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Line transects, covariance functions and set approximation

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Abstract

We introduce a new geometric transform of sets, based upon intersections of the set with lines. Many basic integral-geometric formulae used in stereology can be expressed as identities concerning this transform. It is also closely related to the glance function used by Waksman (1987) to construct a metric for subsets of the plane. We modify that construction to obtain a new metric η for 'regular' sets $S \subset \mathbb{R}^d$. It is shown that η is topologically equivalent to the Hausdorff metric on *convex* sets. We also prove that the covariance function is a continuous functional with respect to this new metric on classes of uniformly bounded regular sets.

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Introduction.

This paper studies the determination of a set $A \subset \mathbb{R}^d$ from information on one-dimensional linear transects $A \cap l$. Three separate issues are discussed:

(a) characterisation: whether a set $A \subset \mathbb{R}^d$ is completely determined by the values of an associated transform (such as the Radon transform or the covariance function);

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(b) stereology: whether geometrical parameters of $A \subset \mathbb{R}^d$ can be statistically estimated from randomly-sampled values of the transforms;
(c) approximation: whether good deterministic or stochastic approximation of transforms ensures closeness of the corresponding sets.

Although problems of this kind have received much attention (see the references mentioned below) it is common for the issues (a)–(c) to be considered separately. In this paper we introduce a construct, the *linear scan transform*, that is serviceable in all three contexts.

Let us briefly sketch the relevant history. The interest in characterisation problems for the set covariance function has recently increased. Nagel (1989) showed that a 'generic' convex plane polygon is uniquely determined by its set covariance. Lešanovský and Rataj (1990) constructed an example of two distinct *non-convex* sets with the same covariance function. For a more restricted class of generic polygons, excluding those of the abovementioned counterexample, there is a reconstruction procedure due to Schmitt (1992).

The related class of problems concerning characterisation of a set from information on linear transects, like the Radon transform, has a much longer history. See e.g. Ambartzumian (1983) and Waksman (1985) and the references therein. For the characterisation of polygons by the distribution of the Radon transform, see Waksman (1985). The famous example of Mallows and Clark (1971) of two non-congruent convex polygons with the same chord length distribution is not a counterexample for the covariance function (see Nagel (1989)).

In the stereological context (b) above, there are several well known integral geometric identities connecting the chord length distribution of a convex set with its interpoint distance distribution and its set covariance. Moreover an identity of Crofton (1885) concerning the moments of chord lengths was generalised to non-convex sets by Miles (1979) and Jensen and Gundersen (1985) in the construction of an estimator for the total volume of particles. See also Goodey and Weil (1991).

Suppose the intersection $l \cap A$ of a line l with a set $A \subset \mathbb{R}^d$ is a finite union of compact intervals, with ordered endpoints $x_1, x_2, \ldots x_{2n}$. Miles (1983) defined the k-linc as

$$[\sigma(l \cap A)^k] = \sum_{i=1}^{2n} \sum_{j=i+1}^{2n} (-1)^{i+j+1}(x_j - x_i)^k, \quad \text{for } k \geq 1.$$

Moreover Waksman (1987) introduced the glance function of A. In our notation its definition can be written as

$$H_{l \cap A}(t) = \sum_{i=1}^{2n} \sum_{j=i+1}^{2n} (-1)^{i+j+1} 1\{x_j - x_i \leq t\}, \quad \text{for } t \geq 0.$$

Waksman used this function to construct a metric on a class of 'regular' subsets of the plane with applications to the approximation problem (c).

The new geometric transform introduced in this paper is also associated with linear transects of a set $A \subset \mathbb{R}^d$. The transform is a minor modification of Waksman's glance function (1987) but turns out to be extremely natural, arising as minus the derivative of the one-dimensional set covariance of a linear transect. It turns out that all the abovementioned integral geometric identities can be rewritten in terms of the linear scan.

The first section of the paper provides necessary preliminaries such as the definition of the covariance function and the specification of 'regular' sets. In section 2 the linear scan is introduced as described above. Section 3 is devoted to the relations of the linear scan transform with integral geometry and stereology. We pursue the approximation problem in section 4 by constructing a metric on 'regular' subsets of $\mathbb{R}^d$ defined as the L^1-distance between their linear scan transforms. For the case of *convex* sets we show η is the L^1-distance between Radon transforms and is topologically equivalent to the Hausdorff metric. An example of different convergence behaviour is given in the non-convex case.

In the final section we gather analytic properties of the set covariance function of $A \subset \mathbb{R}^d$, showing in particular that it is an η-continuous functional on classes of uniformly bounded regular sets A. We also obtain an elementary proof of Lipschitz continuity of d-dimensional volume with respect to η.

1. Preliminaries.

1.1 Covariance functions.

Let $S \subset \mathbb{R}^d$ be a Borel measurable set with finite Lebesgue measure, $\lambda(S) < \infty$.

1.1 Definition. *The covariance function of S is the function $C_S : \mathbb{R}^d \to \mathbb{R}_+$ defined by*

$$C_S(y) := \lambda(S \cap T_y S), \quad y \in \mathbb{R}^d.$$

where we write

$$T_y S = \{y + s : s \in S\}$$

for the translation of S by a vector $y \in \mathbb{R}^d$.

1.2 Remark. The covariance function is measurable as a function of y: the function $g(u, w) = 1_S(u) 1_S(u + w)$ is clearly measurable and integrable on $\mathbb{R}^{2d}$ so Fubini's theorem guarantees measurability of $C_S(\cdot)$.

The following properties are immediate consequences of the definition:

1.3 Lemma.
(a) $C_S(0) = \lambda(S)$.
(b) C_S is compactly supported: $C_S(y) = 0$, for all $y \geq diam(S)$.
(c) C_S is symmetric: $C_S(y) = C_S(-y)$, for all $y \in \mathbb{R}^d$.

1.4 Lemma (Borel's overlap formula). *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a measurable function. Denote by $\|\cdot\|$ Euclidean length. Then*

$$\int_S \int_S f(\|u - v\|)\, du\, dv = \int_{\mathbb{R}^d} f(\|w\|)\, C_S(w)\, dw, \tag{1.1}$$

in the sense that whenever one side of (1.1) exists, so does the other in which case they are equal. In particular

$$\int_{\mathbb{R}^d} C_S(y)\,d\lambda(y) = \lambda(S)^2. \tag{1.2}$$

Proof.

$$\begin{aligned}\int_S\int_S f(\|u-v\|)\,du\,dv &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} 1_S(u)\,1_S(v)\,f(\|u-v\|)\,du\,dv\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} 1_S(u)\,1_S(u+w)\,f(\|w\|)\,du\,dw,\end{aligned}$$

where the last line results from transforming (u,v) to $(u,w)=(u,v-u)$. Now

$$1_S(u+w) = 1_{T_{-w}S}(u)$$

so the inner integral above is

$$\begin{aligned}f(\|w\|)\int 1_S(u)\,1_{T_{-w}S}(u)\,du &= f(\|w\|)\,\lambda(S\cap T_{-w}S)\\ &= f(\|w\|)\,C_S(-w)\\ &= f(\|w\|)\,C_S(w),\end{aligned}$$

yielding the main result. The second formula is the special case $f\equiv 1$. □

Next we give the relation between the distribution function of the distance between two independent uniformly distributed points in a set and the covariance function of that set. This is a direct consequence of the Borel formula (1.1).

1.5 Corollary. *Let X,Y be two independent uniformly distributed points in S. Then*

$$\mathbb{P}\{\|X-Y\|\le\rho\} = \frac{1}{\lambda(S)^2}\int_{B(0,\rho)} C_S(y)\,dy, \quad \textit{for } 0\le\rho\le 2\,diam(S).$$

Denote by $\mathcal{L}$ the class of all one-dimensional lines in $\mathbb{R}^d$, and by μ the normalised invariant measure on $\mathcal{L}$ (see Santaló, 1976, pp. 28, 200). The following result is also well-known:

1.6 Lemma. *For measurable $f:\mathbb{R}_+\to\mathbb{R}$*

$$\int_S\int_S f(\|u-v\|)\,du\,dv = \int_{\mathcal{L}}\int_{l\cap S}\int_{l\cap S} |s-t|^{d-1} f(|s-t|)\,ds\,dt\,d\mu(l), \tag{1.3}$$

in the same sense as Lemma 1.4.

This is a consequence of the Blaschke-Petkantschin formula (see Santaló, 1976, eqn (4.2) p. 46 and eqn (12.23) p. 201).

1.2 Regular sets.

From now on we need to consider the more restricted class $\mathcal{V}$ of inner regular, briefly called regular, subsets of $I\!R^d$, defined below.

Let $\mathcal{D}$ denote the collection of non-empty, open, convex, relatively compact sets. Then the class $\mathcal{C}$ of closures of sets in $\mathcal{D}$

$$\mathcal{C} = \{\bar{D} : D \in \mathcal{D}\}$$

is the class of convex bodies (i.e. convex, compact with non-empty interior).
Let

$$\mathcal{E} = \mathcal{D} \cup \mathcal{C}$$

and let $\mathcal{W}$ be the algebra generated by $\mathcal{E}$, i.e. by finite intersections, unions and differences of subsets of $\mathcal{E}$.

1.7 Definition.

$$\mathcal{V} = \{V \in \mathcal{W} : \; Cl(Int(V)) = V\}.$$

An element of $\mathcal{V}$ will be called a (inner) regular set.

Observe that the elements of $\mathcal{V}$ are closed.

1.8 Definition. *For a regular set S and $l \in \mathcal{L}$, write $n(l \cap S)$ for the number of components of $l \cap S$, and $\sigma(l \cap S)$ for the length (1-dimensional Lebesgue measure) of $l \cap S$.*

The following lemma is a version of a standard result in integral geometry (e.g. Santaló, 1976, pp. 29, 31, 234 and Federer, 1969, pp. 173, 258, 294). Here $\mathcal{H}^{d-1}$ denotes $d-1$ dimensional Hausdorff measure ("surface area", or in $I\!R^2$, "length") and $\kappa_d = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of the unit ball in $I\!R^d$. The proof is given in Appendix A.

1.9 Lemma. *For $S \in \mathcal{V}$*

(a) *$\sigma(l \cap S)$ is a measurable function of $l \in \mathcal{L}$, and*

$$\int_{\mathcal{L}} \sigma(l \cap S) \, d\mu(l) = d\kappa_d \lambda(S). \tag{1.4}$$

(b) *$n(l \cap S)$ is a measurable function of $l \in \mathcal{L}$ and*

$$\int_{\mathcal{L}} n(l \cap S) \, d\mu(l) = \frac{\kappa_{d-1}}{2} \mathcal{H}^{d-1}(\partial S). \tag{1.5}$$

In particular, for μ-almost all lines l, the transect $l \cap S$ is a finite union of bounded line segments.

2. The linear scan transform.

2.1 Definition. *The linear scan transform of a regular set $S \in \mathcal{V}$ is defined for given $l \in \mathcal{L}$ as*

$$G_{l\cap S}(t) = -C'_{l\cap S}(|t|), \quad t \in \mathbb{R}.$$

That $G_{l\cap S}$ is well-defined for almost all l will be shown below.

The following properties are immediate.

2.2 Lemma. *(a) If $l \cap S = \emptyset$, then $C_{l\cap S} \equiv G_{l\cap S} \equiv 0$.*

(b) For a compact convex set K the intersection $l \cap K$ is either empty or a compact interval of length $\sigma(l \cap K) \geq 0$ in which case

$$C_{l\cap K}(t) = (\sigma(l \cap K) - t)_+,$$

$$G_{l\cap K}(t) = 1\{\sigma(l \cap K) > t\} = \begin{cases} 1 & \text{if } \sigma(l \cap K) > t \\ 0 & \text{otherwise} \end{cases}$$

(c) $C_{l\cap S}(t) = G_{l\cap S}(t) = 0$ for all $t \geq diam(S)$.

According to Lemma 1.9 we may assume $l \cap S$ is a finite union of line segments, and compute $C_{l\cap S}, G_{l\cap S}$.

2.3 Lemma. *Represent $l \cap S$ isometrically as a subset of $\mathbb{R}$*

$$J = \bigcup_{i=1}^{n} [x_{2i+1}, x_{2i+2}] \tag{2.1}$$

where $x_1 < x_2 < \cdots < x_{2n} \in \mathbb{R}$ are the coordinates of the endpoints (with respect to an arbitrary origin on l). Then for $t \geq 0$

$$C_{l\cap S}(t) = \sum_{k=1}^{2n} \sum_{i=1}^{2n} (-1)^{i+k+1} (x_k - x_i - t)_+ \tag{2.2}$$

$$G_{l\cap S}(t) = \sum_{k=1}^{2n} \sum_{i=1}^{2n} (-1)^{i+k+1} 1\{x_k - x_i > t\}. \tag{2.3}$$

This representation is independent of the choice of the origin on l.

The fact that it is possible by Lemma 1.9 to represent $l \cap S$ for almost all lines as in (2.1) together with (2.3), shows that the linear scan transform is well-defined.

Proof. First note that for J as in (2.1) and for any Lebesgue integrable $f : \mathbb{R}_+ \to \mathbb{R}$, trivially

$$\int_J f(t)\, dt = \sum_{k=1}^{2n} (-1)^k F(x_k), \tag{2.4}$$

where F is any primitive of f. Now consider

$$C_J(t) = \text{length}(J \cap (J \oplus t))$$

(2.5)
$$= \int_J 1_{J\oplus t}(u)\, du$$

Defining $f(u) = 1_{J\oplus t}(u) = 1_J(u-t)$, the primitive is

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(u)\, du \\ &= \int_{-\infty}^{x} 1_J(u-t)\, du \\ &= \int_{-\infty}^{x-t} 1_J(s)\, ds \end{aligned}$$

(2.6)
$$= \int_J 1_{(-\infty,x-t]}(s)\, ds.$$

Thus we need to apply (2.4) to the function

$$f_1(s) = 1_{(-\infty,x-t]}(s).$$

This has primitive

$$F_1(y) = \int_{-\infty}^{y} f_1(s)\, ds = -(x-t-y)_+$$

So (2.6) becomes

$$F(x) = \sum_{i=1}^{2n} (-1)^i F_2(x_i) = \sum_{i=1}^{2n} (-1)^{i+1}(x - x_i - t)_+.$$

Substituting into (2.5) gives

$$\begin{aligned} C_J(t) &= \sum_{k=1}^{2n} (-1)^k F(x_k) \\ &= \sum_{k=1}^{2n} \sum_{i=1}^{2n} (-1)^{i+k+1}(x_k - x_i - t)_+. \end{aligned}$$

Independence of the choice of the origin on l follows immediately from these representations since they only depend on the *length* of the intervals in $l \cap S$. This proves the Lemma. □

It is an interesting exercise to check that the alternating sum expression for $C_{l\cap S}(0)$ collapses to $\sigma(l \cap S)$. Setting $t = 0$ in (2.2) gives

$$C_{l\cap S}(0) = \sum_i \sum_{k>i} (-1)^{i+k+1}(x_k - x_i);$$

the inner sum collapses to $(x_{i+1} - x_i)$ for left endpoints (i odd) and 0 for right endpoints (i even) so the total is $\sum(x_{2m} - x_{2m-1}) = \sigma(l \cap S)$.

In the sequel we encounter $G_{l\cap S}(t)$ mostly for $t > 0$. In that case we can write (2.3) as

$$G_{l\cap S}(t) = \sum_{i=1}^{2n} \sum_{j=i+1}^{2n} (-1)^{i+j+1} 1\{x_j - x_i > t\}.$$

3. Identities concerning the linear scan transform.

3.1 Basic relation.

First we establish a link between the covariance function of a set S in $\mathbb{R}^d$ and the linear scan transform.

An oriented line $l \in \mathcal{L}$ is determined by its coordinates $(p, \theta), p \in \mathbb{R}$, $\theta \in [0, 2\pi)$, where p is the perpendicular distance from an origin to l and θ the angle formed by the line and a fixed axis (say the x-axis).

Hence the invariant measure can be written as

$$(*) \qquad d\mu = dp \wedge d\theta.$$

For $x \in \mathbb{R}^d$, let $\mathcal{L}_x$ denote the set of all oriented lines in $\mathbb{R}^d$ parallel to the vector x and let μ_x be the measure $d\mu_x = dp$ on $\mathcal{L}_x$.

3.1 Proposition. *For a set $S \in \mathcal{V}$*

$$C_S(y) = \int_{\mathcal{L}_y} \int_{\|y\|}^{\infty} G_{l\cap S}(t)\, dt\, d\mu_y(l), \quad \textit{for all } y \in \mathbb{R}^d.$$

Proof. By definition

$$\begin{aligned} C_S(y) &= \int_{\mathbb{R}^d} 1_S(x) 1_{S-y}(x)\, dx \\ (3.1) \qquad &= \int_{\mathbb{R}^d} 1_S(x) 1_{S-y}(x)\, dx \end{aligned}$$

Consider the transformation

$$x = (x_1, x_2) \to (p, u)$$

transforming a point $x \in \mathbb{R}^d$ to the line through x in a given direction y_θ, parametrised by the perpendicular distance p from the origin to the line and by the coordinate of x on the line w.r.t. the foot of this perpendicular.

Using polar coordinates and the representation $(*)$, we see that (3.1) becomes

$$\int_{\mathcal{L}_y} \int_{\mathbb{R}} 1_{l\cap S}(u) 1_{l\cap S}(u + \|y\|)\, du\, d\mu_y(l) = \int_{\mathcal{L}_y} C_{l\cap S}(\|y\|)\, d\mu_y(l).$$

This proves the Proposition because by definition

$$C_{l\cap S}(\|y\|) = \int_{\|y\|}^{\infty} G_{l\cap S}(t)\,dt.$$

□

As a corollary to the previous proposition, we can write the expression obtained for the interpoint distance in Corollary 1.5 in terms of G.

3.2 Stereological relations.

In this section we first establish a connection with $n(l \cap S)$ and with the so-called k-linc (for "k-th order line section of non-convex domain") introduced by Miles(1983).

3.2 Definition. *(Miles) Let $l \in \mathcal{L}$. The k-linc of a regular set S in $\mathbb{R}^d$, is*

$$[\sigma(l \cap S)^k] = \sum_{i=1}^{2n(l\cap S)} \sum_{j=i+1}^{2n(l\cap S)} (-1)^{i+j+1}(x_j - x_i)^k,$$

where $x_1, x_2, \ldots$ are the ordered endpoints of intercepted intervals as before.

The k-linc is of stereological importance, especially when $d = 2$ and $k = 3$ and also when $d = 3$ and $k = 4$ (see Miles(loc.cit.) and Jensen and Gundersen(1985)).

3.3 Lemma. *Let $l \in \mathcal{L}$. For a regular set S we have for μ-almost all l*
(a) $n(l \cap S) = G_{l\cap S}(0)$.
(b) $[\sigma(l \cap S)^k] = k \int_0^\infty t^{k-1} G_{l\cap S}(t)\,dt$ *for* $k \geq 1$.
(c) $[\sigma(l \cap S)^k] = k(k-1) \int_0^\infty t^{k-2} C_{l\cap S}(t)\,dt$ *for* $k \geq 2$.

Proof. Let $\mathcal{N}$ be the set of lines on which $n(l \cap S)$, $G_{l\cap S}$ and $[\sigma(l \cap S)^k]$ are infinte and *not* well-defined. Throughout the proof we only consider lines in $\mathcal{L} \setminus \mathcal{N}$.

To prove (a), set $t = 0$ in (2.3). This gives

$$G_{l\cap S}(0) = \sum_i \sum_k (-1)^{i+k+1} 1\{x_k > x_i\}$$

and the inner sum collapses to 0 for right endpoints (i even) and 1 for left endpoints (i odd) so that the total is $n(l \cap S)$.

For the second statement observe that by Lemma 2.3

$$G_{l\cap S}(t) = \sum_{i=1}^{2n(l\cap S)} \sum_{j=i+1}^{2n(l\cap S)} (-1)^{i+j+1} 1\{x_j - x_i > t\}.$$

Hence the right-hand side of (b) is

$$\begin{aligned}
&\int_{\mathbb{R}} kt^{k-1} \sum_{i=1}^{2n(l\cap S)} \sum_{j=i+1}^{2n(l\cap S)} (-1)^{i+j+1} 1\{x_j - x_i > t\}\, dt \\
&= \sum_{i=1}^{2n(l\cap S)} \sum_{j=i+1}^{2n(l\cap S)} (-1)^{i+j+1} \int_0^{x_j - x_i} kt^{k-1}\, dt \\
&= \sum_{i=1}^{2n(l\cap S)} \sum_{j=i+1}^{2n(l\cap S)} (-1)^{i+j+1} (x_j - x_i)^k \\
&= [\sigma(l\cap S)^k]
\end{aligned}$$

Finally, the third claim follows from the previous one by integration by parts. □

3.4 Examples.

3.4.1. If S is convex, as when $n(l\cap S) = 1$

$$\begin{aligned}
[\sigma(l\cap S)^k] &= \int_0^\infty kt^{k-1} 1[0, \sigma(l\cap S)]\, dt \\
&= \int_0^{\sigma(l\cap S)} kt^{k-1}\, dt \\
&= \sigma(l\cap S)^k
\end{aligned}$$

3.4.2. If $k = 1$,

$$[\sigma(l\cap S)^1] = \int_0^\infty G_{l\cap S}(t)\, dt = C_{l\cap S}(0) - C_{l\cap S}(\infty) = C_{l\cap S}(0) = \sigma(l\cap S),$$

by part (a) of Lemma 1.3.

3.4.3. From Example 3.4.2 we obtain

$$\sigma(l\cap S) = \int_0^\infty G_{l\cap S}(t)\, dt$$

for almost all l. Consequently

$$\int_{\mathcal{L}} \int_0^\infty G_{l\cap S}(t)\, dt d\mu(l) = \int_{\mathcal{L}} \sigma(l\cap S) d\mu(l) = d\kappa_d \lambda(S),$$

see Lemma 1.9.

3.5 Proposition. *For integrable $f : \mathbb{R}_+ \to \mathbb{R}$ and $S \in \mathcal{V}$*

$$\begin{aligned}
\int_S \int_S f(\|u - v\|)\, du\, dv &= \int_{\mathcal{L}} \int_{\mathbb{R}} |t|^{d-1} f(|t|) C_{l\cap S}(t)\, dt\, d\mu(l) \\
&= 2 \int_{\mathcal{L}} \int_0^\infty t^{d-1} f(t) C_{l\cap S}(t)\, dt\, d\mu(l) \\
&= 2 \int_{\mathcal{L}} \int_0^\infty F_{d-1}(t)\, G_{l\cap S}(t)\, dt\, d\mu(l) \qquad (3.2)
\end{aligned}$$

where $F_{d-1}(t) = \int_0^t s^{d-1} f(s)\, ds$.

Proof. By Lemma 1.6

$$\int_S \int_S f(\|u-v\|)\, du\, dv = \int_{\mathcal{L}} \int_{l\cap S} \int_{l\cap S} |s-t|^{d-1} f(|s-t|)\, dsdt\, d\mu(l).$$

Now we apply Borel's overlap formula (1.1) to the one-dimensional set $l \cap S$ and the function
$|s-t|^{d-1} f(|s-t|)$:

$$\int_{\mathcal{L}} \int_{l\cap S} \int_{l\cap S} |s-t|^{d-1} f(|s-t|)\, dsdt\, d\mu(l) = \int_{\mathcal{L}} \int_{\mathbb{R}} |w|^{d-1} f(|w|) C_{l\cap S}(w)\, dw.$$

By symmetry of the covariance function the latter integral is

$$2\int_0^\infty w^{d-1} f(w) C_{l\cap S}(w)\, dw.$$

Let F_{d-1} be as defined in the statement of the Proposition. Integration by parts yields

$$\begin{aligned}\int_0^\infty w^{d-1} f(w) C_{l\cap S}(w)\, dw &= \int_0^\infty C_{l\cap S}(w)\, dF_{d-1}(w)\\ &= \int_0^\infty F_{d-1}(w) G_{l\cap S}(w)\, dw,\end{aligned}$$

since $C_{l\cap S}(w) = 0$ for w large enough and $F(0) = 0$. This proves the Proposition. □

From Proposition 3.5 and part (b) of Lemma 3.3, we obtain a generalisation of Crofton's formula, relating chord length and interpoint distance for non-convex sets (cf. Santaló(1975), p. 238(14.25)).

3.6 Corollary. For $S \in \mathcal{V}$

$$2\int_{\mathcal{L}} [\sigma(l\cap S)^k] = k(k-1) \int_S \int_S \|u-v\|^{k-d-1}\, dudv, \quad k \geq 2. \tag{3.3}$$

Proof. Take $f(t) = t^{k-d-1}$ in Proposition 3.5. Then

$$\begin{aligned}\int_S \int_S \|u-v\|^{k-d-1}\, dudv &= 2\int_{\mathcal{L}} \int_0^\infty \left(\int_0^t s^{k-2}\, ds\right) dtd\mu(l)\\ &= \frac{2}{k-1} \int_{\mathcal{L}} \int_0^\infty t^{k-1} G_{l\cap S}(t)\, dtd\mu(l).\end{aligned}$$

By Lemma 3.3(b) this is

$$\frac{2}{k(k-1)} \int_{\mathcal{L}} [\sigma(l\cap S)^k]\, d\mu(l)$$

which proves the claim. □

3.7 Remark.
In the convex case we recover Crofton's formula, since by Example 3.4.2

$$[\sigma(l \cap S^k)] = \sigma(l \cap S)^k \quad \text{for convex } S.$$

3.8 Remark.
Taking $k = d + 1$ in Corollary 3.6 or $f \equiv 1$ in Proposition 3.5 we get

$$\begin{aligned}\lambda(S)^2 &= \frac{2}{d(d+1)} \int_{\mathcal{L}} [\sigma(l \cap S)^{d+1}]\, d\mu(l) \\ &= \frac{2}{d} \int_{\mathcal{L}} \int_0^\infty t^d G_{l\cap S}(t)\, dt d\mu(l).\end{aligned}$$

Finally, there is a connection with the chord length distribution

$$F_S(x) = \mathbf{P}\{l \in \mathcal{L} : \sigma(l \cap S) \leq x\},$$

where $\mathbf{P}$ is a properly normalised probability measure on the set of lines that intersect a *convex* set S. Recall from Lemma 2.2(b) that $1\{\sigma(l \cap S) > x\} = G_{l\cap S}(x)$. Hence

$$\mathbf{P}\{\sigma(l \cap S) \leq x\} = 1 - \frac{1}{\mu(S)} \int_{\mathcal{L}} G_{l\cap S}(x)\, d\mu(l).$$

Comparing this with results of Waksman and Pohl, we conclude that $\int_{\mathcal{L}} G_{l\cap S}(x)\, d\mu(l)$ is equivalent to a_S, the so-called *associated* function to S. This function was defined by Pohl(1980) and used by Waksman(1985) to partially solve the problem of characterising convex plane polygons by their chord length distributions. (Extended to a more general class of polygons by Cabo (1989).)

3.3. Glance functions.

Waksman (1987) considered a subclass of *open* subsets of $\mathcal{W}$ with C^2 boundaries made up of finitely many arcs on which the curvature does not change sign. Moreover their diameters are bounded by a fixed constant D. For such a set, he introduced the *glance function*. In our notation, its definition boils down to

$$H_{l\cap S}(t) := \sum_{i=1}^{2n} \sum_{j=i+1}^{2n} (-1)^{i+j+1} 1\{x_j - x_i \leq t\}$$

While the linear scan transform G takes into account only those endpoints which are separated by more than a distance t, the glance function only 'sees' those endpoints that are not more than t apart.

Contrary to the linear scan transform, the glance function depends on the bound D. Furthermore, observe that the glance function has the following unnatural property: it equals zero in the case the line l has an empty intersection with S, but if l hits a vertex of S but has empty intersection with $\mathrm{Int}(S)$, then $H_{l\cap S}$ is non-zero. Another feature of the linear scan transform is that it equals zero whenever the total intersection length is zero.

3.9 Lemma. *Let $t \in [0, D]$. Then*

$$H_{l \cap S}(t) = n(l \cap S) - G_{l \cap S}(t),$$

where $n(l \cap S)$ is the number of components of $l \cap S$.

Proof. Suppose $l \cap S \neq \emptyset$ (otherwise there is nothing to be proved). Then for all $0 \leq t \leq D$

$$\begin{aligned} H_{l \cap S}(t) + G_{l \cap S}(t) &= \sum_{i=1}^{2n} \sum_{j=i+1}^{2n} (-1)^{i+k+1} \left(1\{x_j - x_i \leq t\} + 1\{x_j - x_i > t\}\right) \\ &= \sum_{i=1}^{2n} \sum_{j=i+1}^{2n} (-1)^{i+j+1} 1\{0 \leq x_j - x_i \leq D\} \\ &= \sum_{i=1}^{2n} \sum_{j=i+1}^{2n} (-1)^{i+j+1} 1\{x_j - x_i \geq 0\} \\ &= G_{l \cap S}(0) = n(l \cap S), \quad \text{by part (a) of Lemma 3.3.} \end{aligned}$$

□

4. Metrics.

4.1 A new metric for sets.

On the class of regular sets (introduced in § 1) we define a new 'stereological' metric, so called because it is defined only in terms of the linear scan transform G.

4.1 Definition For $S, T \in \mathcal{V}$, let

$$\eta(S,T) = \int_{\mathcal{L}} \|G_{l \cap S} - G_{l \cap T}\|_1 \, d\mu(l), \tag{4.1}$$

where $\|.\|_1$ denotes the L^1 norm.

4.2 Remark. In fact this definition can be viewed as a modification of the metric defined by Waksman (1987) as the L^1 distance between glance functions. Nevertheless the class of sets on which it was defined was different; e.g. all the sets had to be contained in a fixed disk with bounded diameter, on which the definition depended (see also §3.3).

Measurability and integrability of the linear scan transform are proved in Appendix B; this ensures that η is well-defined.

Let the set of lines that intersect S be denoted by $[S]$:

$$[S] := \{l \in \mathcal{L} : l \cap S \neq \emptyset\}$$

4.3 Proposition. *η is a metric on $\mathcal{V}$.*

Proof. Since η is an L^1-metric, the only non-trivial property we have to check is that $\eta(S,T)=0$ implies $S=T$. Suppose $\eta(S,T)=0$. Observe that

$$\begin{aligned}\int_{\mathcal{L}}|\sigma(l\cap S)-\sigma(l\cap T)|\,d\mu(l) &= \int_{\mathcal{L}}\Big|\int_0^\infty G_{l\cap S}(t)\,dt-\int_0^\infty G_{l\cap T}(t)\,dt\Big|\,d\mu(l)\\ &\le \int_{\mathcal{L}}\int_0^\infty |G_{l\cap S}(t)-G_{l\cap T}(t)|\,dt\,d\mu(l)\\ &= \eta(S,T)\\ &= 0.\end{aligned}$$

Thus

$$\sigma(l\cap S)=\sigma(l\cap T),\qquad \text{for } \mu\text{-almost lines } l.$$

Consequently the Radon transforms of the indicator functions of S and T are equal for almost all lines. By Helgason (1980), Proposition I.7.5, p. 52 this yields equality of these indicators in L^1. Equivalently

$$\lambda(S\triangle T)=0. \tag{4.2}$$

Suppose however there is an $x\in S\setminus T$. Because T is closed, we can find an open ball $B(x,r)$ around x such that

$$\begin{aligned}&B(x,r)\cap T=\emptyset\\ \text{and } &B(x,r)\cap S\neq\emptyset.\end{aligned} \tag{4.3}$$

Recall that by regularity of S, $S=$Cl(Int(S)). Hence, for all $\epsilon>0$, we can find $y\in$Int(S) such that $\|x-y\|<\epsilon$.
Take $\epsilon=\frac{1}{2}r$. Then there is a $\delta<\frac{1}{2}r$ such that

$$B(y,\delta)\subset S.$$

Hence

$$B(y,\delta)\subset B(x,r)\implies B(y,\delta)\cap T=\emptyset,\quad \text{by (4.3)}.$$

However this would imply that $\lambda(S\setminus T)\ge\lambda(B(y,\delta))>0$, contradicting (4.2). Thus we conclude $S\setminus T=\emptyset$ and by symmetry also $T\setminus S=\emptyset$, yielding the desired equality of S and T. □

4.4 Examples.

4.4.1. If $K_1,K_2\in\mathcal{V}$ are both convex

$$\eta(K_1,K_2)=\int_{\mathcal{L}}|\sigma(l\cap K_1)-\sigma(l\cap K_2)|\,d\mu(l),$$

the L^1 distance between their Radon transforms. To see this, observe that for all l

$$G_{l\cap K_1}-G_{l\cap K_2}=1_{[0,\sigma(l\cap K_1)]}(t)-1_{[0,\sigma(l\cap K_2)]}(t),$$

by convexity (see Lemma 2.2) so that

$$\begin{aligned}\|G_{l\cap K_1} - G_{l\cap K_2}\|_1 &= \int_0^\infty \left|1_{[0,\sigma(l\cap K_1)]}(t) - 1_{[0,\sigma(l\cap K_2)]}(t)\right| dt \\ &= |\sigma(l\cap K_2) - \sigma(l\cap K_1)|.\end{aligned}$$

4.4.2. If K_1, K_2 are convex and $K_1 \subseteq K_2$ then

$$\eta(K_1, K_2) = d\kappa_d\{\lambda(K_2) - \lambda(K_1)\}.$$

This follows from (1.4) and the previous example since $\sigma(l\cap K_2) - \sigma(l\cap K_1) \geq 0$.

Apart from this special case, it seems quite hard to give an explicit expression for the metric η. For instance, the other relatively simple case of two non-intersecting convex sets needs results related to the Sylvester problem (see Santaló(1976), p. 63–65). On the other hand, a simple upper bound obtains in the general case.

4.5 Lemma. *Let $S, T \in \mathcal{V}$. Then*

$$\begin{aligned}\eta(S,T) &\leq \int_{\mathcal{L}} \|G_{l\cap S}\|_1\, d\mu(l) + \int_{\mathcal{L}} \|G_{l\cap T}\|_1\, d\mu(l) \\ &\leq diam(S) length(\partial S) + diam(T) length(\partial T).\end{aligned} \tag{4.4}$$

Proof. The first inequality simply follows from the triangle inequality for the L^1 norm.
It is also clear that

$$|G_{l\cap S}(t)| \leq G_{l\cap S}(0) = n(l\cap S) \qquad \text{for all } t. \tag{4.5}$$

Thus we get

$$\begin{aligned}\int_{\mathcal{L}} \|G_{l\cap S}\|_1\, d\mu(l) + \int_{\mathcal{L}} \|G_{l\cap T}\|_1\, d\mu(l) &= \int_{\mathcal{L}}\int_0^\infty |G_{l\cap S}(t)|\, dt\, d\mu(l) + \int_{\mathcal{L}}\int_0^\infty |G_{l\cap T}(t)|\, dt\, d\mu(l) \\ &\leq \text{diam}(S)\int_{\mathcal{L}} G_{l\cap S}(0) d\mu(l) + \text{diam}(T)\int_{\mathcal{L}} G_{l\cap T}(0) d\mu(l).\end{aligned}$$

This yields the Lemma by (1.3). (See Santaló(1976) p. 31.) □

4.6 Remark. This bound is sharp in the following sense:
Let $x_n \in \mathbb{R}^d$ be such that $\|x_n\| \to \infty$, $n \to \infty$. Then $\eta(S, T_{x_n}T)$ tends to the first bound in (4.4).

Proof. Let $R_n := T_{x_n}T$.

$$\begin{aligned}\eta(S,R_n) &= \int_{\mathcal{L}} \|G_{l\cap S} - G_{l\cap R_n}\|_1 \, d\mu(l) \\ &= \int_{[S]\cap[R_n]} \|G_{l\cap S} - G_{l\cap R_n}\|_1 \, d\mu(l) \\ &+ \int_{[S]\setminus[R_n]} \|G_{l\cap S} - G_{l\cap R_n}\|_1 \, d\mu(l) \\ &+ \int_{[R_n]\setminus[S]} \|G_{l\cap S} - G_{l\cap R_n}\|_1 \, d\mu(l) \\ &= \int_{[S]\cap[R_n]} \|G_{l\cap S} - G_{l\cap R_n}\|_1 \, d\mu(l) \\ &+ \int_{[S]\setminus[R_n]} \|G_{l\cap S}\|_1 \, d\mu(l) \\ &+ \int_{[R_n]\setminus[S]} \|G_{l\cap R_n}\|_1 \, d\mu(l)\|.\end{aligned}$$

As n tends to infinity the first integral tends to zero. Moreover

$$\mu([S] \setminus [R_n]) \nearrow \mu([S]).$$

Hence monotone convergence applied to $1_{[S]\setminus[R_n]}\|G_{l\cap S}\|$ yields

$$\int_{\mathcal{L}} 1_{[S]\setminus[R_n]}\|G_{l\cap S}\|_1 \, d\mu(l) \nearrow \int_{\mathcal{L}} \|G_{l\cap S}\| \, d\mu(l).$$

For the last integral observe that by translation invariance of G

$$\int_{[R_n]\setminus[S]} \|G_{l\cap R_n}\|_1 \, d\mu(l) = \int_{[T]\setminus[T_{-x_n}S]} \|G_{l\cap T}\|_1 \, d\mu(l)$$

Then the same argument as above yields

$$\eta(S,R_n) \to \int_{\mathcal{L}} \|G_{l\cap S}\| \, d\mu(l) + \int_{\mathcal{L}} \|G_{l\cap T}\| \, d\mu(l),$$

as n tends to infinity. □

4.7 Example.

Consider the case $d = 2$. Then the second inequality in (4.4) is an equality if S and T are rectangles in the plane of the following form

$$\begin{aligned} S &= [0,x] \times [0,\pi] \\ T &= [5, 5+x] \times [0,\pi], \end{aligned}$$

where $x \approx 1.22049$.

Indeed

$$\int_{\mathcal{L}} \|G_{l\cap S}(t)\|\, d\mu(l) = \int_{\mathcal{L}} \int_0^\infty 1_{[0,\sigma(l\cap S)]}(t)\, dt d\mu(l)$$
$$= 2\pi \operatorname{area}(S) \quad \text{(see Example 4.4.2)}$$
$$= 2\pi^2 x.$$

We find x as the solution of

$$2\pi^2 x = \operatorname{diam}(S) \cdot \operatorname{length}(\partial S)$$
$$= \sqrt{\pi^2 + x^2} \cdot 2(\pi + x).$$

4.2 Connection with the Hausdorff distance.

It is of interest to compare the metric η introduced in the previous section with the well-known Hausdorff distance between sets.

4.8 Definition. *The Hausdorff distance between two nonempty sets $A, B \subset \mathbb{R}^d$ is*

$$\mathcal{H}(A,B) = \max\left\{ \sup_{a\in A} \inf_{b\in B} \|a-b\|, \sup_{b\in B} \inf_{a\in A} \|a-b\| \right\}$$
$$= \inf\{r > 0 : A \subset B^r \text{ and } B \subset A^r\},$$

where $B^r = \{x \in \mathbb{R}^d : \inf_{b\in B} \|x - b\| \le r\}$ denotes the parallel set of B.

The following example shows that the two metrics do not have the same topology.

4.9 Example.

Define

$$X_n = B(0,1) \cup B(x, \frac{1}{n}) \qquad \text{where } n \ge 1 \text{ and } \|x\| = 3.$$

In the Hausdorff metric X_n converges to $B(0,1) \cup \{x\}$. However in η it converges to the unit ball:

$$\eta(X_n, B(0,1)) = \int_{\mathcal{L}} |\sigma(l \cap B(0,1) \cup B(x, \frac{1}{n})) - \sigma(l \cap B(0,1))|\, d\mu(l)$$
$$= \int_{[B(0,1)]\setminus[B(x,\frac{1}{n})]} |\sigma(l \cap X_n) - \sigma(l \cap B(0,1))|\, d\mu(l)$$
$$+ \int_{[B(x,\frac{1}{n})]\setminus[B(0,1)]} |\sigma(l \cap X_n) - \sigma(l \cap B(0,1))|\, d\mu(l)$$
$$+ \int_{[B(0,1)]\cap[B(x,\frac{1}{n})]} |\sigma(l \cap B(0,1) \cup B(x, \frac{1}{n})) - \sigma(l \cap B(0,1))|\, d\mu(l).$$

The first integral is zero because $\sigma(l \cap X_n)$ equals $\sigma(l \cap B(0,1))$ on the domain of integration. For the second integral we have as n tends to infinity

$$\int_{[B(x,\frac{1}{n})]\setminus[B(0,1)]} \sigma(l \cap B(x, \frac{1}{n}))\, d\mu(l) \le \int_{[B(x,\frac{1}{n})]} \sigma(l \cap B(x, \frac{1}{n}))\, d\mu(l) = \frac{d\kappa_d^2}{n^d} \to 0.$$

The third integral tends to zero by observing that the integrand is bounded above by 2 and $\mu([B(0,1)] \cap B(x, \frac{1}{n})) \to \mu([B(0,1)] \cup [\{x\}]) = 0$, for n tending to infinity.

However, for *convex* sets with non-empty interior (i.e. elements of $\mathcal{C}$, see § 1.2) the metrics *are* topologically equivalent. The purpose of the remainder of this section is to prove this.

4. 10 Theorem. *The two metrics η and $\mathcal{H}$ are topologically equivalent on the collection of convex bodies.*

Since the proof of Theorem 4.10 is rather lengthy, we treat the two directions separately. The first part is a straightforward application of Steiner's formula.

4.11 Proposition. *Given $S \in \mathcal{C}$ and $\epsilon > 0$ there exists $\delta > 0$ such that for all $T \in \mathcal{C}$*

$$\mathcal{H}(S,T) < \delta \Rightarrow \eta(S,T) < \epsilon.$$

Proof. Suppose $S, T \in \mathcal{C}$ and $\delta > 0$ satisfy $\mathcal{H}(S,T) < \delta$. Then by definition

$$S \subset T^\delta \quad \text{and } T \subset S^\delta \tag{4.6}$$

Moreover, the convexity of S and T implies the convexity of their parallel sets S^δ and T^δ. The triangle inequality yields

$$\eta(S,T) \leq \eta(S,S^\delta) + \eta(S^\delta,T).$$

Using (4.6) and Example 4.4.2 gives

$$\eta(S,S^\delta) = d\kappa_d\,(\lambda(S^\delta) - \lambda(S))$$
$$\eta(T,S^\delta) = d\kappa_d\,(\lambda(S^\delta) - \lambda(T))$$

Let $W_r(\cdot)$ for $r = 0, \ldots, d$ denote the Minkowski functionals on $\mathcal{C}$ (see e.g. Santaló, 1976, p. 217). In particular $W_0(K) = \lambda(K)$. The Steiner formula (see Santaló, 1976, p. 220; Federer, 1969, p. 271) states that

$$W_r(K^\delta) = \sum_{s=0}^{d-r} \binom{d-r}{s} W_{r+s}(K)\delta^s$$

Applying the case $r = 0$ to $K = S$ yields

$$\lambda(S^\delta) - \lambda(S) = \sum_{s=1}^{d} \binom{d}{s} W_s(S)\delta^s$$

Applying the same to $K = T$

$$\lambda(T^\delta) = \sum_{s=0}^{d} \binom{d}{s} W_s(T)\delta^s.$$

This yields

$$\begin{aligned}\eta(S,T) &\leq \eta(S,S^\delta)+\eta(S^\delta,T)\\ &= d\kappa_d\left(2(\lambda(S^\delta)-\lambda(S))+\lambda(S)-\lambda(T)\right)\\ &= d\kappa_d\left(2\sum_{s=1}^{d}\binom{d}{s}W_s(S)\delta^s+\lambda(S)-\lambda(T)\right)\\ &\leq d\kappa_d\left(2\sum_{s=1}^{d}\binom{d}{s}W_s(S)\delta^s+\lambda(T^\delta)-\lambda(T)\right)\\ &= d\kappa_d\left(2\sum_{s=1}^{d}\binom{d}{s}(W_s(S)+W_s(T))\delta^s\right)\end{aligned}$$

Using the Steiner formula for general r, the inclusion $T\subset S^\delta$ and the monotonicity of $W_r(\cdot)$

$$W_r(T)\leq W_r(S^\delta)=\sum_{k=0}^{d-r}\binom{d-r}{k}W_{r+k}(S)\delta^s.$$

Collecting together we have

$$\eta(S,T)\leq d\kappa_d\left(2\sum_{s=1}^{d}\binom{d}{s}W_s(S)\delta^s+\sum_{s=1}^{d}\binom{d}{s}\sum_{k=0}^{d-s}\binom{d-s}{k}W_k(S)\delta^{s+k}\right)$$

This is a polynomial in δ with zero constant term and finite positive coefficients determined by S. The result follows. □

Next we turn to the second part of the proof of Theorem 4.10.

4.12 Proposition. *Let $\{K_n\}$ be a sequence in $\mathcal{C}$ and let $K\in\mathcal{C}$. If $\eta(K_n,K)\to 0$ then also $\mathcal{H}(K_n,K)\to 0$.*

To prove Proposition 4.12 we need the following result.

4.13 Proposition. *Let K, K_n be in $\mathcal{C}$ and suppose $\eta(K_n,K)\to 0$. Then all members of K_n are contained in a bounded portion of $\mathbb{R}^d$.*

Proof of Proposition 4.12. We assume that K_n tends to K in η, but *not* in $\mathcal{H}$. Then for all $\epsilon>0$, there is a subsequence, K_{n_i} say, such that

$$\mathcal{H}(K_{n_i},K)>\epsilon \quad \text{for all } i. \tag{4.7}$$

Since by assumption $\eta(K_n,K)\to 0$, Proposition 4.13 yields that K_{n_i} is contained in a bounded portion of $\mathbb{R}^d$. However it then follows from Blaschke's selection theorem (see Eggleston(1958)) that there is a sub-subsequence, $K_{n_{i_j}}$ say, that *does* converge in $\mathcal{H}$. Suppose its limit is K^*:

$$\mathcal{H}(K_{n_{i_j}},K^*)\to 0,\quad j\to\infty.$$

But then

$$\eta(K, K^*) \leq \eta(K, K_{n_{ij}}) + \eta(K_{n_{ij}}, K^*) \to 0,$$

by the assumption and Proposition 4.11. This implies $\eta(K_n, K) = 0$. Since η is a metric, $K = K^*$, that is

$$K_{n_{ij}} \xrightarrow{\mathcal{H}} K.$$

This contradicts (4.7), thus proving that convergence in η implies convergence in $\mathcal{H}$ to the same limit. □

In the sequel we consider the inradius and minimal width of a convex set. Their definitions are given below.

4.14 Definition. *(see e.g. Eggleston(1958)). Let K be a compact, convex set in $\mathbb{R}^d$.*

(i) The inradius $r(K)$ of S is the supremum of the radii of all balls contained in K.

(ii) Consider all pairs of parallel support hyperplanes at K. The minimal width $w(K)$ of K is the minimum of the distances between these planes.

4.15 Remark. For a compact set K, $K^{-\epsilon}$ denotes the set of all points that are centers of balls of radius ϵ contained in K:

$$K^{-\epsilon} = \{y : B(y, \epsilon) \subset K\}.$$

$K^{-\epsilon}$ is called the *erosion* (see Serra (1982), p. 39) of K. From the definition it is clear that $r(K) < \epsilon$ is equivalent to $K^{-\epsilon} = \emptyset$.

We now proceed with the proof of Proposition 4.13. It is divided into three Lemmas.

4.16 Lemma. *Suppose $\eta(K_n, K) \to 0$ and suppose $\cup_{n=1}^{\infty} K_n$ is unbounded. Then $r(K_n) \to 0$ as $n \to \infty$.*

Proof. Assume without loss of generality that the entire sequence $\{K_n\}$ diverges in the sense that there are points $x_n \in K_n$ such that $0 < \|x_n\| \uparrow \infty$. We prove the result by contradiction. Suppose (again without loss of generality) that $r(K_n) > \epsilon$ for *all* n, where $\epsilon > 0$. Thus by the Remark 4.15 $K_n^{-\epsilon}$ is not empty for *all* n.

We claim that $D(K, K_n^{-\epsilon/2}) \to \infty$, where

$$D(S, T) := \sup_{y \in T} \inf_{x \in S} \|x - y\|.$$

For, either $D(K, K_n^{-\epsilon}) \to \infty$ (which implies the claim) or $D(K, K_n^{-\epsilon}) < M$ for all n (without loss of generality). In the latter case for every n we can find $y_n \in K_n^{-\epsilon}$ such that $d(y_n, K) < M$, so that $B(y_n, \epsilon) \subset K^M$. Put $C_n := \mathrm{co}(B(y_n, \epsilon) \cup \{x_n\})$. Then C_n is contained in K_n. Enclosing K in the ball $A_n = B(y_n, \rho)$ where $\rho = \epsilon + 2\mathrm{diam}(K) + M$, we have $K_n \setminus K \supseteq C_n \setminus A_n$. Simple trigonometry shows that

$$r(C_n \setminus A_n) \geq \epsilon/2 \quad \text{when } \|x_n - y_n\| > 2\rho + \epsilon.$$

Now $\|x_n - y_n\| \to \infty$ since $d(y_n, K) < M$, so

$$r(C_n \setminus A_n) \geq \epsilon/2 \quad \text{for all sufficiently large } n,$$

which implies the claim.

Consequently we can find balls $B(z_n, \epsilon/2) = B_n \subset K_n$ such that $D(K, B_n) \to \infty$. Now

$$\begin{aligned}\eta(K_n, K) &\geq \int_{[B_n]\setminus[K]} \sigma(l \cap B_n)\, d\mu(l) \\ &= \int_{[B_n]} \sigma(l \cap B_n)\, d\mu(l) - \int_{[B_n]\cap[K]} \sigma(l \cap B_n)\, d\mu(l) \\ &\geq \lambda(B_n) - \epsilon\, \mu([B_n] \cap [K]).\end{aligned}$$

If $B = B(x, \mathrm{diam}(K))$ is the circumsphere of K, then we have

$$\mu([B_n] \cap [B]) = c(R, \epsilon/2, \|z_n - x\|),$$

where $c(r_1, r_2, s)$ is the measure of all lines intersecting two disjoint balls of radii r_1, r_2 with midpoints separated by a distance s. By standard integral geometric arguments it can be shown that for fixed r_1, r_2, $c(r_1, r_2, s) \to 0$ as $s \to \infty$. Hence $\mu([B_n] \cap [B]) \to 0$, i.e.

$$\eta(K_n, K) > \kappa_d (\epsilon/2)^d.$$

This contradiction proves the Lemma. □

4.17 Lemma. *For all $\epsilon > 0$ and for every compact convex set K*

$$r(K) < \epsilon \quad \textit{implies } w(K) < \frac{\epsilon}{c_d},$$

where c_d is a constant depending only on the dimension d.

This is a consequence of the following inequality (see Eggleston(1958), p. 112). For a compact convex set K

$$r(K) \geq c_d \cdot w(K), \quad \text{where } c_d = \begin{cases} \frac{1}{2} d^{-\frac{1}{2}}, & d \text{ odd} \\ \frac{(d+2)^{\frac{1}{2}}}{2(d+1)}, & d \text{ even.} \end{cases}$$

4.18 Lemma. *Let $K \in \mathcal{C}$. Then*

$$\liminf_{L \in \mathcal{C}: w(L) \to 0} \eta(K, L) > 0.$$

Proof. Fix $0 < \alpha < r(K)$. Observe that for all $0 \leq \alpha < r(K)$, $\lambda(K^{-\alpha}) \neq \emptyset$. Let $0 < \eta_0 < \frac{2}{3}\kappa_d \lambda(K^{-\alpha})$, where κ_d is the volume of the unit ball in $\mathbb{R}^d$. Then

$$\lambda(K^{-\alpha}) > \frac{\eta_0}{\frac{2}{3}\kappa_d}. \tag{4.8}$$

Now we take $\delta = \delta(\eta_0, K) \leq \min(\alpha, \frac{\frac{2}{3}\kappa_d \lambda(K^{-\alpha}) - \eta_0)}{2\mu([K])})$ and let L be an element of $\mathcal{C}$ with minimal width smaller than δ.

Choose a direction, $\theta^{\perp}_{\min}$ say, normal to two parallel support hyperplanes containing L that are a distance $w(L)$ apart. Write W for the region bounded by the two supporting hyperplanes mentioned above and let $\mathcal{L}_\alpha$ be the set of lines l intersecting $K^{-\alpha}$ whose directions $\theta(l)$ make an angle with $\theta^{\perp}_{\min}$ that lies in the interval $(-\frac{1}{3}\pi, \frac{1}{3}\pi)$:

$$\mathcal{L}_\alpha := \left\{ l \in [K^{-\alpha}] : \sphericalangle(\theta(l), \theta^{\perp}_{\min}) \in (-\frac{1}{3}\pi, \frac{1}{3}\pi) \right\}.$$

.

Let $l \in \mathcal{L}_\alpha$. Then by the assumption on $\sphericalangle(\theta(l), \theta^{\perp}_{\min})$

$$\sigma(l \cap W) = \frac{w(L)}{\cos(\sphericalangle(\theta, \theta^{\perp}_{\min}))} < 2w(L),$$

where we use the fact that $\cos\phi > \frac{1}{2}$ if $\phi \in (-\frac{1}{3}\pi, \frac{1}{3}\pi)$. Thus also

$$\sigma(l \cap L) < 2w(L). \tag{4.9}$$

Furthermore, observe that for lines intersecting $K^{-\alpha}$,

$$\sigma(l \cap K) \geq 2\alpha. \tag{4.10}$$

Hence by (4.9) and (4.10)

$$\sigma(l \cap K) \geq 2\alpha > 2\delta > 2w(L) > \sigma(l \cap L) \quad \text{for } l \in \mathcal{L}_\alpha.$$

Consequently

$$\begin{aligned} \eta(K, L) &\geq \int_{\mathcal{L}_\alpha} |\sigma(l \cap K) - \sigma(l \cap L)| \, d\mu(l) \\ &= \int_{\mathcal{L}_\alpha} \sigma(l \cap K) \, d\mu(l) - \int_{\mathcal{L}_\alpha} \sigma(l \cap L) \, d\mu(l). \end{aligned}$$

By part (a) of Lemma 1.9 and the definition of $\mathcal{L}_\alpha$

$$\int_{\mathcal{L}_\alpha} \sigma(l \cap K) \, d\mu(l) \geq \int_{\mathcal{L}_\alpha} \sigma(l \cap K^{-\alpha}) d\mu(l) = \frac{2}{3}\kappa_d \lambda(K^{-\alpha}).$$

For the second integral we have by (4.9)

$$\begin{aligned} \int_{\mathcal{L}_\alpha} \sigma(l \cap L) \, d\mu(l) &< 2w(L) \, \mu(\mathcal{L}_\alpha) \\ &\leq 2w(L) \, \mu([K]). \end{aligned}$$

Summarising

$$\begin{aligned} \eta(K, L) &\geq \frac{2}{3}\kappa_d \lambda(K^{-\alpha}) - 2w(L)\mu([K]) \\ &> \frac{2}{3}\kappa_d \lambda(K^{-\alpha}) - 2\delta\mu([K]) \quad \text{by assumption} \\ &> \eta_0 > 0, \quad \text{by(4.8)}. \end{aligned}$$

This implies $\liminf_{w(L)<\delta} \eta(K,L) > 0$ for δ arbitrarily small and proves Lemma 4.18. □

To complete the proof of Proposition 4.13, suppose $\eta(K_n, K) \to 0$ and suppose to the contrary that $\cup_{n=1}^{\infty} K_n$ is unbounded. By Lemma 4.16 the inradius of K_n tends to 0. Then by Lemma 4.17, the same is true for the minimal width of K_n. But by Lemma 4.18, $\eta(K_n, K) > 0$. This contradiction implies that $\cup_{n=1}^{\infty} K_n$ is bounded, proving the Proposition.

5. Continuity results.

5.1 Volume.

5.1 Lemma The mapping $S \longmapsto \lambda(S)$ is Lipschitz-continuous with constant 1 on $(\mathcal{V}, \eta)$.

Proof. Using Proposition 3.1, expressing the covariance function in terms of G

$$\begin{aligned}|\lambda(S) - \lambda(T)| &= |C_S(0) - C_T(0)| \\ &= \left|\int_{\mathcal{L}} \int_0^{\infty} (G_{l\cap S}(t) - G_{l\cap T}(t))\, dt\, d\mu(l)\right| \\ &\leq \int_{\mathcal{L}} \int_0^{\infty} |G_{l\cap S}(t) - G_{l\cap T}(t)|\, dt\, d\mu(l) \\ &= \eta(S,T).\end{aligned}$$

5.2 The covariance function.

As a corollary to the equivalence of η and $\mathcal{H}$ on the collection of convex bodies we obtain pointwise convergence of an η-convergent sequence in $\mathcal{C}$.

5.2 Corollary. Let K, $K_n \in \mathcal{C}$ and suppose $\mathcal{H}(K_n, K) \to 0$. Then

$$C_{K_n}(y) \to C_K(y) \quad \text{pointwise.}$$

Proof. Fix $y \in \mathbb{R}^d$ and write $C_K(y) = \int_{\mathbb{R}^d} 1_{K_n}(x)\, 1_{T_y K_n}(x)\, dx$. The result will follow by applying dominated convergence to $1_{K_n}(x)\, 1_{T_y K_n}(x)$. Since $\mathcal{H}(K_n, K) \to 0$

$$K_n \subset K^{\epsilon_n}, \quad \text{where } \epsilon_n \downarrow 0 \text{ as } n \to \infty.$$

But then by convexity, we also have

$$K^{-\epsilon_n} \subset K_n.$$

(See Matheron(1975).) Hence

$$1_{K^{-\epsilon_n}}(x) \leq 1_{K_n}(x) \leq 1_{K^{\epsilon_n}}(x), \quad \forall x.$$

Since the two bounds converge to $1_K(x)$, $\forall x$ as $n \to \infty$, the same is true for $1_{K_n}(x)$. Thus

$$1_{K_n}(x)\, 1_{T_y K_n}(x) \to 1_K(x)\, 1_{T_y K}(x) \text{ pointwise.}$$

Moreover the functions at the left hand-side are bounded above by 1 and have compact support. Hence dominated convergence yields

$$C_K(y) = \int_{\mathbb{R}^d} 1_{K_n}(x)\, 1_{T_y K_n}(x)\, dx \to \int_{\mathbb{R}^d} 1_K(x)\, 1_{T_y K}(x)\, dx = C_K(y).$$

□

The representation of the covariance function in terms of the function G also enables us to prove continuity results of the covariance function with respect to the metric η. As a first step in that direction the following lemma proves useful.

5.3 Lemma. *For $r > 0$ and $S, T \in \mathcal{V}$*

$$\int_{\|y\|=r} |C_S(y) - C_T(y)|\, d\omega_r \leq \eta(S,T),$$

where ω_r is the spherical measure on a ball with radius r.

Proof. By Proposition 3.1

$$\begin{aligned} |C_S(y) - C_T(y)| &= \left| \int_{\mathcal{L}_y} \int_{\|y\|}^{\infty} G_{l\cap S}(t) - G_{l\cap T}(t)\, dt\, d\mu_y(l) \right| \\ &\leq \int_{\mathcal{L}_y} \int_{\|y\|}^{\infty} |G_{l\cap S}(t) - G_{l\cap T}(t)|\, dt\, d\mu_y(l) \\ &\leq \int_{\mathcal{L}_y} \|G_{l\cap S} - G_{l\cap T}\|\, d\mu_y(l). \end{aligned}$$

Integrating this inequality over all directions yields the Lemma. □

We now have enough tools to prove Lipschitz continuity of the covariance function, for sets bounded by a fixed diameter. Denote by $\mathcal{V}(M)$ the subclass of $\mathcal{V}$ consisting of all sets with diameter bounded by M; i.e.

$$\mathcal{V}(M) := \{S \in \mathcal{V} : \mathrm{diam}(S) \leq M\}.$$

5.4 Theorem. *For all $M > 0$ the mapping $S \longmapsto C_S$ from $(\mathcal{V}(M), \eta)$ into $L^1(\mathbb{R}^d)$ is Lipschitz continuous with constant $\frac{1}{2}M^2$:*

$$\|C_S - C_T\| \leq \frac{1}{2} M^2 \eta(S,T) \qquad \textit{for all } S, T \in \mathcal{V}(M).$$

Proof. Let $S, T \in \mathcal{V}$.

Transforming to polar coordinates we obtain

$$\begin{aligned}\|C_S - C_T\| &= \int_{\mathbb{R}^d} |C_S(y) - C_T(y)|\, d\lambda(y) \\ &= \int_0^\infty r \int_{\|y\|=r} \big|C_S(y) - C_T(y)\big|\, d\omega_r\, dr \\ &= \int_0^M r \int_{\|y\|=r} |C_S(y) - C_T(y)|\, d\omega_r\, dr \\ &\le \eta(S,T) \int_0^M r\, dr \qquad \text{by Lemma 5.3} \\ &= \frac{1}{2} M^2\, \eta(S,T).\end{aligned}$$

This proves the Theorem. □

Lipschitz continuity of the function $C_K : \mathbb{R}^d \to \mathbb{R}_+$ for a fixed convex body has been proved by Matheron (1986).

Denote by S^{d-1} the surface of the d-dimensional unit sphere. Recall that the breadth function $b_K : S^{d-1} \to \mathbb{R}$ is defined as the $d-1$-dimensional volume of the projection of K onto a hyperplane orthogonal to the direction u. For $K \in \mathcal{C}$, b_K is a bounded function.

5.5 Theorem. *(Matheron)*
The mapping $y \longmapsto C_K(y)$ from $\mathbb{R}^d$ to $\mathbb{R}_+$ is Lipschitz continuous for fixed $K \in \mathcal{C}$:

$$|C_K(y) - C_K(x)| \le 2\, b\, \|x - y\|,$$

where b is the supremum over S^{d-1} of the breadth function of K.

5.6 Corollary Let $K_n, K \in \mathcal{C}$ and suppose $\eta(K_n, K) \to 0$. Then

$$C_{K_n} \to C_K \quad \textit{uniformly.}$$

Proof. This is an application of Dini's Theorem:
By Corollary 5.2

$$C_{K_n}(y) \to C_K(y).$$

As in the proof of Corollary 5.2 there is a sequence $\epsilon_n \downarrow 0$ such that

$$K^{-\epsilon_n} \subset K_n \subset K^{\epsilon_n} \quad \text{by convexity.}$$

This implies

$$C_{K^{-\epsilon_n}} \le C_{K_n} \le C_{K^{\epsilon_n}}. \tag{5.1}$$

Observe that the sequences $\{C_{K^{-\epsilon_n}}\}_{n=1}^\infty$ and $\{C_{K^{\epsilon_n}}\}_{n=1}^\infty$ are both monotone. Since $\mathcal{H}(K^{\epsilon_n}, K) = \epsilon_n \downarrow 0$, Theorem 4.10 and Corollary 5.2 yield

$$C_{K^{\epsilon_n}}(y) \downarrow C_K(y), \quad \text{for } y \in \mathbb{R}^d.$$

Moreover if $K^{-\epsilon_n} \uparrow K$ then $\mathcal{H}(K^{-\epsilon_n}, \bar{K}) \to 0$ because $\bar{K}$ is compact (see Matheron (1975) Cor. 3, p. 13). This yields $\mathcal{H}(K^{-\epsilon_n}, K) \to 0$ by compactness of $K(\subset R^d)$. Thus also

$$C_{K^{-\epsilon_n}} \uparrow C_K(y).$$

By Theorem 5.5, all these functions are continuous. Since they all have compact supports, Dini's Theorem asserts that $C_{K^{\epsilon_n}}$ and $C_{K^{-\epsilon_n}}$ converge to C_K *uniformly.* The triangle inequality and other standard arguments together with (5.1) yield the claim. □

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Appendix A. (Proof of Lemma 1.9)

To prove Lemma 1.9, recall the following concepts. As before $\mathcal{H}^m$ denotes m-dimensional Hausdorff measure (see Simon(1983), p. 6).

A.1 Definition. *Let $E \subset \mathbb{R}^d$. Then E is $(\mathcal{H}^m, m)$-rectifiable if $\mathcal{H}^m(E) < \infty$ and there exists a set F containing $\mathcal{H}^m$-almost all of E such that $F = \cup_{n=1}^{\infty} F_n$, where each F_n is the image of a bounded subset of $\mathbb{R}^m$ under a Lipschitz map.*

(See Federer(1969) p. 251, 252 or Simon(1983).)

Federer proved that for an $(\mathcal{H}^m, m)$-rectifiable set E

$$\mathcal{H}^m(E) = c(m,d) \int_{\mathcal{L}} \mathcal{H}^{m+1-d}(l \cap E)\, d\mu(l), \tag{A.1}$$

for a certain constant $c(m,d)$.

It is well known that the boundary of a bounded convex set with non-empty interior in $\mathbb{R}^d$ is $(\mathcal{H}^{d-1}, d-1)$-rectifiable. The following result is easy to prove.

A.2 Lemma. *Every $W \in \mathcal{W}$ (see section 1.2) can be represented (not uniquely) as a disjoint union*

$$W = \bigcup_{j=1}^{m'} \left((D_j \cap C_j) \setminus (\cup_{k=1}^{m_j} E_{jk}) \right),$$

where $D_j \in \mathcal{D}$, $C_j \in \mathcal{C}$, $E_{jk} \in \mathcal{E}$.

A.3 Corollary. For the boundary of a set $W \in \mathcal{W}$

$$\partial W \subset \bigcup_{j=1}^{m} (\partial D_j) \cup (\partial C_j) \cup \bigcup_{k=1}^{m_j} (\partial E_{jk}).$$

Consequently

$$\partial W \subset \bigcup_{i=1}^{n} \partial K_i, \quad \text{where } K_i \in \mathcal{C}.$$

In particular, ∂W is $(\mathcal{H}^{d-1}, d-1)$-rectifiable.

Proof of Lemma 1.9.

Let $\mathfrak{A} := \{l \in \mathcal{L} : m(l \cap S) \neq \frac{1}{2}\mathcal{H}^0(\partial S \cap l)\}$. Since $S \in \mathcal{V}$, ∂S is $(\mathcal{H}^{d-1}, d-1)$-rectifiable, hence we can apply (A.1) with $m = d-1$ and $m+1-d = 0$ yielding

$$\int_{\mathcal{L}} \mathcal{H}^0(\partial S \cap l)\, d\mu(l) = c_d\, \mathcal{H}^{d-1}(\partial S).$$

So we have to prove that $\mu(\mathfrak{A}) = 0$.

Let $l \in \mathfrak{A}$ and let $x \in l$ be such that x is not an endpoint of an interval on $l \cap S$, i.e.

$$x \in S \text{ but } x \notin \partial_l(l \cap S).$$

Equivalently, $x \in \text{Int}_l(l \cap S)$. Here we use the subscript l to indicate that everything is considered with respect to the relative topology induced on $l \cap S$.

By Corollary A.3

$$\partial S \subset \bigcup_{i=1}^{m} \partial K_i, \quad K_i \in \mathcal{C}.$$

Now distinguish two cases:
(a) x belongs to a line segment in l that is completely contained in ∂S;
(b) all points of l in a neighbourhood of x (apart from x itself) are interior points of S.

In case (a) we have for at least one K_i that x belongs to a line segment in l that is contained in ∂K_i (otherwise convexity would imply the segment intersects Int(S)). Thus by convexity

$$l \cap K_i \subset \partial K_i.$$

However it is not difficult to show that $\mu\{l : l \cap K_i \subset \partial K_i\} = 0$.

In case (b) convexity implies that x belongs to an intersection $\partial K_i \cap \partial K_j$ for some i, j, such that no $d-1$-plane through x is a supporting hyperplane of *both* K_i and K_j. All such points x can be covered by a finite union of $d-2$-dimensional rectifiable sets, B say. Hence

$$\mu\{l : l \cap B \neq \emptyset\} = 0.$$

Appendix B.

Lemma. *The function $G_{l\cap S}$ is (measurable and) integrable simultaneously in l and t, for every $S \in \mathcal{V}$.*

To show the Lemma, we first prove measurability of the transect covariance function. This is done by applying the coarea formula (Federer(1969),3.2.22) to the functions and sets defined below.

First let $\mathbb{V} := \mathbb{R}^d \times S^{d-1}$. Define

$$\begin{aligned} f :& \mathbb{V} \times \mathbb{R} \to \mathbb{R}^d \\ & ((x,u),t) \mapsto x + tu, \end{aligned}$$

that is, f maps (x, u, t) onto a point at distance t from x, lying on the line with orientation u through x. Next let

$$\begin{aligned} g :& \mathbb{V} \times \mathbb{R} \to \mathbb{R}^d \times \mathbb{R}^d \\ & ((x,u),t) \mapsto (x, f(x,u,t)). \end{aligned}$$

For any measurable set $A \subset \mathbb{R}^d$ with finite Lebesgue measure, the set

$$A^* := g^{-1}(A \times A) = \{(x,u,t) : x \in A,\ f(x,u,t) \in A\}$$

is clearly measurable. Next consider

$$\begin{aligned} h :& \mathbb{V} \to \mathcal{L} \\ & (x,u) \mapsto \{x + au : a \in \mathbb{R}\}. \end{aligned}$$

Identifying $\mathcal{L}$ as usual (Santaló(1976)) with the cylinder $\mathbb{R} \times [0, 2\pi)$ it is readily seen that h is Lipschitz. Finally we define

$$\begin{aligned} i :& \mathbb{V} \times \mathbb{R} \to \mathcal{L} \times \mathbb{R} \\ & ((x,u),t) \mapsto (h(x,u),t)). \end{aligned}$$

Then the coarea formula implies that

$$\begin{aligned} s(l,t) :&= \int_{i^{-1}(l,t)} 1_{A^*}((x,u),t) d\mathcal{H}^1(x,u) \\ &= 2\, C_{l \cap A}(t) \end{aligned}$$

is measurable in (l,t). Now it is easy to prove Lemma 4.2 for convex bodies. For $K \in \mathcal{C}$

$$G_{l \cap K}(t) = 1\{C_{l \cap K}(t) > 0\}$$

hence G is measurable in (l,t) and integrable since it is compactly supported. The result for regular sets $A \in \mathcal{V}$ now follows using similar arguments applied to the representation in Lemma A.2.