Centrum voor Wiskunde en Informatica
REPORTRAPPORT

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Report CS-R9349 July 1993

Perpetual Reductions in Orthogonal Combinatory Reduction Systems

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Abstract

We design a strategy that for any given term t in an Orthogonal Combinatory Reduction System (OCRS) (that is, a Term Rewriting System with bound variables and substitutions) constructs a longest reduction starting from t if t is strongly normalizable, and constructs an infinite reduction otherwise. We develop a method for finding the least upper bound of lengths of reductions starting from a strongly normalizable term. We study properties of pure substitutions and several kinds of similarity of redexes. We apply these results to construct an algorithm for finding lengths of longest reductions in "strongly persistent" OCRSs. As a corollary, we have an algorithm for finding lengths of longest developments in orthogonal CRSs.

AMS Subject Classification (1991): 68Q42.
CR Subject Classification (1991): F4.2, F4.1.
Keywords & Phrases: orthogonal combinatory reduction systems, perpetual reductions, strong normalization, reduction strategies, strongly persistent combinatory reduction systems.
Note: Part of this work was completed during an enjoyable visit of the author at CWI in the summer of 1993.

1. Introduction

A strategy is *perpetual* if, given a term t, it constructs an infinite reduction starting from t whenever such a reduction exists, that is, whenever t is not strongly normalizable. Perpetual strategies are mostly interesting because termination of a perpetual reduction (constructed according to a perpetual strategy) implies strong normalization of the initial term. For orthogonal (left-linear and non-overlapping) TRSs a very simple perpetual strategy exists — just contract any innermost redex [20]. In fact, any *complete* strategy, i.e., a strategy that in each term contracts a redex that does not erase any other redex, is perpetual. Moreover, one can even reduce redexes all erased arguments of which are strongly normalizable [15].

It is easy to see that in any infinite reduction a redex that itself has an infinite reduction, called an *infinite* redex, is contracted. Thus in order to construct an infinite reduction one should try to retain at least one "potential" infinite redex — a subterm that can become an infinite redex (more precisely, that has a descendant under some reduction that is an infinite redex). Thus any strategy that does not erase potential infinite redexes is perpetual. In OTRSs, any potential infinite redex necessarily has an infinite reduction. That is why all the above strategies are perpetual. In orthogonal Combinatory Reduction Systems (that is, TRSs with bound variables and substitution mechanism [14, 8, 19, 10]) a strongly normalizable subterm may also be a potential infinite redex — after contraction of an outer redex a term can be substituted in it that makes the subterm no more strongly normalizable. Thus innermost reductions and complete reductions are no more perpetual in OCRSs. Therefore one can erase only strongly normalizable arguments in which no substitution of external subterms is possible. For the lambda-calculus, such a strategy was found by Barendregt et al. [3].

Report CS-R9349
ISSN 0169-118X
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

In this paper, we design a perpetual strategy for all orthogonal combinatory reduction systems. Our aim is not only to construct an infinite reduction of any given term t whenever it exist, but also to construct a longest possible one if t is strongly normalizable. Thus we will be able to characterize the complexity of computations of terms. The idea is that, as mentioned above, in order to construct perpetual reductions one should try to avoid erasure of redexes in which substitution of terms is possible during reductions of outer redexes. On the other hand, in order to construct the longest possible reductions one should delay contraction of a redex until it will no more be possible to copy it by reducing an outer redex. The two conditions agree if in each term s one contracts a *limit* redex, which is defined as follows: choose in s an *unabsorbed* redex u_1, i.e., a redex whose descendants never appear inside redex-arguments; choose an erased argument s_1 of u_1 that is not in normal form; choose in s_1 an unabsorbed redex u_2, and so on, as long as possible. The last chosen redex is a limit redex of s.

An unabsorbed redex exists in any term not in a normal form, but there is no general algorithm to find one. So we define some classes of OCRSs, such as non-absorbing, non-left-absorbing, and non-right-absorbing systems, for which the unabsorbed redexes are easy to find. For example, in non-left-absorbing systems, no subterm can be absorbed to the left of the contracted redex, so the leftmost-outermost redexes are unabsorbed. (In particular, the λ-calculus and the combinatory logic are non-left-absorbing.)

We develop a method for proving that the reductions constructed according to our perpetual strategy are indeed the longest, and for finding their lengths. Our method is similar to the Nederpelt's method [18], invented to reduce proofs of strong normalization to proofs of weak normalization (i.e., existence of a normal form). For any OCRS R, we define the corresponding non-erasing OCRS R_μ, called the *μ-extension* of R. We add fresh function symbols μ^n of arity n ($n = 0, 1, \ldots$) in the alphabet of R. For any R-rule $r : t \to s$, we keep the erased arguments of t in the right-hand side of each corresponding R_μ-rule $r_\mu : t' \to s'$ as "μ-erased" arguments of s'. Since this transformation affects the structure of redex-creation in R, and since erasure of arguments of a redex depends not only on the rule, but the arguments themselves, we have to introduce infinitely many R_μ-rules for each R-rule. This helps to have a natural translation of R-reductions into R_μ-reductions and vice-versa. Finally, we keep also all μ-symbols of t' as μ-erased symbols in s', since they can be used as "counters" of steps in longest normalizing reductions. We then show that a term o in an OCRS R is strongly normalizable if it is weakly normalizable in R_μ, and that the least upper bound of lengths of R-reductions starting from o coincides with the number of μ-occurrences in the R_μ-normal form of o.

To find this number, sometimes it is not necessary to do actual transformation of o. We show this for the case of *strongly persistent* CRSs, where creation of redexes is not possible during "pure substitution steps"; creation is only possible during the "TRS part" of reduction steps, and the arguments of a contracted redex and the context in which the reduction takes place do not take part in the creation. This kind of creation we call *generation.* We define several notions to characterize similarity of redexes in OCRSs. The above results relay on the fact that *strongly similar* redexes generate the same number of strongly similar redexes.

2. Orthogonal Combinatory Reduction Systems

Combinatory Reduction Systems have been introduced in Klop [14] to provide a uniform framework for reductions with substitutions, as in the λ-calculus and its extensions [2]. Different formalisms are proposed in Kennaway and Sleep [8] ((Functional) Combinatory Reduction Systems), Khasidashvili [9] (Expression Reduction Systems), and Nipkow [19] (Higher-order Rewrite Systems).

They are extensions of Term Rewriting Systems [5, 15] by means of variable binding and substitution mechanisms. Restricted notions of CRSs were first introduced in Pkhakadze [22] and Aczel [1]. A comparison of some formalisms of rewriting systems with bound variables and substitution mechanism (referred to also as higher order rewrite systems), can be found in van Oostrom and van Raamsdonk [21]. A survey paper is Klop et al. [16]. Here we describe a system of higher order rewriting as defined in Khasidashvili [10]; it is based on the syntaxs of [22].

Definition 2.1 (1) Let Σ be an *alphabet*, comprising *variables* $v_0, v_1, \ldots$; *function symbols*, also called *simple operators*; and *operator signs* or *quantifier signs*. Each function symbol has an *arity* $k \in N$, and each operator sign σ has an *arity* (m, n) with $m, n \neq 0$ such that, for any sequence $x_1, \ldots, x_m$ of pairwise distinct variables, $\sigma x_1 \ldots x_m$ is a *compound operator* or a *quantifier* with *arity* n. Occurrences of $x_1, \ldots, x_m$ in $\sigma x_1 \ldots x_m$ are called *binding variables*. Each quantifier $\sigma x_1 \ldots x_m$, as well as corresponding quantifier sign σ and binding variables $x_1 \ldots x_m$, has a *scope indicator* $(k_1, \ldots, k_l)$ to specify the arguments in which $\sigma x_1 \ldots x_m$ binds all free occurrences of $x_1, \ldots, x_m$. *Terms* are constructed from variables using functions and quantifiers in the usual way.

(2) *Metaterms* are constructed from *terms*, *term metavariables* $A, B, \ldots$, which range over terms, and *object metavariables* $a, b, \ldots$, which range over variables. Apart from the usual rules for term-formation, one is allowed to have *metasubstitutions* — expressions of the form $(A_1/a_1, \ldots, A_n/a_n)A_0$, where a_i are object metavariables and A_j are metaterms. Metaterms that do not contain metasubstitutions are called *simple metaterms*. An *assignment* maps each object metavariable to a variable and each term metavariable to a term over Σ. If t is a metaterm and θ is an assignment, then the θ*-instance* $t\theta$ of t is the term obtained from t by replacing metavariables with their values under θ, and by replacing subterms of the form $(t_1/x_1, \ldots, t_n/x_n)t_0$ by the result of substitution of terms $t_1, \ldots, t_n$ for free occurrences of $x_1, \ldots, x_n$ in t_0.

Definition 2.2 (I) A *Combinatory Reduction System* (CRS) is a pair (Σ, R), where Σ is an *alphabet*, described in Definition 2.1, and R is a set of *rewrite rules* $r : t \to s$, where t and s are metaterms such that t is a simple metaterm and is not a metavariable, and each term metavariable that occurs in s occurs also in t. Further,

(1) The metaterms t and s do not contain variables, and each occurrence of an object metavariable in t and s is bound. The metaterm s may contain occurrences of object metavariables that do not occur in t. They are called *additional object metavariables*.

(2) Each rule $r : t \to s$ has a set of *admissible assignments* $AA(r)$ such that, for any assignment $\theta \in AA(r)$,

(a) occurrences of variables in $s\theta$ that correspond to additional object metavariables of s do not bind variables in subterms that correspond to term metavariables of s.

(b) For any term metavariable A and any object metavariable a occurring in t or s, an occurrence of $A\theta$ in $s\theta$ is in the scope of an occurrence of $a\theta$ in $s\theta$ iff any occurrence of $A\theta$ in $t\theta$ is in the scope of an occurrence of $a\theta$ in $t\theta$.

(c) For any rule $r : t \to s$ in R and any assignment $\theta \in AA(r)$, $t\theta$ is an *r-redex* or an *R-redex*, and $s\theta$ is the *contractum* of $t\theta$. Redexes that are instances of the left-hand side of the same rule (i.e., with the same set of admissible substitutions) are called *weakly similar*.

(II) R is *simple* if right-hand sides of R-rules are simple metaterms.

Example 2.1 Operator signs $\exists$ and $\exists!$ for "there exists" and "there exists exactly one", having the arity $(1, 1)$ and the scope indicator (1), can be defined using Hilbert's operator (sign) τ as follows:

$$\exists aA \to (\tau aA/a)A$$

$$\exists!aA \to \exists aA \wedge \forall a\forall b(A \wedge (b/a)A \Rightarrow a = b)$$

where $\forall$ is the quantifier sign with arity $(1,1)$ and scope indicator (1) for "for any". Any assignment is admissible for the $\exists$-rule. An assignment θ is admissible for the $\exists!$-rule iff $b\theta \notin FV(A\theta)$. Obviously, b is an additional object metavariable.

Remark 2.1 Terms o and e are called *congruent*, notation $o \cong e$, if o is obtained from e by renaming bound variables. The conditions in Definition 2.2 imply that, for any rule $r : t \to s$, if $\theta, \theta' \in AA(r)$, then $t\theta \cong t\theta'$ implies $s\theta \cong s\theta'$. Below we identify all congruent terms.

Notation We use a, b for object metavariables, A, B for term metavariables, c, d for constants, t, s, e, o for terms and metaterms, u, v, w for redexes, σ for operators and operator signs, and P, Q for reductions. We write $s \subseteq t$ if s is a subterm of t. A one-step reduction in which a redex u in a term t is contracted is written as $t \xrightarrow{u} s$ or $t \to s$. We write $P : t \twoheadrightarrow s$ if P denotes a reduction of t to s. $|P|$ denotes the length, i.e., the number of steps, of P. If the last term of P coincides with the initial term of Q, then $P + Q$ denotes the concatenation of P and Q. $\emptyset_t$, or simply $\emptyset$, denotes the empty reduction of a term t; the symbol $\emptyset$ is also used to denote the empty set.

Definition 2.3 A term t in a CRS R is said to be in *normal form (nf)* or to be a nf if it does not contain redexes. If $s \twoheadrightarrow t$ and t is a nf, then t is called a normal form of s. A term is called *weakly normalizable* if it has a nf and is called *strongly normalizable* if it does not possess an infinite reduction. A CRS R is called *weakly normalizing* (resp. *strongly normalizing*) if each term in R is weakly normalizable (resp. strongly normalizable).

Definition 2.4 Let $t \to s$ be a rule in a CRS R and θ be an assignment. Subterms of a redex $v = t\theta$ that correspond to term metavariables of t are the *arguments* of v, and the rest is the *pattern* of v. Subterms of v rooted at the pattern are called the *pattern-subterms* of v. If R is a simple CRS, then arguments, pattern, and pattern-subterms are defined analogously in the contractum $s\theta$ of v.

Definition 2.5 A rewrite rule $t \to s$ in a CRS R is *left-linear* if t is linear, i.e., no term metavariable occurs more than once in t. R is *left-linear* if each rule in R is so. $R = \{r_i \mid i \in I\}$ is *non-ambiguous* or *non-overlapping* if in no term redex-patterns can overlap, i.e., if r_i-redex u contains an r_j-redex u' and $i \neq j$, then u' is in an argument of u, and the same holds if $i = j$ and u' is a proper subterm of u. R is *orthogonal* (OCRS) if it is left-linear and non-overlapping, and if v and w are any R-redexes such that w is in an argument of v and $v \xrightarrow{w} v'$, then v' is also a redex weakly similar to v.

Definition 2.6 The CRS S is a CRS comprising rules of the form

$$S^{n+1}a_1 \dots a_n A_1 \dots A_n A \to (A_1/a_1, \dots, A_n/a_n)A, \ n = 1, 2, \dots,$$

where S^{n+1} is the *operator sign of substitution* with arity $(n, n+1)$ and scope indicator $(n+1)$, and $a_1, \dots, a_n$ and $A_1, \dots, A_n, A$ are pairwise distinct object and term metavariables, respectively. Each assignment is admissible for any rule in S. We call $A_1, \dots, A_n$ the *mobile* arguments of S^{n+1} and call A *immobile*. (In the sequel we omit the superscript in S^{n+1}.)

Definition 2.7 Let $R = \{r_i : t_i \to s_i \mid i \in I\}$ be an OCRS. If R is simple, then $R_{fS} =_{def} R_f =_{def} R$, and otherwise $R_{fS} =_{def} R_f \cup \underline{S}$, where

1. $\underline{S} = \{\underline{S}a_1 \dots a_n A_1 \dots A_n A \to (A_1/a_1, \dots, A_n/a_n)A \mid n = 1, 2, \dots\}$. All assignments are admissible for $\underline{S}$-rules. (We assume that symbols $\underline{S}^{n+1}$ do not occur in the alphabet of R. The *arity* and the *scope indicator* of $\underline{S}^{n+1}$ coincide with that of S^{n+1}).

2. $R_f = \{r'_i : t_i \to s'_i \mid i \in I\}$, where s'_i is obtained from s_i by replacing all metasubstitutions $(t_1/a_1, \ldots, t_n/a_n)t$ with $\underline{S}^{n+1}a_1 \ldots a_n t_1 \ldots t_n t$, respectively.

3. An assignment θ is admissible for an R_f-rule r'_i iff the assignment $\theta_{\underline{S}}$ that to each term metavariable A assigns the $\underline{S}$-normal form of $A\theta$ and that coincides with θ on object metavariables is admissible for r_i.

4. For each step $e = C[t_i\theta] \xrightarrow{u} C[s_i\theta] = o$ in R (corresponding to the rule r_i and an admissible assignment θ) there is a reduction $P : e = C[t_i\theta] \to C[s'_i\theta] \twoheadrightarrow C[s\theta] = o$ in R_{fS}, where $C[s'\theta] \twoheadrightarrow C[s\theta]$ is the rightmost innermost normalizing $\underline{S}$-reduction. We call P the *expansion* of u and denote it by $Ex(u)$. The notion of *expansion* generalizes naturally to arbitrary R-reductions.

Definition 2.8 1. Let $t \xrightarrow{u} s$ in a simple OCRS and e be the contractum of u in s. For each argument t^* of u there are 0 or more arguments of e. We call them $(u-)$*descendants* of t^*. Correspondingly, subterms of t^* have 0 or more *descendants*. The *descendant* of each pattern-subterm of u that is not a variable is e. (We do not define descendants of "variable pattern-subterms", which are binding variables). It is clear what is to be meant under *descendants* of subterms that are not in u. The notion of *descendant* extends naturally to arbitrary reductions in simple OCRSs.

2. Let $t \xrightarrow{u} s$, where $u = Sx_1 \ldots x_n t_1 \ldots t_n t_0$, and let e be the contractum of u in s. For each mobile argument t_i of u $(i = 1, \ldots, n)$ there are substituted occurrences of t_i in e. We call them u-*descendants* of t_i. By definition, they also are u-*descendants* of corresponding free occurrences of the variable x_i in t_0. Subterms in t_i have the same number (possibly none) of *descendants* in s. The *descendant* of u is e. It is clear what is to be meant under *descendants* of subterms that are not in u, or are in t_0 and are not free occurrences of variables $x_1, \ldots, x_n$. The notion of *descendant* extends naturally to S-reductions with 0 or more steps.

3. Let $P : t \twoheadrightarrow s$ in an OCRS R and let $Q = Ex(P)$. It is clear from (1) and (2) what is to be meant under Q-descendants of subterms in t. We call a subterm $o' \in s$ a P-*descendant* of a subterm $o \in t$ if o' is a Q-descendant of o, and call o in this case a P-*ancestor* of o'.

4. Let $t \xrightarrow{u} s$. Descendants of all redexes of t except u are also called *residuals*. By definition, u does not have *residuals* in s. A redex $v \subseteq s$ is a (u)-*new* redex or a *created* redex if it is not a residual of a redex in t. The notion of *residual* of redexes extends naturally to reductions with 0 or more steps.

Definition 2.9 We call the co-initial reductions $P : t \twoheadrightarrow s$ and $Q : t \twoheadrightarrow e$ *strictly equivalent* (written $P \approx_{st} Q$) if $s = e$ and P-descendants and Q-descendants of any subterm of t are the same in s and e.

Notation If F is a set of redexes in t and $P : t \twoheadrightarrow s$, then F/P denotes the set of all residuals of redexes from F in s. If $F = \{u\}$, then we write u/P for $\{u\}/P$. In the following, F will also denote a complete F-development, where the residuals of redexes from F are contracted as long as possible. Similarly, if $u \in t$, then u will also denote the reduction $t \xrightarrow{u} s$.

Definition 2.10 Let $Q : t \twoheadrightarrow s$ and $t \xrightarrow{u} e$. Then the residual Q/u of Q by u is defined modulo permutation of non-overlapping steps by induction on $|Q|$ as follows. If $Q = \emptyset_t$, then $Q/u = \emptyset_e$. If $Q = Q' + v$, then $Q/u = Q'/u + v/(u/Q')$.

Definition 2.11 Let $P : t \twoheadrightarrow s$ and $Q : t \twoheadrightarrow e$. Then the residual P/Q of P by Q and the residual Q/P of Q by P are defined modulo permutation of non-overlapping steps by induction on $|P|$ as follows.

(1) If $P = \emptyset_t$, then $P/Q = \emptyset_e$ and $Q/P = Q$.

(2) If $P = P' + u$, then $P/Q = P'/Q + u/(Q/P')$ and $Q/P = (Q/P')/u$.

We write $P \sqcup Q$ for $P + Q/P$.

Theorem 2.1 (**Strict Church-Rosser** [10]) Let R be an OCRS, and P and Q be co-initial reductions in R. Then $P \sqcup Q \approx_{st} Q \sqcup P$.

3. Properties of S-reductions

In this section, we study some properties of substitutions. In particular, we prove a strengthened version of the Replacement Lemma [12].

Definition 3.1 Let t be a term in an OCRS. We call a subterm s in t *essential* (written $ES(s,t)$) if s has at least one descendant under any reduction starting from t and call it *inessential* (written $IE(s,t)$) otherwise.

The notion of essentiality is a generalization of the notion of *neededness* [7, 17] in a way that it works for all subterms, bound variables in particular. The following two lemmas are valid for all OCRSs; the proofs are similar to the case of orthogonal TRSs [11].

Lemma 3.1 Let $s_0, \ldots, s_k \subseteq t$ be such that $IE(s_i, t)$ for all $i = 0, \ldots, k$. Then there exists a reduction P starting from t such that none of the subterms $s_0, \ldots, s_k$ have P-descendants.
Proof Let P_i be a reduction starting from t such that s_i does not have P_i-descendants (P_i exists since $IE(s_i, t)$). Then, by Theorem 2.1, one can take $P = (\ldots(P_1 \sqcup P_2) \sqcup \ldots \sqcup P_n)$.

Lemma 3.2 Let $P : t \twoheadrightarrow t'$ and $s \subseteq t$. Then $IE(s,t)$ iff any P-descendant s' of s is inessential in t'. In particular, if t' is a normal form, then $ES(s,t)$ iff s has a P-descendant.
Proof ($\Rightarrow$) Let $IE(s,t)$. Then there is some reduction Q starting from t such that s does not have Q-descendants. By Theorem 2.1, $P + Q/P \approx_{st} Q + P/Q$. Hence, s' does not have P/Q-descendants, i.e., $IE(s', t')$. ($\Leftarrow$) If all u-descendants of s are inessential in t', then, by Lemma 3.1, there is some reduction P' starting from t' under which none of them have descendants. Thus s does not have $P + P'$-descendants, i.e., $IE(s,t)$.

Notation Below $EFV_R(t)$ denotes the set of variables having R-essential free occurrences in t and $FV(s)$ denotes the set of variables having free occurrences in s. We write $t = (t_1//e_1, \ldots, t_k//e_k)e$ if t is obtained from e by replacing non-overlapping proper subterms $e_1, \ldots, e_n$ in e with $t_1, \ldots, t_n$, respectively. For any $s \subseteq t$, $BV_R(s)$ denotes the set of free occurrences of s bound by quantifiers belonging to patterns of R-redexes that are outside s.

Definition 3.2 Let $u = Sx_1 \ldots x_n t_1 \ldots t_n t_0$ and t'_0 be an S-normal form of t_0. A subterm e in u is called *u-inessential* (written $IE(u; e)$) if e is in t_i for some $(1 \le i \le n)$ and $x_i \notin FV(t'_0)$.

Lemma 3.3 Let $u = Sx_1 \ldots x_n t_1 \ldots t_n t_0 \subseteq t$. Then $IE_S(u; t_i)$ iff $x_i \notin EFV_S(t_0)$.
Proof By Lemma 3.2, if t'_0 is the S-normal form of t, then $EFV_S(t_0) = FV(t'_0)$.

Lemma 3.4 Let $s \subseteq t$. Then $IE_S(s,t)$ iff $IE_S(u; s)$ for some S-redex u in t.
Proof sketch One can take for u the redex whose residual erases all descendants of s in the rightmost innermost normalizing S-reduction.

Lemma 3.5 Let $e \subseteq s \subseteq t$. Then $ES_S(e,t)$ iff $ES_S(e,s)$ and $ES_S(s,t)$.
Proof ($\Rightarrow$) From Definition 3.1. ($\Leftarrow$) By Lemma 3.4, the redex that would make e inessential can neither occur in s nor contain s in its argument.

Lemma 3.6 Let $s = (s_1//t_1, \ldots, s_n//t_n)t$, where s_i and t_i do not contain S-redexes, and let $ES_S(s_i,s) \Rightarrow BV_S(t_i) \subseteq BV_S(s_i)$ $(i = 1, \ldots, n)$. Further, let s' and t' be any corresponding subterms in s and t that are not in replaced subterms. Then $IE_S(s',s) \Rightarrow IE_S(t',t)$.
Proof By induction on the length of s. If t and s are not S-redexes, then the lemma follows easily from Lemma 3.4 and the induction assumption. So suppose that $t = Sx_1 \ldots x_m e_1 \ldots e_m e_0$, $s = Sx_1 \ldots x_m o_1 \ldots o_m o_0$, $s' \subseteq o_l$, and $IE_S(s',s)$. If $IE_S(s',o_l)$, then by the induction assumption $IE_S(t',e_l)$ and hence $IE_S(t',t)$. Otherwise, by Lemma 3.5, we have $IE_S(o_l,s)$. Hence, by Lemma 3.4, $IE_S(s;o_l)$. Thus, by Lemma 3.3, $x_l \notin EFV_S(o_0)$. Let us show that $x_l \notin EFV_S(e_0)$. By Lemma 3.4, if $s_i \subseteq o_0$, then $ES_S(s_i,s)$ iff $ES_S(s_i,o_0)$. Hence, for any S-essential subterm s_i, $BV_S(t_i) \subseteq BV_S(s_i)$. By the induction assumption, if x_l has an S-essential occurrence in e_0 outside of replaced subterms, then the corresponding occurrence of x_l in o_0 is S-essential. If x_l has an S-essential occurrence in a subterm $t_j \subseteq e_0$, then, by Lemma 3.5, $ES_S(t_j,e_0)$. By the induction assumption, $ES_S(s_j,o_0)$. Hence $BV_S(t_j) \subseteq BV_S(s_j)$. Thus Since $t_j \prec_S s_j$, x_l has a free occurrence in s_j. Since s_j does not contain S-symbols, it follows from Lemma 3.4 that this occurrence is S-essential in s_j and hence, by Lemma 3.5 and $ES_S(s_j,o_0)$, is S-essential in o_0. Hence $x_l \notin EFV_S(e_0)$ and, by Lemma 3.3, $IE_S(t;e_l)$. Therefore, by Definition 3.2 and Lemma 3.4, $IE_S(t;t')$ and $IE_S(t',t)$.

Remark 3.1 The above lemma is a strengthened version of the Replacement Lemma [12].

Definition 3.3 Let $u = C[s_1, \ldots, s_n]$ be a redex with context $C[\,]$ and arguments $s_1, \ldots, s_n$. Further, let $j_1, \ldots, j_k$ be the maximal subsequence of $1, \ldots, n$ such that $s_{j_1}, \ldots, s_{j_k}$ do not have u-descendants, and $i_1, \ldots, i_m$ be the maximal subsequence of $1, \ldots, n$ such that $s_{i_1}, \ldots, s_{i_m}$ do have u-descendants. We call $j_1, \ldots, j_k$ the *erased sequence* of u or the *u-erased sequence*, call $s_{j_1}, \ldots, s_{j_k}$ *(u-)erased arguments*, call $i_1, \ldots, i_m$ the *(u-)main sequence*, and $s_{i_1}, \ldots, s_{i_m}$ *(u-)main arguments.*

Notation Let $C[A_1, \ldots, A_n]$ be the left-hand side of a rule r in an OCRS R, where $C[\,, \ldots, \,]$ is a context and $A_1, \ldots, A_n$ are term metavariables. Sometimes we write $C[\overline{a_1}A_1, \ldots, \overline{a_n}A_n]$ for $C[A_1, \ldots, A_n]$, where $\overline{a_i}$ is the set of metavariables $\{a_{i_1}, \ldots, a_{i_{n_i}}\}$ such that A_i is in the scope of an occurrences of each object metavariable from $\overline{a_i}$. Correspondingly, an r-redex is written $C[\overline{x_1}t_1, \ldots, \overline{x_n}t_n]$, where $t_1, \ldots, t_n$ are arguments and t_i is in the scope of each variable from $\overline{x_i} = \{x_{i_1}, \ldots, x_{i_{n_i}}\}$.

Lemma 3.7 Let $u = C[\overline{x_1}t_1, \ldots, \overline{x_n}t_n]$ and $v = C[\overline{y_1}s_1, \ldots, \overline{y_n}s_n]$ be weakly similar redexes, let $m_1, \ldots, m_l$ be the v-main sequence, $n_1, \ldots, n_k$ be the v-erased sequence, and let for each $j = 1, \ldots, l$: $\overline{x_{m_j}} \cap FV(t_{m_j}) \subseteq \overline{y_{m_j}} \cap FV(s_{m_j})$. Then $t_{n_1}, \ldots, t_{n_k}$ are (not necessarily all) u-erased arguments.
Proof Let $u \to t \twoheadrightarrow o$ and $v \to s \twoheadrightarrow e$ be expansions of u and v, respectively. Then s can be obtained from t by replacing descendants of $t_1, \ldots, t_n$ with $s_1, \ldots, s_n$, respectively. By Definition 3.3 all descendants of $s_{n_1}, \ldots, s_{n_k}$ are $\underline{S}$-inessential. Therefore it follows from conditions (a)-(b) of Definition 2.2 and the assumptions that if t'_i and s'_i are corresponding descendants of t_i and s_i in t and s, then $ES_{\underline{S}}(s'_i,s) \Rightarrow BV_{\underline{S}}(t'_i) \subseteq BV_{\underline{S}}(s'_i)$. Thus the lemma follows from Lemma 3.6.

4. Perpetual strategies in OCRSs

In this section, we design a strategy that for a term t in an OCRS constucts a longest reduction when t is strongly normalizable, and constructs an infinite reduction otherwise. We give also a method to determine the lengths of longest reductions of strongly normalizable terms. Our method is similar to Nederpelt's method [18], by which proving strong normalization in a typed λ-calculus gets reduced to proving weak normalization. Nederpelt's method was reinvented and used by Klop [14] for orthogonal CRSs. Some of the results of this section can also be found in Klop [14]; our proofs are simpler.

Definition 4.1 The *μ-extension* (Σ_μ, R_μ) of an OCRS (Σ, R) is defined as follows:

1. $\Sigma_\mu = \Sigma \cup \{\mu^n \mid n = 0, 1, \ldots\}$, where μ^n is a fresh n-ary function symbol. For any subterm $s = \mu^{n+1}(t_1, \ldots, t_n, t_0)$ of a term t over Σ_μ, the arguments $t_1, \ldots, t_n$, as well as subterms and symbols in $t_1, \ldots, t_n$ and the head-symbol μ itself, are called *μ-erased* or more precisely *μ'-erased*, where μ' is the occurrence of the head symbol of s in t. The argument t_0 is called *μ'-main*. Symbols and subterms in t that are not μ-erased are called *μ-main*. We denote by $[t]_\mu$ the term obtained from t by removing all μ-erased symbols.

2. R_μ is the set of all rules of the form $r_\mu : t' \to s'$ such that
(a) there is a rule $r : t \to s$ in R such that $[t']_\mu = t$;
(b) the term t' is linear (i.e., no term metavariable appears twice or more in t');
(c) the head symbol of t' is not a μ-symbol, i.e., it coincides with the head symbol of t;
(d) the μ-erased arguments of each occurrence μ' of a μ-symbol in t' are term metavariables, and the μ'-main argument is not a term metavariable (i.e., is headed by a function symbol from Σ or a μ-symbol);
(e) Let $A_1, \ldots, A_n$ be the enumeration from left to right of all μ-main term metavariables of t', $B_1, \ldots, B_j$ be the enumeration from left to right of all μ-erased term metavariables of t', let $A_{i_1}, \ldots, A_{i_l}$ be a subsequence of $A_1, \ldots, A_n$, and let k be the number of occurrences of μ-symbols in t. Then

$$s' = \mu^m(\overbrace{\mu^0, \ldots, \mu^0}^{k}, B_1, \ldots, B_j, A_{i_1}, \ldots, A_{i_l}, s)$$

where $m = k + j + l + 1$.

3. An assignment θ is admissible for r_μ iff
(f) the arguments $A_{i_1}\theta, \ldots, A_{i_l}\theta$ of the redex $t'\theta$ do not have descendants under the reduction step $t\theta \to s\theta$; and
(g) the assignment θ_μ such that $A\theta_\mu = [A\theta]_\mu$ and $a\theta_\mu = a\theta$ for any term metavariable A and object metavariable a is admissible for r.

4. We call R_μ and R *μ-corresponding* OCRSs, and call r_μ and r *corresponding* rules.

5. For any r_μ-redex $u = t'\theta$, we call arguments that correspond to $A_{i_1}, \ldots, A_{i_l}$ *quasi-erased* arguments of u, and call the arguments that correspond to other metavariables from $A_1, \ldots, A_n$ *quasi-main*.

Example 4.1 Let $R = \{r : f(c, x) \to d\}$, where c and d are constants. Then R_μ-rules have the form

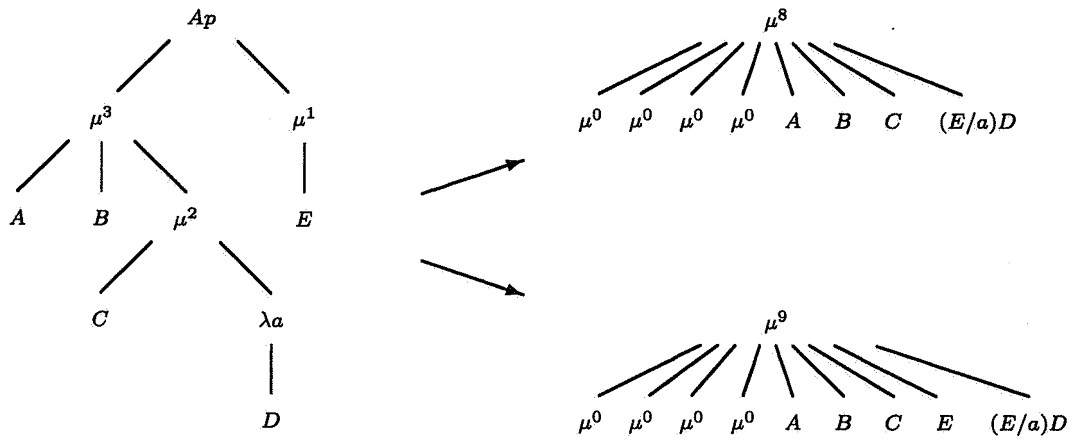
$$f(\mu^k(A_1, \ldots, A_{k-1}, \mu^l(B_1, \ldots, B_{l-1}, \ldots \mu^m(C_1, \ldots, C_{m-1}, c) \ldots)), x) \to \mu(\mu^0, \ldots, \mu^0, A_1, \ldots, A_{k-1}, B_1 \ldots, B_{l-1}, \ldots, C_1 \ldots, C_{m-1}, A, d)$$

For example, $r_\mu : f(\mu^2(B, \mu^2(C, c), A) \to \mu^6(\mu^0, \mu^0, B, C, A, d)$ is an R_μ-rule. For any r_μ-redex $t = f(\mu^2(o, \mu^2(s, c)), e)$, $[t]_\mu = f(c, e)$ is an r-redex, $t' = \mu(\mu^0, \mu^0, o, s, e, d)$ is the contractum of t, and $[t']_\mu = d$ is the contractum of $f(c, e)$.

Example 4.2 Consider the β-rule of the λ-calculus

$$\beta : Ap(\lambda a A, B) \rightarrow (B/a)A$$

On the picture below, two β_μ-rules with the same left-hand side and different right-hand sides are shown.



The arguments A and B are μ-erased and E and D are μ-main. An assignment θ is admissible for the first rule iff $a\theta \in FV(D\theta)$ and is admissible for the second one iff $a\theta \notin FV(D\theta)$.

Lemma 4.1 Let R be an OCRS, t be a term in R_μ, let $[t]_\mu = t'$, let u be a μ-main S-redex in t, u' be its μ-corresponding S-redex in t', let $t \xrightarrow{u} s$, and $t' \xrightarrow{u'} s'$. Then $[s]_\mu = s'$.
Proof Let $e \subseteq s$ be the contractum of u and $e' \subseteq s'$ be the contractum of u'. It is enough to show that $[e]_\mu = e'$. Let $u = Sx_1 \ldots x_n t_1 \ldots t_n t_0$. Then $e = \mu t_{i_1} \ldots t_{i_k} t'_0$, where $t'_0 = (t_1/x_1, \ldots, t_n/x_n)t_0$ and $t_{i_1}, \ldots, t_{i_k}$ are erased arguments of u. Thus $[e]_\mu = [t'_0]_\mu$, $u' = [u]_\mu = Sx_1 \ldots x_n [t_1]_\mu \ldots [t_n]_\mu [t_0]_\mu$, and $e' = ([t_1]_\mu/x_1, \ldots, [t_n]_\mu/x_n)[t_0]_\mu$. An occurrence o in $[e]_\mu = [t'_0]_\mu$ is μ-erased iff it is in a substituted occurrence t'_i of t_i and is μ-erased in t'_i, or o is outside of substituted occurrences of $t_1, \ldots, t_n$ in t'_0 and there is a μ-occurrence μ' outside of these substituted occurrences such that o is μ'-erased. In the first case the ancestor of o is μ-erased in t_i and in the second case the ancestor of o is μ-erased in t_0. Thus $[e]_\mu = [t'_0]_\mu = ([t_1]_\mu/x_1, \ldots, [t_n]_\mu/x_n)[t_0]_\mu = e'$.

Lemma 4.2 Let t be a term over Σ_μ the head-symbol of which is not a μ-symbol, and let $[t]_\mu = s$. Then t is an r_μ-redex iff s is an r-redex, where r_μ and r are corresponding rules in R_μ and R, respectively. Moreover, if t' is the contractum of t in R_μ and s' is the contractum of s in R, then $[t']_\mu = s'$.
Proof From Definition 4.1 and Lemma 4.1.

Corollary 4.1 Let R be an OCRS and $s_0 \xrightarrow{u_0} s_1 \xrightarrow{u_1} \ldots$ be a reduction in R. Then, for any term t_0 in R_μ such that $[t_0]_\mu = s_0$, there is a reduction $t_0 \xrightarrow{v_0} t_1 \xrightarrow{v_1} \ldots$ in R_μ such that $[t_i]_\mu = s_i$, and u_i and v_i are corresponding subterms in s_i and t_i $(i = 0, 1, \ldots)$.

Notation $\|t\|_\mu$ denotes the number of occurrences of μ-symbols in t.

Proposition 4.1 (1) Let R be an OCRS, u and v be R_μ-redexes such that u is in an argument of v, and let $v \xrightarrow{u} w$ in R_μ. Then w is an R_μ-redex weakly similar to v, and quasi-main sequences of v and w coincide.

(2) Let u be a redex in an OCRS R and v be an R_μ-redex such that $[v]_\mu = u$ and the sets of free variables of quasi-main arguments of v coincide with that of corresponding arguments of u. Then an argument of v is quasi-erased iff the corresponding argument of u is erased.

(3) Let u be a redex in an OCRS R and v be an R_μ-redex such that $[v]_\mu = u$. Then the corresponding argument of any quasi-erased argument of v is u-erased.

Proof The proposition is a corollary of Lemma 3.7.

Lemma 4.3 If R is an orthogonal CRS, then so is R_μ.

Proof Any overlap of patterns of two R_μ-redexes in a term t over Σ_μ causes also an overlap of patterns of corresponding R-redexes in $[t]_\mu$. Hence, since R is non-overlapping, so is R_μ. If u and v are R_μ-redexes such that v is in an argument of u and $u \xrightarrow{v} w$, then, by Proposition 4.1.(1), w is weakly similar to u. Therefore, R_μ is orthogonal (because it is left-linear).

Lemma 4.4 Let R be an OCRS. Then R_μ is Church-Rosser.

Proof From Lemma 4.3 and Theorem 2.1.

Lemma 4.5 Let t be a term in an OCRS R. If t is weakly normalizable in R_μ, then t is strongly normalizable in R_μ and R.

Proof Let s be an R_μ-nf of t and $t \to t_1 \to \ldots$ be an R_μ-reduction. By Lemma 4.4, $t_i \twoheadrightarrow s$ for all $i = 1, 2, \ldots$. It is easy to see that $i \leq \|t_i\|_\mu \leq \|s\|_\mu$. Thus t is strongly normalizable in R_μ. Hence, by Corollary 4.1, t is strongly normalizable in R.

Definition 4.2 A rule $r : t \to s$ in an OCRS R is called *non-erasing* if each term-metavariable occurring in t has an occurrence in s that is not in a mobile subterm of a metasubstitution. R is *non-erasing* if each R-rule is so.

It is easy to check that R is non-erasing iff, for any R-reduction step $e \xrightarrow{u} o$, each argument of u has a descendant in o, and the latter holds iff $FV(e) = FV(o)$.

Lemma 4.6 Let R be a non-erasing OCRS, $P : t_0 \to t_1 \to \ldots \to t_n$ be a reduction in R, and $Q : s_0 = t_0 \to s_1 \to \ldots \to s_n$ be its μ-corresponding reduction in R_μ. Then

(1) All redexes in s_i are μ-main ($i = 1, \ldots, n$).

(2) If P is normalizing, then so is Q.

Proof (1) Since R is non-erasing, μ-symbols in s_i are occurrences of μ^1. Thus no subterms of s_i, and hence no redexes, are μ-erased.

(2) From (1) and $[s_n]_\mu = t_n$.

Theorem 4.1 (Extension of Church's theorem, Klop [14]) Let R be a non-erasing OCRS and t be a weakly normalizable term in R. Then t is strongly normalizable.

Proof From Lemma 4.6 and Lemma 4.5.

Definition 4.3 We call a subterm s of a term t *unabsorbed in a reduction* $P : t \twoheadrightarrow e$ if the descendants of s do not appear inside redex-arguments of terms in P, and call s *absorbed in* P otherwise. We call s *unabsorbed in* t if it is unabsorbed in any reduction starting from t, and *absorbed in* t otherwise.

Definition 4.4 1. Let u_l be a redex in a term t defined as follows: choose an unabsorbed redex u_1 in t; choose an erased argument s_1 of u_1 that is not in normal form (if any); choose in s_1 an unabsorbed redex u_2, and so on, as long as possible. Let $u_1, s_1, u_2, \ldots, u_l$ be such a sequence. Then we call u_l a *limit redex* and call $u_1, s_1, u_2, \ldots, u_l$ a *limit sequence* of t.

2. We call a reduction *limit* if each contracted redex in it is limit, and call a strategy *limit* if in any term not in normal form it contracts a limit redex.

Similarly to the case of OTRSs [11], it can be shown that in any term not in normal form there is an unabsorbed redex, hence a limit redex as well.

Lemma 4.7 Let u be a limit redex in t and $P : t \twoheadrightarrow e$. Then there is no new redex in e that contains a descendant of u in its argument.
Proof Let $u_1, s_1, u_2, \ldots, u_l$ be the limit sequence of t with $u_l = u$. We prove by induction on $|P|$ that (a): descendants of redexes $u_1, \ldots, u_l$ do not appear inside arguments of new redexes. If $|P| = 0$, then (a) is obvious. So let $P : t \twoheadrightarrow e' \xrightarrow{v} e$, let o be a descendant of u in e, and o' be its ancestor in e'. It follows from the induction assumption that each redex $u_i \, (i = 1, \ldots, l-1)$ has exactly one residual u_i' in e' (because contraction of a residual of any of the redexes $u_1, \ldots, u_{l-1}$ erases the descendant of u), there is no new redex in e' that contains o' in its argument, and o is the only descendant of u. Thus if there is a new redex w in e that contains the residual u_i'' of some u_i in its argument, then it must be created by v. If $v \not\subseteq u_1'$, then w contains u_i'' in its argument iff it contains the residual of u_1' in its argument, but this is impossible since u_1 is unabsorbed. Thus $v \subseteq u_1'$. Let k be the maximal number such that v is in u_k' and let s_k' be the descendant of s_k in e'. Then v is in s_k' and contains u_{k+1}'. Let $Q : s_k \twoheadrightarrow s_k''$ consist of steps of P that are made in descendants of s_k. Then the residual of u_{k+1} is in an argument of the new redex $w \subseteq s_k''$. But this is impossible since u_{k+1} is unabsorbed in s_k. Thus (a) is valid and the lemma is proved.

Lemma 4.8 Let (Σ, R) be an OCRS, $P : t_0 \xrightarrow{u_0} t_1 \xrightarrow{u_1} \ldots \rightarrow t_n$ be a limit reduction in R, and P_μ : $s_0 = t_0 \xrightarrow{v_0} t_1 \xrightarrow{v_1} \ldots \rightarrow s_n$ be its μ-corresponding reduction in R_μ. Then
(1) for each k $(0 \le k \le n)$, the following holds:
$(a)_k$: $\|s_k\|_\mu = k$;
$(b)_k$: each redex $v_k' \subseteq s_k$ is μ-main in s_k.
$(c)_k$: in quasi-main arguments of any redex v_k'' in s_k there are no μ-symbols.
(2) if P is normalizing, then so is P_μ.
Proof (1) $(a)_0 - (c)_0$ are obvious. Suppose that $(a)_k - (c)_k$ hold and let us show $(a)_{k+1} - (c)_{k+1}$. Let $u_k = C[o_1, \ldots, o_q]$ and $v_k = C'[e_1, \ldots, e_q, e_1', \ldots, e_m']$, where $e_1, \ldots, e_q$ are μ-main arguments of v_k (which correspond to arguments $o_1, \ldots, o_q$ of u_k, respectively) and $e_1', \ldots, e_m'$ are μ-erased arguments of v_k. Since u_k and v_k are corresponding redexes in t_k and s_k, we have $[v_k]_\mu = u_k$ and hence (α): $[e_i]_\mu = o_i$ for all $i = 1, \ldots, q$. Let $e_{i_1}, \ldots, e_{i_l}$ be the v_k-quasi-erased arguments and $e_{j_1}, \ldots, e_{j_p}$ be the v_k-quasi-main arguments. Then the contractum of v_k in R_μ has the following form: $o' = \mu\mu^0 \ldots \mu^0 e_1' \ldots e_m' e_{i_1} \ldots e_{i_l} o$. By Proposition 4.1.(3), $o_{i_1}, \ldots, o_{i_l}$ are u_k-erased, and since u_k is limit, (β): $o_{i_1}, \ldots, o_{i_l}$ are in R-nf. By $(c)_k$, (γ): there are no occurrences of μ-symbols in $e_{j_1}, \ldots, e_{j_p}, o$. (Hence o coincides with the contractum of u_k). It follows from $(\alpha), (\beta), (b)_k$, and Lemma 4.2 that (δ): $e_{i_1}, \ldots, e_{i_l}$ are in R_μ-nf.

By (γ), $\|o'\|_\mu = \|v_k\|_\mu + 1$. Hence $\|s_{k+1}\|_\mu = \|s_k\|_\mu + 1 = k + 1$, i.e., $(\alpha)_{k+1}$ holds.

If $v_{k+1}' \not\subseteq o'$, then $(b)_k$ implies that v_{k+1}' is μ-main. If $v_{k+1}' \subseteq o'$, then by $(b)_k$, $v_{k+1}' \not\subseteq e_1', \ldots, e_m'$ (since ancestors of $e_1', \ldots, e_m'$ are μ-erased arguments of v_k) and by $(\delta), v_{k+1}' \not\subseteq e_{i_1}, \ldots, e_{i_l}$. Hence $v_{k+1}' \subseteq o$ and by (γ), v_{k+1}' is μ-main. Now $(b)_{k+1}$ is proved.

If $o' \cap v_{k+1}'' = \emptyset$, then $(c)_{k+1}$ follows immediately from $(c)_k$. If $v_{k+1}'' \subseteq o'$, then as we have shown above (for v_{k+1}'), $v_{k+1}'' \subseteq o$ and $(c)_{k+1}$ follows from (γ). Suppose now that o' is a proper subterm of v_{k+1}'' and v_{k+1}'' has an v_k-ancestor v_k^* in s_k for which v_{k+1}'' is a residual. Let u_k^* be corresponding redex of v_k^* in t_k (it exists because, by $(b)_k$, v_k^* is μ-main). Obviously, u_k is a proper subterm of u_k^* and since u_k is limit, it must be in an erased argument of u_k^*. By $(c)_k$, v_k^*-quasi-main arguments do not contain μ-symbols. Thus the sets of free variables of v_k^*-quasi-main arguments coincide with that of corresponding arguments of u_k^*. Hence, by Proposition 4.1.(2), v_k is in a quasi-erased argument of v_k^*. Therefore, by Proposition 4.1.(1), o' is in a quasi-erased argument of v''_{k+1} and the quasi-main arguments of v''_{k+1} coincide with the corresponding quasi-main arguments of v_k^*. Thus, by $(c)_k$, in the quasi-main arguments of v''_{k+1} there are no occurrences of μ-symbols. To prove c_{k+1}, it remains to consider the case then o' is a proper subterm of v''_{k+1} and v''_{k+1} is created by v_k. If in quasi-main arguments of v''_{k+1} there are μ-symbols, then in main arguments of corresponding redex u''_{k+1} in s_{k+1}, which is also an u_k-new redex, there are descendants of redexes contracted in P. But each redex contracted in P is a limit redex. Thus, by Lemma 4.7, their descendants can not occur in arguments of new redexes. Hence, also in this case, there are no μ-symbols in quasi-main arguments of v''_{k+1}, and $(c)_{k+1}$ is valid.

(2) By Lemma 4.2 and $(b)_n$.

Theorem 4.2 A limit strategy is perpetual in OCRSs. Moreover, if a term t in an OCRS R is strongly normalizable, then a limit strategy constructs a longest normalizing reduction starting from t, and its length coincides with the μ-norm of R_μ-nf of t.

Proof If a limit R-reduction P starting from t is normalizing, then by Lemma 4.8 its corresponding R_μ-reduction also is normalizing. Hence, by Lemma 4.5, t is strongly normalizable in R. Thus, the limit strategy is perpetual. Now, if t is strongly normalizable, Q is a normalizing R-reduction, and s is an R_μ-normal form of t, then $|Q|$ =(by Corollary 4.1)= $|Q_\mu| \leq$ (by the CR property of R_μ)$\leq \|s\|_\mu$ =(by Lemma 4.8)= $|P|$. Thus, P has the maximal length among all reductions of t to normal form.

Proposition 4.2 (Klop [14]) A term t in an OCRS R is strongly normalizable iff t is weakly normalizable in R_μ.

Proof ($\Rightarrow$) From Lemma 4.8. ($\Leftarrow$) From Lemma 4.5.

The following example shows that, despite the claim of Klop [14] (p. 181, Remark 6.2.5), if R is strongly normalizing, then R_μ does not need to be strongly normalizing.

Example 4.3 Let $R = \{r : f(\tau a(c, A)) \to g((\tau a(A, A)/a)A)\}$, where c is a constant, a is an object metavariable, and τ is a quantifier sign of arity $(1, 2)$ and scope indicator $(1, 2)$. During r-step creation inside contractum is only possible when, say in the case $a = x$, A has a subterm $f(s)$, i.e., $A = C[f(x)]$, and $\tau a(A, A) = \tau x(c, c)$, i.e., $A = c$, or $A = C[f(\tau yx)]$ with $y \neq x$ and $\tau a(A, A) = c$, but this of course never happens. During r-step creation of a redex is not possible also outside contractum, because in this case the outermost g of the contractum should belong to the pattern of a new redex, but this is impossible because g is not a pattern-symbol. Thus no redex creation is possible in R and hence R is strongly normalizing, while contraction of the redex $v = f(\tau x(\mu^2(f(x), c), \mu^2(f(x), c))$ in R_μ creates itself: $v \to \mu^2(f(x), g(\mu^2(v, c)))$, and it is easy to see that v is not normalizable in R_μ. (Recall that, for the case of OTRSs, R is strongly normalizing iff R_μ is weakly normalizing [14, 12].)

Similarly to the case of OTRSs [12], one can define subclasses of OCRSs for which the unabsorbed redexes can be found efficiently; hence the limit strategy is efficient.

Definition 4.5 (1) We call an OCRS *non-absorbing* if, for any reduction step $t \xrightarrow{u} s$, arguments of any new redex in s are in the contractum of u.

(2) We call an OCRS *non-left-absorbing* (resp. *non-right-absorbing*) if, for any reduction step $t \xrightarrow{u} s$, any argument of a created redex in s is inside the contractum of u or to the right (resp. to the left) of it.

Proposition 4.3 (1) Let t be a term in a non-absorbing OCRS. Then any outermost redex in t is unabsorbed.

(2) Let t be a term in a non-left absorbing (resp. non-right-absorbing) OCRS. Then the leftmost-outermost (resp. the rightmost outermost) redex in t is unabsorbed.

Proof From Definitions 4.3 and Definition 4.5.

Remark 4.1 It is easy to see that the leftmost redexes in λ-terms are unabsorbed. Therefore the perpetual strategy of Barendregt et al. [3] is a limit strategy. The proof presented in Barendregt [2] uses only unabsorbness of the leftmost redexes and therefore generalizes easily to the case of OCRSs. The proof of the Conservation Theorem [2] also remain valid for OCRSs: if a term t has an infinite reduction and $t \xrightarrow{u} s$, where u is a non-erasing redex, then s has also an infinite reduction. Bergstra and Klop [4] gave a characterization of erased redexes (i.e. K-redexes) for which the Conservation Theorem in λ-calculus still is valid. Another extension of the Conservation Theorem that can be used for strong normalization proofs of several typed λ-calculi can be found in de Groote [6]. We leave this questions for OCRSs to a future investigation.

The direct proof of the fact that the perpetual reductions are the longest in OTRSs, presented in [12], cannot be generalized to the case of OCRSs (complete redexes does not necessarily remain complete because main arguments may become erased after some steps in the main arguments).

5. Longest reductions in strongly persistent OCRSs

In this section, we design an algorithm for finding the lengths of longest reductions in *strongly persistent* CRSs; as a corollary we obtain an algorithm for finding exact upper bounds of lengths of developments in orthogonal CRSs. To this end, we introduce and study *strong similarity* of redexes.

Without restricting the class of OCRSs, we can assume that in right-hand sides of rewrite rules the immobile argument of each metasubstitution is a term-metavariable or a metasubstitution. For example, we can replace the metasubstitution $f((B/a)g(A))$ by the "equivalent" metasubstitution $f(g((B/a)A))$, and the metasubstitution $(A_1/a_1, A_2/a_2)\sigma a A_0$, where σ is a quantifier sign of arity $(1,1)$ and $a \neq a_1, a_2$, by $\sigma a((A_1/a_1, A_2/a_2)A_0)$. (If $a = a_1$ or $a = a_0$, then we first rename the bound object metavariables). Hence, we can have the following definition.

Definition 5.1 (1) Let $t \xrightarrow{u} s$ in an OCRS R, let $t \to t' \twoheadrightarrow s$ be its expansion, and let v be a new redex in s. We call v *generated* if v is a residual of a redex of t' whose pattern is in the pattern of the contractum of u in R_f.

(2) We call an OCRS R *persistent* (written PCRS) if, for any R-reduction step, each created redex is generated.

(3) We call an OCRS R *strongly persistent* (written SPCRS) if R_{fS} is persistent.

(4) We call an OCRS R *left-canonical* if, for any R-rule $t \to s$, the pattern of t consists of one operator, i.e., t has the form $\sigma a_1 \ldots a_m A_1 \ldots A_n$, where σ is an operator sign of arity (m,n) (σ is a function iff $m, n = 0$).

(5) We call an OCRS R *non-creating* if no redex-creation is possible during reduction steps in R.

Remark 5.1 In [13], we call left-canonical CRSs *Higher Order Recursive Program Schemes.* It is easy to see that a non-simple OCRS R is strongly persistent iff it is left canonical [13].

Lemma 5.1 If R is a strongly persistent OCRS, then so is R_μ.

Proof If R is simple, then the lemma is obvious. Otherwise, by the above remark, R is left canonical and so is R_μ; thus R_μ is strongly persistent.

Remark 5.2 Example 4.3 shows that if R is persistent, then R_μ does not need to be persistent (in the reduction $v = f(\tau x(\mu^2(f(x),c),\mu^2(f(x),c)) \xrightarrow{v} \mu^2(f(x),g(\mu^2(v,c))$, the pattern of the created v contains argument-symbols of the contracted v). On the other hand, a PCRS R such that R_μ also is persistent does not need to be strongly persistent. For example, let $R = \{r_1 : \exists aA \to f((\tau aA/a)A), r_2 : g(f(x)) \to c\}$. Since $\{r_1\}$ is left-canonical, $\{r_{1\mu}\}$ also is left-canonical, hence persistent. Since $\{r_2\}$ is simple and is persistent, $\{r_{2\mu}\}$ is also persistent. Hence, because of "independence" of r_1 and r_2, R_μ is persistent. But R is not strongly persistent, since it is non-simple and is not left-canonical.

Definition 5.2 Let $C[\overline{a_1}A_1,\ldots,\overline{a_n}A_n]$ be the left-hand side of a rewrite rule r in an OCRS R and let $C[\overline{x_1}t_1,\ldots,\overline{x_n}t_n]$ be an r-redex.

(1) The *characteristic system* of u (written $CS(u)$) is the set of pairs (a_{i_j},A_i) such that $x_{i_j} \in FV(t_i)$ $(i = 1,\ldots,n, j = 1,\ldots,n_i)$. In this case, u is an $(r, CS(u))$*-redex*. A *characterized rule* (*C-rule* for short) is a pair $(r, CS(r))$, where $CS(r)$ is a characteristic system for some R-redex. The *main characteristic system* $MCS(u)$ of u is the subset of $CS(u)$ containing a pair (a_{i_j},A_i) iff i-th argument of u is main.

(2) The *strong characteristic system* of u (written $SCS(u)$) is the set of triples of the form (a_{i_j},A_i,n_{i_j}) such that $(a_{i_j},A_i,n_{i_j}) \in SCS(u)$ iff x_{i_j} has n_{i_j} free occurrences in t_i. In this case, u is an $(r, SCS(u))$*-redex*. A *strongly characterized rule* (*SC-rule* for short) is a pair $(r, SCS(r))$, where $SCS(r)$ is a strong characteristic system for some R-redex. The *main strong characteristic system* $MSCS(u)$ of u is the subset of $SCS(u)$ containing a triple (a_{i_j},A_i,n_{i_j}) iff i-th argument of u is main.

(3) We call weakly similar redexes u and v respectively *similar*, *m-similar*, *strongly similar*, or *strongly m*-similar if $CS(u) = CS(v)$, $MCS(u) = MCS(v)$, $SCS(u) = SCS(v)$, or $MSCS(u) = MSCS(v)$.

Proposition 5.1 Let u and v be m-similar redexes in an OCRS R. Then an argument of u is main iff its corresponding argument in v is main.
Proof The proposition is a corollary of Lemma 3.7.

Definition 5.3 Let $u = C[\overline{x_1}t_1,\ldots,\overline{x_n}t_n]$ and $v = C[\overline{y_1}s_1,\ldots,\overline{y_n}s_n]$ with $\overline{x_i} = \{x_{i_1},\ldots,x_{i_{n_i}}\}$ and $\overline{y_i} = \{y_{i_1},\ldots,y_{i_{n_i}}\}$ be weakly similar redexes. We call u and v *strongly* $\underline{S}$*-essentially similar* (resp. *strongly* $\underline{S}$*-essentially m-similar* if x_{i_j} and y_{i_j} have the same number of $\underline{S}$-essential occurrences in t_i and s_i for all $i = 1,\ldots,n; j = 1,\ldots,n_i$ (resp. for all $i = 1,\ldots,n; j = 1,\ldots,n_i$ such that t_i and s_i are main arguments of u and v, respectively). (Note that if the arguments of u and v do not contain $\underline{S}$-redexes, then strong $\underline{S}$-essential (m-)similarity and strong (m-)similarity of u and v coincide.)

Lemma 5.2 Let R be a SPCRS, let $u = C[\overline{x_1}t_1,\ldots,\overline{x_n}t_n]$ and $v = C[\overline{x_1}s_1,\ldots,\overline{x_n}s_n]$ be strongly m-similar R_μ-redexes whose arguments are in nf and are not variables. Further, let $P : u = o_0 \xrightarrow{u} o_1 \xrightarrow{u_1} \ldots$ and $Q : v = e_0 \xrightarrow{v} e_1 \xrightarrow{v_1} \ldots$ be expansions of rightmost R_μ-reductions of u and v that are infinite or end at normal forms. Then it is possible to define one-to-one *correspondence* between the following *occurrences* of o_i and e_i:

(1) $\underline{S}$-essential redexes and their arguments;

(2) $\underline{S}$-essential descendants of redexes;

(3) $\underline{S}$-essential descendants of arguments of u and v, called *argument subterms*; and

(4) $\underline{S}$-essential descendants* of free occurrences of variables $\overline{x_j}$ in t_j and s_j $(j = 1,\ldots,n)$, called *context variables*, where notion of descendant* is defined similarly to that of descendant with

the exception that, during $\underline{S}$-steps $\underline{S}x_1 \dots x_n t_1 \dots t_n t_0 \to (t_1/x_1, \dots, t_n/x_n)t_0$, free occurrences of $x_1, \dots, x_n$ in t_0 do *not* have descendants*.

Furthermore, for each i ($i = 0, 1, \dots$), the following conditions hold:

$(a)_i$: corresponding $\underline{S}$-essential redexes in o_i and e_i are strongly $\underline{S}$-essentially m-similar (in fact, strongly $\underline{S}$-essentially similar if $m > 1$), and u_i and v_i are corresponding $\underline{S}$-essential redexes if one of them is $\underline{S}$-essential.

$(b)_i$: if o^* and e^*, as well as o'' and e'', are corresponding $\underline{S}$-essential occurrences in o_i and e_i, then $o^* \subseteq o''$ iff $e^* \subseteq e''$.

Proof By induction on i. The case $i = 0$ is obvious from the assumptions. Suppose that we have defined the corresponding $\underline{S}$-essential occurrences in o_m and e_m in such a way that $(a)_m$ and $(b)_m$ hold. Assume first that $u_m = \underline{S}y_1 \dots y_k t'_1 \dots t'_k t'_0$ and $v_m = \underline{S}y_1 \dots y_k s'_1 \dots s'_k s'_0$ are $\underline{S}$-redexes. If u_m and v_m are $\underline{S}$-inessential, then all $\underline{S}$-essential occurrences of o_m and e_m are outside u_m and v_m, and each of them has exactly one $\underline{S}$-essential descendant in o_{m+1} and e_{m+1} respectively. Hence descendants of $\underline{S}$-essential corresponding occurrences of o_m and v_m form pairs of corresponding occurrences in o_{m+1} and e_{m+1}. So suppose that both u_m and v_m are $\underline{S}$-essential. It follows from $(a)_m$ and $(b)_m$ that (α): y_i has the same number of corresponding $\underline{S}$-essential occurrences in t'_0 and s'_0 and in each pair of corresponding $\underline{S}$-essential subterms of t'_0 and s'_0. It follows from Corollary 3.5 and Definition 3.1 that descendants of $\underline{S}$-essential occurrences of o_m and v_m are $\underline{S}$-essential in o_{m+1} and e_{m+1} iff they are substituted for $\underline{S}$-essential context-variables. Thus corresponding $\underline{S}$-essential subterms in o_m and e_m have the same number of $\underline{S}$-essential descendants, and corresponding $\underline{S}$-essential context-variables have the same number of $\underline{S}$-essential descendants* in o_{m+1} and e_{m+1}; they form pairs of corresponding $\underline{S}$-essential occurrences in o_{m+1} and e_{m+1}. Since argument-subterms in o_m and e_m are not variables, different subterms have different descendents. Thus the correspondence between these subterms in o_{m+1} and e_{m+1} remains one-to-one. Since, by Lemma 5.1, R_μ is persistent, no new redexes are created in these steps. Thus $(a)_{m+1}$ follows from $(a)_m$ and from the fact that the context-variables form pairs of corresponding occurrences. $(b)_{m+1}$ follows from (α) and $(b)_m$.

Suppose now that u_m and v_m are $R_{\mu f}$-redexes. In this case, there are no $\underline{S}$-redexes in o_m and e_m. Obviously, the contractum of u_m can be obtained from the contractum of v_m by replacing descendants of arguments of v_m with the corresponding arguments of u_m. Since, by Lemma 5.1, R_μ is persistent, for each new redex w in o_{m+1} there is a unique new redex w' in e_{m+1}. All the descendants of the occurrences that are outside u_m and v_m are $\underline{S}$-essential. Apart from these occurrences, descendants of only occurrences that are in main arguments of u_m and v_m can be $\underline{S}$-essential in o_{m+1} and e_{m+1}. By $(a)_m$, u_m and v_m are strongly m-similar. Hence, it follows from conditions (a)-(b) of Definition 2.2 and Lemma 3.6 that corresponding new redexes in o_{m+1} and e_{m+1} are either both essential or both are inessential, the same holds for corresponding arguments of the corresponding redexes, and corresponding occurrences in o_m and e_m have the same number of $\underline{S}$-essential descendants in o_{m+1} and e_{m+1}; together with corresponding $\underline{S}$-essential new redexes they form pairs of corresponding $\underline{S}$-essential occurrences in o_{m+1} and e_{m+1}; the correspondence remains one-to-one. Since variables bound by quantifiers belonging to patterns of w and w' can only occur in the descendants of arguments of u_m and v_m, and $\underline{S}$-essential (in o_{m+1} and e_{m+1}) occurrences of context variables form pairs of corresponding $\underline{S}$-essential occurrences in corresponding arguments of w and w', it follows from Lemma 3.5 that w and w' are strongly $\underline{S}$-essentially similar. Hence $(a)_{m+1}$ follows from $(a)_m$. $(b)_{m+1}$ follows easily from $(b)_m$.

Definition 5.4 Let R be an SPTRS.

(1) Let t be a term in R_μ, let s be a non-variable subterm of t, and let $P : t \twoheadrightarrow e$ be the rightmost innermost normalizing R_μ-reduction. Then, by definition, $Mult_\mu(s, t)$ is the number of

P-descendants of s in e.

(2) Let $u = C[e_1, \ldots, e_n]$ be an r-redex in R_μ, let s' be a non-variable subterm in e_i, let $v = C[o_1, \ldots, o_n]$ be an r-redex strongly m-similar to u whose arguments $o_1, \ldots, o_n$ are in R_μ-normal form and are not variables, and let $Q : v \twoheadrightarrow o$ be the rightmost innermost normalizing R_μ-reductions. Then, by definition, $mult_\mu(u,i) = mult_\mu(u,s') = mult_\mu(r, SCS(u), i) = Mult_\mu(o_i, v)$, and $mult_\mu(u) = mult_\mu(r, SCS(u))$ is a number of μ-subterms in o that appear during Q, i.e., that are not descendants of μ-subterms from (the pattern and arguments of) v. Numbers $mult_\mu(u,i)$ and $mult_\mu(r, SCS(r), i)$ are called *proper μ-indices* of u and $(r, SCS(u))$, and numbers $mult_\mu(u)$ and $mult_\mu(r, SCS(u))$ are called *μ-indices* of u and $(r, SCS(u))$.

The correctness of the above definition follows from Lemma 5.2.

Lemma 5.3 Let t be a strongly normalizable term in a PCRS R_μ, $e \subseteq s \subseteq t$ and e and s be non-variable R_μ-nfs. Then $Mult_\mu(s,t) = Mult_\mu(e,t)$.

Proof Let $t = t_0 \to t_1 \to \ldots \to t_n$ be the expansion of the rightmost normalizing R_μ-reduction. Let us define pairs (s_i^j, e_i^j) of descendants of s and e in t_i (if any) for each $i = 1, \ldots$ in such a way that $(\alpha)_i$: there is no redex in t_i that contains e_i^j and does not contain s_i^j. Obviously, the pair (s, e) satisfies $(\alpha)_0$. Suppose that pairs of descendants of s and e are defined in t_m in such a way that $(\alpha)_m$ holds and let $t_m \overset{u_m}{\to} t_{m+1}$. If u_m is not an S-redex or an $\underline{S}$-redex, then it ie clear that for any pair (s_m^k, e_m^k) of corresponding descendants of s and e in t_m, s_m^k and e_m^k have the same number of descendants, and these descendants form the pairs of corresponding occurrences in t_{m+1}. Since R_μ is persistent, new redexes in t_{m+1} (if any) are not inside the descendants of s_m^k in t_{m+1}. Hence $(\alpha)_{m+1}$ follows immediately from $(\alpha)_m$ in this case. Suppose now that $u_m = Sx_1 \ldots x_l s_1 \ldots s_l s_0$ is an S-redex or an $\underline{S}$-redex. By $(\alpha)_m$, there is no pair (s_m^k, e_m^k) in t_m such that $e_m^k \subseteq u_m \subseteq s_m^k$. Thus either both s_m^k and e_m^k are in the same argument of u_m or non of them is in u_m. Hence s_m^k and e_m^k have the same number of descendants and they form pairs of corresponding occurrences in t_{m+1}. Since R_μ is persistent, there are no new redexes in t_{m+1}. Thus $(\alpha)_{m+1}$ follows immediately from $(\alpha)_m$. Now it is clear that s and e have the same number of descendants in t_n, i.e., $Mult_\mu(s,t) = Mult_\mu(e,t)$.

Notation $L(t)$ denotes the length of a longest reduction starting from t.

Lemma 5.4 Let t be a strongly normalizable term in an SPCRS R and $u_1, \ldots, u_n$ be all redexes in t. Then

$$L(t) = \sum_{i=1}^{n} Mult_\mu(u_i, t) mult_\mu(u_i)$$

Proof Let $P : t \twoheadrightarrow o$ be the rightmost innermost normalizing R_μ-reduction and let $u_1, \ldots, u_n$ be the enumeration of redexes in t from right to left. In the fragment of P in which (the residual of) u_i is reduced to R_μ-nf, $mult_\mu(u_i)$ new μ-symbols appear (in the beginning of the fragment, all arguments of u_i are already in R_μ-nf). By Lemma 5.3, during the rest of P each of these μ-occurrences is copied $Mult_\mu(u_i, t)$-times. Hence

$$\|o\|_\mu = \sum_{i=1}^{n} Mult_\mu(u_i, t) mult_\mu(u_i)$$

and the lemma follows from Theorem 4.2.

Lemma 5.5 Let t be a strongly normalizable term in an PCRS R_μ and $u_1, \ldots, u_n$ be all redexes in t that contain a non-variable subterm s in their arguments. Suppose that s is in m_i-th argument of u_i $(i = 1, \ldots, n)$. Then

$$Mult_\mu(s,t) = \prod_{i=1}^{n} mult_\mu(u_i, s) = \prod_{i=1}^{n} mult_\mu(u_i, m_i)$$

Proof Let $P : t \twoheadrightarrow o$ be the rightmost innermost normalizing R_μ-reduction. It follows from Lemma 5.3 that, in the fragment of P in which (the residual of) u_i is reduced to R_μ-nf, each descendant of s is copied $mult_\mu(u_i, s) = mult_\mu(u_i, m_i)$-times. Thus the lemma follows from persistency of R_μ.

Lemma 5.6 Let $u = C[e_1, \ldots, e_k]$ be an r-redex whose arguments $e_1, \ldots, e_k$ are not variables and are in normal form, in an SPCRS R. Then, for all $i = 1, \ldots, k$,

$$mult_\mu(u,j) = mult_\mu(r, SCS(r), j) = \sum_{i=1}^{m_j} Mult_\mu(e_{j_i}, o),$$

$$mult_\mu(u) = mult_\mu(r, SCS(r)) = \sum_{i=1}^{m} Mult_\mu(u_i, o) mult_\mu(u_i) + 1,$$

where o is the contraction of u in R_μ, $e_{j_1}, \ldots, e_{j_{m_j}}$ are all descendants of e_j in o, and $u_1, \ldots, u_m$ are all redexes in o.
Proof From Definition 5.4 and Theorem 4.2.

Lemma 5.7 Let u and v be strongly m-similar redexes in an SPCRS R, let $u \xrightarrow{u} o$ and $v \xrightarrow{v} e$. Then u and v create the same number of strongly similar redexes.
Proof If in the Lemma 5.2 one takes for P the expansion of u and for Q the expansion of v, then it follows from Lemma 5.2 that for each u-new redex in o there is exactly one strongly similar v-new redex in e.

Corollary 5.1 Let u and v be strongly m-similar redexes in an OCRS R, let $u \xrightarrow{u} o$ and $v \xrightarrow{v} e$. Then u and v generate the same number of strongly similar redexes.

Definition 5.5 We call a sequence of SC-rules $(r_0, SCS(r_0)), (r_1, SCS(r_1)), \ldots$ an $(r_0, SCS(r_0))$-*chain* if an $(r_{i+1}, SCS(r_{i+1}))$-redex is generated by contraction of any $(r_i, SCS(r_i))$-redex. For any $(r_0, SCS(r_0))$-redex u, we also call an $(r_0, SCS(r_0))$-chain an u-*chain.*

The correctness of the above definition follows from Lemma 5.7. In [13], we used C-rules instead of SC-rules to define chains of redexes, but it is easy to see that for each chain of C-rules there is a chain of SC-rules with the same length, and vice versa. Therefore, the following theorem from [13] remains valid for the above definition of chains of redexes.

Theorem 5.1 ([13]) A term t in a PCRS R is strongly normalizable iff all chains of redexes in t are finite.

Theorem 5.2 Let t be a term in an SPCRS R. Then the least upper bound $L(t)$ of lengths of reductions starting from t can be found using the following

Algorithm 5.1 Let $(r_1, SCS(r_1)), \ldots, (r_n, SCS(r_n))$ be all strongly characterized rules such that an $(r_i, SCS(r_i))$-redex has an occurrence in t $(i = 1, \ldots, n)$. If an $(r_i, SCS(r_i))$-chain is infinite for at least one i, then $L(t) = \infty$. Otherwise, using Lemmas 5.6 and 5.5, find the μ-indices and the proper μ-indices of all rules $(r_i, SCS(r_i))$. Finally, using Lemmas 5.5 and 5.4, find $L(t)$.

Proof From Theorems 5.1 and 4.2, and Lemmas 5.4-5.7.

Remark 5.3 It is easy to see that the above results remain valid if we use "main" μ-indices and "main" proper μ-indices $MSCS(\ )$ instead of μ-indices and proper μ-indices $SCS(\ )$.

5.1 The least upper bound of lengths of developments

Let $R = \{r_i : t_i \to s_i \mid i \in I\}$ be an OCRS and let $\underline{R} = \{\underline{r}_i : \underline{t}_i \to s_i \mid i \in I\}$, where $\underline{t}_i$ is obtained from t_i by underlining its head-symbol. *Terms* in $\underline{R}$ are constructed in the usual way with the restriction that underlined symbols may only occur as head-symbols of redexes. Then, for each development $P : e_0 \to e_1 \to \ldots \to e_n$ of e_0 in R (in which only residuals of redexes from e_0 are contracted), there is a reduction $P' : e'_0 \to e'_1 \to \ldots \to e'_n$ in $\underline{R}$ such that e'_i is obtained from e_i by underlining head-symbols of residuals of redexes from e_0. Obviously, $\underline{R}$ is persistent, since no creation of redexes is possible in it. Thus, to find least upper bounds of developments in R, one can use Algorithm 5.1, which becomes simpler in this case: for any strongly characterized rule $(\underline{r}, SCS(\underline{r}))$, $mult_\mu(\underline{r}, SCS(\underline{r})) = 1$, $mult_\mu(\underline{r}, SCS(\underline{r}), i) = 1$ if the i-th argument o_i of an $(R, SCS(r))$-redex does not have a descendant, and $mult_\mu(\underline{r}, SCS(\underline{r}), i)$ coincides with the number of descendants of o_i otherwise.

Acknowledgments I enjoyed discussions with H. Barendregt, J. W. Klop, J.-J. Lévy, L. Maranget, G. Mints, V. van Oostrom, Sh. Pkhakadze, Kh. Rukhaia, and V. Sazonov. I also would like to thank G. Gonthier, G. Tagviashvili, K. Urbaitis, and F. J. de Vries for their help in preparation of this paper.

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