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Generalizing Finiteness Conditions of Labelled Transition Systems

F. van Breugel

Computer Science/Department of Software Technology

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Franck van Breugel

CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands
Vrije Universiteit
P.O. Box 7161, 1007 MC Amsterdam, The Netherlands

Abstract

A labelled transition system is provided with some additional structure by endowing the configurations and the labels with a complete metric. In this way, a so-called metric labelled transition system is obtained. The additional structure on a metric labelled transition system makes it possible to generalize the finiteness conditions finitely branching and image finite to compactly branching and image compact, respectively.

Some topological properties of the operational semantic models and the so-called higher order transformations induced by labelled transition systems satisfying one of the finiteness conditions are discussed. These results are generalized for metric labelled transition systems satisfying one of the generalized finiteness conditions. The generalized results are shown to be useful for studying semantics of programming languages. For example, a proof principle for relating different semantic models for a given language based on the results is presented.

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Introduction

In the field of *semantics of programming languages*, various mathematical structures are used nowadays. Since the late sixties, complete lattices and complete partial orders play a primary role in this field. In the late seventies, complete metric spaces entered the scene. The last five years, there is a growing interest in nonwellfounded sets.[1]

In this paper, we will concentrate on semantic models for programming languages based on *complete metric spaces*. Main parts of the theory in this area have been developed by Arnold and Nivat and their co-workers ([AN80]), the Programming Research Group of Oxford University ([Ree89]), and the Amsterdam Concurrency Group ([BR92]). In this paper, we will introduce some new - what we think are important - concepts which enable us to generalize some of the existing theory. The generalizations give rise to simplifications of proofs of some already known results. Furthermore, we are able to model more advanced programming language notions by means of the new concepts.

We will focus on semantics defined by means of *labelled transition systems*. The use of labelled transition systems for defining so-called *operational semantics* seems to originate with Keller ([Kel76]). The standard work on operational semantics is Plotkin's [Plo81]. Not only in operational semantics,

[1] Of the above mentioned structures, complete partial orders, complete metric spaces, and nonwellfounded sets have been put into a unifying categorical framework by Rutten and Turi in [RT92].

but also in *denotational semantics* labelled transition systems and their theory have turned out be useful (see, e.g., Rutten's [Rut92]).

It is well-known that an operational semantic model induced by a labelled transition system satisfying some *finiteness condition* has a corresponding *topological property*. For example, the so-called linear operational semantics[2] induced by a *finitely branching* labelled transition system is *compact*, and the linear operational semantics induced by an *image finite* labelled transition system is *closed*. Similar results hold for the so-called branching operational semantics[3].

If an operational semantics has one of the above mentioned topological properties, then we can possibly use the *unique fixed point proof principle* in order to relate the semantics to another semantics. This proof principle has been introduced by Kok and Rutten in [KR90]. It is based on *Banach's fixed point theorem* ([Ban22]): a contractive mapping from a complete metric space to itself has a unique fixed point. The proof principle has been applied successfully to relate semantic models for various programming language notions (see, e.g., the theses [AR89a, Eli91, Hor93, Kok89]). An order-theoretic version of the proof principle has been introduced by Hennessy and Plotkin in [HP79].

In order to model advanced notions like Baeten and Bergstra's real time integration ([BB91]) and higher order communication as, e.g., in Thomson's CHOCS ([Tho90]), we sometimes need to deal with labelled transition systems which do not satisfy the above mentioned finiteness conditions. In order to *generalize* the finiteness conditions of labelled transition systems (such that the induced operational semantic models still have the desired topological properties), we will supply the labelled transition system with some additional structure. The structure is added by endowing the configurations and the labels with a complete metric. We will call such an enriched labelled transition system a *metric labelled transition system*. The additional structure enables us to generalize the finiteness conditions finitely branching and image finite to *compactly branching* and *image compact*, respectively[4].

Already in the early sixties, the problem what structure to add to an abstract machine - like a labelled transition system - to obtain a topological machine was formulated by Ginsburg in [Gin62]. In [Shr64], Shreider introduced a particular topological machine - a so-called compact automaton - in order to study dynamic programming. A general and detailed study of topological machines can be found in Brauer's [Bra70]. Our metric labelled transition systems are a special case of Brauer's topological machines. However, the results presented in this paper cannot be found (in some possibly more general form) in Brauer's paper. In [Ken87], Kent studied so-called metrical transition systems. A metrical transition system is a labelled transition system the configurations of which are endowed with an ultraquasimetric (the labels are not provided with any additional structure). In Kent's paper, semantics induced by a metrical transition systems are not addressed. Structures related to labelled transition systems, like abstract reduction systems, have also been provided with additional structure by endowing certain sets with metrics (cf., e.g., Kennaway's metric abstract reduction systems in [Ken92]).

The present paper can be divided into two parts. In the first and main part, a short survey of some theory on labelled transition systems is given and subsequently the theory is generalized. In the second part, we present six applications of the theory developed in the first part to provide some evidence of its usefulness.

We will consider an operational semantics induced by a (metric) labelled transition system to be a mapping from the configurations of the (metric) labelled transition system to some mathematical structure built from the labels of the (metric) labelled transition system. In the first section of this paper, we will define five spaces built from the (complete metric space of) labels by means of recursive

[2] In Van Glabbeek's linear time - branching time spectrum ([Gla90]) this semantics is called the infinitary completed trace semantics.

[3] Van Glabbeek uses the term bisimulation semantics for this semantics.

[4] Compact is a topological generalization of finite. For example, every compact subset of a metric space is the limit of a sequence of finite sets.

domain equations. In the (systems of) domain equations, we will encounter the compact or closed power set (being the metric counterpart of the Smyth, Plotkin, and Hoare power domains as has been shown by Bonsangue and Kok in [BK94]). So far, these spaces - called domains in the sequel - have only been studied in case the labels are endowed with the discrete metric.

In the second section, we will present the definitions of labelled transitions system, linear operational semantics, and the finiteness conditions finitely branching and image finite. Furthermore, the topological properties of the linear operational semantics induced by a labelled transition system satisfying one of the finiteness conditions will be discussed.

All the - already known - results from the second section are generalized in the third section by going from labelled transition systems to metric labelled transition systems and from finitely branching and image finite to compactly branching and image compact.

In the fourth section, we will study so-called *higher order transformations*. Higher order transformations play an important role in the formulation of the already mentioned unique fixed point proof principle. A higher order transformation assigns to a semantics of a programming language another semantics of the language. A semantics of a programming language PL is considered to be a mapping from the language - the set of statements of the language possibly provided with some additional information - to some mathematical structure MS. A corresponding higher order transformation is of the form $\Phi : (PL \rightarrow MS) \rightarrow (PL \rightarrow MS)$. In case MS is a complete metric space, also $PL \rightarrow MS$ can be turned into a complete metric space, and hence Φ is a mapping from a complete metric space to itself. If the higher order transformation Φ is contractive, then Φ has a unique fixed point according to Banach's fixed point theorem.

In *proof by uniqueness of fixed point*, we have two semantic models for a programming language PL, viz $\mathcal{S}_1 : PL \rightarrow MS$ and $\mathcal{S}_2 : PL \rightarrow MS$, we want to prove to be equivalent. Suppose that we can turn the mathematical structure MS into a complete metric space. Assume we can find a contractive higher order transformation $\Phi : (PL \rightarrow MS) \rightarrow (PL \rightarrow MS)$ such that both $\mathcal{S}_1$ and $\mathcal{S}_2$ are fixed point of Φ. Then we can conclude that $\mathcal{S}_1$ and $\mathcal{S}_2$ must be equal.

In this paper, we will focus on higher order transformations induced by (metric) labelled transition systems. That is, the semantic models to be transformed are mappings from the configurations of the labelled transition system to some domain built from the labels of the labelled transition system, and the transformation is driven by the transition relation of the labelled transition system. First, we will introduce a so-called *linear higher order transformation* induced by a metric labelled transition system. The linear operational semantics induced by a metric labelled transition system will be shown to be fixed point of the corresponding linear higher order transformation. If the metric labelled transition system satisfies one of the generalized finiteness conditions, then the induced linear higher order transformation will be proved to be a contractive mapping from a complete metric space to itself. Consequently, we can use the unique fixed point proof principle to relate the linear operational semantics to another semantics as sketched above. Second, we will define a *branching operational semantics* induced by a metric labelled transition system satisfying one of the generalized finiteness conditions. The operational semantics is defined as the unique fixed point of the so-called *branching higher order transformation* induced by the metric labelled transition system. Finally, we will relate the linear and branching higher order transformations and their unique fixed points, viz the linear and branching operational semantics. In establishing this relation, we will use the fact that the codomain of a branching operational semantics - a so-called branching domain - can be viewed as a metric labelled transition system (cf. [Acz88]). The induced operational semantics is an abstraction operator from the branching domain to a so-called linear domain - the codomain of the linear operational semantics.

In the fifth section, we will provide the reader with six examples showing how the theory developed can be used. In the first example, we will use a compactly branching metric labelled transition system in order to model a real time process algebra introduced by Baeten and Bergstra in [BB91]. By means of an image compact metric labelled transition system, a language with the so-called iteration statement will be modelled in the second example. In the third example, we will describe how De Bakker and Van Breugel ([BB93]) have used a metric labelled transition system in order to link an operational and a denotational semantics for a language with higher order communication. Rutten's processes as terms approach ([Rut92]) will be considered in the fourth example. In the setting of complete metric spaces, the approach will elaborated and extended. The fifth and sixth example are related to the above mentioned abstraction operator linking a linear and a branching domain. In the fifth example, an abstraction operator introduced by De Bakker, Bergstra, Klop, and Meyer in [BBKM84] will be shown to coincide with one of the abstraction operators to be introduced in Section 4. By means of the theory of this paper, we will be able to improve some of the results presented in Appendix B of [BBKM84] on this abstraction operator. In the sixth example, an abstraction operator introduced by Rutten in [Rut90] will be shown to be well-defined using the theory developed in this paper, and so providing an alternative proof for the results of Appendix II of [Rut90].

Novel in the present paper are

* the introduction of a metric labelled transition system,
* the generalizations of the finiteness conditions finitely branching and image finite to compactly branching and image compact,
* the study of operational semantic models and higher order transformations induced by metric labelled transition systems satisfying one of the generalized finiteness conditions,
* the linear and branching domains built from a label set endowed with an arbitrary complete metric rather than the discrete metric,
* the semantic study of the iteration statement,
* the elaboration and extension of the processes as terms approach in the setting of complete metric spaces,
* the improvement of the results in Appendix B of [BBKM84], and
* the alternative proof for the results in Appendix II of [Rut90].

All in all, we hope to convince the reader of the usefulness of the generalizations of the finiteness conditions of labelled transition systems in order to give semantics of programming languages.

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1. Linear and branching domains

As already mentioned in the introduction, we consider an operational semantics induced by a labelled transition system to be a mapping from the configurations of the labelled transition system to a *domain* built from the labels of the labelled transition system. We study two classes of these domains: the so-called *linear* and *branching* domains. (Other domains have been studied by, e.g., De Bakker and Warmerdam ([BW91]) and Rutten ([Rut88]).) The elements of a linear domain can be regarded as sets of *sequences* of labels. The elements of a branching domain can be viewed as *trees* the edges of which are indexed by labels.

The linear and branching domains are defined by means of (systems of) *domain equations*. A category theoretic technique to solve these domain equations has been presented by America and Rutten in [AR89b] (cf. Edalat and Smyth's [ES92]). By defining the domains by means of domain equations, we can easily define the domains parametric with respect to the metric space of labels of the (metric) labelled transition system.

In the domain equations, we use the following operations on 1-bounded complete metric spaces: Cartesian product $\times$, disjoint union $+$, nonexpansive function space $\to^1$, (nonempty and) compact power set $\mathcal{P}_{co}$ ($\mathcal{P}_{nco}$), nonempty and closed power set $\mathcal{P}_{ncl}$, and $(\,)_{\frac{1}{2}}$ multiplying the metric by a half (cf. Definition A.2). Furthermore, we encounter the set of labels L of the labelled transition system endowed with a 1-bounded complete metric (in Section 2, we will use the discrete metric on the label set, in Section 3 and 4, the labels will be endowed with arbitrary 1-bounded complete metrics) and the nonempty power set $\mathcal{P}_n$.

Definition 1.1

1. The domain L^∞ is defined by the domain equation

$$L^\infty \cong \{\varepsilon\} + (L \times (L^\infty)_{\frac{1}{2}}).$$

2. The *linear* domains $\mathcal{L}_0\,[L]$, $\mathcal{L}_1\,[L]$, and $\mathcal{L}_2\,[L]$ are defined by

$$\begin{aligned} &\mathcal{L}_0\,[L] = \mathcal{P}_n\,(L^\infty) \\ &\mathcal{L}_1\,[L] = \mathcal{P}_{nco}\,(L^\infty) \\ &\mathcal{L}_2\,[L] = \mathcal{P}_{ncl}\,(L^\infty) \end{aligned}$$

3. The *branching* domains $\mathcal{B}_1\,[L]$ and $\mathcal{B}_2\,[L]$ are defined by the domain equations

$$\begin{aligned} &\mathcal{B}_1\,[L] \cong \mathcal{P}_{co}\,(L \times (\mathcal{B}_1\,[L])_{\frac{1}{2}}) \\ &\mathcal{B}_2\,[L] \cong L \to^1 \mathcal{P}_{co}\,((\mathcal{B}_2\,[L])_{\frac{1}{2}}) \end{aligned}$$

The linear domain $\mathcal{L}_0\,[L]$, the set of nonempty subsets of L^∞ (endowed with the Hausdorff metric), is a pseudometric space but not a metric space. If we restrict the subsets to compact or closed subsets - resulting in $\mathcal{L}_1\,[L]$ or $\mathcal{L}_2\,[L]$ - we obtain a complete metric space (cf. Theorem A.4 and A.5). Also the branching domains $\mathcal{B}_1\,[L]$ and $\mathcal{B}_2\,[L]$ are complete metric spaces.

The domain L^∞ can be viewed as the set of finite and infinite sequences of labels. The empty sequence corresponds to ε and $(l_0, (l_1, \varepsilon))$, written as $l_0 l_1$ in the sequel, corresponds to the sequence $l_0 l_1$. If we endow the label set L with the discrete metric, then we obtain the usual metric space of sequences (as used by, e.g., Nivat in [Niv79]). The linear domains $\mathcal{L}_0\,[L]$, $\mathcal{L}_1\,[L]$, $\mathcal{L}_2\,[L]$ can be seen as sets of label sequences.

The branching domains $\mathcal{B}_1\,[L]$ and $\mathcal{B}_2\,[L]$ with L endowed with the discrete metric have been introduced by De Bakker and Zucker in [BZ83] and Van Breugel in [Bre93], respectively. In [Bre93], it has been shown that the domains $\mathcal{B}_1\,[L]$ and $\mathcal{B}_2\,[L]$ can be regarded as (absorptive, commutative, and closed) indexed trees. The domain $\mathcal{B}_2\,[L]$ has been introduced since it can be used to model a larger class of programming language notions than $\mathcal{B}_1\,[L]$, and it does not give rise to difficulties in modelling basic notions like sequential composition as a third branching domain $\mathcal{B}_3\,[L]$ (cf. Subsection 5.5) introduced by De Bakker and Zucker in [BZ82] does. The domains $\mathcal{B}_1\,[L]$ and $\mathcal{B}_2\,[L]$ can be viewed as labelled transition systems (cf. Lemma 4.8). The corresponding bisimilarity relations can be shown to coincide with equality (as has been proved by Van Glabbeek and Rutten in [GR89] for the branching domain $\mathcal{B}_3[L]$ with the label set L endowed with the discrete metric).

The linear and branching domains presented in this section are used to define the so-called linear and branching operational semantics in the following sections.

2. Labelled transition systems

The so-called *(linear) operational semantics* induced by a *labelled transition system* is a mapping from the configurations of the labelled transition system to the *linear* domain $\mathcal{L}_0[L]$ with L the label set of the labelled transition system endowed with the discrete metric. An operational semantics can be viewed as a mapping assigning to a configuration a set of label sequences. The assignment is driven by the transition relation of the labelled transition system. A label sequence σ is an element of the set assigned to a configuration c if this sequence σ records the labels of a sequence of transitions starting from the configuration c.

Definition 2.1 A *labelled transition system* is a triple $(C, L, \rightarrow)$ consisting of

* a set of *configurations* C,
* a set of *labels* L, and
* a *transition relation* $\rightarrow \subseteq C \times L \times C$.

Instead of $(c, l, c') \in \rightarrow$ we write $c \xrightarrow{l} c'$. If for a configuration c there exist a label l and a configuration c' such that $c \xrightarrow{l} c'$, then we write $c \rightarrow$. Otherwise, we write $c \not\rightarrow$.

Definition 2.2 The *(linear) operational semantics* induced by a labelled transition system $(C, L, \rightarrow)$ is the mapping $\mathcal{O} : C \rightarrow \mathcal{L}_0[L]$ defined by

$$\mathcal{O}(c) = \{\, l_1 l_2 \cdots l_n \mid c = c_0 \xrightarrow{l_1} c_1 \xrightarrow{l_2} \cdots \xrightarrow{l_n} c_n \not\rightarrow \,\} \cup \{\, l_1 l_2 \cdots \mid c = c_0 \xrightarrow{l_1} c_1 \xrightarrow{l_2} \cdots \,\}.$$

Two *topological properties* on operational semantic models are introduced in

Definition 2.3

1. An operational semantics $\mathcal{O} : C \rightarrow \mathcal{L}_0[L]$ is called *compact* if $\mathcal{O} \in C \rightarrow \mathcal{L}_1[L]$.
2. An operational semantics $\mathcal{O} : C \rightarrow \mathcal{L}_0[L]$ is called *closed* if $\mathcal{O} \in C \rightarrow \mathcal{L}_2[L]$.

Every compact operational semantics is closed, but a closed operational semantics is in general not compact.

Not every labelled transition system induces a compact or closed operational semantics. However, if we restrict ourselves to labelled transition systems satisfying one of the *finiteness conditions* introduced in the following definition, then we do obtain compact or closed operational semantics.

Definition 2.4

1. A labelled transition system $(C, L, \rightarrow)$ is called *finitely branching* if, for all $c \in C$, the set

$$\mathcal{FB}(c) = \{ (l, c') \mid c \xrightarrow{l} c' \}$$

is finite.

2. A labelled transition system $(C, L, \to)$ is called *image finite* if, for all $c \in C$ and $l \in L$, the set

$$\mathcal{IF}(c)(l) = \{ c' \mid c \xrightarrow{l} c' \}$$

is finite.

Every finitely branching labelled transition system is image finite, but an image finite labelled transition system is not necessarily finitely branching.

Theorem 2.5

1. *The operational semantics induced by a finitely branching labelled transition system is compact.*
2. *The operational semantics induced by an image finite labelled transition system is closed.*

The above theorem, relating the topological properties and the finiteness conditions, seems to be folklore ([Arn93]).[5]

3. Metric labelled transition systems

In this section, we generalize the results of the previous section. For this purpose, we supply a labelled transition system with some additional structure by endowing the configurations and the labels with a 1-bounded complete metric. In this way, we obtain a so-called *metric labelled transition system.*

Definition 3.1 A *metric labelled transition system* is a triple $(C, L, \to)$ consisting of

* a 1-bounded complete metric space of *configurations* C,
* a 1-bounded complete metric space of *labels* L, and
* a *transition relation* $\to \subseteq C \times L \times C$.

A metric labelled transition system induces an operational semantics along the lines of Definition 2.2. The operational semantics induced by a metric labelled transition system $(C, L, \to)$ is a mapping from the complete metric space of configurations C to the domain $\mathcal{L}_0[L]$ with L the complete metric space of labels.

By means of the additional structure, we can *generalize* the finiteness conditions finitely branching and image finite to *compactly branching* (and nonexpansive) and *image compact* (and binonexpansive), respectively.

Definition 3.2

1. A metric labelled transition system $(C, L, \to)$ is called *compactly branching and nonexpansive* if the mapping $\mathcal{CB} : C \to \mathcal{P}(L \times C)$ defined by

[5] So far, the author has not been able to locate the original proofs of Theorem 2.5.1 and 2.5.2. Both theorems are based on König's lemma ([Kön26]). Proofs of related results can be found in, e.g., [Arn83, BMOZ88].

$$\mathcal{CB}(c) = \{ (l, c') \mid c \xrightarrow{l} c' \}$$

is an element of $C \to^1 \mathcal{P}_{nco}(L \times (C)_{\frac{1}{2}})$.

2. A metric labelled transition system $(C, L, \to)$ is called *image compact and binonexpansive* if the mapping $\mathcal{IC} : C \to L \to \mathcal{P}(C)$ defined by

$$\mathcal{IC}(c)(l) = \{ c' \mid c \xrightarrow{l} c' \}$$

is an element of $C \to^1 L \to^1 \mathcal{P}_{nco}((C)_{\frac{1}{2}})$.

If we endow the configurations and the labels of a finitely branching labelled transition system both with the discrete metric, then we obtain a compactly branching and nonexpansive metric labelled transition system. Similarly, an image finite labelled transition system with the configurations and labels endowed with the discrete metric gives rise to an image compact and binonexpansive metric labelled transition system.

A compactly branching and nonexpansive metric labelled transition system is in general not finitely branching. Also an image compact and binonexpansive metric labelled transition system is not necessarily image finite.

Hence, we can conclude that the above definition generalizes the finiteness conditions of the previous section. Furthermore, we can generalize Theorem 2.5 by proving that the operational semantics induced by a metric labelled transition system satisfying one of the generalized finiteness conditions still has the corresponding topological property (and is nonexpansive).

Theorem 3.3

1. *The operational semantics induced by a compactly branching and nonexpansive metric labelled transition system is compact and nonexpansive.*

2. *The operational semantics induced by an image compact and binonexpansive metric labelled transition system is closed and nonexpansive.*

A proof of Theorem 3.3.1 can be found in Appendix B. The proof of the theorem contains the main ingredients for proofs of most of the other theorems of this paper.

The results of this section and their relation with the results of the previous section are depicted in the following diagram.

$$\begin{tikzcd}
 & \substack{\text{compact}\\ \text{and nonexpansive}} \arrow[rr] & & \substack{\text{closed}\\ \text{and nonexpansive}} \\
\substack{\text{compactly branching}\\ \text{and nonexpansive}} \arrow[ur] & & \substack{\text{image compact}\\ \text{and binonexpansive}} \arrow[ur] & \\
 & \text{compact} \arrow[uu] \arrow[rr] & & \text{closed} \arrow[uu] \\
\text{finitely branching} \arrow[uu] \arrow[ur] \arrow[rr] & & \text{image finite} \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

There is no arrow from compactly branching and nonexpansive to image compact and binonexpansive, since a compactly branching and nonexpansive metric labelled transition system is image compact but not necessarily binonexpansive.

4. Higher order transformations

In order to relate the operational semantic models studied in the foregoing sections to other semantic models, we introduce two classes of *higher order transformations.* The higher order transformations are induced by metric labelled transition systems. The semantic models to be transformed are mappings from the configurations of the metric labelled transition system to one of the linear and branching domains of Section 1. The transformation is driven by the transition relation of the metric labelled transition system. First, we focus on the so-called *linear* higher order transformations: mappings transforming semantic models the codomain of which is a *linear* domain.

Definition 4.1 A *(linear) higher order transformation* induced by a metric labelled transition system $(C, L, \rightarrow)$ is a mapping $\Phi : (C \rightarrow \mathcal{L}_0\,[L]) \rightarrow (C \rightarrow \mathcal{L}_0\,[L])$ defined by

$$\Phi\,(\phi)(c) = \begin{cases} \{\varepsilon\} & \text{if } c \not\rightarrow \\ \{\, l\sigma \mid c \xrightarrow{l} c' \wedge \sigma \in \phi\,(c')\,\} & \text{otherwise} \end{cases}$$

The mapping Φ transforms a semantics $\phi : C \rightarrow \mathcal{L}_0\,[L]$ to the semantics $\Phi\,(\phi) : C \rightarrow \mathcal{L}_0\,[L]$. The semantics $\Phi\,(\phi)$ assigns to a configuration c, with $c \not\rightarrow$, the singleton set consisting of the empty sequence ε. To a configuration c, with $c \rightarrow$, the semantics $\Phi\,(\phi)$ assigns the set of sequences $l\sigma$ obtained from the label l of a transition from the configuration c to some configuration c' and the sequence σ of $\phi\,(c')$.

Property 4.2 *The operational semantics $\mathcal{O}$ induced by a metric labelled transition system is fixed point of the higher order transformation Φ induced by the metric labelled transition system, i.e.*

$$\mathcal{O} = \Phi\,(\mathcal{O}).$$

In order to turn the higher order transformation induced by a metric labelled transition system into a contractive mapping from a complete metric space to itself, we restrict ourselves to compact or closed (and nonexpansive) semantic models.

Definition 4.3

1. A higher order transformation $\Phi : (C \rightarrow \mathcal{L}_0\,[L]) \rightarrow (C \rightarrow \mathcal{L}_0\,[L])$ is called *compactness and nonexpansiveness preserving* if $\Phi \in (C \rightarrow^1 \mathcal{L}_1\,[L]) \rightarrow (C \rightarrow^1 \mathcal{L}_1\,[L])$.
2. A higher order transformation $\Phi : (C \rightarrow \mathcal{L}_0\,[L]) \rightarrow (C \rightarrow \mathcal{L}_0\,[L])$ is called *closedness and nonexpansiveness preserving* if $\Phi \in (C \rightarrow^1 \mathcal{L}_2\,[L]) \rightarrow (C \rightarrow^1 \mathcal{L}_2\,[L])$.

A higher order transformation satisfying one of the above introduced topological properties is a mapping from a complete metric space to itself. Furthermore, such a higher order transformation can be shown to be contractive.

Not every metric labelled transition system induces a compactness or closedness and nonexpansiveness preserving higher order transformation.

Lemma 4.4 [6]

1. *The higher order transformation induced by a compactly branching and nonexpansive metric labelled transition system is compactness and nonexpansiveness preserving.*
2. *The higher order transformation induced by an image compact and binonexpansive metric labelled transition system is closedness and nonexpansiveness preserving.*

The operational semantics $\mathcal{O}$ induced by a compactly branching and nonexpansive metric labelled transition system is compact and nonexpansive according to Theorem 2.5. Together with Property 4.2 this gives us that the operational semantics $\mathcal{O}$ is fixed point of the higher order transformation Φ induced by the metric labelled transition system. Since Φ is contractive, $\mathcal{O}$ is the unique fixed point of Φ - denoted by $\mathit{fix}(\Phi)$ - according to Banach's fixed point theorem (Theorem A.3).

Theorem 4.5 *The operational semantics $\mathcal{O}$ induced by a compactly branching and nonexpansive or an image compact and binonexpansive metric labelled transition system is the unique fixed point of the higher order transformation Φ induced by the metric labelled transition system, i.e.*

$$\mathcal{O} = \mathit{fix}(\Phi).$$

Let $(C, L, \rightarrow)$ be a compactly branching and nonexpansive metric labelled transition system. If we can show that the semantics $\mathcal{S} : C \rightarrow \mathcal{L}_1[L]$ is fixed point of the induced higher order transformation, then we can conclude that the induced operational semantics is equal to the semantics $\mathcal{S}$ according to the unique fixed point proof principle.

The above theorem can be turned into a definition, that is, we can define the operational semantics induced by, e.g., a compactly branching and nonexpansive metric labelled transition system as the unique fixed point of the induced higher order transformation. According to the unique fixed point property, the operational semantics $\mathcal{O}$ induced by the compactly branching and nonexpansive metric labelled transition system $(C, L, \rightarrow)$ is the unique mapping $\mathcal{O} : C \rightarrow^1 \mathcal{L}_1[L]$ satisfying

$$\mathcal{O}(c) = \begin{cases} \{\varepsilon\} & \text{if } c \not\rightarrow \\ \{\, l\sigma \mid c \xrightarrow{l} c' \wedge \sigma \in \mathcal{O}(c')\,\} & \text{otherwise} \end{cases}$$

Theorem 4.5 generalizes the - already known - result that the operational semantics induced by a finitely branching or an image finite labelled transition system is the unique fixed point of the induced higher order transformation (see, e.g., [KR90]).

Second, we discuss the so-called *branching* higher order transformations.

Property 4.6

1. A compactly branching and nonexpansive metric labelled transition system $(C, L, \rightarrow)$ induces a *branching higher order transformation* $\Phi : (C \rightarrow^1 \mathcal{B}_1[L]) \rightarrow (C \rightarrow^1 \mathcal{B}_1[L])$ defined by

$$\Phi(\phi)(c) = \{\, (l, \phi(c')) \mid c \xrightarrow{l} c' \,\}.$$

[6] A compactly branching and nonexpansive metric labelled transition system induces not necessarily a compactness preserving higher order transformation Φ. In proving that $\Phi(\phi)$ is compact, the nonexpansiveness of ϕ is needed. A similar remark applies to a higher order transformation induced by an image compact and binonexpansive metric labelled transition system. Since the semantic models to be transformed have to be nonexpansive, we restricted the operational semantic models in Section 3 to nonexpansive mappings.

2. An image compact and binonexpansive metric labelled transition system $(C, L, \rightarrow)$ induces a *branching higher order transformation* $\Phi : (C \rightarrow^1 \mathcal{B}_2\,[L]) \rightarrow (C \rightarrow^1 \mathcal{B}_2\,[L])$ defined by

$$\Phi\,(\phi)(c) = \lambda l.\{\, \phi\,(c') \mid c \xrightarrow{l} c' \,\}.$$

We restrict ourselves to metric labelled transition systems satisfying one of the generalized finiteness conditions, because for arbitrary metric labelled transition systems the above property is in general not valid. The branching higher order transformation introduced in Property 4.6.1 has already been studied for finitely branching labelled transition systems. The branching higher order transformation presented in Property 4.6.2 has not been considered in the context of labelled transition systems. Both higher order transformations are contractive mappings from a complete metric space to itself. According to Banach's theorem, the higher order transformations have unique fixed points.

Just as linear operational semantic models are the unique fixed points of linear higher order transformations, *branching operational semantic models* are defined as the unique fixed points of branching higher order transformations.

Definition 4.7

1. The *branching operational semantics* induced by a compactly branching and nonexpansive metric labelled transition system $(C, L, \rightarrow)$ is a mapping $\mathcal{O} : C \rightarrow \mathcal{B}_1\,[L]$ defined by

   $$\mathcal{O} = \mathit{fix}\,(\Phi),$$

   where Φ is the branching higher order transformation induced by the metric labelled transition system.

2. The *branching operational semantics* induced by a image compact and binonexpansive metric labelled transition system $(C, L, \rightarrow)$ is a mapping $\mathcal{O} : C \rightarrow \mathcal{B}_2\,[L]$ defined by

   $$\mathcal{O} = \mathit{fix}\,(\Phi),$$

   where Φ is the branching higher order transformation induced by the metric labelled transition system.

According to the unique fixed point property, the branching operational semantics $\mathcal{O}$ induced by, e.g., a compactly branching and nonexpansive metric labelled transition system $(C, L, \rightarrow)$ is the unique mapping $\mathcal{O} : C \rightarrow^1 \mathcal{B}_1\,[L]$ satisfying

$$\mathcal{O}\,(c) = \{\, (l, \mathcal{O}\,(c')) \mid c \xrightarrow{l} c' \,\}.$$

We conclude this section with relating the linear and branching higher order transformations and their unique fixed points. We first link the linear and branching domains before relating the corresponding higher order transformations. We link the branching domains $\mathcal{B}_1\,[L]$ and $\mathcal{B}_2\,[L]$ to the linear domains $\mathcal{L}_1\,[L]$ and $\mathcal{L}_2\,[L]$ by means of operators abstracting from the branching structure. For this purpose, it is convenient to view the branching domains as metric labelled transition systems satisfying one of the generalized finiteness conditions.

LEMMA 4.8

1. *The metric labelled transition system* $(\mathcal{B}_1[L], L, \rightarrow)$, *with* $\beta \xrightarrow{l} \beta'$ *if* $(l, \beta') \in \beta$, *is compactly branching and nonexpansive.*

2. *The metric labelled transition system* $(\mathcal{B}_2[L], L, \rightarrow)$, *with* $\beta \xrightarrow{l} \beta'$ *if* $\beta' \in \beta(l)$, *is image compact and binonexpansive.*

The operational semantics induced by, e.g., the metric labelled transition system introduced in Lemma 4.8.1 - denoted by *trace* in the sequel - is a nonexpansive mapping from the branching domain $\mathcal{B}_1[L]$ to the linear domain $\mathcal{L}_1[L]$ according to Theorem 3.3.1. This operational semantics is the above mentioned abstraction operator from $\mathcal{B}_1[L]$ to $\mathcal{L}_1[L]$. The abstraction operators can be defined as follows.

DEFINITION 4.9

1. The mapping $\mathit{trace} : \mathcal{B}_1[L] \rightarrow^1 \mathcal{L}_1[L]$ is the unique mapping satisfying

$$\mathit{trace}(\beta) = \begin{cases} \{\varepsilon\} & \text{if } \beta = \emptyset \\ \{\, l\sigma \mid (l, \beta') \in \beta \wedge \sigma \in \mathit{trace}(\beta')\,\} & \text{otherwise} \end{cases}$$

2. The mapping $\mathit{trace} : \mathcal{B}_2[L] \rightarrow^1 \mathcal{L}_2[L]$ is the unique mapping satisfying

$$\mathit{trace}(\beta) = \begin{cases} \{\varepsilon\} & \text{if } \beta = \lambda l.\emptyset \\ \{\, l\sigma \mid \beta' \in \beta(l) \wedge \sigma \in \mathit{trace}(\beta')\,\} & \text{otherwise} \end{cases}$$

By means of the abstraction operators *trace*, we can relate the linear and branching higher order transformations.

THEOREM 4.10 *For the linear and branching higher order transformations* Φ_l *and* Φ_b *induced by a compactly branching and nonexpansive or an image compact and binonexpansive metric labelled transition system we have that*

$$\mathit{fix}(\Phi_l) = \mathit{trace} \circ \mathit{fix}(\Phi_b).$$

Suppose $(C, L, \rightarrow)$ is a compactly branching and nonexpansive metric labelled transition system. If the semantics $\mathcal{S} : C \rightarrow^1 \mathcal{B}_1[L]$ is a fixed point of the induced higher order transformation, then we can conclude that the induced linear operational semantics $\mathcal{O}$ is related to the semantics $\mathcal{S}$ by $\mathcal{O} = \mathit{trace} \circ \mathcal{S}$ by uniqueness of fixed point.

Combining Theorem 4.5, Definition 4.7, and Theorem 4.10, we arrive at

THEOREM 4.11 *For the linear and branching operational semantics* $\mathcal{O}_l$ *and* $\mathcal{O}_b$ *induced by a compactly branching and nonexpansive or an image compact and binonexpansive metric labelled transition system we have that*

$$\mathcal{O}_l = \mathit{trace} \circ \mathcal{O}_b.$$

5. APPLICATIONS

In this fifth and final section, we present six applications of the theory developed in the preceding sections.

5.1 Real time integration

In [BB91], Baeten and Bergstra have introduced the *real time* process algebras ACPρ and ACPrρ. In [Bre91], Van Breugel has studied semantic models for a sequential fragment of ACPrρ. In this fragment, timed actions and *integration* are the time dependent notions. A timed action $a[r]$, with a an element of some set of actions and r a nonnegative real number, denotes that the action a has to be performed r time units after its enabling. Integration is an alternative composition over a continuum. For example, the integration $\int_{t \in [0.5,\, 1.0]} a[t]$, with t an element of some set of time variables, denotes that the action a has to be performed between 0.5 and 1.0 time units after the enabling of the integration.

In [Bre91], an operational semantics induced by a modification of the labelled transition system for ACPrρ of [BB91] has been presented. (The labelled transition system has been modified along the lines of Klusener's modification of the labelled transition system for ACPρ of [BB91] in [Klu91].) The labelled transition system is not image finite and, a fortiori, not finitely branching, since, e.g., for all $r \in [0.5,\, 1.0]$,

$$\int_{t \in [0.5,\, 1.0]} a[1] \cdot a[t] \xrightarrow{a[1]} a[r].$$

By endowing the configurations and the labels with suitable metrics, the labelled transition system can be turned into a compactly branching and nonexpansive metric labelled transition system. These metrics are based on the 1-bounded (topologically) equivalent of the Euclidean metric defined by

$$d\,(r, r') = \frac{|r - r'|}{|r - r'| + 1}.$$

Closed intervals of the real numbers - being part of the integration construct - are compact with respect to this metric.

In [Bre91], also a denotational semantics for the fragment of ACPrρ has been presented. Furthermore, the operational and denotational semantic models have been proved to be equivalent by means of the unique fixed point proof principle using some of the results of the previous sections.

5.2 Iteration

An operational semantics and a denotational semantics for a simple programming language built from assignments and operators like sequential composition and conditionals have been presented by De Bakker and Meyer in [BM88]. Furthermore, the semantic models have been related by means of the unique fixed point proof principle. Since the labelled transition system inducing the operational semantics is finitely branching, the generalized theory developed in this paper is not needed.

Now, we add to the language the so-called *iteration statement* $s^{\dagger}$, with s an arbitrary statement. The execution of the statement $s^{\dagger}$ amounts to first choosing the number of iterations of the statement and second executing the statement s the chosen number of iterations. The number of iterations can be any natural number or infinity.

The configurations of the labelled transition system of [BM88] are pairs of the form $[s, \varsigma]$ where s is a statement and ς is a state, i.e. a mapping from variables to values. The labels of the labelled transition system are states.

In order to model the language extended with the iteration statement operationally, we introduce the auxiliary statements s^n, with $n \in \mathbb{N}$, and s^{∞}. The execution of the statement s^n (s^{∞}) amounts to executing n (an infinite number of) times the statement s. Furthermore, we add some rules to the transition system specification inducing the labelled transition system (as described by, e.g., Groote and Vaandrager in [GV92]). Below, we use so-called zero-step transitions of the form $s \rightarrow_0 s'$ for denoting rules of the form

$$\frac{[s, \varsigma] \xrightarrow{\varsigma'} [s'', \varsigma'']}{[s', \varsigma] \xrightarrow{\varsigma'} [s'', \varsigma'']}.$$

We add the following rules:

* $s^\dagger \rightarrow_0 s^n$, for all $n \in \mathbb{N}$
* $s^\dagger \rightarrow_0 s^\infty$
* $s^0 \rightarrow_0 \mathrm{E}$
* $s^{n+1} \rightarrow_0 s \,;\, s^n$, for all $n \in \mathbb{N}$
* $s^\infty \rightarrow_0 s \,;\, s^\infty$

where the empty statement E denotes termination and ';' denotes sequential composition. The obtained labelled transition system is no longer finitely branching, not even image finite. However, the labelled transition system can be turned into an image compact and binonexpansive metric labelled transition system by endowing the configurations and the labels with a complete metric using the compact metric on $\mathbb{N} \cup \{\infty\}$ defined by

$$d(k, k') = \begin{cases} 0 & \text{if } k = k' \\ 2^{-\min\{k,k'\}} & \text{otherwise} \end{cases}$$

Also the denotational semantics of [BM88] can be extended to deal with the iteration statement. By means of the theory developed in the foregoing sections, the operational and denotational semantic models can be related. The details will appear in [Bre94].

5.3 Second order communication

In [BB93], De Bakker and Van Breugel have presented a linear operational semantics and a branching denotational semantics for a language with *second order communication*. Recall that in a CSP-like language value-passing communication is expressed by the two statements $c \,!\, e$ and $c \,?\, v$, for c a channel, e some expression, and v an individual variable, occurring in two parallel components, and synchronized execution of these statements results in the transmission of the current value of e to v. A second order variant of this is the pair of communication constructs $c \,!\, s$ and $c \,?\, x$, for c a channel, s a statement, and x a statement variable. Now a higher order value is passed at the moment of synchronized execution. In the operational semantics the statement s is passed, whereas in the denotational semantics the (semantic) meaning of s is transmitted. In order to link the operational and denotational semantic models, a branching operational semantics is introduced. This operational semantics is induced by a labelled transition system not satisfying one of the finiteness conditions. By endowing the configurations and the labels with suitable complete metrics, the labelled transition system can be turned into a compactly branching and nonexpansive metric labelled transition system. The branching operational and denotational semantic models are related by means of the unique fixed point proof principle.

5.4 Metric processes as terms

As we have already mentioned, from a transition system specification one can derive a labelled transition system and hence an operational semantics. In [Rut92], Rutten has shown that also an equivalent denotational semantics can be derived from a transition system specification provided that the transition system specification is in the so-called BSOS format. Crucial in this so-called *processes as terms* approach is the use of elements of some (semantic) mathematical structure - called processes - in the terms of the transition system specification. All this has been carried out in a setting using nonwellfounded sets as mathematical structure for the semantic models. In the final section of the paper, Rutten has argued that also complete metric spaces instead of nonwellfounded sets can be used. In that case, the complete metric space $\mathcal{B}_1[L]$, with the set L endowed with the discrete metric, is the collection of processes used as terms. Since the metric processes are used as terms in the transition system specification, we encounter them as configurations of the labelled transition system. Because this labelled transition system is not finitely branching nor image finite, we have to consider metric labelled transition systems (cf. Lemma 4.8.1). The (metric) processes as terms approach can be extended to deal with the complete metric spaces $\mathcal{B}_1[L]$ and $\mathcal{B}_2[L]$, where the set L is endowed with an arbitrary complete metric, using the theory developed in this paper.

5.5 The trace operator of [BBKM84]

De Bakker, Bergstra, Klop, and Meyer have presented a linear and a branching denotational semantics for a simple language in [BBKM84]. The linear semantics uses the linear domain $\mathcal{L}_2[L]$ and the branching semantics uses the branching domain $\mathcal{B}_3[L]$ defined by the domain equation

$$\mathcal{B}_3[L] \cong \mathcal{P}_{cl}(L \times (\mathcal{B}_3[L])_{\frac{1}{2}}),$$

where $\mathcal{P}_{cl}$ denotes the closed power set and the set L is endowed with the discrete metric. In order to relate the semantic models, an abstraction operator *trace* is introduced. In case the set L is finite, *trace* is shown to be a continuous mapping from $\mathcal{B}_3[L]$ to $\mathcal{L}_2[L]$. We can improve this result by proving that *trace* is a nonexpansive mapping from $\mathcal{B}_3[L]$ to $\mathcal{L}_1[L]$. If the set L is finite, we have that $\mathcal{B}_3[L] \cong \mathcal{B}_1[L]$ (cf. [Bre93, BW93]). The abstraction operator of [BBKM84] coincides with the operator *trace* introduced in Definition 4.9.1.

5.6 The abstr operator of [Rut90]

In [Rut90], Rutten has presented a linear operational semantics and a branching denotational semantics for Philips' parallel object-oriented language POOL. The semantic models have been related by means of an abstraction operator *abstr*. The well-definedness proof of *abstr* is far from trivial (cf. Appendix II of [Rut90]). The branching domain used in the denotational semantics is similar to - although much more complicated than - the branching domain $\mathcal{B}_1[L]$. Also in this case, the branching domain can be viewed as a compactly branching and nonexpansive metric labelled transition system. The abstraction operator *abstr* turns out to be the induced linear operational semantics and the well-definedness of *abstr* is an immediate consequence of Theorem 3.3.

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A. Metric spaces

We present some definitions and theorems of (metric) topology. For further details on topology, we refer the reader to [Eng89].

Definition A.1 Let (X, d_X) and $(X', d_{X'})$ be metric spaces.

1. A mapping $f : X \to X'$ is called *contractive* if there exists an ϵ, with $0 \le \epsilon < 1$, such that, for all x, $x' \in X$,

$$d_{X'}(f(x), f(x')) \le \epsilon \cdot d_X(x, x').$$

2. A mapping $f : X \to X'$ is called *nonexpansive* if, for all x, $x' \in X$,

$$d_{X'}(f(x), f(x')) \le d_X(x, x').$$

Definition A.2 Let (X, d_X) and $(X', d_{X'})$ be 1-bounded metric spaces. Let Y be a set.

1. A metric on the Cartesian product of X and X', $X \times X'$, is defined by

$$d_{X \times X'}((x, x'), (\bar{x}, \bar{x}')) = \max\{ d_X(x, \bar{x}), d_{X'}(x', \bar{x}') \}.$$

2. A metric on the disjoint union of X and X', $X + X'$, is defined by

$$d_{X+X'}(x, \bar{x}) = \begin{cases} d_X(x, \bar{x}) & \text{if } x, \bar{x} \in X \\ d_{X'}(x, \bar{x}) & \text{if } x, \bar{x} \in X' \\ 1 & \text{otherwise} \end{cases}$$

3. A metric on the collection of mappings from Y to X, $Y \to X$, is defined by

$$d_{Y \to X}(f, f') = \sup\{ d_X(f(y), f'(y)) \mid y \in Y \}.$$

4. A metric on the collection of nonexpansive mappings from X to X', $X \to^1 X'$, is defined by

$$d_{X \to^1 X'}(f, f') = \sup\{ d_{X'}(f(x), f'(x)) \mid x \in X \}.$$

5. The Hausdorff metric on the set of nonempty and compact subsets of X, $\mathcal{P}_{nco}(X)$, and on the set of nonempty and closed subsets of X, $\mathcal{P}_{ncl}(X)$, is defined by

$$\begin{aligned} d_{\mathcal{P}(X)}(A, A') = \max\{ & \sup\{ \inf\{ d_X(x, x') \mid x' \in A' \} \mid x \in A \}, \\ & \sup\{ \inf\{ d_X(x', x) \mid x \in A \} \mid x' \in A' \} \} \end{aligned}$$

6. A metric on the set of compact subsets of X, $\mathcal{P}_{co}(X)$, is defined by

$$\mathcal{P}_{co}(X) = \mathcal{P}_{nco}(X) + \{\emptyset\}.$$

A metric on the set of closed subsets of X, $\mathcal{P}_{cl}(X)$, is defined by

$$\mathcal{P}_{cl}(X) = \mathcal{P}_{ncl}(X) + \{\emptyset\}.$$

7. A new metric on X is defined by

$$d_{(X)_{\frac{1}{2}}}(x, \bar{x}) = \tfrac{1}{2} \cdot d_X(x, \bar{x}).$$

THEOREM A.3 (BANACH'S THEOREM) *Let (X, d_X) be a complete metric space. If $f : X \to X$ is a contraction then f has a unique fixed point $\mathit{fix}(f)$. For all $x \in X$,*

$$\lim_n f^n(x) = \mathit{fix}(f)$$

where

$$f^0(x) = x \text{ and } f^{n+1}(x) = f(f^n(x)).$$

PROOF See Theorem II.6 of [Ban22]. □

THEOREM A.4 (KURATOWSKI'S THEOREM) *If (X, d_X) is a 1-bounded complete metric space, then $(\mathcal{P}_{nco}(X), d_{\mathcal{P}(X)})$ is a 1-bounded complete metric space.*

PROOF See Lemma 3 of [Kur56]. □

THEOREM A.5 (HAHN'S THEOREM) *If (X, d_X) is a 1-bounded complete metric space, then $(\mathcal{P}_{ncl}(X), d_{\mathcal{P}(X)})$ is a 1-bounded complete metric space.*

PROOF See §9.6 and §18.10 of [Hah48]. □

THEOREM A.6 (MICHAEL'S THEOREM) *Let (X, d_X) be a 1-bounded metric space.*

1. *If $\mathcal{A} \in \mathcal{P}_{co}(\mathcal{P}_{co}(X))$ then $\bigcup \mathcal{A} \in \mathcal{P}_{co}(X)$.*
2. *The mapping $\bigcup : \mathcal{P}_{co}(\mathcal{P}_{co}(X)) \to \mathcal{P}_{co}(X)$ is nonexpansive.*

PROOF See Theorem 2.5 of [Mic51]. □

B. PROOF OF THEOREM 3.3.1

As an illustration how to prove the results of this paper, we present a proof of Theorem 3.3.1. The proof of this theorem contains the main ingredients for proofs of most of the other theorems.

THEOREM 3.3.1 *The operational semantics induced by a compactly branching and nonexpansive metric labelled transition system is compact and nonexpansive.*

We prove the theorem in two steps. First, we show that the operational semantics induced by a compactly branching and nonexpansive metric labelled transition system is compact (Theorem B.3). Second, we demonstrate that the compact operational semantics induced by a compactly branching and nonexpansive metric labelled transition system is nonexpansive (Theorem B.4). In the proof of Theorem B.3, we use the following two lemmas.

LEMMA B.1 *For a compactly branching and nonexpansive metric labelled transition system $(C, L, \rightarrow)$, for all c, $c' \in C$,*

$$\text{if } c \rightarrow \text{ and } c' \not\rightarrow \text{ then } d(c, c') = 1.$$

PROOF Let c, $c' \in C$. Assume $c \rightarrow$ and $c' \not\rightarrow$. Because $\mathcal{CB}(c) \neq \emptyset$ and $\mathcal{CB}(c') = \emptyset$ and the metric labelled transition system is compactly branching and nonexpansive,

$$1 = d(\mathcal{CB}(c), \mathcal{CB}(c')) \leq d(c, c').$$

□

LEMMA B.2 *For a compactly branching and nonexpansive metric labelled transition system* $(C, L, \rightarrow)$, *for all* $c \in C$ *and* $n \in \mathbb{N}$, *the set*

$$\{ (c_0, l_1, c_1, l_2, \ldots, l_n, c_n) \mid c = c_0 \xrightarrow{l_1} c_1 \xrightarrow{l_2} \cdots \xrightarrow{l_n} c_n \}$$

is compact.

PROOF We prove this lemma by induction on n. Obviously, the set is compact for $n = 0$. Let $n > 0$. Because the metric labelled transition system is compactly branching and nonexpansive, for all $c_{n-1} \in C$, the set

$$\{ (l_n, c_n) \mid c_{n-1} \xrightarrow{l_n} c_n \}$$

is compact. Consequently, for all c_0, c_1, $\ldots$, $c_{n-1} \in C$ and l_1, l_2, $\ldots$, $l_{n-1} \in L$, the set

$$\{ (c_0, l_1, c_1, l_2, \ldots, l_n, c_n) \mid c_{n-1} \xrightarrow{l_n} c_n \}$$

is also compact. Since the metric labelled transition system is compactly branching and nonexpansive, the mapping

$$\lambda(c_0, l_1, c_1, l_2, \ldots, l_{n-1}, c_{n-1}).\{ (c_0, l_1, c_1, l_2, \ldots, l_n, c_n) \mid c_{n-1} \xrightarrow{l_n} c_n \}$$

is continuous. By induction, the set

$$\{ (c_0, l_1, c_1, l_2, \ldots, l_{n-1}, c_{n-1}) \mid c = c_0 \xrightarrow{l_1} c_1 \xrightarrow{l_2} \cdots \xrightarrow{l_{n-1}} c_{n-1} \}$$

is compact. Because the continuous image of a compact set is compact,

$$\left\{ \{ (c_0, l_1, c_1, l_2, \ldots, l_n, c_n) \mid c_{n-1} \xrightarrow{l_n} c_n \} \;\middle|\; c = c_0 \xrightarrow{l_1} c_1 \xrightarrow{l_2} \cdots \xrightarrow{l_{n-1}} c_{n-1} \right\}$$

is a compact set of compact sets. From Michael's theorem (Theorem A.6.1), the compactness of the set

$$\bigcup \left\{ \{ (c_0, l_1, c_1, l_2, \ldots, l_n, c_n) \mid c_{n-1} \xrightarrow{l_n} c_n \} \;\middle|\; c = c_0 \xrightarrow{l_1} c_1 \xrightarrow{l_2} \cdots \xrightarrow{l_{n-1}} c_{n-1} \right\}$$

can be concluded.

□

THEOREM B.3 *The operational semantics induced by a compactly branching and nonexpansive metric labelled transition system is compact.*

PROOF Let $(C, L, \rightarrow)$ be a compactly branching and nonexpansive metric labelled transition system. We will prove that the induced operational semantics $\mathcal{O}$ is compact, i.e., for all $c \in C$, the set $\mathcal{O}(c)$ is compact.

Let $c \in C$. Let $(\sigma_i)_i$ be a sequence in $\mathcal{O}(c)$. We will show that there exists a subsequence $(\sigma_{f(i)})_i$ of $(\sigma_i)_i$ converging to some $\sigma \in \mathcal{O}(c)$.

The subsequence $(\sigma_{f(i)})_i$ will be constructed from a collection of subsequences $(\sigma_{f_n(i)})_i$ satisfying

$$(\forall n \in \mathbb{N} : Q(n)) \vee (\exists m \in \mathbb{N} : \forall 0 \leq n < m : Q(n) \wedge R(m)) \tag{B.1}$$

where

$$\begin{aligned} Q(n) \Leftrightarrow \forall i : \;& \sigma_{f_n(i)} = l_{1,f_n(i)} l_{2,f_n(i)} \cdots l_{n,f_n(i)} \sigma_{n,f_n(i)} \wedge \\ & c = c_{0,f_n(i)} \xrightarrow{l_{1,f_n(i)}} c_{1,f_n(i)} \xrightarrow{l_{2,f_n(i)}} \cdots \xrightarrow{l_{n,f_n(i)}} c_{n,f_n(i)} \rightarrow \wedge \\ & \sigma_{n,f_n(i)} \in \mathcal{O}(c_{n,f_n(i)}) \wedge \\ & \forall 1 \leq j \leq n : \lim_i l_{j,f_n(i)} = l_j \wedge \\ & \forall 0 \leq j \leq n : \lim_i c_{j,f_n(i)} = c_j \wedge \\ & c = c_0 \xrightarrow{l_1} c_1 \xrightarrow{l_2} \cdots \xrightarrow{l_n} c_n \rightarrow \end{aligned}$$

and

$$\begin{aligned} R(n) \Leftrightarrow \forall i : \;& \sigma_{f_n(i)} = l_{1,f_n(i)} l_{2,f_n(i)} \cdots l_{n,f_n(i)} \wedge \\ & c = c_{0,f_n(i)} \xrightarrow{l_{1,f_n(i)}} c_{1,f_n(i)} \xrightarrow{l_{2,f_n(i)}} \cdots \xrightarrow{l_{n,f_n(i)}} c_{n,f_n(i)} \not\rightarrow \wedge \\ & \forall 1 \leq j \leq n : \lim_i l_{j,f_n(i)} = l_j \wedge \\ & \forall 0 \leq j \leq n : \lim_i c_{j,f_n(i)} = c_j \wedge \\ & c = c_0 \xrightarrow{l_1} c_1 \xrightarrow{l_2} \cdots \xrightarrow{l_n} c_n \not\rightarrow . \end{aligned}$$

The existence of the subsequences $(\sigma_{f_n(i)})_i$ is verified by proving

$$P(k) \Leftrightarrow (\forall 0 \leq n \leq k : Q(n)) \vee (\exists 0 \leq m \leq k : \forall 0 \leq n < m : Q(n) \wedge R(m))$$

by induction on k.

To prove $P(0)$ it suffices to show $Q(0) \vee R(0)$. By definition, each subsequence $(\sigma_{f_0(i)})_i$ satisfies $Q(0) \vee R(0)$.

Let $k > 0$. To prove $P(k-1) \Rightarrow P(k)$ it suffices to show $Q(k-1) \Rightarrow Q(k) \vee R(k)$. If $Q(k-1)$, then

$$\begin{aligned} \forall i : \;& ((\sigma_{f_{k-1}(i)} = l_{1,f_{k-1}(i)} l_{2,f_{k-1}(i)} \cdots l_{k,f_{k-1}(i)} \sigma_{k,f_{k-1}(i)} \wedge \\ & \quad c = c_{0,f_{k-1}(i)} \xrightarrow{l_{1,f_{k-1}(i)}} c_{1,f_{k-1}(i)} \xrightarrow{l_{2,f_{k-1}(i)}} \cdots \xrightarrow{l_{k,f_{k-1}(i)}} c_{k,f_{k-1}(i)} \rightarrow \wedge \\ & \quad \sigma_{k,f_{k-1}(i)} \in \mathcal{O}(c_{k,f_{k-1}(i)})) \vee \\ & \quad (\sigma_{f_{k-1}(i)} = l_{1,f_{k-1}(i)} l_{2,f_{k-1}(i)} \cdots l_{k,f_{k-1}(i)} \wedge \\ & \quad c = c_{0,f_{k-1}(i)} \xrightarrow{l_{1,f_{k-1}(i)}} c_{1,f_{k-1}(i)} \xrightarrow{l_{2,f_{k-1}(i)}} \cdots \xrightarrow{l_{k,f_{k-1}(i)}} c_{k,f_{k-1}(i)} \not\rightarrow)) \wedge \\ & \forall 1 \leq j \leq k-1 : \lim_i l_{j,f_{k-1}(i)} = l_j \wedge \\ & \forall 0 \leq j \leq k-1 : \lim_i c_{j,f_{k-1}(i)} = c_j \wedge \\ & c = c_0 \xrightarrow{l_1} c_1 \xrightarrow{l_2} \cdots \xrightarrow{l_{k-1}} c_{k-1} \rightarrow . \end{aligned}$$

By Lemma B.2, there exists a subsequence

$$(c_{0,g_{k-1}(i)}, l_{1,g_{k-1}(i)}, c_{1,g_{k-1}(i)}, l_{2,g_{k-1}(i)}, \ldots, l_{k,g_{k-1}(i)}, c_{k,g_{k-1}(i)})_i$$

of the sequence

$$(c_{0,f_{k-1}(i)}, l_{1,f_{k-1}(i)}, c_{1,f_{k-1}(i)}, l_{2,f_{k-1}(i)}, \ldots, l_{k,f_{k-1}(i)}, c_{k,f_{k-1}(i)})_i,$$

which converges to $(c_0, l_1, c_1, l_2, \ldots, l_k, c_k)$ such that

$$c = c_0 \xrightarrow{l_1} c_1 \xrightarrow{l_2} \cdots \xrightarrow{l_k} c_k.$$

According to Lemma B.1, if $c_k \rightarrow$ ($c_k \not\rightarrow$) then there exists a subsequence

$$(c_{0,f_k(i)}, l_{1,f_k(i)}, c_{1,f_k(i)}, l_{2,f_k(i)}, \ldots, l_{k,f_k(i)}, c_{k,f_k(i)})_i$$

of the sequence

$$(c_{0,g_{k-1}(i)}, l_{1,g_{k-1}(i)}, c_{1,g_{k-1}(i)}, l_{2,g_{k-1}(i)}, \ldots, l_{k,g_{k-1}(i)}, c_{k,g_{k-1}(i)})_i$$

satisfying $c_{k,f_k(i)} \rightarrow$ ($c_{k,f_k(i)} \not\rightarrow$). Consequently $Q(k)$ ($R(k)$).

From the subsequences $(\sigma_{f_n(i)})_i$ satisfying (B.1) we next construct the subsequence $(\sigma_{f(i)})_i$ distinguishing the following two cases.

1. If $\forall n \in \mathbb{N} : Q(n)$, then we define $f(i) = f_i(i)$. In this case, the sequence $(\sigma_{f(i)})_i$ converges to $\sigma = l_1 l_2 \cdots$ in $\mathcal{O}(c)$.
2. If $\exists m \in \mathbb{N} : \forall 0 \leq n < m : Q(n) \wedge R(m)$, then we define $f(i) = f_m(i)$. The sequence $(\sigma_{f(i)})_i$ converges to $\sigma = l_1 l_2 \cdots l_m$ in $\mathcal{O}(c)$.

$\square$

THEOREM B.4 *The compact operational semantics induced by a compactly branching and nonexpansive metric labelled transition system is nonexpansive.*

PROOF Let $(C, L, \rightarrow)$ be a compactly branching and nonexpansive metric labelled transition system. We will prove that the induced operational semantics $\mathcal{O}$ is nonexpansive.

To prove the nonexpansiveness of $\mathcal{O}$, a sequence $(\mathcal{O}_i)_i$ of nonexpansive mappings converging to $\mathcal{O}$, is introduced. Because nonexpansiveness is a closed property, $\mathcal{O}$ is nonexpansive.

The mapping $\mathcal{O}_i : C \rightarrow^1 \mathcal{L}_1[L]$ is defined by

$$\mathcal{O}_i(c) = \{ l_1 l_2 \cdots l_j \mid c = c_0 \xrightarrow{l_1} c_1 \xrightarrow{l_2} \cdots \xrightarrow{l_j} c_j \not\rightarrow \wedge j \leq i \} \cup \{ l_1 l_2 \cdots l_i \mid c \xrightarrow{l_1} c_1 \xrightarrow{l_2} \cdots \xrightarrow{l_i} c_i \rightarrow \}.$$

The well-definedness of these mappings is proved by induction on i. Obviously, $\mathcal{O}_0$ is well-defined. Let $i > 0$.

First, we prove that, for all $c \in C$, the set $\mathcal{O}_i(c)$ is compact. Let $c \in C$. By definition,

$$\mathcal{O}_i(c) = \begin{cases} \{\varepsilon\} & \text{if } c \not\rightarrow \\ \{ l\sigma \mid c \xrightarrow{l} c' \wedge \sigma \in \mathcal{O}_{i-1}(c') \} & \text{otherwise} \end{cases}$$

Because the metric labelled transition system is compactly branching and nonexpansive, the set

$$\{ c' \mid c \xrightarrow{l} c' \}$$

is compact. By induction, $\mathcal{O}_{i-1}$ delivers compact sets. One can easily verify that, for all $l \in L$ and $c' \in C$, the set $\{ l\sigma \mid \sigma \in \mathcal{O}_{i-1}(c') \}$ is compact. By induction, the mapping $\mathcal{O}_{i-1}$ is nonexpansive and hence continuous. Because the continuous image of a compact set is compact, we can derive that

$$\left\{ \{ l\sigma \mid \sigma \in \mathcal{O}_{i-1}(c') \} \;\middle|\; c \xrightarrow{l} c' \right\}$$

is a compact set of compact sets. According to Michael's theorem (Theorem A.6.1),

$$\bigcup \left\{ \{ l\sigma \mid \sigma \in \mathcal{O}_{i-1}(c') \} \;\middle|\; c \xrightarrow{l} c' \right\}$$

is compact. Hence, the set $\mathcal{O}_i(c)$ is compact.

Second, we show that $\mathcal{O}_i$ is nonexpansive. We will prove that, for all c, $\bar{c} \in C$,

$$d(\mathcal{O}_i(c), \mathcal{O}_i(\bar{c})) \leq d(c, \bar{c}). \tag{B.2}$$

Let c, $\bar{c} \in C$. We distinguish the following three cases.

1. If $c \not\rightarrow$ and $\bar{c} \not\rightarrow$, then (B.2) is vacuously true.
2. If $c \not\rightarrow$ and $\bar{c} \rightarrow$ or $c \rightarrow$ and $\bar{c} \not\rightarrow$, then $d(c, \bar{c}) = 1$ according to Lemma B.1. Consequently, (B.2) is also valid in this case.
3. If $c \rightarrow$ and $\bar{c} \rightarrow$, then

$$d(\mathcal{O}_i(c), \mathcal{O}_i(\bar{c})) = d(\bigcup \left\{ \{ l\sigma \mid \sigma \in \mathcal{O}_{i-1}(c') \} \;\middle|\; c \xrightarrow{l} c' \right\}, \bigcup \left\{ \{ \bar{l}\bar{\sigma} \mid \bar{\sigma} \in \mathcal{O}_{i-1}(\bar{c}') \} \;\middle|\; \bar{c} \xrightarrow{\bar{l}} \bar{c}' \right\}).$$

   Because the metric labelled transition system is compactly branching and nonexpansive,

$$d(\{ (l, c') \mid c \xrightarrow{l} c' \}, \{ (\bar{l}, \bar{c}') \mid \bar{c} \xrightarrow{\bar{l}} \bar{c}' \}) \leq d(c, \bar{c}).$$

   By induction, $\mathcal{O}_{i-1}$ is nonexpansive. Combining the above, one can verify that

$$d(\left\{ \{ l\sigma \mid \sigma \in \mathcal{O}_{i-1}(c') \} \;\middle|\; c \xrightarrow{l} c' \right\}, \left\{ \{ \bar{l}\bar{\sigma} \mid \bar{\sigma} \in \mathcal{O}_{i-1}(\bar{c}') \} \;\middle|\; \bar{c} \xrightarrow{\bar{l}} \bar{c}' \right\}) \leq d(c, \bar{c}).$$

   According to Michael's theorem (Theorem A.6.2),

$$d(\bigcup \left\{ \{ l\sigma \mid \sigma \in \mathcal{O}_{i-1}(c') \} \;\middle|\; c \xrightarrow{l} c' \right\}, \bigcup \left\{ \{ \bar{l}\bar{\sigma} \mid \bar{\sigma} \in \mathcal{O}_{i-1}(\bar{c}') \} \;\middle|\; \bar{c} \xrightarrow{\bar{l}} \bar{c}' \right\}) \leq d(c, \bar{c}).$$

□