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On Interpolating Random Fields using a Finite Number of Observations

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Abstract

The best in the mean square sense linear interpolator for a zero mean weakly stationary random field is expressed in terms of orthonormal polynomials associated with the spectral distribution function of this field.

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1 Introduction

In this report we present formulas for interpolating a zero mean weakly stationary random field $\{X_u\}_{u\in\mathbb{Z}^d}$ under the assumption that the second order characteristics of the random field are known. Let $\{X_{\omega_m}\}_{m\in W}$ be a finite set of random variables, with $W = \{0,\ldots,N\}$. Fix subsets $\{X_{\omega_s}\}_{s\in S}$ and $\{X_{\omega_t}\}_{t\in W\setminus S}$ with S a non-empty proper subset of W, and suppose that only observations from the last set of random variables are available. Our aim is now to find the best (in the least squares sense) linear interpolators $\{\hat{X}_{\omega_s}\}_{s\in S}$ based on the observations available. It is well known that the interpolators $\hat{X}_{\omega_s}$ can be determined by solving a $(|W|-|S|)\times(|W|-|S|)$ linear system of equations, no matter whether the random field is stationary or not.

In this report we exploit the fact that the random field is stationary: by the stationarity assumption we can recurrently construct orthonormal polynomials $\{\phi_{\omega,n}\}_{n=0}^N$ as described in [Dzhaparidze and Janssen]. The coefficients of the linear interpolators $\{\hat{X}_{\omega_s}\}_{s\in S}$ can now be expressed in terms of the coefficients of the orthonormal polynomials (cf. Theorem 1). Notice that the procedure described here requires the inversion of a $|S|\times|S|$ matrix.

This report is organized as follows: in Chapter 2 basic notions are presented and the (least squares) minimization problem is formulated. In Chapter 3 the minimization problem is solved in terms of orthonormal polynomials. In Chapter 4 the results of Chapter 3 are applied to a zero mean real Gaussian random field.

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2 Interpolation

2.1 Basic Notions

In this subsection some basic notions are introduced; see [Dzhaparidze and Janssen] for more details.

By using the usual multi-index notations, z to the power η for $z = (z_1, z_2, \dots, z_d) \in \mathbb{C}^d$ and $\eta = (\eta_1, \eta_2, \dots, \eta_d) \in \mathbb{Z}^d$ is defined by

$$z^\eta := \prod_{k=1}^{d} z_k^{\eta_k} \in \mathbb{C},$$

so that

$$z^\eta \cdot z^\nu = z^{\eta+\nu}$$

for all $\eta, \nu \in \mathbb{Z}^d$. Once powers of the variable z are known, monomials can be defined. A monomial is then a function $\mathbb{C}^d \to \mathbb{C}$ such that

$$z \longmapsto z^\eta$$

for some $\eta \in \mathbb{Z}^d$. This mapping will be denoted by e_η. Notice that

$$e_\eta \cdot e_\nu = e_{\eta+\nu}$$

for all $\eta, \nu \in \mathbb{Z}^d$. Finally a polynomial is a finite linear combination of monomials.

The ρ^{th} - derivative of the polynomial p in z with $\rho = (\rho_1, \dots, \rho_d) \in \mathbb{N}^d$ is defined by

$$\frac{d^\rho}{dz^\rho} p(z) := \frac{\partial^{\rho_1}}{\partial z_1^{\rho_1}} \cdots \frac{\partial^{\rho_d}}{\partial z_d^{\rho_d}} p(z).$$

So we have

$$\frac{d^\rho}{dz^\rho} e_\rho(z) = \rho!$$

with

$$\rho! := \prod_{k=1}^{d} \rho_k!$$

Using the notation

$$0_d := (0, \dots, 0) \in \mathbb{N}^d,$$

we get for all $\eta \in \mathbb{N}^d$ that if $\eta \neq \rho$, then

$$\frac{d^\rho}{dz^\rho} e_\eta(z) \,|_{z=0_d} = 0.$$

Let $(\mathrm{P}^d, \mathbb{C})$ denote the complex vector space consisting of the polynomials. On this vector space, an inner product will be defined. Let $F : \Lambda \to \mathbb{R}$ be a spectral distribution function, where $\Lambda = (-\pi, \pi]^d$. Then *it is assumed that* the mapping $< \cdot ; \cdot >: \mathrm{P}^d \times \mathrm{P}^d \to \mathbb{C}$ given by

$$< f; g > := \int\limits_{\Lambda} f(e^{i\lambda})\overline{g(e^{i\lambda})}\ dF(\lambda)$$

is an inner product, where the numbers $e^{i\lambda}$ for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{R}^d$ are defined by

$$e^{i\lambda} := (e^{i\lambda_1}, e^{i\lambda_2}, \ldots, e^{i\lambda_d}) \in \mathbb{C}^d.$$

The norm $\| \cdot \| : \mathrm{P}^d \to \mathbb{R}$ induced by the inner product is given by

$$\| \cdot \| = < \cdot ; \cdot >^{\frac{1}{2}} .$$

We can now introduce systems of orthonormal polynomials: let

$$e_{\omega_0}, e_{\omega_1}, \ldots, e_{\omega_N}$$

be an ordered system of monomials, with $\omega = (\omega_0, \omega_1, \ldots, \omega_N) \in \mathbb{Z}^d \times \mathbb{Z}^d \times \cdots \times \mathbb{Z}^d$ and $\omega_i \neq \omega_j$ for all $i \neq j$. Then a unique system of polynomials $\{\phi_{\omega,n}\}_{n=0}^N$ exists such that

- $\phi_{\omega,n}$ is a linear combination of the monomials $e_{\omega_0}, \ldots, e_{\omega_n}$.
- the coefficient of e_{ω_n} in $\phi_{\omega,n}$ is a positive real number.
- the polynomials $\{\phi_{\omega,n}\}_{n=0}^N$ are orthonormal, i.e.

$$< \phi_{\omega,n}; \phi_{\omega,m} > = \delta_{n,m} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

cf. [Grenander and Szegö]. In this report the sequences ω belong either to the set

$$\begin{aligned} \Omega_N = \{(\omega_0, \omega_1, \ldots, \omega_N) \ \ | & \ \ \omega_i \in \mathbb{Z}^d, \omega_0 = 0_d \\ , & \ \ \forall i,j \in \{0, \ldots, N\} : (i \neq j) \Rightarrow (\omega_i \neq \omega_j)\} \end{aligned} \tag{1}$$

or to the set

$$\begin{aligned} \Omega_N^+ = \{(\omega_0, \omega_1, \ldots, \omega_N) \ \ | & \ \ \omega_i \in \mathbb{N}^d, \omega_0 = 0_d \\ , & \ \ \forall i,j \in \{0, \ldots, N\} : (i \neq j) \Rightarrow (\omega_i \neq \omega_j)\}. \end{aligned}$$

Notice that for each component ω_i from $\omega = (\omega_0, \omega_1, \ldots, \omega_N) \in \Omega_N^+$, we can take the ω_i^{th} - derivative of an arbitrary polynomial.

Fix now $\omega \in \Omega_N$. For a fixed $a \in \mathbb{C}^d$, define the kernel polynomial $s_\omega(a, \cdot) : \mathbb{C}^d \to \mathbb{C}$ by

$$s_\omega(a, \cdot) = \sum_{k=0}^{N} \overline{\phi_{\omega,k}(a)}\phi_{\omega,k}(\cdot)$$

Notice that this polynomial can also be represented as

$$s_\omega(a,\cdot) = \overline{(e_{\omega_0}(a),\ldots,e_{\omega_N}(a))}\,(\mathrm{H}_\omega)^{-1}\begin{pmatrix} e_{\omega_0}(\cdot) \\ \vdots \\ e_{\omega_N}(\cdot)\end{pmatrix},$$

with

$$\mathrm{H}_\omega = \begin{pmatrix} < e_{\omega_0}; e_{\omega_0} > & < e_{\omega_0}; e_{\omega_1} > & \ldots & < e_{\omega_0}; e_{\omega_N} > \\ < e_{\omega_1}; e_{\omega_0} > & < e_{\omega_1}; e_{\omega_1} > & \ldots & < e_{\omega_1}; e_{\omega_N} > \\ \ldots & \ldots & \ldots & \ldots \\ < e_{\omega_N}; e_{\omega_0} > & < e_{\omega_N}; e_{\omega_1} > & \ldots & < e_{\omega_N}; e_{\omega_N} > \end{pmatrix}.$$

cf. [Dzhaparidze and Janssen].

2.2 Minimization problem

Let $\{X_t\}_{t\in\mathbb{Z}^d}$ be a complex valued zero mean weakly stationary random field with given covariances, i.e.

- $\forall t \in \mathbb{Z}^d$ $\mathbb{E}(X_t) = 0$ and $\mathbb{E}(|X_t|^2) < \infty$.
- $\forall s,t \in \mathbb{Z}^d$

  $$\mathrm{Cov}(X_s, X_t) = \mathbb{E}(X_s\overline{X_t})$$

  depends only on $s-t$.

Due to stationarity, there exists a spectral distribution function $F : \Lambda \to \mathbb{R}$ with $\Lambda = (-\pi,\pi]^d$ such that for all $s,t \in \mathbb{Z}^d$

$$\mathbb{E}(X_s\overline{X}_t) = \int_\Lambda (e^{i\lambda})^{s-t}\, dF(\lambda) =< e_s; e_t > .$$

From now on we fix $N+1$ random variables in 'a window'. This window $\omega = (\omega_0,\ldots,\omega_N)$ of indices of the random variables belongs to the set Ω_N. For $\omega \in \Omega_N$ the covariance matrix of $X_{\omega_0},\ldots,X_{\omega_N}$ is given by

$$\Gamma_\omega = \mathbb{E}\begin{pmatrix} X_{\omega_0} \\ \vdots \\ X_{\omega_N}\end{pmatrix}\overline{(X_{\omega_0},\ldots,X_{\omega_N})} = \mathrm{H}_\omega.$$

Assume that we miss observations on $X_{\omega_{s_0}},\ldots,X_{\omega_{s_q}}$ with $s_0 < \ldots < s_q$. Surely the set of indices $S = \{s_0,\ldots,s_q\}$ belongs to $W = \{0,\ldots,N\}$. For a fixed $s \in S$, the best *linear* interpolator for X_{ω_s}, based on $\{X_{\omega_t} | t \in W\setminus S\}$, is given by

$$\hat{X}_{\omega_s} = \sum_{t\in W\setminus S} c_t^{(s)} X_{\omega_t},$$

where the numbers $c_t^{(s)} \in \mathbb{C}$ are chosen so that the mean squared error

$$\mathbb{E}|X_{\omega_s} - \hat{X}_{\omega_s}|^2$$

is minimized. Using the notation $W \setminus S = \{t_0, \ldots, t_{N-q-1}\}$ with $t_0 < \ldots < t_{N-q-1}$, we can write

$$\hat{X}_{\omega_s} = \sum_{n=0}^{N-q-1} c_{t_n}^{(s)} X_{\omega_{t_n}}. \tag{2}$$

It is well known that the coefficients $c_{t_n}^{(s)}$ minimizing the mean squared error, satisfy the system

$$(c_{t_0}^{(s)}, \ldots, c_{t_{N-q-1}}^{(s)})\Gamma_{W\setminus S} = (\mathbb{E}(X_{\omega_s}\overline{X}_{\omega_{t_0}}), \ldots, \mathbb{E}(X_{\omega_s}\overline{X}_{\omega_{t_{N-q-1}}})), \tag{3}$$

in which

$$\Gamma_{W\setminus S} = \mathbb{E}\begin{pmatrix} X_{\omega_{t_0}} \\ \vdots \\ X_{\omega_{t_{N-q-1}}} \end{pmatrix} \overline{(X_{\omega_{t_0}}, \ldots, X_{\omega_{t_{N-q-1}}})}. \tag{4}$$

In view of stationarity the mean squared error can be written in the following form:

$$\mathbb{E}|X_{\omega_s} - \hat{X}_{\omega_s}|^2 = \int_\Lambda |(e^{i\lambda})^{\omega_s} - \sum_{t\in W\setminus S} c_t^{(s)} (e^{i\lambda})^{\omega_t}|^2 \, dF(\lambda).$$

If the polynomial

$$\pi_\omega^{(s)} = e_{\omega_s} + \sum_{t\in W\setminus S} a_t^{(s)} e_{\omega_t}$$

minimizes $\|\cdot\|$ over $\mathrm{span}(e_{\omega_t} | t \in (W \setminus S) \cup \{s\})$ under the restriction that the coefficient of e_{ω_s} equals one, then the numbers $c_t^{(s)}$ are given by

$$c_t^{(s)} = -a_t^{(s)}.$$

In the next section we will construct the polynomials $\pi_\omega^{(s)}$ for $s \in S$.

3 Interpolation Polynomials

To find the coefficients of the interpolator $\hat{X}_{\omega_s}$, we have to solve the system (3), i.e. we have to invert the $(N-q)\times(N-q)$ matrix $\Gamma_{W\setminus S}$. Alternatively, we can proceed as follows: by the stationarity assumption we can recurrently construct the orthonormal polynomials $\{\phi_{\omega,n}\}_{n=0}^{N}$ as described in [Dzhaparidze and Janssen]. Coefficients of these polynomials will then be used to construct matrices A_S and Φ_S given by (5) and (8). Finally, the interpolation polynomials $\pi_\omega^{(s)}$ for $s \in S$ are constructed by using (11) and (12) (at this step we have to invert the $(q+1)\times(q+1)$ matrix Φ_S). The coefficients of $\pi_\omega^{(s)}$ determine the coefficients of the best linear interpolator $\hat{X}_{\omega_s}$, as was already noted at the end of the previous section.

Fix $N \in \mathbb{N}$ and $\omega = (\omega_0,\ldots,\omega_N) \in \Omega_N$. Along with $W = \{0,\ldots,N\}$ and $S = \{s_0,\ldots,s_q\} \subset W$ where $s_0 < \ldots < s_q$, we use the notation $Q = \{0,\ldots,q\}$. For a fixed $r \in Q$ we denote by $\pi_\omega^{(s_r)}$ the polynomial that minimizes $\|\cdot\|$ over $\mathrm{span}(e_{\omega_t}|t \in (W\setminus S)\cup\{s_r\})$ under the restriction that the coefficient of $e_{\omega_{s_r}}$ equals one.

Theorem 1 shows how the polynomial $\pi_\omega^{(s_r)}$ can be constructed by using the matrices A_S and Φ_S, which are defined as follows: let the orthonormal polynomials $\{\phi_{\omega,n}\}_{n=0}^{N}$ be given by

$$\phi_{\omega,n} = \sum_{m=0}^{n} \varphi_{n,m}\, e_{\omega_m},$$

in which $\varphi_{n,n} = k_n \in \mathbb{R}^+$ (cf. [Dzhaparidze and Janssen]). We set $\varphi_{n,m} = 0$ when $m > n$. Then the matrix $[(\mathrm{A}_S)_{\mu,\nu}|0 \le \mu \le N - s_0, 0 \le \nu \le q]$ is given by

$$(\mathrm{A}_S)_{\mu,\nu} = \varphi_{s_0+\mu,s_\nu}.$$

So

$$\mathrm{A}_S = \begin{pmatrix} \varphi_{s_0,s_0} & 0 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_{s_1-1,s_0} & 0 & \vdots & 0 \\ \varphi_{s_1,s_0} & \varphi_{s_1,s_1} & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 \\ \varphi_{s_q,s_0} & \varphi_{s_q,s_1} & \vdots & \varphi_{s_q,s_q} \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_{N,s_0} & \varphi_{N,s_1} & \vdots & \varphi_{N,s_q} \end{pmatrix} \tag{5}$$

If $\omega \in \Omega_N^+$, then

$$(\mathrm{A}_S)_{\mu,\nu} = \frac{1}{\omega_{s_\nu}!}\frac{d^{\omega_{s_\nu}}}{dz^{\omega_{s_\nu}}}\phi_{\omega,s_0+\mu}(z)|_{z=0_d}. \tag{6}$$

The $(q+1)\times(q+1)$ matrix $[(\Phi_S)_{u,v}|0\le u,v\le q]$ is given by

$$(\Phi_S)_{u,v} := \sum_{k=s_0}^{N} \overline{\varphi}_{k,s_u}\varphi_{k,s_v} = \sum_{k=0}^{N} \overline{\varphi}_{k,s_u}\varphi_{k,s_v}. \tag{7}$$

Hence

$$\Phi_S = \mathrm{A}_S^*\mathrm{A}_S, \tag{8}$$

where A_S^* is the conjugate transpose of A_S. The matrix Φ_S is invertible, since the columns of the matrix A_S are linearly independent ($\varphi_{s,s}\in\mathbb{R}^+$ for all $s\in S$).

If $\omega\in\Omega_N^+$, then

$$\begin{aligned}(\Phi_S)_{u,v} &= \sum_{k=0}^{N} \overline{\varphi}_{k,s_u}\varphi_{k,s_v}\\ &= \sum_{k=0}^{N}\left[\overline{\frac{1}{\omega_{s_u}!}\frac{d^{\omega_{s_u}}}{da^{\omega_{s_u}}}\phi_{\omega,k}(a)|_{a=0_d}}\right]\left[\frac{1}{\omega_{s_v}!}\frac{d^{\omega_{s_v}}}{da^{\omega_{s_v}}}\phi_{\omega,k}(z)|_{z=0_d}\right]\\ &= \overline{\frac{1}{\omega_{s_u}!}\frac{\partial^{\omega_{s_u}}}{\partial a^{\omega_{s_u}}}}\frac{1}{\omega_{s_v}!}\frac{\partial^{\omega_{s_v}}}{\partial z^{\omega_{s_v}}}s_\omega(a,z)|_{a=z=0_d}\end{aligned}$$

Using the notation

$$\nabla_{S,z} = \left(\frac{1}{\omega_{s_0}!}\frac{\partial^{\omega_{s_0}}}{\partial z^{\omega_{s_0}}},\ldots,\frac{1}{\omega_{s_q}!}\frac{\partial^{\omega_{s_q}}}{\partial z^{\omega_{s_q}}}\right),$$

we can rewrite the last identity as follows:

$$\Phi_S = \nabla_{S,a}^*\nabla_{S,z}\, s_\omega(a,z)|_{a=z=0_d}. \tag{9}$$

For $\omega\in\Omega_N$ we can also write

$$\begin{aligned}(\Phi_S)_{u,v} &= \sum_{k=0}^{N} \overline{\varphi}_{k,s_u}\varphi_{k,s_v}\\ &= \sum_{k=0}^{N}\left[\frac{1}{(2\pi)^d}\int_\Lambda (e^{i\lambda})^{\omega_{s_u}}\overline{\phi_k(e^{i\lambda})}\,d\lambda\right]\left[\frac{1}{(2\pi)^d}\int_\Lambda (e^{-i\mu})^{\omega_{s_v}}\phi_k(e^{i\mu})\,d\mu\right]\\ &= \frac{1}{(2\pi)^{2d}}\int_\Lambda\int_\Lambda (e^{i\lambda})^{\omega_{s_u}}(e^{-i\mu})^{\omega_{s_v}}s_\omega(e^{i\lambda},e^{i\mu})d\lambda d\mu,\end{aligned}$$

which yields

$$\Phi_S = \frac{1}{(2\pi)^{2d}}\int_\Lambda\int_\Lambda \begin{pmatrix}(e^{i\lambda})^{\omega_{s_0}}\\ \vdots\\ (e^{i\lambda})^{\omega_{s_q}}\end{pmatrix}\overline{((e^{i\mu})^{\omega_{s_0}},\ldots,(e^{i\mu})^{\omega_{s_q}})}\, s_\omega(e^{i\lambda},e^{i\mu})d\lambda d\mu. \tag{10}$$

Theorem 1 *Fix $r \in Q$. Let $\pi_\omega^{(s_r)}$ be the unique polynomial that minimizes $\|\cdot\|$ over span$(e_{\omega_t} | t \in (W \setminus S) \cup \{s_r\})$ under the restriction that the coefficient of $e_{\omega_{s_r}}$ equals one. Then*

$$\pi_\omega^{(s_r)} = \sum_{n=s_0}^{N} v_n^{(s_r)} \phi_{\omega,n}, \tag{11}$$

where $v^{(s_r)} = (v_{s_0}^{(s_r)}, \ldots, v_N^{(s_r)})$ is given by

$$v^{(s_r)} = \chi_{q,r}^{\mathrm{T}} \Phi_S^{-1} \mathrm{A}_S^*. \tag{12}$$

The minimum itself equals $(\Phi_S^{-1})_{r,r}$. Here $\chi_{q,r}$ is the unit vector of length $q+1$ with 1 at the $(r+1)^{th}$ place. Therefore formula (12) is equivalent to

$$\begin{pmatrix} v_{s_0}^{(s_0)} & \ldots & v_N^{(s_0)} \\ \ldots & \ldots & \ldots \\ v_{s_0}^{(s_q)} & \ldots & v_N^{(s_q)} \end{pmatrix} = \Phi_S^{-1} \mathrm{A}_S^*.$$

<u>Proof:</u> Let $g_\omega^{(s_r)}$ be an element of span$(e_{\omega_t} | t \in (W \setminus S) \cup \{s_r\})$, with the coefficient of $e_{\omega_{s_r}}$ equal to one, i.e.

$$g_\omega^{(s_r)} = \sum_{n=0}^{N} b_n^{(s_r)} \phi_{\omega,n},$$

with the constraint that for all $\nu \in Q$

$$\sum_{n=0}^{N} b_n^{(s_r)} \varphi_{n,s_\nu} = \delta_{\nu,r} = \begin{cases} 1 & \text{if } \nu = r \\ 0 & \text{if } \nu \neq r \end{cases} \tag{13}$$

Since $\varphi_{n,m} = 0$ for $m > n$, formula (13) reduces to

$$\sum_{n=s_0}^{N} b_n^{(s_r)} \varphi_{n,s_\nu} = \delta_{\nu,r} \tag{14}$$

for all $\nu \in Q$, or in vector notation

$$(b_{s_0}^{(s_r)}, \ldots, b_N^{(s_r)}) \mathrm{A}_S = \chi_{q,r}^{\mathrm{T}}. \tag{15}$$

Notice that (15) does not depend on $b_n^{(s_r)}$ for $n < s_0$. So when minimizing

$$\|g_\omega^{(s_r)}\|^2 = \sum_{n=0}^{N} |b_n^{(s_r)}|^2 = \sum_{n=0}^{s_0-1} |b_n^{(s_r)}|^2 + \sum_{n=s_0}^{N} |b_n^{(s_r)}|^2,$$

under the constraint (15), we have to take $b_n^{(s_r)} = 0$ for $n < s_0$. This means that

$$\pi_\omega^{(s_r)} = \sum_{n=s_0}^{N} v_n^{(s_r)} \phi_{\omega,n},$$

where $v^{(s_r)} = (v_{s_0}^{(s_r)}, \ldots, v_N^{(s_r)})$ is the particular solution of (15) with

$$\sum_{n=s_0}^{N} |v_n^{(s_r)}|^2$$

as small as possible. Thus $[v^{(s_r)}]^{\mathrm{T}}$ has to satisfy

$$\mathrm{A}_S^{\mathrm{T}}[v^{(s_r)}]^{\mathrm{T}} = \chi_{q,r}$$

(cf. (15)) and $[v^{(s_r)}]^{\mathrm{T}}$ has to belong to the column space of $(\mathrm{A}_S^{\mathrm{T}})^*$. This yields

$$v^{(s_r)} = \chi_{q,r}^{\mathrm{T}}(\mathrm{A}_S^*\mathrm{A}_S)^{-1}\mathrm{A}_S^* = \chi_{q,r}^{\mathrm{T}}\Phi_S^{-1}\mathrm{A}_S^*.$$

The minimum itself equals

$$v^{(s_r)}[v^{(s_r)}]^* = \chi_{q,r}^{\mathrm{T}}\Phi_S^{-1}[\chi_{q,r}^{\mathrm{T}}]^* = (\Phi_S^{-1})_{r,r}.$$

□

Remark: If $\omega \in \Omega_N^+$, then (6), (9), (11) and (12) yield

$$\pi_\omega^{(s_r)}(\cdot) = \chi_{q,r}^{\mathrm{T}}(\nabla_{S,a}^*\nabla_{S,z}\, s_\omega(a,z)|_{a=z=0_d})^{-1}\nabla_{S,a}^*\, s_\omega(a,\cdot)|_{a=0_d}$$

Corollary 1 *If S contains only one element, i.e. $q = 0$ and $S = \{s_0\}$, then the polynomial $\pi_\omega^{(s_0)}$ is given by*

$$\pi_\omega^{(s_0)} = \frac{\sum_{n=s_0}^{N} \overline{\varphi}_{n,s_0}\phi_{\omega,n}}{\sum_{n=s_0}^{N} \overline{\varphi}_{n,s_0}\varphi_{n,s_0}}.$$

This polynomial is called a 'one-point interpolation polynomial'. If moreover $s_0 = 0$, then

$$\pi_\omega^{(0)} = \frac{\sum_{n=0}^{N} \bar{l}_n\phi_{\omega,n}}{\sum_{n=0}^{N} |l_n|^2},$$

where $l_n = \varphi_{n,0}$. If $\omega \in \Omega_N^+$, then the polynomial $\pi_\omega^{(s_0)}$ is given by

$$\pi_\omega^{(s_0)}(\cdot) = \frac{\overline{\frac{\partial^{\omega_{s_0}}}{\partial a^{\omega_{s_0}}}} s_\omega(a,\cdot)|_{a=0_d}}{\frac{1}{\omega_{s_0}!}\overline{\frac{\partial^{\omega_{s_0}}}{\partial a^{\omega_{s_0}}}}\frac{\partial^{\omega_{s_0}}}{\partial z^{\omega_{s_0}}} s_\omega(a,z)|_{a=z=0_d}}$$

□

Remark: Notice that the adapted Gram-Schmidt orthogonalization procedure defined in [Dzhaparidze and Janssen] by

$$e_{\omega_N}, \ldots, e_{\omega_0} \xrightarrow{GS\bullet} u_{\omega,N},$$

leads to the identity

$$u_{\omega,N} = \pi_\omega^{(0)},$$

with $\pi_\omega^{(0)}$ defined in Corollary 1. Hence, using again the notations from [Dzhaparidze and Janssen] we have

$$\frac{\phi^*_{V_N(\omega),N}}{k_{V_N(\omega),N}} = \frac{\sum_{n=0}^{N} \bar{l}_n \phi_{\omega,n}}{\sum_{n=0}^{N} |l_n|^2}. \tag{16}$$

The next theorem shows that the interpolation polynomials $\pi_\omega^{(s)}$ can be expressed in terms of the one-point interpolation polynomials as defined in Corollary 1.

Theorem 2 *For $u \in Q$ define*

$$\hat{\pi}_\omega^{(s_u)} = \frac{\sum_{n=s_u}^{N} \overline{\varphi}_{n,s_u} \phi_{\omega,n}}{\sum_{n=s_u}^{N} \overline{\varphi}_{n,s_u} \varphi_{n,s_u}}$$

(cf. Corollary 1). Define the $(q+1) \times (q+1)$ matrix $[(\hat{\Phi}_S)_{\alpha,\beta} | 0 \le \alpha, \beta \le q]$ by

$$(\hat{\Phi}_S)_{\alpha,\beta} = (\Phi_S)_{\alpha,\beta}\, \delta_{\alpha,\beta}, \tag{17}$$

in which the matrix Φ_S is given by (8). Then

$$\Phi_S \begin{pmatrix} \pi_\omega^{(s_0)} \\ \vdots \\ \pi_\omega^{(s_q)} \end{pmatrix} = \hat{\Phi}_S \begin{pmatrix} \hat{\pi}_\omega^{(s_0)} \\ \vdots \\ \hat{\pi}_\omega^{(s_q)} \end{pmatrix} \tag{18}$$

<u>Proof:</u> Fix $r \in Q$ and let $d^{(s_r)} = (d_0^{(s_r)}, \ldots, d_q^{(s_r)})$ be given by

$$d^{(s_r)} = \chi_{q,r}^{\mathrm{T}} \Phi_S^{-1} \hat{\Phi}_S. \tag{19}$$

Then (7), (11) and (12) yield

$$\begin{aligned} \pi_\omega^{(s_r)} &= \chi_{q,r}^{\mathrm{T}} \Phi_S^{-1} \mathrm{A}_S^*(\phi_{\omega,s_0}, \ldots, \phi_{\omega,N})^{\mathrm{T}} \\ &= \chi_{q,r}^{\mathrm{T}} \Phi_S^{-1} \hat{\Phi}_S \hat{\Phi}_S^{-1} \mathrm{A}_S^*(\phi_{\omega,s_0}, \ldots, \phi_{\omega,N})^{\mathrm{T}} \\ &= d^{(s_r)} (\hat{\pi}_\omega^{(s_0)}, \ldots, \hat{\pi}_\omega^{(s_q)})^{\mathrm{T}}. \end{aligned}$$

So

$$\pi_\omega^{(s_r)} = \sum_{u=0}^{q} d_u^{(s_r)} \hat{\pi}_\omega^{(s_u)}.$$

This yields (18). □

4 Gaussian Random Fields

In conclusion, we present the following application of results to a zero mean Gaussian random field.

Theorem 3 *Assume that $\{X_t\}_{t\in\mathbb{Z}^d}$ is a real zero mean Gaussian random field. Then the conditional probability density function of $X_{\omega_{s_r}}, r \in Q$ given $X_{\omega_t} = x_{\omega_t}, t \in W \setminus S$ is*

$$p(x_{\omega_{s_0}}, \dots, x_{\omega_{s_q}} | x_{\omega_t}, t \in W \setminus S) = \frac{1}{(\sqrt{2\pi})^{q+1}\sqrt{\det(\Phi_S^{-1})}} e^{-\frac{1}{2}\mathbf{x}^T\Phi_S\mathbf{x}}, \tag{20}$$

where

$$\mathbf{x}^T = ([x_{\omega_{s_0}} + \sum_{t\in W\setminus S} a_t^{(s_0)} x_{\omega_t}], \dots, [x_{\omega_{s_q}} + \sum_{t\in W\setminus S} a_t^{(s_q)} x_{\omega_t}]),$$

with the coefficients satisfying the following identities:

$$\pi_\omega^{(s_r)} = e_{\omega_{s_r}} + \sum_{t\in W\setminus S} a_t^{(s_r)} e_{\omega_t}$$

for all $r \in Q$.

Proof: Recall that for a real Gaussian random field $\{X_t\}_{t\in\mathbb{Z}^d}$, the best linear interpolator for X_{ω_s} $(s \in S)$ based on $\{X_{\omega_t} | t \in W \setminus S\}$, is the conditional expectation

$$\hat{X}_{\omega_s} = \mathbb{E}[X_{\omega_s} | X_{\omega_t}, t \in W \setminus S] \text{ almost surely.}$$

This determines the form of the vector $\mathbf{x}$ in (20). It remains therefore to verify that for $k, l \in Q$

$$\mathbb{E}[X_{\omega_{s_k}} - \hat{X}_{\omega_{s_k}}]\overline{[X_{\omega_{s_l}} - \hat{X}_{\omega_{s_l}}]} = (\Phi_S^{-1})_{k,l}.$$

This is indeed true, because by using (12) we have

$$\begin{aligned}
\mathbb{E}[X_{\omega_{s_k}} - \hat{X}_{\omega_{s_k}}]\overline{[X_{\omega_{s_l}} - \hat{X}_{\omega_{s_l}}]} &= < \pi_\omega^{(s_k)}; \pi_\omega^{(s_l)} > \\
&= v^{(s_k)}[v^{(s_l)}]^* \\
&= \chi_{q,k}^{\mathrm{T}} \Phi_S^{-1} \mathrm{A}_S^* \mathrm{A}_S \Phi_S^{-1} \chi_{q,l} \\
&= (\Phi_S^{-1})_{k,l}.
\end{aligned}$$

□

Remark: If $X_{\omega_t} = x_{\omega_t} = 0$ for all $t \in W \setminus S$, then by (10) the quadratic form $\mathbf{x}^T\Phi_S\mathbf{x}$ in (20) can be expressed as follows:

$$\mathbf{x}^T\Phi_S\mathbf{x} = \frac{1}{(2\pi)^{2d}} \int_\Lambda\int_\Lambda \overline{\left(\sum_{u=0}^{q} x_{\omega_{s_u}} (e^{-i\lambda})^{\omega_{s_u}}\right)} \left(\sum_{v=0}^{q} x_{\omega_{s_v}} (e^{-i\mu})^{\omega_{s_v}}\right) s_\omega(e^{i\lambda}, e^{i\mu}) d\lambda d\mu. \tag{21}$$

It is perhaps worthwhile to compare the result of Theorem 3 with the following assertion:

Theorem 4 [Dobrushin] *Suppose*

$$\frac{dF(\lambda)}{d\lambda} = f(\lambda)$$

exists and

$$\int_\Lambda \frac{1}{f(\lambda)} d\lambda < \infty,$$

(notice that $f \geq 0$). Define the matrix $T(\frac{1}{(2\pi)^{2d} f})$ by

$$T(\frac{1}{(2\pi)^{2d} f}) = \int_\Lambda \begin{pmatrix} (e^{i\lambda})^{\omega_{s_0}} \\ \vdots \\ (e^{i\lambda})^{\omega_{s_q}} \end{pmatrix} \overline{((e^{i\lambda})^{\omega_{s_0}}, \ldots, (e^{i\lambda})^{\omega_{s_q}})} \frac{1}{(2\pi)^{2d} f(\lambda)} d\lambda. \tag{22}$$

Then the conditional probability density function of $X_{\omega_{s_r}}, r \in Q$ given $X_m = 0$, $m \notin \{\omega_{s_r} \mid r \in Q\}$ is

$$p(x_{\omega_{s_0}}, \ldots, x_{\omega_{s_q}}) = \frac{1}{(\sqrt{2\pi})^{q+1} \sqrt{\det(T(\frac{1}{(2\pi)^{2d} f})^{-1})}} e^{-\frac{1}{2}\mathbf{x}^T T(\frac{1}{(2\pi)^{2d} f})\mathbf{x}}, \tag{23}$$

with

$$\mathbf{x}^T T(\frac{1}{(2\pi)^{2d} f})\mathbf{x} = \frac{1}{(2\pi)^{2d}} \int_\Lambda \frac{|\sum_{u=0}^{q} x_{\omega_{s_u}} (e^{-i\lambda})^{\omega_{s_u}}|^2}{f(\lambda)} d\lambda. \tag{24}$$

□

Notice that (21) and (24) yield

$$0 \leq \mathbf{x}^T T(\frac{1}{(2\pi)^{2d} f})\mathbf{x} - \mathbf{x}^T \Phi_S \mathbf{x} =$$

$$\frac{1}{(2\pi)^{2d}} \int_\Lambda \int_\Lambda \frac{\overline{\left(\sum_{u=0}^{q} x_{\omega_{s_u}} (e^{-i\lambda})^{\omega_{s_u}}\right)} \left(\sum_{v=0}^{q} x_{\omega_{s_v}} (e^{-i\mu})^{\omega_{s_v}}\right)}{f(\lambda) f(\mu)} \zeta_\omega(\lambda, \mu)\, dF(\lambda)\, dF(\mu) \tag{25}$$

in which the function ζ_ω is determined by the relation

$$\int_\Lambda p(e^{i\lambda}) \zeta_\omega(\lambda, \mu)\, dF(\lambda) = \begin{cases} 0 & \text{if } p \in \text{span}\{e_{\omega_0}, \ldots, e_{\omega_N}\} \\ p(e^{i\mu}) & \text{if } p \in \text{span}\{e_{\omega_0}, \ldots, e_{\omega_N}\}^\perp \end{cases}$$

Loosely speaking, we can say that

$$\zeta_\omega(\lambda, \mu) = \frac{\delta(\lambda - \mu)}{f(\lambda)} - \overline{s_\omega(e^{i\mu}, e^{i\lambda})}.$$

References

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