Sc}(Y)$ if $x \in \omega(m_1)(o)$ then $m_2(x) \subseteq m_1(x) \subseteq o$. Therefore for every $o \in \mathbf{O}_{Sc}(Y)$

$$\omega(m_1)(o) = \{x | m_1(x) \subseteq o\} \subseteq \{x | m_2(x) \subseteq o\} = \omega(m_2)(o),$$

Let $\pi_1, \pi_2 \in \mathbf{O}_{Sc}(Y) \rightarrow_M \mathbf{O}_d(X)$ be such that $\pi_1(o) \subseteq \pi_2(o)$ for every $o \in \mathbf{O}_{Sc}(Y)$. Then $\{o | x \in \pi_1(o)\} \subseteq \{o | x \in \pi_2(o)\}$ for every $x \in X$. Therefore

$$\omega^{-1}(\pi_1)(x) = \bigcap\{o | x \in \pi_1(o)\} \supseteq \bigcap\{o | x \in \pi_2(o)\} = \omega^{-1}(\pi_2)(x),$$

that is $\omega^{-1}(\pi_1) \sqsubseteq_S \omega^{-1}(\pi_2)$.

Notice that for every $\pi \in \mathbf{O}_{Sc}(Y) \rightarrow_M \mathbf{O}_d(X)$ and for every $x \in X$ the function $\omega^{-1}(\pi)(x) \in \mathcal{S}(Y)$. Indeed, Let $y_1 \sqsubseteq_Y y_2 \in Y$ with $y_1 \in \omega^{-1}(\pi)(x)$. Then $y_1 \in \bigcap\{o | x \in \pi(o)\}$ and hence for every $o \in \mathbf{O}_{Sc}(Y)$ if $x \in \pi(o)$ then $y_1 \in o$. But $o \in \mathbf{O}_{Sc}(Y)$, thus if $y_1 \in o$ then $y_2 \in y_1\!\uparrow \subseteq o$. Therefore $y_2 \in \bigcap\{o | x \in \pi(o)\} = \omega^{-1}(\pi)(x)$ and hence, being upper closed, $\omega^{-1}(\pi)(x) \in \mathcal{S}(Y)$.

Next we prove ω and ω^{-1} are inverses of each other:

($\omega^{-1} \circ \omega = id_{(X \rightarrow \mathcal{S}(Y))}$))

Let $m \in X \rightarrow \mathcal{S}(Y)$. Then

$$\omega^{-1}(\omega(m))(x)$$

$=$ { definition of ω^{-1} }

$$\bigcap\{o | x \in (\omega(m))(o)\}$$

$=$ { definition of ω }

$$\bigcap\{o | m(x) \subseteq o\}$$

Clearly $m(x) \subseteq \bigcap\{o | m(x) \subseteq o\}$, thus it remains to prove $\bigcap\{o | m(x) \subseteq o\} \subseteq m(x)$. But $m(x)$ is an upper closed set of Y, thus $m(x) \in \mathbf{O}_{Al}(Y) = \mathbf{O}_{Sc}(Y)^{\cap}$, say $m(x) = \bigcap_I o_i$ with $o_i \in \mathbf{O}_{Sc}(Y)$. Hence, for every $i \in I$, $m(x) \subseteq o_i$, that means $\{o_i | i \in I\} \subseteq \{o | m(x) \subseteq o\}$. Thus $m(x) = \bigcap\{o_i | \in I\} \supseteq \bigcap\{o | m(x) \subseteq o\}$. Therefore $m(x) = \bigcap\{o | m(x) \subseteq o\}$.

($\omega \circ \omega^{-1} = id_{(\mathbf{O}_{Sc}(Y) \rightarrow_M \mathbf{O}_d(X))}$))

Let $\pi \in \mathbf{O}_{Sc}(Y) \rightarrow_M \mathbf{O}_d(X)$. Then

$$\omega(\omega^{-1}(\pi))(\hat{o})$$

$=$ { definition of ω }

$$\{x | \omega^{-1}(\pi)(x) \subseteq \hat{o}\}$$

$=$ { definition of ω^{-1} }

$$\{x | \bigcap\{o | x \in \pi(o)\} \subseteq \hat{o}\}$$

$=$ { stability lemma 3.6 }

$$\{x | x \in \pi(\hat{o})\}$$

$$= \pi(\hat{o}).$$

□

Again we consider this isomorphism the basic building block for the other isomorphisms by considering restrictions on the left and the right hand sides. First we consider liveness predicate transformers that satisfy the 'excluded miracles law'.

Lemma 5.9 *Let X and Y be two algebraic dcpo's. Then*

$$X \to \mathcal{S}^+(Y) \cong \mathbf{O}_{Sc}(Y) \to_{sM} \mathbf{O}_d(X).$$

Proof: Let $m \in X \to \mathcal{S}^+(Y)$. Then $\omega(m)(\emptyset) = \{x | m(x) \subseteq \emptyset\} = \emptyset$, and hence $\omega(m)$ is strict.

Let now $\pi \in \mathbf{O}_{Sc}(Y) \to_{sM} \mathbf{O}_d(X)$ and assume $\omega^{-1}(\pi)(x) = \bigcap\{o | x \in \pi(o)\} = \emptyset$ for some $x \in X$. Then we get following contradiction:

$$x \in \bigcap_{x \in \pi(o)} \pi(o) = \pi(\emptyset) = \emptyset.$$

□

The Smyth 'compact' state transformers corresponds to continuous and multiplicative liveness predicate transformers:

Lemma 5.10 *Let X and Y be two algebraic dcpo's. Then*

$$X \to \mathcal{S}_{co}(Y) \cong \mathbf{O}_{Sc}(Y) \to_{cM} \mathbf{O}_d(X).$$

Proof: Let $m \in X \to \mathcal{S}_{co}(Y)$ and let $S \subseteq \mathbf{O}_{Sc}(Y)$ be a directed set. Since $\omega(m)$ is multiplicative, it is also monotone, and hence we have

$$\bigcup_{o \in S} \omega(m)(o) \subseteq \omega(m)(\bigcup S).$$

Consider $x \in \omega(m)(\bigcup S)$, that is $m(x) \subseteq \bigcup S$. As $m(x)$ is compact in $\mathbf{O}_{Sc}(Y)$ there exists a finite subset $S' \subseteq S$ such that $m(x) \subseteq \bigcup S'$. But S is directed, thus for every finite $S' \subseteq S$ there exists an upper bound $o \in S$. Hence $m(x) \subseteq o$ and $x \in \bigcup_{o \in S} \omega(m)(o)$. Therefore $\omega(m)(\bigcup S) = \bigcup_{o \in S} \omega(m)(o)$.

Let now $\pi \in \mathbf{O}_{Sc}(Y) \to_{cM} \mathbf{O}_d(X)$. We have to prove $\omega^{-1}(\pi)(x)$ compact in $\mathbf{O}_{Sc}(Y)$ for every $x \in X$. Let $S \subseteq \mathbf{O}_{Sc}(Y)$ be a directed set and $x \in X$. Assume $\omega^{-1}(\pi)(x) \subseteq \bigcup S$. Then $\bigcap\{o \in \mathbf{O}_{Sc}(Y) | x \in \pi(o)\} \subseteq \bigcup S$ and since $\bigcup S \in \mathbf{O}_{Sc}(Y)$, by stability lemma 3.6 we have $x \in \pi(\bigcup S)$. But π is continuous, hence $x \in \pi(\bigcup S) = \bigcup_{s \in S} \pi(s)$, that is, there exists $\hat{s} \in S$ such that $x \in \pi(\hat{s})$. Again by stability lemma 3.6 we obtain

$$\omega^{-1}(\pi)(x) = \bigcap\{o \in \mathbf{O}_{Sc}(Y) | x \in \pi(o)\} \subseteq \hat{s}.$$

Therefore, by lemma 2.2 $\omega^{-1}(\pi)(x)$ is Scott compact. □

If we consider monotonic Smyth state transformers, then the liveness predicate transformers have Alexandroff open sets as codomain. Moreover these predicate transformers are intersection extensible.

Lemma 5.11 *Let X and Y be two algebraic dcpo's. Then*

$$X \to_m \mathcal{S}(Y) \cong \mathbf{O}_{Sc}(Y) \to_M \mathbf{O}_{Al}(X).$$

Proof: Let $m \in X \to_m \mathcal{S}(Y)$ and $x_1, x_2 \in X$ be such that $x_1 \in \omega(m)(o)$ and $x_1 \sqsubseteq x_2$. Then $m(x_1) \sqsubseteq_S m(x_2)$, that is, $m(x_2) \subseteq m(x_1) \subseteq o$, thus $x_2 \in \omega(m)(o)$. Therefore $\omega(m)(o)$ is upper-closed, and hence $\omega(m)(o) \in \mathbf{O}_{Al}(X)$.

Let now $\pi \in \mathbf{O}_{Sc}(Y) \to_M \mathbf{O}_{Al}(X)$ and $x_1, x_2 \in X$ be such that $x_1 \in \pi(o)$ and $x_1 \sqsubseteq x_2$. Then $x_2 \in x_1\uparrow \subseteq \pi(o) \in \mathbf{O}_{Al}(X)$. Thus $\{o | x_1 \in \pi(o)\} \subseteq \{o | x_2 \in \pi(o)\}$ and hence

$$\omega^{-1}(\pi)(x_1) = \bigcap\{o | x_1 \in \pi(o)\} \supseteq \bigcap\{o | x_2 \in \pi(o)\} = \omega^{-1}(\pi)(x_2),$$

that is, $\omega^{-1}(\pi)(x_1) \sqsubseteq_S \omega^{-1}(\pi)(x_2)$. □

When we consider liveness predicate transformers that are multiplicative and with Scott open sets as codomain, we obtain a subset of the continuous Smyth state transformers carrying a notion of β-algebraicity. The collection of β-algebraic functions need not to be a dcpo.

Lemma 5.12 *Let X and Y be two algebraic dcpo's and $\beta = \{B\uparrow \in \mathcal{S}(Y) | B \subseteq B_Y\}$. Then*

$$X \to_{alg(\beta)} \mathcal{S}(Y) \cong \mathbf{O}_{Sc}(Y) \to_M \mathbf{O}_{Sc}(X).$$

Proof: Let $m \in X \to_{alg(\beta)} \mathcal{S}(Y)$. Then it is continuous and hence also monotone. Thus $\omega(m)(o) \in \mathbf{O}_{Al}(X)$ for every $o \in \mathbf{O}_{Sc}(Y)$, that is $\omega(m)(o)$ is upper closed. Let now $S \subseteq X$ be a directed set and $\bigsqcup S \in \omega(m)(o)$ for $o \in \mathbf{O}_{Sc}(Y)$. Then $m(\bigsqcup S) = \bigsqcup_{x \in S} m(x) \subseteq o$ because m is continuous. But $o \in \beta = \{B\uparrow | B \subseteq B_Y\}$ and m is β-algebraic, thus there exists an $x_k \in S$ such that $m(\bigsqcup S) = \bigsqcup_{x \in S} m(x) \subseteq m(x_k) \subseteq o$, that is, $x_k \in \omega(m)(o)$. This proves $\omega(m)(o) \in \mathbf{O}_{Sc}(X)$.

Let now $\pi \in \mathbf{O}_{Sc}(Y) \to_M \mathbf{O}_{Sc}(X)$, we have to prove that $\omega^{-1}(\pi)$ is β-algebraic. Since $\pi(o) \in \mathbf{O}_{Sc}(X) \subseteq \mathbf{O}_{Al}(X)$ we have $\omega^{-1}(\pi)$ monotone. Thus

$$\bigsqcup_{x \in S} \omega^{-1}(\pi)(x) \sqsubseteq_S \omega^{-1}(\pi)(\bigsqcup S),$$

for every directed set $S \subseteq X$. Consider $y \in \bigsqcup_{x \in S} \omega^{-1}(\pi)(x) = \bigcap \omega^{-1}(\pi)(x)$ then

$$\forall x \in S, \forall o \in \mathbf{O}_{Sc}(Y) : x \in \pi(o) \Rightarrow y \in o.$$

But if $\bigsqcup S \in \pi(o)$ then there exists $x_k \in S$ such that $x_k \in \pi(o)$ because it is Scott open. Hence by above we have:

$$\forall o \in \mathbf{O}_{Sc}(Y) : \bigsqcup S \in \pi(o) \Rightarrow \exists x_k \in \pi(o) \Rightarrow y \in o.$$

Hence $y \in \bigcap\{o | \bigsqcup S \in \pi(o)\} = \omega^{-1}(\pi)(\bigsqcup S)$, which means $\omega^{-1}(\pi)(\bigsqcup S) \supseteq \bigsqcup_{x \in S} \omega^{-1}(\pi)(x)$, or equivalently $\omega^{-1}(\pi)(\bigsqcup S) \sqsubseteq_S \bigsqcup_{x \in S} \omega^{-1}(\pi)(x)$.

It remains to prove that $\omega^{-1}(\pi)$ is β-algebraic. Let $o' \in \beta = \mathbf{O}_{Sc}(Y)$, let $S \subseteq X$ be a directed set and assume $o' \sqsubseteq_S \omega^{-1}(\pi)(\bigsqcup S)$. Then $\bigcap\{o | \bigsqcup S \in \pi(o)\} \subseteq o'$ and since π is multiplicative we obtain

$$\bigsqcup S \in \bigcap_{\bigsqcup S \in \pi(o)} \pi(o) \subseteq \pi(o').$$

But $\pi(o')$ is Scott open, thus $\bigsqcup S \in \pi(o')$ implies there exists $x_k \in S$ such that $x_k \in \pi(o')$. By stability lemma 3.6 we obtain the required result:

$$\omega^{-1}(\pi)(x_k) = \bigcap\{o | x_k \in \pi(o)\} \subseteq o'.$$

□

Next we consider only predicate transformers that are intersection extensibles. They correspond to the continuous and stabilizing Smyth state transformers. This is a severe restriction because in general the identity predicate transformer $\pi(o) = o$ for every $o \in \mathbf{O}_{Sc}(Y)$ is not intersection extensible (the Scott open sets need not to be closed under arbitrary intersections).

Lemma 5.13 *Let X and Y be two algebraic dcpo's. Then*

$$X \to_z \mathcal{S}(Y) \cong \mathbf{O}_{Sc}(Y) \to_{IM} \mathbf{O}_{Sc}(X).$$

Proof: Let $m \in X \to_{c_s} \mathcal{S}(Y)$. Since m is continuous and stabilizing, then it is also β-algebraic with $\beta = \{B\uparrow \in \mathcal{S}(Y) | B \subseteq B_Y\}$. Hence $\omega(m)(o) \in \mathbf{O}_{Sc}(X)$ for every $o \in \mathbf{O}_{Sc}(Y)$. Thus for every $P \subseteq \mathbf{O}_{Sc}(Y)$, and for every $o \in P$, $\omega(m)(o) \in \mathbf{O}_{Sc}(X)$, that is $\omega(m)(o)$ is upper closed. Hence also $\bigcap_{o \in P} \omega(m)(o)$ is upper closed. Let now $S \subseteq X$ be a directed set such that $\bigsqcup S \in \bigcap_{o\in P} \omega(m)(o)$. Then $m(\bigsqcup S) = \bigsqcup_{x \in S} m(x) \subseteq o$ for every $o \in P$ because m is continuous. Thus $\bigsqcup_{x\in S} m(x) \subseteq \bigcap_{o \in P} o$, and, since m is also stabilizing, hence there exists $\hat{x} \in S$ such that $\bigsqcup_{x \in S} m(x) = m(\hat{x}) \subseteq \bigcap_{o\in P} o \subseteq o$ for every $o \in P$. Therefore $\hat{x} \in \omega(m)(o)$ for every $o \in P$, and hence $\hat{x} \in \bigcap_{o \in P} \omega(m)(o)$. This proves $\bigcap_{o\in P} \omega(m)(o) \in \mathbf{O}_{Sc}(X)$.

Let now $\pi \in \mathbf{O}_{Sc}(Y) \to_{IM} \mathbf{O}_{Sc}(X)$. We have to prove that $\omega^{-1}(\pi)$ is continuous and stabilizing. Since $\pi \in (o) \in \mathbf{O}_{Sc}(Y) \to_M \mathbf{O}_{Sc}(X)$ we have $\omega^{-1}(\pi)$ β-algebraic, and hence continuous. Thus

$$\bigsqcup_{x \in S} \omega^{-1}(\pi)(x) = \omega^{-1}(\pi)(\bigsqcup S),$$

for every directed set $S \subseteq X$. It remains to prove that $\omega^{-1}(\pi)$ stabilizes for every directed set $S \subseteq X$. Since π is intersection extensible, we have $\bigsqcup S \in \bigcap_{\bigsqcup S \in \pi(o)} \pi(o) \in \mathbf{O}_{Sc}(X)$. Hence there exists $x_k \in S$ such that $x_k \in \bigcap_{\bigsqcup S\in\pi(o)} \pi(o)$. Hence $x_k \in \pi(o)$ for every $o \in \mathbf{O}_{Sc}(Y)$ such that $\bigsqcup S \in \pi(o)$, and by stability lemma 3.6 we obtain $\bigcap\{o|x_k \in \pi(o)\} \subseteq o$. Thus

$$\bigcap\{o|x_k \in \pi(o)\} \subseteq \bigcap_{\bigsqcup S \in \pi(o)} o = \bigcap\{o | \bigsqcup S \in \pi(o)\},$$

that is, $\omega^{-1}(\pi)(\bigsqcup S) \sqsubseteq_S \omega^{-1}(\pi)(x_k)$ for some $x_k \in S$. But $\omega^{-1}(\pi)$ is monotone, and $x_k \sqsubseteq \bigsqcup S$, thus $\omega^{-1}(\pi)(x_k) \sqsubseteq_S \omega^{-1}(\pi)(\bigsqcup S)$ and hence $\omega^{-1}(\pi)(x_k) = \omega^{-1}(\pi)(\bigsqcup S)$. □

5.3 Safety - Liveness Predicate Transformers and Plotkin State Transformers

Finally we relate the Plotkin state transformers with pairs of safety and liveness predicate transformers:

Theorem 5.14 *Let X and Y be two algebraic dcpo's. We have the following isomorphisms between partial orders:*

1. $X \to \mathcal{P}(Y) \cong (\mathbf{O}_{Sc}(Y) \to_M \mathbf{O}_d(X)) \boxtimes (\mathbf{C}_{Sc}(Y) \to_M \mathbf{C}_d(X))$
2. $X \to \mathcal{P}^+(Y) \cong (\mathbf{O}_{Sc}(Y) \to_{sM} \mathbf{O}_d(X)) \boxtimes (\mathbf{C}_{Sc}(Y) \to_{sM} \mathbf{C}_d(X))$,
3. $X \to \mathcal{P}_{co}(Y) \cong (\mathbf{O}_{Sc}(Y) \to_{cM} \mathbf{O}_d(X)) \boxtimes (\mathbf{C}_{Sc}(Y) \to_M \mathbf{C}_d(X))$,
4. $X \to_m \mathcal{P}(Y) \cong (\mathbf{O}_{Sc}(Y) \to_M \mathbf{O}_{Al}(X)) \boxtimes (\mathbf{C}_{Sc}(Y) \to_M \mathbf{C}_{Al}(X))$,
5. $X \to_c \mathcal{P}_{co}(Y) \cong (\mathbf{O}_{Sc}(Y) \to_{cM} \mathbf{O}_{Sc}(X)) \boxtimes (\mathbf{C}_{Sc}(Y) \to_M \mathbf{C}_{Sc}(X))$,
6. $X \to_z \mathcal{P}(Y) \cong (\mathbf{O}_{Sc}(Y) \to_{IM} \mathbf{O}_{Sc}(X)) \boxtimes (\mathbf{C}_{Sc}(Y) \to_{UM} \mathbf{C}_{Sc}(X))$.

In all cases the isomorphism is given by the function η:

$$\eta(m)(o,c) = (\{x|m(x)\uparrow \subseteq o\}, \{x|\overline{m(x)} \subseteq c\})$$

The inverse of η is given by $\eta^{-1}((\pi,\rho))(x) = \bigcap\{o|x \in \pi(o)\} \cap \bigcap\{c|x \in \rho(c)\}$.

Notice that we have a restricted version of item 5. (different from item 5. of Theorem 5.7) because we consider only continuous state transformers and the compact version of the Plotkin power domain. In this case the state transformers are the B-algebraic functions with $B = \{A^*|A \subseteq_{fin} B_Y\}$. We could also treat the case without the restriction to compactness, but then we should introduce a new notion of algebraicity of functions depending on two sets.

We start with an useful lemma:

Lemma 5.15 *For every $A \subseteq Y, o \in \mathbf{O}_{Al}(Y)$ and $c \in \mathbf{C}_{Sc}(Y)$ we have*

1. $A\uparrow \subseteq o \Leftrightarrow A \subseteq o$,
2. $\overline{A} \subseteq c \Leftrightarrow A \subseteq c$.

Proof: Trivial. □

Next we prove the first item of Theorem 5.14. The other items are combinations of Theorem 5.1 and Theorem 5.7 and are left to the reader.

Lemma 5.16 *Let X and Y be two algebraic dcpo's. Then*

$$X \to \mathcal{P}(Y) \cong (\mathbf{O}_{Sc}(Y) \to_M \mathbf{O}_d(X)) \boxtimes (\mathbf{C}_{Sc}(Y) \to_M \mathbf{C}_d(X)).$$

Proof: We begin by proving for every $m : X \to \mathcal{P}(Y)$ that $\eta(m)$ is well defined. Indeed let $\eta_1(m)(o) = \{x|m(x)\uparrow \subseteq o\}$ and $\eta_2(m)(c) = \{x|\overline{m(x)} \subseteq c\}$ where $o \in \mathbf{O}_{Sc}(Y)$ and $c \in \mathbf{C}_{Sc}(Y)$. Then by Theorem 5.7 and Theorem 5.1 we have $\eta_1(m) \in \mathbf{O}_{Sc}(Y) \to_M \mathbf{O}_d(X)$ and also $\eta_2(m) : \mathbf{C}_{Sc}(Y) \to_M \mathbf{C}_d(X)$. Consider a collection of predicates $P, P' \subseteq \mathbf{O}_{Sc}(Y)$ and $Q, Q' \subseteq \mathbf{C}_{Sc}(Y)$ such that $\bigcap P \cap \bigcap Q \subseteq \bigcap P' \cap \bigcap Q'$. We have to prove $\bigcap_{p \in P} \eta_1(m)(p) \cap \bigcap_{q \in Q} \eta_2(m)(q) \subseteq \bigcap_{p' \in P'} \eta_1(m)(p') \cap \bigcap_{q' \in Q'} \eta_2(m)(q')$.

$$\bigcap_{p \in P} \eta_1(m)(p) \cap \bigcap_{q \in Q} \eta_2(m)(q)$$

$= \{$ definition of $\eta = (\eta_1, \eta_2)$ $\}$

$$\bigcap_{p\in P} \{x | m(x)\!\uparrow \subseteq p\} \cap \bigcap_{q \in Q} \{x | \overline{m(x)} \subseteq q\}$$

$= \{$ calculation $\}$

$$\{x | m(x)\!\uparrow \subseteq \bigcap P\} \cap \{x | \overline{m(x)} \subseteq \bigcap Q\}$$

$= \{$ calculation $\}$

$$\{x | m(x)\!\uparrow \subseteq \bigcap P \ \wedge \ \overline{m(x)} \subseteq \bigcap Q\}$$

$= \{$ Lemma 5.15 $\}$

$$\{x | m(x) \subseteq \bigcap P \ \wedge \ m(x) \subseteq \bigcap Q\}$$

$=$

$$\{x | m(x) \subseteq \bigcap P \cap \bigcap Q\}$$

$\subseteq \{\bigcap P \cap \bigcap Q \subseteq \bigcap P' \cap \bigcap Q'\}$

$$\{x | m(x) \subseteq \bigcap P' \cap \bigcap Q'\}$$

$\subseteq \{ \bigcap P' \cap \bigcap Q' \subseteq \bigcap P'$ and $\bigcap P' \cap \bigcap Q' \subseteq \bigcap Q' \}$

$$\{x | m(x) \subseteq \bigcap P'\} \cap \{x | m(x) \subseteq \bigcap Q'\}$$

$= \{$ Lemma 5.15 $\}$

$$\{x | m(x)\!\uparrow \subseteq \bigcap P'\} \cap \{x | \overline{m(x)} \subseteq \bigcap Q'\}$$

$=$

$$\bigcap_{p'\in P'} \{x | m(x)\!\uparrow \subseteq p'\} \cap \bigcap_{q'\in Q'} \{x | \overline{m(x)} \subseteq q'\}$$

$= \{$ definition of $\eta = (\eta_1, \eta_2)$ $\}$

$$\bigcap_{p'\in P'} \eta_1(m)(p') \cap \bigcap_{q'\in Q'} \eta_2(m)(q').$$

Next we prove the function η is strictly monotone: Let $m_1, m_2 \in X \to \mathcal{P}(Y)$ be such that $m_1 \sqsubseteq_P m_2$, that is $m_1(x)\!\uparrow \supseteq m_2(x)\!\uparrow$ and $\overline{m_1(x)} \subseteq \overline{m_2(x)}$. Thus, for every $o \in \mathbf{O}_{Sc}(Y)$ we have

$$\eta_1(m_1)(o) = \{x | m_1(x)\!\uparrow \subseteq o\} \subseteq \{x | m_2(x)\!\uparrow \subseteq o\} = \eta_1(m_2)(o),$$

and for every $c \in \mathbf{C}_{Sc}(Y)$

$$\eta_2(m_2)(c) = \{x | \overline{m_2(x)} \subseteq c\} \subseteq \{x | \overline{m_1(x)} \subseteq c\} = \eta_2(m_1)(c),$$

that is $(\eta_1(m_1), \eta_2(m_1)) \sqsubseteq_N (\eta_1(m_2), \eta_2(m_2))$.

Suppose now $m_1 \not\sqsubseteq_P m_2$. Then there exists an $x \in X$ such that $m_1(x){\uparrow} \not\supseteq m_2(x){\uparrow}$ or $\overline{m_1(x)} \not\subseteq \overline{m_2(x)}$. But $\overline{m_2(x)} \in \mathbf{C}_{Sc}(Y)$ and $m_1(x){\uparrow} \in \mathbf{O}_{Al}(Y)$, say $m_1(x){\uparrow} = \bigcap_I o_i$ with $o_i \in \mathbf{O}_{Sc}(Y)$ for every $i \in I$. Hence, either

$$\eta_2(m_2)(\overline{m_2(x)}) = \{x | \overline{m_2(x)} \subseteq \overline{m_2(x)}\} \not\subseteq \{x | \overline{m_1(x)} \subseteq \overline{m_2(x)}\} = \eta_2(m_1)(\overline{m_2(x)}),$$

or, for some $i \in I$,

$$\eta_1(m_1)(o_i) = \{x | m_1(x){\uparrow} \subseteq o_i\} \not\subseteq \{x | m_2(x){\uparrow} \subseteq o_i\} = \eta_1(m_2)(o_i).$$

Now we prove for every (π, ρ) in $\mathbf{O}_{Sc}(Y) \to_M \mathbf{O}_d(X) \bowtie (\mathbf{C}_{Sc}(Y) \to_M \mathbf{C}_d(X))$ and for every x in X that the function $\eta^{-1}((\pi,\rho))(x)$ is an element of $\mathcal{P}(Y)$, that is we have to prove that $\eta^{-1}((\pi,\rho))(x)$ is *-closed. By definition of *-closure we have

$$\eta^{-1}((\pi,\rho))(x) \subseteq (\eta^{-1}((\pi,\rho))(x))^*.$$

Let $y \in (\eta^{-1}((\pi,\rho))(x))^*$. Then by definition

$$(\exists \hat{y} \in \eta^{-1}((\pi,\rho))(x) : \hat{y} \sqsubseteq y) \ \wedge \ (\forall b \in B_Y : b \sqsubseteq Y \Rightarrow \exists y_b \in \eta^{-1}((\pi,\rho))(x) : b \sqsubseteq y_b).$$

Using the definition of $\eta^{-1}((\pi,\rho))(x)$ we obtain

$$\forall o \in \mathbf{O}_{Sc}(Y) : x \in \pi(o) \Rightarrow \hat{y} \in o,$$

and also

$$\forall c \in \mathbf{C}_{Sc}(Y) : x \in \rho(c) \Rightarrow y_b \in c.$$

But $y \in \hat{y}{\uparrow} \subseteq o$ as $o \in \mathbf{O}_{Sc}(Y)$ and hence

$$\forall o \in \mathbf{O}_{Sc}(Y) : x \in \pi(o) \Rightarrow y \in o,$$

that is $y \in \bigcap\{o | x \in \pi(o)\}$. Moreover $b' \sqsubseteq y_b$ implies $b' \in y_b \downarrow \subseteq c$ because $c \in \mathbf{C}_{Sc}(Y)$, hence

$$\forall c \in \mathbf{C}_{Sc}(Y) : x \in \rho(c) \ \wedge \ b \sqsubseteq y \Rightarrow b \in c.$$

But Y is an algebraic dcpo, hence $y = \bigsqcup\{b | b \in B_Y \ \wedge \ b \sqsubseteq_Y y\}$ and all these finite elements are elements of c. Since c is Scott closed we obtain $y = \bigsqcup\{b | b \in B_Y \ \wedge \ b \sqsubseteq_Y y\} \in c$. This means $y \in \bigcap\{c | x \in \rho(c)\}$. Therefore

$$y \in \bigcap\{o | x \in \pi(o)\} \cap \bigcap\{c | x \in \rho(c)\} = \eta^{-1}(x).$$

Finally we prove that η^{-1} is the left and right inverse of $\eta = (\eta_1, \eta_2)$.

$(\eta^{-1} \circ \eta = id_{(X \to \mathcal{P}(Y))}\ ))$

Let $m \in X \to \mathcal{P}(Y)$. Then

$$\eta^{-1}(\eta_1(m),\eta_1(m))(x)$$

$= \{$ definition of η^{-1} $\}$

$$\bigcap\{o|x \in (\eta_1(m))(o)\} \cap \bigcap\{c|x \in (\eta_2(m))(c)\}$$

$= \{$ definition of η $\}$

$$\bigcap\{o|m(x)\uparrow\subseteq o\} \cap \bigcap\{c|\overline{m(x)} \subseteq c\}$$

$=$

$$m(x)\uparrow \cap\overline{m(x)}$$

$= \{$ because $m(x)$ is *-closed $\}$

$$m(x)$$

$(\eta \circ \eta^{-1} = id_{(\mathbf{O}_{Sc}(Y)\to_M \mathbf{O}_d(X))\bowtie(\mathbf{C}_{Sc}(Y)\to_M \mathbf{C}_d(X))}\ ))$

Let $(\pi,\rho) \in (\mathbf{O}_{Sc}(Y) \to_M \mathbf{O}_d(X))\bowtie(\mathbf{C}_{Sc}(Y) \to_M \mathbf{C}_d(X))$. Then

$$\eta(\eta^{-1}(\pi,\rho))(\hat{o},\hat{c})$$

$= \{$ definition of η $\}$

$$(\{x|\eta^{-1}(\pi,\rho)(x) \subseteq \hat{o}\},\{x|\eta^{-1}(\pi,\rho)(x) \subseteq \hat{c}\})$$

$= \{$ definition of η^{-1} $\}$

$$(\{x|(\bigcap\{o|x \in \pi(o)\} \cap \bigcap\{c|x \in \rho(c)\})\uparrow\subseteq \hat{o}\},$$
$$\{x|\overline{(\bigcap\{o|x \in \pi(o)\} \cap \bigcap\{c|x \in \rho(c)\})} \subseteq \hat{c}\})$$

$= \{$ Lemma 5.15 $\}$

$$(\{x|\bigcap\{o|x \in \pi(o)\} \cap \bigcap\{c|x \in \rho(c)\} \subseteq \hat{o}\},$$
$$\{x|\bigcap\{o|x \in \pi(o)\} \cap \bigcap\{c|x \in \rho(c)\} \subseteq \hat{c}\})$$

$= \{$ stability lemma 3.8 $\}$

$$(\{x|x \in \pi(\hat{o})\},\{x|x \in \pi(\hat{c})\})$$

$=$

$$(\pi(\hat{o}),\pi(\hat{c})).$$

□

6 Adding deadlock as empty set

In this section we discuss what should be done to introduce deadlock in state transformers and predicate transformers. Deadlock is represented by the empty set.

First we show how to add the empty set to the power domains. These extended power domains can then be used in state transformers. The empty set is usually added to the Plotkin power domain by means of a smash-product [MM79],[Plo81] and also [Abr91]. The same is done [BK92] for the Smyth power domain in the flat case. We have the following:

Definition 6.1 *Let X be an algebraic dcpo. Define*

1. *the Hoare-deadlock power domain* $\mathcal{H}\delta(X) = \langle\{A | A \subseteq X \ \wedge \ A = \overline{A}\}, \sqsubseteq_{H\delta}\rangle$, *where* $A \sqsubseteq_{H\delta} B$ *if* $(A = \{\bot\}) \vee (A = \emptyset \Rightarrow B = \emptyset) \vee (A \neq \emptyset \wedge A \sqsubseteq_H B)$,
2. *the Smyth-deadlock power domain* $\mathcal{S}\delta(X) = \langle\{A | A \subseteq X \ \wedge \ A = A\uparrow\}, \sqsubseteq_{S\delta}\rangle$, *where* $A \sqsubseteq_{S\delta} B$ *if* $(A = X) \vee (A = \emptyset \Rightarrow B = \emptyset) \vee (B \neq \emptyset \wedge A \sqsubseteq_S B)$.
3. *the Plotkin-deadlock power domain* $\mathcal{P}\delta(X) = \langle\{A | A \subseteq X \ \wedge \ A = A^*\}, \sqsubseteq_{P\delta}\rangle$, *where* $A \sqsubseteq_{P\delta} B$ *if* $(A = \{\bot\}) \vee (A = \emptyset \Rightarrow B = \emptyset) \vee (A \neq \emptyset \wedge A \sqsubseteq_P B)$.

We have that $\mathcal{P}\delta_{co}(X)$ coincides with the standard way of adding the empty set to the Plotkin power domain [MM79], [Plo81] and also [Abr91].

Next we turn to the predicate transformers. We modify the implication order both on safety and liveness predicate transformers to preserve deadlock states (states that are the image of the empty set).

Definition 6.2 *Let X and Y be algebraic dcpo's and let $\pi_1, \pi_2 \in \mathbf{O}_{Sc}(Y) \to_M \mathbf{O}(X)$ be two liveness predicate transformers. Define $\pi_1 \sqsubseteq_{LD} \pi_2$ if*

$$(\pi_1(\emptyset) \subseteq \pi_2(\emptyset)) \ \wedge \ (\forall o \in \mathbf{O}_{Sc}(Y) : o \neq Y \Rightarrow \pi_1(o) \setminus \pi_1(\emptyset) \subseteq \pi_2(o) \setminus \pi_2(\emptyset)).$$

For every safety predicate transformers $\rho_1, \rho_2 \in \mathbf{C}_{Sc}(Y) \to_M \mathbf{C}(X)$, define $\rho_1 \sqsubseteq_{SD} \rho_2$ if

$$(\rho_1(\emptyset) \subseteq \rho_2(\emptyset)) \ \wedge \ (\forall c \in \mathbf{C}_{Sc}(Y) : c \neq \emptyset \Rightarrow \rho_1(c) \supseteq \rho_2(c) \setminus \rho_1(\emptyset)).$$

The isomorphisms of the previous sections also hold if we substitute the deadlock versions for the the Hoare, Smyth and Plotkin power domains when we use the safety and liveness predicate transformers ordered with the new deadlock orders.

First we show that the isomorphisms of Theorem 5.1 hold if we substitute $\mathcal{H}\delta(Y)$ for $\mathcal{H}(Y)$ and replace the the superset order by the deadlock order for the safety predicate transformers. Since $\mathcal{H}(Y)$ and $\mathcal{H}\delta(Y)$ differ only in the orderings, we need only to prove the following two lemmas:

Lemma 6.3 *Let $m_1, m_2 \in X \to \mathcal{H}\delta(Y)$. If $m_1 \sqsubseteq_{H\delta} m_2$ then $\gamma(m_1) \sqsubseteq_{SD} \gamma(m_2)$.*

Proof: Let $m_1 \sqsubseteq_{H\delta} m_2$. If $x \in \gamma(m_1)(\emptyset)$ then $m_1(x) \subseteq \emptyset$, that is $m_1(x) = \emptyset$. But then also $m_2(x) = \emptyset$ and hence $x \in \gamma(m_2)(\emptyset)$. If $x \in \gamma(m_2)(c)$ for some non-empty $c \in \mathbf{C}_{Sc}(Y)$ but $x \notin \gamma(m_1)(\emptyset)$ then $m_1(x) \neq \emptyset$ and $m_2(x) \subseteq c$. Thus either $m_1(x) = \{\bot\}$ or $m_1(x) \subseteq m_2(x)$. In each case $m_1(x) \subseteq c$ because either $\bot \in c$ since every nonempty Scott closed set contains it, or $m_1(x) \subseteq m_2(x) \subseteq c$. Therefore $x \in \gamma(m_1)(c)$. □

Lemma 6.4 *Let $\rho_1, \rho_2 \in \mathbf{C}_{Sc}(Y) \to_M \mathbf{C}(X)$. If $\rho_1 \sqsubseteq_{SD} \rho_2$ then $\gamma^{-1}(\rho_1) \sqsubseteq_{H\delta} \gamma^{-1}(\rho_2)$.*

Proof: Let $\rho_1 \sqsubseteq_{SD} \rho_2$ and $x \in X$. Consider the following three cases:

1. If $x \notin \rho_1(\emptyset)$ and $x \in \rho_2(\emptyset)$ then for every Scott-closed set $c \neq \emptyset$, $\bot \in c$ and $x \in \rho_2(c) \setminus \rho_1(\emptyset) \subseteq \rho_1(c)$. Hence $\gamma^{-1}(\rho_1)(x) = \bigcap\{c | x \in \rho_1(c)\} = \{\bot\}$, that is $\gamma^{-1}(\rho_1)(x) \sqsubseteq_{H\delta} \gamma^{-1}(\rho_2)$.
2. If $x \in \rho_1(\emptyset)$ then $x \in \rho_2(\emptyset)$ because $\rho_1(\emptyset) \subseteq \rho_2(\emptyset)$. Hence $\gamma^{-1}(\rho_1)(x) = \bigcap\{c|x \in \rho_1(c)\} = \emptyset$ which implies also $\gamma^{-1}(\rho_2)(x) = \bigcap\{c|x \in \rho_2(c)\} = \emptyset$, that is $\gamma^{-1}(\rho_1)(x) \sqsubseteq_{H\delta} \gamma^{-1}(\rho_2)$.
3. If $x \notin \rho_1(\emptyset)$ and $x \notin \rho_2(\emptyset)$, then for every Scott closed $c \neq \emptyset$ such $x \in \rho_2(c)$ since $x \notin \rho_1(\emptyset)$ we have $x \in \rho_2(c) \setminus \rho_1(\emptyset) \subseteq \rho_1(c)$, that implies $\{c|x \in \rho_1(c)\} \supseteq \{c|x \in \rho_2(c)\}$. Therefore $\gamma^{-1}(\rho_1)(x) \neq \emptyset$, because otherwise by stability lemma 3.6 $x \in \rho_1(\emptyset)$ contradicting the hypothesis, and

   $$\gamma^{-1}(\rho_1)(x) = \bigcap\{c|x \in \rho_1(c)\} \subseteq \bigcap\{c|x \in \rho_2(c)\} = \gamma^{-1}(\rho_2)(x),$$

   that is $\gamma^{-1}(\rho_1)(x) \sqsubseteq_{H\delta} \gamma^{-1}(\rho_2)$.

□

Next we show that the isomorphisms of Theorem 5.7 hold if we substitute $\mathcal{S}\delta(Y)$ for $\mathcal{S}(Y)$ and replace the subset order by the deadlock order for liveness predicate transformers. Since $\mathcal{S}(Y)$ and $\mathcal{S}\delta(Y)$ differ only in their order, we again need only to prove the following two lemmas:

Lemma 6.5 *Let $m_1, m_2 \in X \to \mathcal{S}\delta(Y)$. If $m_1 \sqsubseteq_{S\delta} m_2$ then $\omega(m_1) \sqsubseteq_{LD} \omega(m_2)$.*

Proof: Let $m_1 \sqsubseteq_{S\delta} m_2$. If $x \in \omega(m_1)(\emptyset)$ then $m_1(x) \subseteq \emptyset$, that is $m_1(x) = \emptyset$. But then also $m_2(x) = \emptyset$ and hence $x \in \omega(m_2)(\emptyset)$. If $x \in \omega(m_1)(o)$ for some $Y \neq o \in \mathbf{O}_{Sc}(Y)$, but $x \notin \omega(m_1)(\emptyset)$ then $m_1(x) \neq \emptyset$ and $m_1(x) \subseteq o \neq Y$. Since $m_1 \sqsubseteq_{S\delta} m_2$ the only possible case is $m_2(x) \neq \emptyset$ and $m_2(x) \subseteq m_1(x) \subseteq o$. Therefore $x \in \omega(m_2)(o) \setminus \omega(m_2)(\emptyset)$. □

Lemma 6.6 *Let $\pi_1, \pi_2 \in \mathbf{O}_{Sc}(Y) \to_M \mathbf{O}(X)$. If $\pi_1 \sqsubseteq_{LD} \pi_2$ then $\omega^{-1}(\pi_1) \sqsubseteq_{S\delta} \omega^{-1}(\pi_2)$.*

Proof: Let $\pi_1 \sqsubseteq_{LD} \pi_2$ and $x \in X$. If $\omega^{-1}(\pi_1)(x) = Y$ then clearly $\omega^{-1}(\pi_1)(x) \sqsubseteq_{S\delta} \omega^{-1}(\pi_2)(x)$. If $\omega^{-1}(\pi_1)(x) = \emptyset$ then by stability lemma 3.6 we have $x \in \pi_1(\emptyset)$ and since $\pi_1(\emptyset) \subseteq \pi_2(\emptyset)$ we obtain $x \in \pi_2(\emptyset)$. But then again by stability lemma 3.6 $\omega^{-1}(\pi_2)(x) = \emptyset$. Otherwise, $\omega^{-1}(\pi_1)(x) \neq \emptyset$ and $\omega^{-1}(\pi_1)(x) \neq Y$, that is, $x \notin \pi_1(\emptyset)$. In this case, for every $o \in \mathbf{O}_{Sc}(Y)$ if $x \in \pi_1(o)$ then $x \in \pi_2(o)$, because either $o = Y$ and hence clearly $x \in \pi_2(Y)$ (π_2 is top preserving) or $o \neq Y$ and $x \in \pi_1(o) \setminus \pi_1(\emptyset) \subseteq \pi_2(o) \setminus \pi_2(\emptyset)$. Therefore, if $x \notin \pi_1(\emptyset)$ and $\omega^{-1}(\pi_1)(x) \neq Y$, $\{o|x \in \pi_1(o)\} \subseteq \{o|x \in \pi_2(o)\}$ and $x \notin \pi_2(\emptyset)$. Hence

$$\omega^{-1}(\pi_1)(x) = \bigcap\{o|x \in \pi_1(o)\} \supseteq \bigcap\{o|x \in \pi_2(o)\} = \omega^{-1}(\pi_2)(x)$$

and $\omega^{-1}(\pi_2)(x) \neq \emptyset$. Summarizing, we have $\omega^{-1}(\pi_1)(x) \sqsubseteq_{S\delta} \omega^{-1}(\pi_2)(x)$. □

Finally we show that the isomorphisms of Theorem 5.14 hold if we replace $\mathcal{P}(Y)$ by $\mathcal{P}\delta(Y)$ and replace the subset-superset orderings by the deadlock order for pairs of liveness and safety predicate transformers. The following lemma is enough to prove this result since we combine the results of the four lemmas above.

Lemma 6.7 *Let X be an algebraic domain and $A, B \in \mathcal{P}\delta(X)$. Then*

$$A \sqsubseteq_{P\delta} B \Leftrightarrow A\!\uparrow \sqsubseteq_{S\delta} B\!\uparrow \ \wedge\ \overline{A} \sqsubseteq_{H\delta} \overline{B} \Leftrightarrow A\!\uparrow \sqsubseteq_{S} B\!\uparrow \ \wedge\ \overline{A} \sqsubseteq_{H\delta} \overline{B}$$

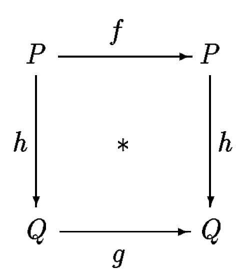
Proof: Trivial. □

7 Fixed Point Transformation Technique

In [BK92] a backtrack operator $\diamond_S$ from $(X \to \mathcal{S}(X))^2$ to $X \to \mathcal{S}(X)$ is defined (where X and Y are flat dcpo's) and it is not monotone with respect to the lifting of $\sqsubseteq_S$ to the function space. Also the sequential composition operator $;_{S\delta}$ from $(X \to \mathcal{S}\delta(X))^2$ to $X \to \mathcal{S}\delta(X)$ is not monotone with respect to the lifting of $\sqsubseteq_{S\delta}$ to the function space. In order to be able to calculate least fixed points of equations in which we use these non monotonic functions, a transfer lemma is used in [BK92]. This lemma requires the existence of a function between $\mathcal{P}\delta(X)$ and $\mathcal{S}(X)$ (or $\mathcal{S}\delta(X)$). In this section we show that also for non-flat ω-algebraic dcpo's X there exists a function between the Plotkin power domain $\mathcal{P}\delta_{co}(X)$ and the Smyth power domain $\mathcal{S}_{co}(X)$ (or $\mathcal{S}\delta_{co}(X)$) which satisfies the requirements of the transfer lemma.

Let P be a dcpo. For $f : P \to P$, we denote by $\mu.f$ the *least fixed point* of f, that is, $f(\mu.f) = \mu.f$ and for every other $x \in P$ with $f(x) = x$ we have $\mu.f \sqsubseteq x$. For a monotone function $f : P \to_m P$ the least fixed point of f always exists and it can be calculated by iteration: there exists an ordinal λ such that $\mu.f = f^\lambda$, where the α-iteration of f is defined by $f^\alpha = f(\bigsqcup_{k<\alpha} f^k)$ for every ordinal α [HP72]. If f is also continuous then $\lambda \leq \omega_0$. Via a fixed point transformation technique we can show that in certain cases also non-monotonic functions have least fixed points that can be calculated by iteration:

Theorem 7.1 *Let P be a dcpo and Q be a partially ordered set, $f : P \to_m P$ be a monotone function, $h : P \to_c Q$ be an onto and continuous function and $g : Q \to Q$ be a (possibly non-monotone) function such that the following diagram commutes:*



Then for every ordinal α the α-iteration from the bottom element g^α exists. Moreover, if for each $y \in Q$ the partially ordered set $h^{-1}(y) \subseteq P$ is finite or contains either the bottom or the top element then the smallest fixed point $\mu.g$ exists and $\mu.g = h(\mu.f)$.

The function $g : Q \to Q$ is a said a *representation as quotient* of $f : P \to_m P$ via $h : P \to_c Q$. To apply this theorem we need to establish relationships between the different power domains. The next two lemmas paves the way for these relationships.

Let X be an algebraic dcpo and $A \subseteq X$. Define the set of *minimal elements* of A by

$$min(A) = \{x | x \in A \ \wedge\ (\forall y \in A : x \sqsubseteq_X y \Rightarrow x = y)\}.$$

Minimal elements are used in [Kni93b] and [Kni93a] for the construction of the power domains $\mathcal{S}^+_{co}(X)$ and $\mathcal{P}^+_{co}(X)$ from an ω-algebraic dcpo's X. Define the set of *maximal elements* of A by

$$max(A) = \{x \in \overline{A} | x \in B_X \Rightarrow (\forall b \in A \cap B_X : b \sqsubseteq_X x \Rightarrow b = x)\}.$$

It is not always true that $max(A) \subseteq A$.

Lemma 7.2 *Let X be an algebraic dcpo and $A \subseteq X$. Then $min(A)\!\uparrow = A\!\uparrow$ and for every other $B \subseteq X$ such that $B\!\uparrow = A\!\uparrow$ then $min(A) \subseteq B$. Also, $\overline{max(A)} = \overline{A}$ and for every *-closed $B \subseteq X$ such that $\overline{B} = \overline{A}$ then $max(A) \subseteq B$.*

We use the lemma in the following:

Lemma 7.3 *Let X be an algebraic dcpo and $A \subseteq X$. Then*

1. $\overline{(\overline{A}\!\uparrow \cap \overline{A})} = \overline{A}$ *and for every other* $B \in \mathcal{P}(X)$ *such that* $\overline{B} = \overline{A}$ *we have* $(\overline{A}\!\uparrow \cap \overline{A}) \sqsubseteq_P B$.
2. $\overline{(max(A)\!\uparrow \cap \overline{A})} = \overline{A}$ *and for every other* $B \in \mathcal{P}(X)$ *such that* $\overline{B} = \overline{A}$ *we have* $B \sqsubseteq_P (max(A)\!\uparrow \cap \overline{A})$.
3. $(A\!\uparrow \cap \overline{min(A)})\!\uparrow = A\!\uparrow$ *and for every other* $B \in \mathcal{P}(X)$ *such that* $B\!\uparrow = A\!\uparrow$ *we have* $(A\!\uparrow \cap \overline{min(A)}) \sqsubseteq_P B$.
4. $(A\!\uparrow \cap \overline{A})\!\uparrow = A\!\uparrow$ *and for every other* $B \in \mathcal{P}(X)$ *such that* $B\!\uparrow = A\!\uparrow$ *we have* $B \sqsubseteq_P (A\!\uparrow \cap \overline{A})$.

Proof:

1. By Lemma 4.1.1 we have $\overline{(\overline{A}\!\uparrow \cap \overline{A})} = \overline{A}$. Moreover $(\overline{A}\!\uparrow \cap \overline{A}) \in \mathcal{P}(X)$ because the intersection of an upper-closed and a Scott closed set is *-closed. Let now $B \in \mathcal{P}(X)$ be such that $\overline{B} = \overline{A}$. Then $B \subseteq \overline{B} = \overline{A}$ implies $B\!\uparrow \subseteq \overline{A}\!\uparrow$. Therefore $B = (B\!\uparrow \cap \overline{B}) \subseteq (\overline{A}\!\uparrow \cap \overline{A})$, implies $B\!\uparrow \subseteq (\overline{A}\!\uparrow \cap \overline{A})\!\uparrow$. But $\overline{B} = \overline{A} = \overline{(\overline{A}\!\uparrow \cap \overline{A})}$, hence $(\overline{A}\!\uparrow \cap \overline{A}) \sqsubseteq_P B$.
2. Since $\overline{max(A)} = \overline{A}$, by Lemma 4.1.1 we have $\overline{(max(A)\!\uparrow \cap \overline{A})} = \overline{A}$. Let now $B \in \mathcal{P}(X)$ be such that $\overline{B} = \overline{A}$. Then by Lemma 7.2 $max(A) \subseteq B$ and hence $max(A)\!\uparrow \subseteq B\!\uparrow$. But by Lemma 4.1.3 $(max(A)\!\uparrow \cap \overline{A})\!\uparrow = max(A)\!\uparrow \subseteq B\!\uparrow$ and $\overline{(max(A)\!\uparrow \cap \overline{A})} = \overline{A} = \overline{B}$, therefore $B \sqsubseteq_P (max(A)\!\uparrow \cap \overline{A})$.
3. Since $min(A)\!\uparrow = A\!\uparrow$, by Lemma 4.1.2 we have $(A\!\uparrow \cap \overline{min(A)})\!\uparrow = A\!\uparrow$. Let now $B \in \mathcal{P}(X)$ be another set such that $B\!\uparrow = A\!\uparrow$. Then $min(A) \subseteq B$ by Lemma 7.2 and hence $\overline{min(A)} \subseteq \overline{B}$. But by Lemma 4.1.3 $\overline{(A\!\uparrow \cap \overline{min(A)})} = \overline{min(A)} \subseteq \overline{B}$ and $(A\!\uparrow \cap \overline{min(A)})\!\uparrow = A\!\uparrow = B\!\uparrow$, therefore, $(A\!\uparrow \cap \overline{min(A)}) \sqsubseteq_P B$.
4. By Lemma 4.1.2 we have $(A\!\uparrow \cap \overline{A\!\uparrow})\!\uparrow = A\!\uparrow$. Let now $B \in \mathcal{P}(X)$ be another set such that $B\!\uparrow = A\!\uparrow$. Then $B \subseteq B\!\uparrow = A\!\uparrow$ implies $\overline{B} \subseteq \overline{A\!\uparrow}$. Therefore $B = (B\!\uparrow \cap \overline{B}) \subseteq (A\!\uparrow \cap \overline{A\!\uparrow})$, implies $\overline{B} \subseteq \overline{(A\!\uparrow \cap \overline{A\!\uparrow})}$. But $B\!\uparrow = A\!\uparrow = (A\!\uparrow \cap \overline{A\!\uparrow})\!\uparrow$, hence $B \sqsubseteq_P (A\!\uparrow \cap \overline{A\!\uparrow})$.

□

If $A \subseteq X$ is Scott compact, then also $A\!\uparrow$ is Scott compact and hence also $(A\!\uparrow \cap \overline{min(A)})$ and $(A\!\uparrow \cap \overline{A\!\uparrow})$ are Scott compact (the intersection of a Scott compact set with a Scott closed set is Scott compact). Also $(\overline{A}\!\uparrow \cap \overline{A})$ is Scott compact for every $A \subseteq X$ because $\bot$ is an element of it. We give now the relationships:

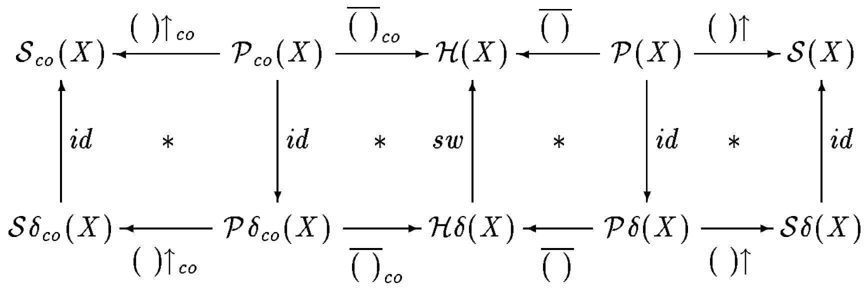


Figure 1: Relationships among the power domains

Theorem 7.4 *Let X be an algebraic dcpo. Then*

- *the function $\overline{(\cdot)} : \mathcal{P}(X) \to \mathcal{H}(X)$ defined by $\overline{(A)} = \overline{A}$ for every $A \in \mathcal{P}(X)$ is onto, monotone, and for every $B \in \mathcal{H}(X)$ the inverse $\overline{(B)}^{-1} \subseteq \mathcal{P}(X)$ has a minimal and a maximal element (with respect to $\sqsubseteq_P$) given by $(B\!\uparrow \cap B)$ and $(max(B)\!\uparrow \cap B)$, respectively.*
- *the function $(\cdot)\!\uparrow : \mathcal{P}(X) \to \mathcal{S}(X)$ defined by $(A)\!\uparrow = A\!\uparrow$ for every $A \in \mathcal{P}(X)$ is onto, monotone, and for every $B \in \mathcal{S}(X)$ the inverse $\overline{(S)}^{-1} \subseteq \mathcal{P}(X)$ has a minimal and a maximal element (with respect to $\sqsubseteq_P$) given by $(B \cap \overline{min(B)})$ and by $(B \cap \overline{B})$, respectively.*
- *the function $\overline{(\cdot)}_{co} : \mathcal{P}_{co}(X) \to \mathcal{H}(X)$ defined by $\overline{(A)} = \overline{A}$ for every $A \in \mathcal{P}_{co}(X)$ is onto, monotone, and for every $B \in \mathcal{H}(X)$ the inverse $\overline{(B)}^{-1} \subseteq \mathcal{P}_{co}(X)$ has a minimal element (with respect to $\sqsubseteq_P$) given by $(B\!\uparrow \cap B)$.*
- *the function $(\cdot)\!\uparrow_{co} : \mathcal{P}_{co}(X) \to \mathcal{S}_{co}(X)$ defined by $(A)\!\uparrow_{co} = A\!\uparrow$ for every $A \in \mathcal{P}_{co}(X)$ is onto, monotone, and for every $B \in \mathcal{S}_{co}(X)$ the inverse $\overline{(S)}^{-1} \subseteq \mathcal{P}_{co}(X)$ has a minimal and a maximal element (with respect to $\sqsubseteq_P$) given by $(B \cap \overline{min(B)})$ and by $(B \cap \overline{B})$, respectively.*

Proof: Follows from Lemma 7.3. □

The theorem also holds if we substitute the power domains $\mathcal{H}\delta(X), \mathcal{S}\delta(X)$ and $\mathcal{P}\delta(X)$ for $\mathcal{H}(X), \mathcal{S}(X)$ and $\mathcal{P}(X)$. Moreover, if X is an ω-algebraic dcpo, then $\overline{(\cdot)}_{co}$ and $(\cdot)\!\uparrow_{co} : \mathcal{P}_{co}(X) \to \mathcal{S}_{co}(X)$ are continuous function (see [Smy83] and also [Plo81]). This means that given an ω-algebraic dcpo and a function $f : \mathcal{P}_{co}(X) \to_m \mathcal{P}_{co}(X)$, every representation as quotient of f via $(\cdot)\!\uparrow_{co}$ or $\overline{(\cdot)}_{co}$ has least fixed point that can be calculated by iteration by Theorem 7.1.

Define $sw : \mathcal{H}\delta(X) \to \mathcal{H}(X)$ by

$$sw(A) = \begin{cases} \emptyset & \text{if } A = \{\bot\} \\ \{\bot\} & \text{if } A = \emptyset \\ A & \text{otherwise.} \end{cases}$$

It is clearly continuous. The diagram in Figure 1 summarizes the situation.

8 Conclusions and Future Work

We have proposed a formal definition of safety and liveness predicates and of predicate transformers following the line of [Smy83, Kwi91]. Furthermore we have given generalizations of the standard definitions of power domains and of state transformers, which give us a complete series of isomorphisms between predicate and state transformers (including the Plotkin state transformers).

Future work includes: a generalization of the results to arbitrary topological spaces and applications of predicate transformers to non-flat domains for concurrency and communication and the design of an associated logic.

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