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# Multiparameter Quantum Groups and Multiparameter $R$-matrices

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**Abstract**

There exists an $\binom{n}{2}+1$ parameter quantum group deformation of $GL_n$ which has been constructed independently by several (groups of) authors. In this note I give an explicit $R$-matrix for this multiparameter family. This gives additional information on the nature of this family and facilitates some calculations. This explicit $R$-matrix satisfies the Yang-Baxter equation. The centre of the paper is section 3 which describes all solutions of the YBE under the restriction $r^{ab}_{cd}=0$ unless $\{a,b\}=\{c,d\}$. One kind of the most general constituents of these solutions precisely corresponds to the $\binom{n}{2}+1$ parameter quantum group mentioned above. I describe which solutions extend to an enhanced Yang-Baxter operator and hence define link invariants. The paper concludes with some preliminary results on these link invariants

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## 0. Introduction and statement of main results

This paper is concerned with multiparameter $R$-matrices and corresponding quantum groups and knot and link invariants. The starting point is an $\binom{n}{2}+1$ parameter deformation of the bialgebra of polynomials on the $n\times n$ matrices

$$K[t^1_n;t^2_n;t^2_n;\dots;t^n_1,\dots,t^n_n]=K[t],\ t^i_j\mapsto t^i_k\otimes t^k_j,\epsilon(t^i_j)=\delta^i_j,$$

where $\delta^i_j$ is the Kronecker delta. Here $K$ is an arbitrary ground field and the Einstein summation convention is in force, i.e. $t^i_k\otimes t^k_j$ stands for $\sum_{k=1}^n t^i_k\otimes t^k_j$. This $\binom{n}{2}+1$ parameter deformation has apparently been independently constructed in various ways by many (groups of) authors, published and unpublished, all more or less in the winter of 1990/1991. I know of several (including myself) and the construction is so natural that quite likely there are more, [1,3,5,9,11,14,15,16,17,18,19,20,21]. (Not all these papers deal with the full family and [3] in fact describes a quantum group which does not fit in this family at all).

Perhaps the most natural point of view is to take two "most general" $n$-dimensional quantum spaces

$$A=K\langle X^1,\dots,X^n\rangle/(X^iX^j=q^{ij}X^jX^i),\ B=K\langle Y_1,\dots,Y_n\rangle/Y_iY_j=q_{ij}Y_jY_i.$$

Here the $q^{ij}=(q^{ij})^{-1}$, $q^{ii}=1$, $q_{ji}=(q_{ij})^{-1},q_{ii}=1$ are arbitrary parameters (viewed as elements of $K$ or as (Laurent) variables). Now look for a maximal quotient $K\langle t\rangle/I$, of $K\langle t\rangle$, $t^i_j\mapsto t^i_k\otimes t^k_j$, to co-act on the left on $A$ and on the right on $B$ by the standard formulas

$$X^i\mapsto t^i_k\otimes X^k,\ \ Y_j\mapsto Y_k\otimes t^k_j$$

For the resulting bialgebra $K\langle t\rangle/I$ to be nice, in the sense that the underlying algebra is PBW (Poincaré-Birkhoff-Witt) certain relations must hold between the $q^{ij}$ and $q_{kl}$, viz. that after a possible permutation of the $1,\dots,n$ (a renumbering of the variables), $q^{ij}q_{ij}=\lambda\neq-1$, for all $i\neq j$. This material from [1] and other papers is recalled in sections 1 and 2 below

The heart of the paper is section 3. In it I consider the Yang-Baxter equation

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12},\quad R=(r^{ab}_{cd})\tag{0.1}$$

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and describe all invertible solutions which satisfy the additional condition

$$r^{ab}_{cd} = 0 \text{ unless } \{a,b\} = \{c,d\} \tag{0.2}$$

These solutions consist of several kinds of blocks which are fitted together in certain noy entirely trivial ways. Cf Theorem 2.44 for a precise discription. Two of the three types of blocks (blocks of type I and blocks of size 1) are rather trivial. The third type of block looks as follows in the size 3 case (from which the general case is easily guessed; cf again theorem 3.44 for the exact description)

$$\begin{array}{c|ccc|ccc|ccc|}
 & 11 & 12 & 13 & 21 & 22 & 23 & 31 & 32 & 33 \\
11 & \rho_1 & & & & & & & & \\
12 & & x_{21}^{-1}z & & y & & & & & \\
13 & & & x_{31}^{-1}z & & & & y & & \\
\hline
21 & & & & x_{21} & & & & & \\
22 & & & & & \rho_2 & & & & \\
23 & & & & & & x_{32}^{-1}z & & y & \\
\hline
31 & & & & & & & x_{31} & & \\
32 & & & & & & & & x_{32} & \\
33 & & & & & & & & & \rho_3
\end{array} \tag{0.3}$$

where the $\rho_1, \rho_2, \rho_3$ are all three solutions of $X^2 = yX + z$ (but not necessarily all three equal). Given any $n^2 \times n^2$ matrix $R$ there is a natural bialgebra $k\langle t\rangle/I(R)$, $t^i_j \mapsto t^i_k \otimes t^k_j$. Here $I(R)$ is the ideal generated by the fundamental commutation relations (FCR) of [6]

$$RT_1T_2 = T_2T_1R \tag{0.4}$$

where $T = (t^i_j)$, $T_1 = T \otimes I_n$, $T_2 = I_n \otimes T$

Multiplying a solution of (0.1) with an invertible scalar produces another solution and does not affect the relations defined by (0.4). Thus the parameter $z$ in (0.3) (or rather its $n^2 \times n^2$ generalization) can be normalized to 1 (by multiplying with $(\sqrt{z})^{-1}$). The two roots of $X^2 = yX + z$ are then $q, -q^{-1}$. If all the $\rho_i$ are now equal to $q$ the invertible $n^2 \times n^2$ matrix like (0.3) precisely defines the $\binom{n}{2} + 1$ parameter deformation of sections 1 and 2. This is the main result of section 4. Having an explicit invertible $R$-matrix, that satisfies the YBE (0.1), for this $\binom{n}{2} + 1$ parameter quantum matrix algebra has a number of considerable advantages. For instance it immediately follows that the rewriting rules (0.4) are confluent which greatly simplifies the proof that this $\binom{n}{2} + 1$ parameter quantum matrix algebra is a PBW algebra. It also helps with the matter of defining a quantum determinant and the definition of an antipode on the bialgebra obtained by making the quantum determinant invertible thus obtaining an $\binom{n}{2} + 1$ parameter quantum group. This is not further explored here, but see [4,12,13,6].

It also seems from (0.3) that $\binom{n}{2}$ parameters of the $\binom{n}{2} + 1$, viz. the $x_{ij}, i > j$, are rather trivial and that there is only one real parameter viz. $y$ (or $q; y = q - q^{-1}$ if $z = 1$). This does not mean that the general quantum matrix algebra ($z = 1, x_{ij}$ arbitrary) and the classical one ($z = 1 = x_{ij}$) are isomorphic; they are not. All the same the $x_{ij}$ do seem less basic than $q$. I do not know how to make this intuition more precise except in the case of the link invariants defined by the enhanced Yang Baxter operator that is associated to (0.3), cf below.

Each block of a solution of (0.1) (assuming (0.2)) defines a scalar. If all those scalars are equal (and only then) the solution gives rise to an enhanced Yang Baxter operator $(\tau R, \nu, \alpha, \beta)$ in the sense of [22] and hence gives rise to a link invariant. In this setting the $\binom{n}{2}$ extra parameters $x_{ij}, i > j$ are indeed trivial. They do not show up in the link invariant in the sense that if the $n^2 \times n^2$ generalization of (0.3) (even with both $q$ and $-q^{-1}$ occurring for the $\rho_i$; we are taking $z = 1$) is extended to an enhanced Yang Baxter operator, which can always be done, than the resulting link invariant is the same as one obtained with all $x_{ij} = 1 = z$ (but possibly a different $n$). This "triviality of the $x_{ij}$" result only applies to "one type II block" solutions of (0.1). Even in the case of a two size 1 block solution of (0.1), nontrivially fitted together, a nontrivial link invariant appears. Though, of course, the two constituents themselves give nothing. (An $n = 1$ solution of (0.1) always defines a trivial link invariant.) Mixing and fitting together different blocks of both different and the same types seems to promise a rich collection of probably new link invariants. This matter remains to be explored.

## 1. Generalized quantum space $\mathbb{A}_q^n$

The coordinate ring is $K\langle X^1, X^2, \ldots, X^n\rangle / I_n$, where $I_n$ is the ideal generated by the elements

$$X^a X^b - q^{ab} X^b X^a \tag{1.1}$$

where $q^{ab} = (q^{ba})^{-1}$ and $q^{aa} = 1$ if for all $a, b \in \{1, \ldots, n\}$. Thus, depending on one's point of view, $\mathbb{A}_q^n$ is a family of algebras parametrized by $\binom{n}{2}$ parameters or an algebra over $K[q^{ab}, (q^{ab})^{-1}; a > b]$, the ring of commutative Laurent polynomials in $\binom{n}{2}$ variables $q^{ab}$, $a > b$

For $q^{ab} = 1$ all $a, b$, one refinds the coordinate ring $K[X^1, X^2, \ldots, X^n]$. The algebra $\mathbb{A}_q^n$ is graded and it is a graded deformation of $A_0^n = K[X^1, \ldots, X^n]$ in the sense that $\dim(A_q^n)_m = \dim(A_0^n)_m$ for all $q$ where a lower $m$ indicates the homogeneous part of degree $m$. Also $\mathbb{A}_q^n$ is a PBW algebra in the sense that the monomials

$$(X^1)^{i_1} \cdots (X^n)^{i_n}, \quad i_j \in \mathbb{N} \cup \{0\} \tag{1.2}$$

form a basis of $\mathbb{A}_q^n$. Indeed it is obvious from (1.1) that every element can be written as a sum of elements of the form (1.2); to prove the other half it suffices by the diamond lemma, [2], to prove that all the "overlaps"

$$X^a(X^b X^c), \quad (X^a X^b)X^c$$

are confluent, i.e. give the same results when using the rewriting rules (1.1). Now

$$\begin{aligned} X^a(X^b X^c) &= q^{bc}(X^a X^c)X^b = q^{bc}q^{ac}X^c(X^a X^b) = q^{bc}q^{ac}q^{ab}X^c X^b X^a \\ (X^a X^b)X^c &= q^{ab}X^b(X^a X^c) = \quad q^{ab}q^{ac}X^c X^c X^a = q^{ab}q^{ac}q^{bc}X^c X^b X^a \end{aligned}$$

## 2. Generalized matrix quantum algebras

Consider the left-coaction of

$$k\langle t\rangle = k\langle t_1^1, \ldots, t_n^1; \ldots, t_1^n, \ldots, t_n^n\rangle$$

on

$$k\langle X\rangle = k\langle X^1, \ldots, X^n\rangle$$

given by the usual formula

$$X^i \mapsto t_k^i \otimes X^k$$

(summation implied).

Now look at what relations are needed between the $t$'s in order that this becomes a co-action of some quotient of $k\langle t\rangle$ on $\mathbb{A}_q^n$. This means that the relations $X^a X^b = q^{ab} X^b X^a$ must be preserved. The image of $X^{ab} - q^{ab} X^b X^a$ under (2.1) is

$$t^a_{r_1}t^b_{r_2}\otimes X^{r_1}X^{r_2}-q^{ab}t^b_{s_1}t^a_{s_2}\otimes X^{s_1}X^{s_2} \tag{2.2}$$

The coefficient of $X^rX^r$ in (2.2) is

$$t^a_rt^a_r-q^{ab}t^b_rt^a_r \tag{2.3}$$

and the coefficient of $X^rX^s$, $r<s$ in (2.2) is

$$t^a_rt^b_s-q^{ab}t^b_rt^a_s+(q^{rs})^{-1}t^a_st^b_r-(q^{rs})^{-1}q^{ab}t^b_st^a_r \tag{2.4}$$

Lets count the number of independent relations.
(i) For $a=b$ no relations arise from (2.3)
(ii) If $a\neq b$, then the relations (2.3) fall in groups of two

$$\begin{aligned} t^a_rt^b_r &= q^{ab}t^b_rt^a_r \\ t^b_rt^a_r &= q^{ba}t^a_rt^b_r \end{aligned} \tag{2.5}$$

which are equivalent because $q^{ba}=(q^{ab})^{-1}$. Thus there are precisely

$$n\binom{n}{2}=\frac{1}{2}n^2(n-1)$$

relations resultng from (2.3). And these are independent.

(iii) If $a=b$ in (2.4) no relations result

(iv) If $r=s$ in (2.4) the relations (2.4) are implied by (2.3)

(v) For $a\neq b$, $r\neq s$, the relations (2.4) fall into groups of four (or groups of two if one takes $r<s$), viz.

$$\begin{aligned} &t^a_rt^b_s-q^{ab}t^b_rt^a_s+(q^{rs})^{-1}t^a_st^b_r-q^{ab}(q^{rs})^{-1}t^b_st^a_r=0 \\ &t^b_rt^a_s-q^{ba}t^a_rt^b_s+(q^{rs})^{-1}t^b_st^a_r-q^{ba}(q^{rs})^{-1}t^a_st^b_r=0 \\ &t^a_st^b_r-q^{ab}t^b_st^a_r+(q^{sr})^{-1}t^a_rt^b_s-q^{ab}(q^{sr})^{-1}t^b_rt^a_s=0 \\ &t^b_st^a_r-q^{ba}t^a_st^b_r+(q^{sr})^{-1}t^b_rt^a_s-q^{ba}(q^{sr})^{-1}t^a_rt^b_s=0 \end{aligned} \tag{2.6}$$

These four relations are all the same. E.g. the second is obtained from the first by multiplication of the first by $-q^{ba}$ and the fourth results from the first by multiplication of the first by $(-q^{ba})(q^{sr})^{-1}$. These relations only involve the four products $t^a_rt^b_s, t^b_rt^a_s, t^a_st^b_r, t^b_st^a_r$ and they are the only relations in which these four (for given $a,b,r,s$) are involved Thus there are precisely

$$\frac{n^2(n-1)^2}{4}$$

independent relations of this type. In total we therefore have

$$\frac{1}{2}n^2(n-1)+\frac{1}{4}n^2(n-1)^2=\frac{1}{4}n^2(n^2-1)$$

quadratic relations.

To make the dimension of the degree two part of $k\langle t\rangle/I$ equal to that of the degree two part of $k[t]$ we need

$$n^4-(n^2+\frac{n^2(n^2-1)}{2})=\frac{1}{2}n^2(n^2-1)$$

relations, so that precisely half of them are missing. There are a variety of ways to add the missing relations. An extremely elegant one is to make $k\langle t\rangle/I$ also act on the right on the dual of the quantum pace $\mathbb{A}_q^n$, [16]. This, however, does not result in the most general quantum matrix algebra. To obtain that, consider a second, a priori completely different, quantum space

$$\mathbb{B}_q^n = K\langle X_q, \ldots, X_n\rangle/(X_bX_a = q_{ba}X_aX_b, a, b \in \{1, \ldots, n\}) \tag{2.7}$$

On which a suitable quotient of $k\langle t\rangle$ is supposed to act on the right by

$$X_i \mapsto X_j \otimes t_i^j \tag{2.8}$$

where, of course, $q_{ba} = q_{ab}^{-1}$, $q_{aa} = 1$, $q_{ab} \neq 0$.

(NB, the $q_{ba}$ are second set of variables, which have, a priori, nothing to do with the $q^{ab}$.) The requirement that the action (2.8) be compatible with the commutation relations $X_bX_a = q_{ba}X_aX_b$ of $\mathbb{B}_q^n$, gives necessary relations on the $t_j^i$ which are completely analogous to those produced by having $k\langle t_j^i\rangle$ act on the left on $k\langle X^a\rangle$ as above. They are

$$t_a^r t_b^r = q_{ab}t_b^r t_a^r \tag{2.9}$$

$$t_a^r t_b^s - q_{ab}t_b^r t_a^s + (q_{rs})^{-1}t_a^s t_b^r - q_{ab}(q_{rs})^{-1}t_b^s t_a^r = 0 \tag{2.10}$$

In case $q_{ab} = -(q^{ab})^{-1}$, relations (2.4) and (2.10) coincide. But generically they are independent.

2.11. LEMMA. *Let $I_L$ in $k\langle t\rangle$ be the two sided ideal generated by the elements (2.3) and let $I_R$ be the two sided ideal generated by the relations (2.9) and (2.10). Both $I_L$ are $I_R$ are bialgebra ideals in $K\langle t\rangle$ and hence so is $I$, the two sided ideal generated by $I_L$ and $I_R$ together.*

The proof of this is contained in Appendix 1.

REMARK. There is also a more elegant way to see that $I_L$ and $I_R$ are bialgebra ideals. Let $A = \mathbb{A}_q^n$. The dual space is $A^! = K\langle X_1, \ldots, X_n\rangle/J$ where $J$ is generated by $X_j^2$, $X_iX_j = -q^{ij}X_jX_i$. It is now a simple mather to check that $A^! \bullet A$ as defined in [16] is precisely $k\langle t\rangle/I_L$. Now $A^! \bullet A$ is always a bialgebra ([16], section 5), for any quadratic algebra $A$. The results above now brings the additional bit of information that $A^! \bullet A$ is in fact the largest quotient of $k\langle t\rangle$ which coacts on the left on $A_q^n$.

Assume from now on that $q^{ab} + q_{ba}^{-1} \neq 0$ for all $a, b$. Then the relations (2.4) and (2.10) combine to give

$$\begin{aligned} t_s^b t_r^a = {} & (q^{sr} + q_{sr}^{-1})^{-1}(q^{ba}q^{sr} - q_{sr}^{-1}q_{ba}^{-1})t_s^a t_r^b \\ & +(q^{sr} + q_{sr}^{-1})^{-1}(q^{ba} + q_{ba}^{-1})t_r^a t_s^b \end{aligned} \tag{2.12}$$

Now order the $t_b^a$ as follows. Choose an ordering on the set of indices $\{1, \ldots, n\}$

$$t_b^a < t_d^c \Leftrightarrow \begin{cases} a < c \\ \text{or } a = c \text{ and } b < d \end{cases} \tag{2.13}$$

Then it follows from $t_r^a t_s^a = q_{rs}t_s^a t_r^a$ and (2.12) that every monomial in $K\langle t\rangle$ can be written modulo $I$ in the form

$$t_{j_1}^{i_1} t_{j_2}^{i_2} \ldots t_{j_m}^{i_m} \quad t_{j_1}^{i_1} \leq t_{j_2}^{i_2} \leq \ldots \leq t_{j_m}^{i_m} \tag{2.14}$$

2.15. DEFINITION. An algebra $A$ over $K$ is a PBW algebra if there are elements $x_1, \ldots, x_m$ in $A$ such that the monomials

$$x_1^{r_1} x_2^{r_2} \ldots x_m^{r_m}, \quad r_i \in \mathbb{N} \cup \{0\}$$

from a basis of $A$ over $K$.

It does not yet follow that $K\langle t\rangle/I$ is a PBW algebra. All we know so far is that (for any ordering of the indices $a, b, \ldots$) the monomials (2.14) generate the algebra and that the monomials of degree 2

$$t^{i_1}_{j_1} t^{i_2}_{j_2} \quad t^{i_1}_{j_1} \leq t^{i_2}_{j_2}$$

are independent (as they should be for a PBW algebra).

2.16. EXAMPLE OF A PBW. Let $g$ be a Lie algebra over $K$ and $Ug$ its universal enveloping algebra. Let $x_1, \ldots, x_m$ be a basis over $K$ for $g \subset Ug$ (as a vector space). Then by the PBW-theorem (Poincaré-Birkhoff-Witt). The

$$x_1^{r_1} \ldots x_m^{r_m}, \quad r_i \in \mathbb{N} \cup \{0\}$$

are a basis for $Ug$ over $K$. Thus $Ug$ is a PBW algebra. This is of course the result which suggested the phrase "PBW-algebra". If $g$ is abelian then $Ug = Sg$ the symmetric algebra of $g$ over $K$, viz.

$$Sg = K[x_1, \ldots, x_m]$$

2.17. THEOREM, [1]. *Let $K$, $q_{ab}, q^{ab}$, $t, I$ be as before. Then $K\langle t\rangle/I$ is a PBW algebra with generators $t^i_j$, $i, j = 1, \ldots, n$ if and only if $q^{ab} + q^{-1}_{ab} \neq 0$ for all $a, b$ and there is a total ordering on the index set $I$ (possibly different from $1 < 2 < \ldots < n$) such that*

$$q_{ab}/q_{ba} = q^{cd}/q_{dc} = \rho \neq -1 \text{ for all } a < b,\ c < d \tag{2.18}$$

Thus we get an $\binom{n}{2}+1$ parameter family of PBW deformations of the polynomial algebra $K[t^1_1, \ldots t^n_n]$. Note that $I$ is a graded ideal so that $M_q = K\langle t\rangle/I$ is also graded. Give the $t^i_j$ degree 1. Then

$$\dim(M_q)_r = \#\{(r_1, \ldots, r_m) : r_i \in \mathbb{N} \cup \{0\}, \sum_{i=1}^{m} r_i = r\}$$

$$= \dim K[t^1_1, \ldots, t^n_n]_r$$

where $m = n^2$, and $A_r$ denotes the homogeneous component of degree $r$ of a graded algebra $A$. The Hilbert-Poincaré series of a graded algebra $A$ is by definition equal to

$$H_A(t) = \sum_{r=1}^{\infty} \dim(A_r) t^r \tag{2.19}$$

Thus the Hilbert-Poncaré series of every $K\langle t\rangle/I$ satisfying (2.18) is equal to that of the polynomial algebra $K[t]$ and the $M_q = K\langle t\rangle/I$ are a deformation of the graded algebra $K[t]$ in the sense of graded algebras.

2.20. PROOF OF THE NECESSITY OF(2.18). By the remark just below 2.10 we already know that we must have

$$q^{ab} + q^{-1}_{ab} \neq 0$$

to get the right amount of monomials of degree 2.

Take $s = a$, $r = b$ in (2.12) to get

$$t^b_a t^a_b = q^{ba} q_{ab} t^a_b t^b_a \tag{2.21}$$

Now use (2.21) and (2.12) and $t^a_r t^b_r = q^{ab} t^b_r t^a_r$, $t^a_r t^a_s = q_{rs} t^a_s t^a_r$ to calculate $t^c_a t^b_a t^a_b$ in two ways for $a \neq b \neq c \neq a$

$$\begin{aligned}
t^c_a(t^b_a t^a_b) &= q^{ba}q_{ab}(t^c_a t^a_b)t^b_a \\
&= q^{ba}q_{ab}(q^{ba}+q^{-1}_{ab})^{-1})q^{ca}q^{ab}-q^{-1}_{ab}q^{-1}_{ca})t^a_a(t^c_b t^b_a) \\
&+q^{ba}q_{ab}(q^{ab}+q^{-1}_{ab})^{-1}(q^{ca}+q^{-1}_{ca})t^a_b(t^c_a t^b_a) \\
&= q^{ba}q_{ab}(q^{ab}+q^{-1}_{ab})^{-1}(q^{ca}q^{ab}-q^{-1}_{ab}q^{-1}_{ca})(q^{ba}+q^{-1}_{ba})^{-1}(q^{cb}q^{ba}-q^{-1}_{cb}q^{-1}_{ba})t^a_a t^b_b t^c_a \\
&+q^{ba}q_{ab}(q^{ab}+q^{-1}_{ab})^{-1}(q^{ca}q^{ab}-q^{-1}_{ab}q^{-1}_{ca})(q^{ba}+q^{-1}_{ba})^{-1}(q^{cb}-q^{-1}_{cb})t^a_a t^b_a t^c_b \\
&+q^{ba}q_{ab}(q^{ab}+q^{-1}_{ab})^{-1}(q^{ca}+q^{-1}_{ca})q^{cb}t^a_b t^b_a t^c_a
\end{aligned}$$

On the other hand

$$\begin{aligned}
(t^c_a t^b_a)t^a_b &= q^{cb}t^b_a(t^c_a t^a_b) \\
&= q^{cb}(q^{ab}+q^{-1}_{ab})^{-1}(q^{ca}q^{ab}-q^{-1}_{ca}q^{-1}_{ab})(t^b_a t^a_a)t^c_b \\
&+q^{cb}(q^{ab}+q^{-1}_{ab})^{-1}(q^{ca}+q^{-1}_{ca})(t^b_a t^a_b)t^c_a \\
&= q^{cb}(q^{ab}+q^{-1}_{ab})^{-1}(q^{ca}q^{ab}-q^{-1}_{ca}q^{-1}_{ab})q^{ba}t^a_a t^b_a t^c_b \\
&= q^{cb}(q^{ab}+q^{-1}_{ab})^{-1}(q^{ca}+q^{-1}_{ca})q^{ba}q_{ab}t^a_b t^b_a t^c_a
\end{aligned}$$

It follows that the coefficient of $t^a_a t^a_b t^c_a$ must be zero which gives

$$q^{ca}q^{ab}-q^{-1}_{ab}q^{-1}_{ca}=0 \text{ or } q^{cb}q^{ba}-q^{-1}_{cb}q^{-1}_{ba}=0 \tag{2.22}$$

Let $\rho_{ab}=q^{ab}q_{ab}=q^{ab}q^{-1}_{ba}$. Then (2.22) says

$$\rho_{ab}=\rho_{ac} \text{ or } \rho_{ab}=\rho_{cd} \tag{2.23}$$

(This holds for all triples $a\neq b\neq c\neq a$). Choose a fixed $i,j$ say $i=1, j=2$ and let $\rho=\rho_{ij}$. Then (2.23) implies

$$\rho_{ab}=\rho \text{ or } \rho_{ab}=\rho^{-1} \text{ for all } a,b \tag{2.24}$$

(but (2.24) is strictly weaker than (2.23)).
If $\rho=\rho^{-1}$ (i.e. $\rho=\pm 1$) then for all $a,b$

$$\rho_{ab}=\frac{q^{ab}}{q_{ba}}=\rho$$

and any ordering works. If $\rho\neq\rho^{-1}$ define

$$i>j \Leftrightarrow \rho_{ij}=\rho \tag{2.25}$$

Then $i>j$, $j>k \Rightarrow \rho_{ij}=\rho$ and $\rho_{jk}=\rho$ so that by (2.23) (with $a=i$, $b=k$, $c=j$) $\rho_{ik}=\rho$, i.e. $i>k$, proving that the order defined by (2.25) is transitive. For this order we have

$$\frac{q^{ij}}{q_{ji}}=\rho_{ij}=\rho \text{ for } i>j$$

This finishes the proof of the necessity of theorem 2.17. The sufficiency can now be handled by the Diamond lemma, [2], which says, in this case, that if all the overlaps

$$(t^a_b t^r_s)t^u_v \;-\; t^a_b(t^r_s t^u_v)$$

are zero, then the monomials (2.14) are a basis. Through there is a good deal of symmetry which can be exploited, this still involves quite a number of cases and rather lenghty calculations for each case. We shall use a different approach, cf Corollary 4.25.

## 3. A RATHER GENERAL CANDIDATE $R$-MATRIX

Let $R = (r^{ab}_{cd})$ be an $n^2 \times n^2$ matrix over $K$. In this section we examine a fairly general $R$-matrix whose form is inspired by the kind of commutation relations of section 2 and study when it satisfies the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{3.1}$$

Here, $R: V \otimes V \to V \otimes V$ with basis $e^1, \ldots, e^n$ is given by

$$R(e^i \otimes e^j) = r^{ij}_{kl} e^k \otimes e^l$$

$R_{12} = R \otimes Id, R_{23} = Id \otimes R$ and $R_{13}(e^i \otimes e^j \otimes e^k) = r^{ik}_{mn} e^m \otimes e^j \otimes e^n$. In terms of the entries $r^{ab}_{cd}$ of $R$ the equation (3.1) says

$$r^{ab}_{k_1k_2} r^{k_1c}_{uk_3} r^{k_2k_3}_{vw} = r^{bc}_{l_1l_2} r^{al_2}_{l_3w} r^{l_3l_1}_{uv} \tag{3.2}$$

for all $a, b, c, u, v, w \in \{1, 2, \ldots, n\}$.

Now consider a general $R$-matrix with the requirement

$$r^{ab}_{cd} = 0 \qquad \text{unless} \qquad \{a, b\} = \{c, d\} \tag{3.3}$$

Thus the only nonzero entries are of the form $r^{ab}_{ab}, r^{ab}_{ba}, r^{aa}_{aa}$ (and $r^{ba}_{ba}, r^{ba}_{ab}$), $a \neq b$.
This is more or less inspired by the commutation relations of section 2 and, as we shall see in section 4, it is possible to choose the $r^{ab}_{cd}$ such that the commutation relations of section 2 are reproduced. It is somewhat remarkable that the requirement that an $R$-matrix of type (3.3) satisfy $YB$ is practically (but not quite) equivalent to the requirement that it gives the right number of relations in degree 2 and that then these are precisely the commutation relations of section 2 above.

The following lemma drastically reduces the number of equations (32) that must be examined (from $n^6$ to 25).

3.4. LEMMA. *Let $R$ be an $n^2 \times n^2$ matrix satisfying (3.3). Then both sides of (3.2) are zero unless* $\{a, b, c\} = \{u, v, w\}$.

PROOF. If a term of the left hand side of (3.2) is zero we must have $\{a, b\} = \{k_1, k_2\}$, $k_3 \in \{k_1, c\}$ so $\{k_1, k_2, k_3\} \subset \{a, b, c\}$. Further $u \in \{k_1, c\}, v, w \in \{k_2, k_3\}$ so $\{u, v, w\} \subset \{k_1, c, k_2, k_3\} = \{k_1, k_2, k_3\} \subset \{a, b, c\}$.

Similarly $\{k_2, k_3\} = \{v, w\}, k_1 \in \{u, k_3\}$ so $\{k_1, k_2, k_3\} \subset \{u, v, w\}$; $\{a, b\} = \{k_1, k_2\}, c \in \{u, k_3\}$ so $\{a, b, c\} \subset \{k_1, k_2, k_3, u\} \subset \{u, v, w\}$.

The argument that for a nonzero term on the right hand side we must have $\{a, b, c\} = \{u, v, w\} = \{l_1, l_2, l_3\}$ is quite similar. Indeed $\{b, c\} = \{l_1, l_2\}, l_3 \in \{a, l_2\}$ so $\{l_1, l_2, l_3\} \subset \{a, b, c\}$; $\{u, v\} = \{l_1, l_3\}$, $w \in \{a, l_2\}$, so $\{u, v, w\} \subset \{l_1, l_2, l_3, a\} \subset \{a, b, c\}$; and $\{l_1, l_3\} = \{u, v\}$, $l_2 \in \{l_3, w\}$, so $\{l_1, l_2, l_3\} \subset \{u, v, w\}$; $\{b, c\} = \{l_1, l_2\}, a \in \{l_3, w\}$ so $\{a, b, c\} \subset \{l_1, l_2, l_3, w\} \subset \{u, v, w\}$.

3.5. LEMMA. *Let $R$ be an $n^2 \times n^2$ matrix satisfying 3.3. Then*

$$\det(R) = \prod_{i=1}^{n} r^{ii}_{ii} \prod_{i<j} (r^{ij}_{ij} r^{ji}_{ji} - r^{ij}_{ji} r^{ji}_{ij})$$

PROOF. Immediate.

3.6. THE $R$-EQUATIONS. Many of the equations (3.2), assuming (3.3), are automatically satisfied. Take for example $a \neq b \neq c \neq a, u = a, v = b, w = c$. Then the nonzero lefthand terms must have $k_1 = a = u, k_3 = c$ and hence $k_2 = b$ so the $LHS$ is equal to $r^{ab}_{ab} r^{ac}_{ac} r^{bc}_{bc}$. For the $RHS$ we must have

$l_3 = a, l_2 = c$, hence $l_1 = b$ and so the $RHS$ is $r^{bc}_{bc} r^{sc}_{ac} r^{ab}_{ab}$ and so this equation is automatically satisfied. As it turns out there remain the following equations

$$r^{bc}_{bc}(r^{ab}_{ba} r^{ac}_{ca}) = r^{bc}_{bc}(r^{ab}_{ba} r^{bc}_{cb} + r^{ac}_{ca} r^{cb}_{bc}) \qquad (a \neq b \neq c \neq a,\ u = b, v = c, w = a) \tag{R1}$$

$$r^{ab}_{ab}(r^{ac}_{ca} r^{ba}_{ab} + r^{ab}_{ba} r^{bc}_{cb}) = r^{ab}_{ab}(r^{bc}_{cb} r^{ac}_{ca}) \qquad (a \neq b \neq c \neq a,\ u = c,\ v = a,\ w = b) \tag{R2}$$

$$r^{ab}_{ab} r^{ba}_{ba} r^{ac}_{ca} + r^{ab}_{ba} r^{ab}_{ba} r^{bc}_{cb} = r^{bc}_{bc} r^{cb}_{cb} r^{ac}_{ca} + r^{bc}_{cb} r^{bc}_{cb} r^{ab}_{ba} \qquad (a \neq b \neq c \neq a,\ u = c,\ v = b,\ w = a) \tag{R3}$$

$$r^{ac}_{ac} r^{ac}_{ca} r^{ca}_{ac} = 0 \qquad (a = b \neq c,\ u = a,\ v = c,\ w = a) \tag{R4}$$

$$r^{aa}_{aa} r^{aa}_{aa} r^{ac}_{ca} = r^{aa}_{aa} r^{ac}_{ca} r^{ac}_{ca} + r^{ac}_{ac} r^{ac}_{ca} r^{ca}_{ca} \qquad (a = b \neq c,\ u = c, v = w = a) \tag{R5}$$

$$r^{aa}_{aa} r^{aa}_{aa} r^{ca}_{ac} = r^{aa}_{aa} r^{ca}_{ac} r^{ca}_{ac} + r^{ca}_{ca} r^{ac}_{ac} r^{ca}_{ac} \qquad (a \neq b = c,\ u = b,\ v = b,\ w = a) \tag{R6}$$

$$r^{ab}_{ba} r^{ba}_{ab} r^{ab}_{ba} = r^{ab}_{ba} r^{ba}_{ab} r^{ba}_{ab} \qquad (a = c \neq b,\ u = a,\ v = b,\ w = a) \tag{R7}$$

All the other cases either give nothing or give back one of these six types of equations. For the complete detailed analysis, cf Appendix 2.

3.7. A SOLUTION FAMILY

Take

$$r^{ij}_{ij} = x_{ij} \text{ for } i > j,\ r^{ij}_{ij} = x^{-1}_{ji} \lambda^u \lambda_d \text{ for } i < j$$

$$r^{ii}_{ii} = \lambda^u,\ r^{ij}_{ji} = \lambda^u - \lambda_d \text{ if } i < j,\ r^{ij}_{ji} = 0 \text{ for } i > j$$

It is a straightforward matter to check that these $r$'s satisfy (R1) - (R7).

There are $\binom{n}{2}$ parameters $x_{ij}, i < j$ and two more parameters $\lambda^u, \lambda_d$. One of these can be eliminated by dividing all parameters by an arbitrary number.

We have here thus an $\binom{n}{2} + 1$ parameter family and this is in fact the $\binom{n}{2} + 1$ parameter family of section 2 above. The connections are

$$q^{ab} = x^{-1}_{ab} \lambda^u,\ q_{ba} = x^{-1}_{ab},\quad a > b. \tag{3.8}$$

3.9. PARTIAL ORDENING $\{1, \dots, n\}$

We assume that $R$ is invertible. Define for $a, b \in \{1, \dots, n\}$.

$$a \leq b \Leftrightarrow r^{ab}_{ba} \neq 0 \tag{3.10}$$

3.11. LEMMA. *The relation defined by (3.10) is a partial order*

PROOF. We have to show transitivity. Let $r^{ab}_{ba} \neq 0 \neq r^{bc}_{cb}$, i.e. $a \leq b, b \leq c$ and we have to show $r^{ac}_{ca} \neq 0$ (which is $a \leq c$).

By (R7) here are four cases to consider

$$r^{ab}_{ba} \neq 0,\ r^{ba}_{ab} = 0,\ r^{bc}_{cb} \neq 0,\ r^{cb}_{bc} = 0 \tag{3.11.1}$$

$$r^{ab}_{ba} \neq 0,\ r^{ba}_{ab} = 0,\ r^{bc}_{cb} = r^{cb}_{bc} \neq 0 \tag{3.11.2}$$

$$r^{ab}_{ba} = r^{ba}_{ab} \neq 0,\ r^{bc}_{cb} \neq 0,\ r^{cb}_{bc} = 0 \tag{3.11.3}$$

$$r^{ab}_{ba} = r^{ba}_{ab} \neq 0,\ r^{bc}_{cb} = r^{cb}_{bc} \neq 0 \tag{3.11.4}$$

In case (1) by the invertibility of $R$ (cf Lemma 3.5), also $r^{ab}_{ab} \neq 0 \neq r^{ba}_{ba}$. Hence by (R2)

$$r^{ab}_{ba}r^{bc}_{cb} = r^{bc}_{cb}r^{ac}_{ca}$$

and hence

$$r^{ac}_{ca} = r^{ab}_{ba} \neq 0.$$

In case (2), also $r^{ab}_{ab} \neq 0 \neq r^{ba}_{ba}$ and using (R2) with $a$ and $b$ transposed gives

$$r^{bc}_{cb}r^{ab}_{ba} = r^{ac}_{ca}r^{bc}_{cb}$$

so that again

$$r^{ac}_{ca} = r^{ab}_{ba} \neq 0$$

In case (3), by the invertibility of $R, r^{bc}_{bc} \neq 0 \neq r^{cb}_{cb}$ and hence by (R1)

$$r^{ab}_{ba}r^{ac}_{ca} = r^{ab}_{ba}r^{bc}_{cb}$$

and hence $r^{ac}_{ca} = r^{bc}_{cb} \neq 0$

In case (4), suppose that $r^{ac}_{ca} = 0$. Then, by invertibility of $R, r^{ac}_{ac} \neq 0 \neq r^{ca}_{ca}$.

Now use (R3) with $b$ and $c$ transposed to obtain

$$r^{ac}_{ac}r^{ca}_{ca}r^{ab}_{ba} + r^{ac}_{ca}r^{ac}_{ca}r^{cb}_{bc} = r^{cb}_{cb}r^{bc}_{bc}r^{ab}_{ba} + r^{cb}_{bc}r^{cb}_{bc}r^{ac}_{ca}$$

By (R4), $r^{cb}_{cb} = r^{bc}_{bc} = 0$ (because $r^{bc}_{cb}r^{cb}_{bc} \neq 0$); hence this would give

$$r^{ac}_{ac}r^{ca}_{ca}r^{ab}_{ba} = 0, \text{ i.e. } r^{ab}_{ba} = 0,$$

a contradiction. Hence $r^{ac}_{ca} \neq 0$, concluding the proof of the lemma.

3.12. BLOCKS
Still assuming that $R$ is invertible define two indices $a, b \in \{1, \ldots, n\}$ to be connected if $a \leq b$ or $b \leq a$ in the ordening of 3.9 above.

3.13. LEMMA. *Connectedness is an equivalent relation.*

REMARK. This is not immediately implied by Lemma 3.11. It adds information e.g. to the case $a \leq b,\ a \leq c$, by stating that then $b, c$ are comparable.

PROOF OF LEMMA 3.13. There are four cases to consider

$$r^{ab}_{ba} \neq 0,\ r^{bc}_{cb} \neq 0. \text{ Then } a \leq b,\ b \leq c, \text{ hence } a \leq c \text{ and } r^{ac}_{ca} \neq 0 \qquad (3.13.1)$$

$$r^{ba}_{ab} \neq 0,\ r^{cb}_{bc} \neq 0. \text{ Then } b \leq a,\ c \leq b, \text{ hence } c \leq a \text{ and } r^{ca}_{ac} \neq 0 \qquad (3.13.2)$$

The other two cases involve more work

$$r^{ab}_{ba} \neq 0,\ r^{cb}_{bc} \neq 0 \qquad (3.13.3)$$

As in the case of the proof of lemma 3.11 there are (by (R7)) four possible subcases to consider.

$$r^{ab}_{ba} \neq 0,\ r^{ba}_{ab} = 0,\ r^{cb}_{bc} \neq 0,\ r^{bc}_{cb} = 0 \qquad (3.13.3.1)$$

$$r^{ab}_{ba} \neq 0,\ r^{ba}_{ab} = 0,\ r^{cb}_{bc} = r^{bc}_{cb} \neq 0 \qquad (3.13.3.2)$$

$$r^{ab}_{ba} = r^{ba}_{ab} \neq 0,\ r^{cb}_{bc} \neq 0,\ r^{bc}_{cb} = 0 \qquad (3.13.3.3)$$

$$r^{ab}_{ba} = r^{ba}_{ab} \neq 0,\ r^{cb}_{bc} = r^{bc}_{cb} \neq 0 \tag{3.13.3.4}$$

In the last three subcases lemma 3.11 is immediately applicable. It remains to deal with (3.13.3.1). In this case $r^{bc}_{bc} \neq 0$ by invertibility.

Now use (R2) after the permutation $b \mapsto c \mapsto b,\ a \mapsto a$ to find

$$r^{ac}_{ac}(r^{ab}_{ba}r^{ca}_{ac} + r^{ac}_{ca}r^{cb}_{bc}) = r^{ac}_{ac}(r^{cb}_{bc}r^{ab}_{ba}) \tag{3.13.3.5}$$

Now if $r^{ac}_{ca} = r^{ca}_{ac} = 0$, $r^{ac}_{ac} \neq 0$ by invertibility. Hence the *RHS* of (3.13.3.5) is $\neq 0$ so that also $r^{ca}_{ac}$ or $r^{ac}_{ca}$ must be $\neq 0$.

The final case is

$$r^{ba}_{ab} \neq 0,\ r^{bc}_{cb} \neq 0. \tag{3.13.4}$$

Again there are four subcases

$$r^{ba}_{ab} \neq 0,\ r^{ab}_{ba} = 0,\ r^{bc}_{cb} \neq 0,\ r^{cb}_{bc} = 0. \tag{3.13.4.1}$$

$$r^{ba}_{ab} \neq 0,\ r^{ab}_{ba} = 0,\ r^{bc}_{cb} = r^{cb}_{bc} \neq 0. \tag{3.13.4.2}$$

$$r^{ba}_{ab} = r^{ab}_{ba} \neq 0,\ r^{bc}_{cb} \neq 0,\ r^{cb}_{bc} = 0. \tag{3.13.4.3}$$

$$r^{ba}_{ab} = r^{ab}_{ba} \neq 0,\ r^{bc}_{cb} = r^{cb}_{bc} \neq 0. \tag{3.13.4.4}$$

Again, lemma 3.11 immediately takes care of (3.13.4.2) - (3.13.4.4) and only (3.13.4.1) remains. In this case if $r^{ac}_{ca} = r^{ca}_{ac} = 0$, $r^{ac}_{ac} \neq 0$ which by (R1) would imply $r^{ba}_{ab}r^{ba}_{cb} = 0$ contradicting (3.13.4.1). Hence $r^{ac}_{ca} \neq 0$ or $r^{ca}_{ac} \neq 0$ and we are done.

3.14. Definition. *An equivalence class $B \subset \{1, \ldots, n\}$ under the equivalence relation of connectedness will be called a block.*

3.15. Structure of blocks I

In this subsection and the next the structure of blocks is examined. More precisely if $B$ is a block, the submatrix $R_B = (r^{ab}_{cd})_{a,b,c,d,\in B}$ is determined. After that we will examine how blocks can fit together. The first kind of block is (as it will turn out) the trivial one. The result is:

3.16. Proposition. *Let $B$ be a block. Suppose that there are $a \neq b$, $a, b \in B$ such that $a \leq b$ and $b \leq a$. Then there is a $\lambda \neq 0$ such that for all $c, d \in B$*

$$r^{cc}_{cc} = r^{dd}_{dd} = r^{cd}_{dc} = r^{dc}_{cd} = \lambda,\ r^{cd}_{cd} = r^{dc}_{dc} = 0 \qquad \forall c, d \in B \tag{3.17}$$

Proof. By assumption $r^{ab}_{ba} \neq 0 \neq r^{ba}_{ab}$. Hence $r^{ab}_{ab} = r^{ba}_{ba} = 0$ by (R4), and $\lambda = r^{ab}_{ba} = r^{ba}_{ab}$ by (R6). Putting this in (R5) gives

$$r^{aa}_{aa}r^{ab}_{ba} = r^{aa}_{aa}r^{ab}_{ba}r^{ab}_{ba} \tag{3.18}$$

By invertibility (cf Lemma 3.5), $r^{aa}_{aa} \neq 0$. Hence $r^{aa}_{aa} = r^{ab}_{ba} = \lambda$ and switching $a, b$ also $r^{bb}_{bb} = \lambda$. Hence (3.17) holds for these particular $a, b \in B$. Now let $c \in B$, $a \neq c \neq b$. If now suffices to prove

$$r^{cc}_{cc} = \lambda = r^{ac}_{ca},\ r^{ac}_{ac} = r^{ca}_{ca} = 0 \tag{3.19}$$

and for this, in turn, it suffices, by the previous argument, to show that

$$r^{ac}_{ca} \neq 0 \neq r^{ca}_{ac} \tag{3.20}$$

We already know

$$r^{ac}_{ca} \neq 0 \quad \text{or} \quad r^{ca}_{ac} \neq 0.$$

Suppose $r^{ac}_{ca} \neq 0$, then $a \leq c$, $b \leq a$, so also $b \leq c$ and $r^{bc}_{cb} \neq 0$. If $r^{cb}_{bc} \neq 0$, also $c \leq b \leq a$, so $r^{ca}_{ac} \neq 0$. Hence it remains (assuming $r^{ac}_{ca} \neq 0$) to analyze the case

$$r^{ac}_{ca} \neq 0 \neq r^{bc}_{cb},\ r^{ca}_{ac} = 0 = r^{cb}_{bc} \tag{3.21}$$

i.e. the case $a \sim b < c$ when $a \sim b$ means $a \leq b$ and $b \leq a$.
It follows from (3.21) that

$$r^{bc}_{bc} \neq 0 \neq r^{cb}_{cb},\ r^{ac}_{ac} \neq 0 \neq r^{ca}_{ca} \tag{3.22}$$

Now use (R1) to see that

$$r^{ac}_{ca} = r^{bc}_{cb} = \mu \tag{3.23}$$

(defining $\mu$). Now replace $a$ by $b$ in (R5) to find

$$\lambda^2 \mu = \lambda \mu^2 + r^{bc}_{bc} r^{cb}_{cb} \mu \tag{3.24}$$

On the other hand (R3) gives

$$\lambda^2 \mu + \lambda^2 \mu = r^{bc}_{bc} r^{cb}_{cb} \mu + \mu^2 \lambda. \tag{3.25}$$

It follows that $\lambda^2 \mu = 0$, i.e. $\mu = 0$ a contradiction. So $r^{ca}_{ac} \neq 0$. The case $r^{ca}_{ac} \neq 0$ (but $r^{ac}_{ca} = 0$) is disposed of similarly and (3.20) is established so that the proposition is proved.

3.26. STRUCTURE OF OF BLOCKS II
Now let $B$ be a block such that there are no $a, b \in B$ for which $a \leq b$ and $b \leq a$. Then by the previous proposition for all $c \neq d$, $c, d \in B$ it holds that not $c \leq d$, $d \leq c$, i.e.

$$r^{ab}_{ba} r^{ba}_{ab} = 0 \qquad \text{for all} \qquad a, b \in B. \tag{3.27}$$

We also know that all $a, b \in B$ are comparable, so that the induced order on $B$ must be total. Hence we have the situation

$$r^{ab}_{ba} \neq 0,\ r^{ba}_{ab} = 0,\ r^{ab}_{ab} \neq 0 \neq r^{ba}_{ba} \qquad \text{all} \qquad a < b,\ a, b \in B \tag{3.28}$$

3.29. COROLLARY. *A block $B$ is one of the following two types:*
*Type I: For all $a, b \in B$, $a \sim b$ and its structure is given by (3.17)*
*Type II: $B$ is linearly ordered and no two elements $a, b$ satisfy $a \sim b$.*

The next proposition gives the structure of a block of type II

3.30. PROPOSITION. *Let $B$ be a block of type II. Then there are $y \neq 0$, $z \neq 0$ such that for all $a, b \in B$.*

$$r^{ba}_{ab} = 0,\ r^{ab}_{ba} = y \quad \text{for} \quad a < b,\ r^{ab}_{ab} r^{ba}_{ba} = z \quad \text{for} \quad a \neq b \tag{3.31}$$

$$r^{aa}_{aa} = \lambda \quad \text{or} \quad \mu \quad \text{where} \quad \lambda \quad \text{and} \quad \mu \tag{3.32}$$

are the two solutions of

$$X^2 = yX + z. \tag{3.33}$$

PROOF. Take $a < b < c$ (where $a < b$ means $a \leq b$ and not $b \leq a$). Then using (3.28) and (R2) it follows that

$$r^{ac}_{ca} = r^{bc}_{cb} \tag{3.34}$$

This defines $y$ and establishes the first part of (3.31). Using this in (R3) gives

$$r^{ab}_{ab}r^{ba}_{ba}y + y^3 = r^{bc}_{bc}r^{cb}_{cb}y + y^3$$

so that for all $a < b < c$

$$r^{ab}_{ab}r^{ba}_{ba} = r^{bc}_{bc}r^{cb}_{cb} \tag{3.35}$$

Switching $b$ and $c$ in (R3) now gives

$$r^{ac}_{ac}r^{ca}_{ca}y = r^{cb}_{cb}r^{bc}_{ba}y$$

and, as $y \neq 0$, this, together with (3.35), establishes the second part of (3.31). The last part of proposition (3.30) now follows directly from (R5) and (R6).

3.36. THE DIAGONAL BLOCK
A block consisting of a single element is (trivially) both of type I and type II. These blocks behave slightly different when connecting blocks together then blocks of type I or type II of size $> 1$. It is therefore sometimes convenient to group all the blocks of size 1 together in one diagonal block $D \subset \{1, \ldots, n\}$. For this group of indices we therefore have

$$d \in D \quad \Rightarrow \quad r^{da}_{ad} = r^{ad}_{da} = 0 \quad \text{for all} \quad a \in \{1, \ldots, n\}$$

and hence

$$r^{ad}_{ad} \neq 0 \neq r^{da}_{da}, \; d \in D, \; a \in \{1, \ldots, n\} \tag{3.38}$$

3.39. SIMPLIFYING (R3)
Both the second term on the left of (R3), i.e. $r^{ab}_{ba}r^{ab}_{ba}r^{bc}_{cb}$, and the second term on the right of (R3), i.e. $r^{bc}_{bc}r^{bc}_{bc}r^{ab}_{ba}$ are zero unless $a \leq b$ and $b \leq c$. But then $r^{ab}_{ba} = r^{bc}_{cb}$ so that these terms are equal. Thus (R3) simplifies to

$$r^{ab}_{ab}r^{ba}_{ba}r^{ac}_{ca} = r^{bc}_{bc}r^{cb}_{cb}r^{ac}_{ca} \tag{R3'}$$

3.40. PROPOSITION. *Let $B_1, \ldots, B_m$ be the blocks of $\{1, \ldots, n\}$. Then there are $z_{st}$, $s, t \in \{1, \ldots, m\}$, $z_{st} = z_{ts}$, $z_{ss} = 0$ if $B_s$ is a block of type I, such that for all $a \neq b$.*

$$r^{ab}_{ab}r^{ba}_{ba} = z_{st} \quad \text{if} \quad a \in B_s, \; b \in B_t, \; a \neq b. \tag{3.41}$$

PROOF. If $s = t$ and $B_s$ is of type I, then $r^{ab}_{ab} = 0 = r^{ba}_{ba}$, and if $B_s$ is of type II then the result is part of the structure proposition of blocks of type II. We can therefore assume that $s \neq t$. Choose $c \in B_s$, $d \in B_t$ and set

$$z_{st} = r^{cd}_{cd}r^{dc}_{dc} \tag{3.42}$$

If $\#B_s = \#B_t = 1$ there is nothing more to prove. If $\#B_s = 1$, $\#B_t > 1$, let $b \in B_t$, $b \neq d$. Then $r^{bd}_{db} \neq 0$ or $r^{db}_{bd} \neq 0$ and in both cases (R3$'$) gives

$$r^{bc}_{bc} r^{cb}_{cb} = r^{cd}_{cd} r^{dc}_{dc} \tag{3.43}$$

establishing the result in this case. The case $\#B_s > 1$, $\#B_t = 1$ goes the same. Finally if $a \neq c$, $a \in B_s$, $b \neq d$, $b \in B_t$ then we get again (3.34) and also because $r^{ac}_{ca} \neq 0$ or $r^{ca}_{ac} \neq 0$

$$r^{ab}_{ab} r^{ba}_{ba} = r^{bc}_{bc} r^{cb}_{cb}$$

which combined with (3.42) gives (3.41).

It will now turn out that the various properties which have been derived in above are in fact also sufficient to guarantee a solution of the $YBE$.

This leads to the following description of all solutions of the $YBE$ under the restriction $r^{ab}_{cd} = 0$ unless $\{a,b\} = \{c,d\}$.

3.44. THEOREM. *Divide the set of indices $\{1,\dots,n\}$ into blocks. Assign to each block type I or type II (or both in the case of a block of size 1). Further choose numbers $\in K$ as follows*

*(i) For a block $B_s$ of size 1 choose $\lambda_s \in K$, $\lambda_s \neq 0$*
*(ii) For a block of type I choose $\lambda_s \in K$, $\lambda_s \neq 0$*
*(iii) For a block of type II choose $y_s \in K$, $z_s \in K$, $z_s \neq 0$, $y_s \neq 0$*
*(iv) For each two blocks $B_s, B_t$, $s \neq t$ choose $z_{st} \in K$, $z_{st} \neq 0$, $z_{st} = z_{ts}$.*
*(v) For each $a > b$ such that $a, b$ are not in the same block of type I, choose $x_{ab} \in K$, $x_{ab} \neq 0$*

*Now define the $r^{ab}_{cd}$ as follows*

*(vi) If $a \in B_s$, $\#B_s = 1$, $r^{aa}_{aa} = \lambda_s$*
*(vii) If $a, b \in B_s$, $B_s$ of type I, $r^{aa}_{aa} = r^{bb}_{bb} = r^{ab}_{ba} = r^{ba}_{ab}\lambda = \lambda_s, r^{ab}_{ab} = r^{ba}_{ba} = 0$*
*(viii) If $a, b \in B_s$, $B_s$ of type II, $a < b$, $r^{ab}_{ba} = y_s, r^{ba}_{ab} = 0, r^{ab}_{ab} = z_s x^{-1}_{ba}$, $r^{ba}_{ba} = x_{ba}$, $r^{aa}_{aa} = \lambda_s, \mu_s$, $r^{bb}_{bb} = \lambda_s, \mu_s$ where $\lambda_s, \mu_s$ are the two solutions of $X^2 = Xy_s + z_s$ (It is allowed that $r^{aa}_{aa} \neq r^{bb}_{bb}$)*
*(ix) If $a, b$ are in different blocks, $a > b$, $a \in B_s$, $b \in B_t$, $r^{ab}_{ab} = x_{ab}$, $r^{ba}_{ba} = z_{st} x^{-1}_{ab}$, $r^{ab}_{ba} = r^{ba}_{ab} = 0$*
*(x) $r^{ab}_{cd} = 0$ unless $\{a,b\} = \{c,d\}$.*

*Then the $r^{ab}_{cd}$ thus specified constitute a solution of the $YBE$.*
*Moreover up to a permutation of $\{1,\dots,n\}$ (non-unique as a rule) every solution satisfying $(x)$ is thus obtained.*

PROOF. After a permutation of indices if necessary, the partial order defined by $a \leq b \Leftrightarrow r^{ab}_{ba} \neq 0$ is compatible with the natural order of $\{1,\dots,n\}$. The statement that all solutions under the restriction $(x)$ are obtained by the recipe *(i) - (ix)* above is now the context of the lemmas and formulas (3.10)-(3.43).

It remains to show that if $R = (r^{ab}_{cd})$ is constructed by this recipe then it is indeed a solution. This is a fairly straightforward verification of (R1)-(R7).

*The six equations* (R1). If $a, b, c$ do not all belong to the same block at most one of the three pairs $r^{ab}_{ba}, r^{ba}_{ab}$; $r^{bc}_{cb}, r^{cb}_{bc}$; $r^{ac}_{ca}, r^{ca}_{ac}$ can be nonzero. As each term in an (R1) equation involves a product of elements from different pairs, all terms in an (R1) equation are zero in this case. It remains to check the case that $a, b, c$ all belong to the same block. If this block is of type I $r^{ab}_{ab} = r^{bc}_{bc} = r^{ac}_{ac} = 0$ and again all terms are zero. So let $a \neq b \neq c \neq a$ all belong to the same block $B_s$ of type II. If $a < b < c$, $r^{cb}_{bc} = 0$ and $r^{ac}_{ca} = r^{bc}_{cb} = y_s$; if $a < c < b$, $r^{bc}_{cb} = 0$ and $r^{ab}_{ba} = y_s = r^{cb}_{bc}$; if $b < a < c$, $r^{ab}_{ba} = 0 = r^{cb}_{bc}$; if $b < c < a$, $r^{ac}_{ca} = 0 = r^{ab}_{ba}$; if $c < a < b$, $r^{ac}_{ca} = 0 = r^{bc}_{cb}$; if $c < b < a$, $r^{ab}_{ba} = 0 = r^{ac}_{ca}$; so (R1) holds in all six cases (or, equivalently for a given $abc$ all six (R1) equations hold).

*The six equations* (R2). As in the case of (R1) if $a, b, c$, $a \neq b \neq c \neq a$, do not all belong to the same block, all terms are zero, and, also again, if $a, b, c$ all belong to the same block of type I then $r^{ab}_{ab} = r^{bc}_{bc} = r^{ac}_{ac} = 0$. Thus it remains to deal with the case that $a, b, c$ belong all to a block $B_s$ of type II. If $a < b < c$, $r^{ba}_{ab} = 0$ and $r^{ab}_{ba} = y_s = r^{ac}_{ca}$; if $a < c < b$, $r^{bc}_{cb} = 0 = r^{ba}_{ab}$; if $b < a < c$, $r^{ab}_{ba} = 0$,

$r^{ba}_{ab} = y_s = r^{bc}_{cb}$; if $b < c < a$, $r^{ac}_{ca} = 0 = r^{ab}_{ba}$; if $c < a < b$, $r^{ac}_{ca} = 0 = r^{bc}_{cb}$; if $c < b < a$; $r^{ac}_{ca} = 0 = r^{ab}_{ba}$. Thus (R2) holds in all six cases.

*The six equations* (R3). These can first be simplified to (R3′). Indeed, unless, $a, b, c$ belong to the same block, the second term on the left and the second term on the right are both zero. If all these belong to the same block of type I both these terms are equal to $\lambda_s^3$; and finally if $a, b, c$ all belong to a block of type II both terms are equal to zero unless $a < b < c$ and then both are equal to $y_s^3$.

It remains to check (R3′). Both terms in (R3′) are zero unless $a < c$. Let $B_s$ be the block of $a, c$. If $b$ is also in $B_s$ and $B_s$ is of type II, $r^{ab}_{ab} r^{ba}_{ba} = z_s = r^{bc}_{bc} r^{cb}_{cb}$ by (viii). If $b \in B_s$ is of type I, $r^{ab}_{ab} = 0 = r^{bc}_{bc}$. If $b \in B_t \neq B_s$, then if $a > b$, $r^{ab}_{ab} r^{ba}_{ba} = x_{ab} z_{st} x^{-1}_{ab} = z_{st}$ and if $b > a$, $r^{ba}_{ba} r^{ab}_{ab} = x_{ba} z_{ts} x^{-1}_{ba} = z_{ts} = z_{st}$; similarly $r^{bc}_{bc} r^{cb}_{cb} = z_{st}$ because $b \in B_t$, $c \in B_s$. Thus (R3) holds.

*The two equations* (R4). If $a, c$ are not in the same block $r^{ac}_{ca} = 0$. If they are in the same block of type I, $r^{ac}_{ac} = 0$; if they are in the same block of type II $r^{ac}_{ca} r^{ca}_{ac} = 0$.

*The two equations* (R5) If $a, c$ are not both in the same block $r^{ac}_{ca} = 0$ and all terms are zero. If $a, c$ are in the same block $B_s$ of type I, $r^{aa}_{aa} = \lambda_s = r^{ac}_{ca}$ and $r^{ac}_{ac} = r^{ca}_{ca} = 0$ so that (R5) holds. Finally, if $a, c$ are in a block $B_s$ of type II, all terms are zero unless $a > c$ and then $r^{aa}_{aa} = \lambda_s, \mu_s; r^{ac}_{ca} = y_s$, $r^{ac}_{ac} r^{ca}_{ca} = z_s$ by (viii) and (R5) holds because $\lambda_s, \mu_s$ both solve $X^2 = X y_s + z_s$.

*The two equations* (R6). Exactly the same argument as (R5).

*The two equations* (R7). $r^{ab}_{ba} r^{ba}_{ab} = 0$ unless $a, b$ belong to the same block $B_s$ of type I and then $r^{ab}_{ba} = r^{ba}_{ab} = \lambda_s$.

3.45. Some examples

$$
\begin{array}{c|ccc|ccc|ccc}
 & 11 & 12 & 13 & 21 & 22 & 23 & 31 & 32 & 33 \\
11 & \lambda & & & & & & & & \\
12 & & zx_{21}^{-1} & & y & & & & & \\
13 & & & zx_{31}^{-1} & & & & y & & \\
\hline
21 & & & & x_{21} & & & & & \\
22 & & & & & \lambda & & & & \\
23 & & & & & & zx_{32}^{-1} & & y & \\
\hline
31 & & & & & & & x_{31} & & \\
32 & & & & & & & & x_{32} & \\
33 & & & & & & & & & \mu
\end{array}
$$

$n = 3$; one block of type II ($\lambda^2 = \lambda y + z$; $\mu^2 = \mu y + z$; $\lambda, \mu, x_{ij}, z$ all $\neq 0$; $p = 4$)

$$
\begin{array}{c|ccc|ccc|ccc}
 & 11 & 12 & 13 & 21 & 22 & 23 & 31 & 32 & 33 \\
11 & \lambda & & & & & & & & \\
12 & & & & \lambda & & & & & \\
13 & & & & & & & \lambda & & \\
\hline
21 & & \lambda & & & & & & & \\
22 & & & & & \lambda & & & & \\
23 & & & & & & & & \lambda & \\
\hline
31 & & & \lambda & & & & & & \\
32 & & & & & & \lambda & & & \\
33 & & & & & & & & & \lambda
\end{array}
$$

$n = 3$; one block of type I ($\lambda \neq 0$; $p = 1$)

$$
\begin{array}{c cccc|cccc|cccc|cccc}
 & 11 & 12 & 13 & 14 & 21 & 22 & 23 & 24 & 31 & 32 & 33 & 34 & 41 & 42 & 43 & 44 \\
11 & \lambda_1 & & & & & & & & & & & & & & & \\
12 & & z_1 x_{21}^{-1} & & & y_1 & & & & & & & & & & & \\
13 & & & z_{12} x_{31}^{-1} & & & & & & & & & & & & & \\
14 & & & & z_{12} x_{41}^{-1} & & & & & & & & & & & & \\
\cline{2-17}
21 & & & & & x_{21} & & & & & & & & & & & \\
22 & & & & & & \lambda_1 & & & & & & & & & & \\
23 & & & & & & & z_{12} x_{32}^{-1} & & & & & & & & & \\
24 & & & & & & & & z_{12} x_{42}^{-1} & & & & & & & & \\
\cline{2-17}
31 & & & & & & & & & x_{31} & & & & & & & \\
32 & & & & & & & & & & x_{32} & & & & & & \\
33 & & & & & & & & & & & \lambda_2 & & & & & \\
34 & & & & & & & & & & & & z_2 x_{43}^{-1} & & & y_2 & \\
\cline{2-17}
41 & & & & & & & & & & & & & x_{41} & & & \\
42 & & & & & & & & & & & & & & x_{42} & & \\
43 & & & & & & & & & & & & & & & x_{43} & \\
44 & & & & & & & & & & & & & & & & \mu_2
\end{array}
$$

$n = 4$; two block of type II, both of size 2; $\lambda_1^2 = \lambda_1 y_1 + z_1$, $\lambda_2^2 = \lambda_2 y_2 + z_2$, $\mu_2^2 = \mu_2 y_2 + z_2$; $x_{ij}, \lambda_i, \mu_2, z_i, z_{12}$ all $\neq 0$ $p = 12$

$$
\begin{array}{c ccc|ccc|ccc}
 & 11 & 12 & 13 & 21 & 22 & 23 & 31 & 32 & 33 \\
11 & \lambda_1 & & & & & & & & \\
12 & & z_{12} x_{21}^{-1} & & & & & & & \\
13 & & & z_{13} x_{31}^{-1} & & & & & & \\
\cline{2-10}
21 & & & & x_{21} & & & & & \\
22 & & & & & \lambda_2 & & & & \\
23 & & & & & & z_{23} x_{32}^{-1} & & & \\
\cline{2-10}
31 & & & & & & & x_{31} & & \\
32 & & & & & & & & x_{32} & \\
33 & & & & & & & & & \lambda_3
\end{array}
$$

($n = 3$, there blocks of size 1, $p = g$, all parameter $\neq 0$; if all blocks are of size 1, $R$ is simple any invertible diagonal matrix)

$$
\begin{array}{c|cccc|cccc|cccc|cccc}
 & 11 & 12 & 13 & 14 & 21 & 22 & 23 & 24 & 31 & 32 & 33 & 34 & 41 & 42 & 43 & 44 \\
\hline
11 & \lambda_1 & & & & & & & & & & & & & & & \\
12 & & z_1x_{21}^{-1} & & & y_1 & & & & & & & & & & & \\
13 & & & z_{12}x_{31}^{-1} & & & & & & & & & & & & & \\
14 & & & & z_{13}x_{41}^{-1} & & & & & & & & & & & & \\
\hline
21 & & & & & x_{21} & & & & & & & & & & & \\
22 & & & & & & \lambda_1 & & & & & & & & & & \\
23 & & & & & & & z_{12}x_{32}^{-1} & & & & & & & & & \\
24 & & & & & & & & z_{13}x_{42}^{-1} & & & & & & & & \\
\hline
31 & & & & & & & & & x_{31} & & & & & & & \\
32 & & & & & & & & & & x_{32} & & & & & & \\
33 & & & & & & & & & & & \lambda_2 & & & & & \\
34 & & & & & & & & & & & & z_{23}x_{43}^{-1} & & & & \\
\hline
41 & & & & & & & & & & & & & x_{41} & & & \\
42 & & & & & & & & & & & & & & x_{42} & & \\
43 & & & & & & & & & & & & & & & x_{43} & \\
44 & & & & & & & & & & & & & & & & \mu_3
\end{array}
$$

($n = 4$, 1 block of type II of size 2, 2 blocks of size 1, $\lambda_1^2 = \lambda_1 y_1 + z_1$, $p = 13$, $\lambda_i, z_{ij}, z_1, x_{ij} \neq 0$)

$$
\begin{array}{c|cccc|cccc|cccc|cccc}
 & 11 & 12 & 13 & 14 & 21 & 22 & 23 & 24 & 31 & 32 & 33 & 34 & 41 & 42 & 43 & 44 \\
\hline
11 & \lambda_1 & & & & & & & & & & & & & & & \\
12 & & z_1x_{21}^{-1} & & & y_1 & & & & & & & & & & & \\
13 & & & z_{12}x_{31}^{-1} & & & & & & & & & & & & & \\
14 & & & & z_{12}x_{41}^{-1} & & & & & & & & & & & & \\
\hline
21 & & & & & x_{21} & & & & & & & & & & & \\
22 & & & & & & \lambda_1 & & & & & & & & & & \\
23 & & & & & & & z_{12}x_{32}^{-1} & & & & & & & & & \\
24 & & & & & & & & z_{12}x_{42}^{-1} & & & & & & & & \\
\hline
31 & & & & & & & & & x_{31} & & & & & & & \\
32 & & & & & & & & & & x_{32} & & & & & & \\
33 & & & & & & & & & & & \lambda_2 & & & & & \\
34 & & & & & & & & & & & & & & & \lambda_2 & \\
\hline
41 & & & & & & & & & & & & & x_{41} & & & \\
42 & & & & & & & & & & & & & & x_{42} & & \\
43 & & & & & & & & & & & & \lambda_2 & & & & \\
44 & & & & & & & & & & & & & & & & \lambda_2
\end{array}
$$

($n = 4$, 1 block of type II of size 2, 1 block of type I of size 2; $\lambda_1^2 = \lambda_1 y_1 + z_1$ $\lambda_i, x_{ij}$,
$z_1$, $z_{12}$ all $\neq 0$; $p = 8$)

In the examples above $p$ is the number of parameters. In the "irreducible" case of one block of type II of size $n$, $p = \binom{n}{2} + 2$; in the "irreducible" case of one block of type I of size $n$, $p = 1$. In the "reducible" cases the number of parameters can increase drastically to a maximum of $n^2$; in that case there are $n$ blocks of size 1 and $R$ is simply any invertible diagonal matrix; this is, in a way, the most degenerate case.

3.46. Concluding comments for section 3

Any solution of the YBE, in fact any $n^2 \times n^2$ matrix $R$, can be used to define a bialgebra by commutation relations $RT_2T_2 = T_2T_1R$, cf below. The "standard" quantum group of type $A_{n-1}$ corresponds to the case of one block of type II of size $n$ with $y = q - q^{-1}$, $r^{aa}_{aa} = \lambda = q$ for all $a$, $z = 1$, $x_{ab} = 1$ for all $a > b$.

As we shall see, the irreducible case of type II, with $r^{aa}_{aa}$ for all $a$ equal to the same solution $\lambda$ of $X^2 = yX + z$ corresponds to the $\binom{n}{2} + 1$ multiparameter quantum group of section 2. In this case there are $p = \binom{n}{2} + 2$ parameters, but one is superfluous because multiplication by a scalar is irrelevant both for the YBE and for the commutation relations defined by an $R$.

The structure of the $R$-matrix for the $\binom{n}{2} + 1$ parameter quantum group is illuminating. There are $\binom{n}{2}$ "diagonal parameters" and these define what in several ways seems to be a rather nonessential though definitely not trivial in the technical sense) deformation of the matrix algebra. The phrase "rather nonessential" here is intuitive and should be given precise meaning. One fact in this direction is that the extra $\binom{n}{2}$ parameters (the $x_{ij}$) do not appear to give any more sensitive Turaev-type knot invariants; they simple drop out of the defining trace formula even though the relevant braid group representations are different.

The irreducible type II $R$-matrix with mixed $r^{aa}_{aa}$, meaning that some of the $r^{aa}_{aa}$ are equal to one solution of $X^2 = (q - q^{-1})X + 1$ and some to the other one, give rise to bialgebras with nilpotents (so not quantum groups in the accepted sense of the word); they also give the same polynomial Turaev-type knot invariants (for a lower size $R$-matrix).

The known classical $R$-matrices of type $B^1, C^1, D^1, A^2$ do not arise as special cases of those of theorem 3.44. These classical $R$-matrices do, however, satisfy a very similar condition to the one considered here. Let $\sigma$ be the involution on $\{1, \ldots, n\}$ given by $\sigma(i) = n + 1 - i$. Then these $R$ matrices of type $B^1, C^1, D^1, A^2$ satisfy

$$r^{ab}_{cd} = 0 \quad \text{unless} \quad \{a, b\} = \{c, d\} \quad \text{or} \quad b = \sigma(a),\ d = \sigma(c) \tag{3.47}$$

It looks possible to extend the analysis of this section to the case of all solutions of the YBE satisfying (3.47).

It seems likely that the $\binom{n}{2} + 1$ parameter $R$-matrix quantum is maximal though this remains to be proved. Possibly it will thus be possible to find the maximal families for type $B^1, C^1, C^1, A^2$ as well.

Work on all these matters is in progress.

4. The $R$-matrix bialgebras defined by the fairly general $R$-matrix of section 3

Let $R$ be again be any matrix satisfying

$$R^{ab}_{cd} = 0 \quad \text{unless} \quad \{a, b\} = \{c, d\} \tag{4.1}$$

We investigate the commutation relations defined by

$$RT_1T_2 = T_2T_1R \tag{4.2}$$

where

$$T = \begin{pmatrix} t^1_1 & \cdots & t^1_n \\ \vdots & & \vdots \\ t^n_1 & \cdots & t^n_n \end{pmatrix}, \quad T_1 = T \otimes I_n, \quad T_2 = I_n \otimes T.$$

Then the relations (4.2) written out become

$$r^{ab}_{i_1 i_2} t^{i_1}_c t^{i_2}_d = r^{j_1 j_2}_{cd} t^b_{j_2} t^a_{j_1} \tag{4.3}$$

Let $I(R)$ be the ideal in $k\langle t\rangle$ generated by the relations (4.3). Then $I(R)$ is a bialgebra ideal, cf e.g. [10].

4.4. THEOREM *Let $R$ be a solution of the YBE consisting of one type II block of size $n$ such that moreover $r_{aa}^{aa}$ = constant for all $a \in \{1,\dots,n\}$, Then $R$ defines a multiparameter quantum matrix algebra as described in section 2 above.*

PROOF. Recall that the quantum matrix algebra in question arises by taking the maximal quotient of $K\langle t_1^1,\dots,t_n^n\rangle$ that acts from the left on a quantum space $K\langle X^1,\dots,X^n\rangle$, $X^iX^j = q^{ij}X^jX^i$ by the usual matrix action and from the right on a quantum space $K\langle Y_1,\dots,Y_n\rangle$, $Y_kY_l = q_{kl}Y_lY_k$, where $q^{ii}=1$, $q^{ij}=(q^{ij})^{-1}$, $q_{kk}=1$, $q_{kl}=(q_{lk})^{-1}$ and the $q^{ij}$ and $q_{kl}$ are related by

$$q^{ij}q_{ij} = \rho \neq -1 \tag{4.5}$$

and the relations defining the quantum matrix algebra are

$$t_a^r t_b^r = q_{ab} t_b^r t_a^r \tag{4.6}$$

$$t_a^r t_b^s - q_{ab} t_b^r t_a^s + (q_{rs})^{-1} t_a^s t_b^r - q_{ab}(q_{rs})^{-1} t_b^s t_a^r = 0 \tag{4.7}$$

$$t_a^r t_a^s = q^{rs} t_a^s t_b^r \tag{4.8}$$

$$t_a^r t_b^s - q^{rs} t_a^s t_b^r + (q^{ab})^{-1} t_b^r t_a^s - (q^{rs})(q^{ab})^{-1} t_b^s t_a^r = 0 \tag{4.9}$$

Choose $y, z, x_{ij}, i<j$, as in theorem 3.44. Let $\lambda^u, -\lambda_d$ be the two solution of $X^2 = Xy + z$ and take

$$\begin{aligned} &r_{aa}^{aa} = \lambda^u,\ r_{ab}^{ab} = x_{ab} \text{ for } a>b,\ r_{ba}^{ba} = \lambda^u\lambda_d x_{ab}^{-1} \text{ for } a>b \\ &r_{ba}^{ab} = \lambda^u - \lambda_d \text{ for } a<b,\ r_{ba}^{ab} = 0 \text{ for } a<b \end{aligned} \tag{4.10}$$

as described by theorem 3.44. (One can also take $r_{aa}^{aa} = -\lambda_d$ for all $a$; that gives an isomorphic matrix algebra).

The nontrivial relations resulting from 4.3 are

$$a=b,\ c=d,\quad r_{aa}^{aa} t_c^a t_c^a = r_{cc}^{cc} t_c^a t_c^a \tag{4.11}$$

$$a=b,\ c\neq d,\quad r_{aa}^{aa} t_c^a t_d^a = r_{cd}^{cd} t_d^a t_c^a + r_{cd}^{dc} t_c^a t_d^a \tag{4.12}$$

$$a\neq b,\ c=d,\quad r_{ab}^{ab} t_c^a t_c^b + r_{ba}^{ab} t_c^b t_c^a = r_{cc}^{cc} t_c^b t_c^a \tag{4.13}$$

$$a\neq b,\ c\neq d,\quad r_{ab}^{ab} t_c^a t_d^b + r_{ba}^{ab} t_c^b t_d^a = r_{cd}^{cd} t_d^b t_c^a + r_{cd}^{dc} t_c^b t_d^a \tag{4.14}$$

Because $r_{aa}^{aa} = r_{cc}^{cc} = \lambda^u$, (4.11) holds. Now take

$$q^{ab} = x_{ab}(\lambda^u)^{-1}, \quad q_{ba} = x_{ab}\lambda_d^{-1} \quad \text{for} \quad a<b \tag{4.15}$$

Notice that indeed $q^{ab}q_{ab} = x_{ab}(\lambda^u)^{-1}(x_{ab}^{-1}\lambda_d) = \lambda_d(\lambda^u)^{-1} = \rho =$ constant. Substituting the values of (4.10) in (4.12) we obtain for $d<c$

$$\lambda^u t_c^a t_d^a = x_{cd} t_d^a t_c^a + (\lambda^u - \lambda_d) t_c^a t_d^a$$

so that indeed

$$t_c^a t_d^a = \lambda_d^{-1} x_{cd} t_d^a t_c^a = \lambda_d^{-1} x_{dc} t_d^a t_c^a = q_{cd} t_d^a t_c^a \tag{4.16}$$

which is (4.6). And for $c < d$ we get

$$\lambda^u t^a_c t^a_d = \lambda^u \lambda_d x^{-1}_{cd} t^a_d t^a_c$$

which gives

$$t^a_c t^a_d = \lambda_d x^{-1}_{cd} t^a_d t^a_c = q^{-1}_{dc} t^a_d t^a_c = q_{cd} t^a_c t^a_d$$

which is the same as (4.16).

Now substitute the values of (4.10) in (4.13). There are again two cases to consider.

If $a < b$ we find

$$\lambda_d \lambda^u x^{-1}_{ab} t^a_c t^b_c + (\lambda^u - \lambda_d) t^b_c t^a_c = \lambda^u t^b_c t^a_c$$

which gives (using 4.15)

$$t^a_c t^b_c = (\lambda^u)^{-1} x_{ab} t^b_c t^a_c = q^{ab} t^b_c t^a_c$$

which is (4.8).

If $a > b$ we find

$$x_{ab} t^a_c t^b_c = \lambda^u t^b_c t^a_c$$

which gives

$$t^b_c t^a_c = (\lambda^u)^{-1} x_{ab} t^a_c t^b_c = q^{ba} t^a_c t^b_c$$

Finally substitute the values of (4.10) in (4.14). Note that (4.14) really embodies four equations between the $t^a_c t^b_d$, $t^a_d t^b_c$, $t^b_c t^a_d$, $t^b_d t^a_c$; namely, the one written down and the three obtained by switching $a$ and $b$, switching $c$ and $d$, and switching both.

Taking $a < b$, $c < d$ we find

$$\lambda^u \lambda_d x^{-1}_{ab} t^a_c t^b_d + (\lambda^u - \lambda_d) t^b_c t^a_d = \lambda^u \lambda_d x^{-1}_{cd} t^b_d t^a_c \tag{4.17}$$

Switching $a$ and $b$ in (4.14) and then substituting gives

$$x_{ab} t^b_c t^a_d = \lambda^u \lambda_d x^{-1}_{cd} t^a_d t^b_c \tag{4.18}$$

$$\lambda^u \lambda_d x^{-1}_{ab} t^a_d t^b_c + (\lambda^u - \lambda_d) t^b_d t^a_c = x_{cd} t^b_c t^a_d + (\lambda^u - \lambda_d) t^b_d t^a_c \tag{4.19}$$

Finally, switching both $a, b$ and $c, d$ and then substituting gives

$$x_{ab} t^b_d t^a_c = x_{cd} t^a_c t^b_d + (\lambda^u - \lambda_d) t^a_d t^b_c \tag{4.20}$$

Observe that (4.18) and (4.19) are identical. It is easily checked that

$$x_{ab}(\lambda^u \lambda_d)^{-1}\ (4.17)\ + (x^{-1}_{cd})\ (4.20) - (\lambda^{-1}_d - (\lambda^u)^{-1})\ (4.18)$$

has equal left and right hand sides. Thus (4.17)-(4.20) are equivalent to (4.17)-(4.18).

Multiply (4.17) by $x_{ab}(\lambda^u \lambda_d)^{-1}$ to find

$$t^a_c t^b_d + x_{ab} \lambda^{-1}_d t^b_c t^a_d - x_{ab} \lambda^{-1}_u t^b_c t^a_d - x_{ab} x^{-1}_{cd} t^b_d t^a_c = 0 \tag{4.21}$$

and now use (4.18) to rewrite the third term to find

$$t_c^a t_d^b + x_{ab}\lambda_d^{-1} t_c^b t_d^a - \lambda_d x_{cd}^{-1} t_d^a t_c^b - x_{ab} x_{cd}^{-1} t_d^b t_c^a = 0 \tag{4.22}$$

Because $a < b$, $c < d$ we have by (4.15) that

$$q_{ab}^{-1} = q_{ba} = x_{ab}\lambda_d^{-1}, \quad q_{cd} = (q_{dc})^{-1} = (x_{cd}\lambda_d^{-1})^{-1} = \lambda_d x_{cd}^{-1},$$

$$q_{ab}^{-1} q_{cd} = x_{ab}\lambda_d^{-1}\lambda_d x_{cd}^{-1} = x_{ab} x_{cd}^{-1}$$

so that (4.22) is identical with (4.9).

Now use (4.18) to rewrite the second term in (4.21). This gives

$$t_c^a t_d^b + \lambda^u x_{cd}^{-1} t_d^a t_c^b - x_{ab}\lambda_u^{-1} t_c^b t_d^a - x_{ab} x_{cd}^{-1} t_d^b t_c^a = 0 \tag{4.23}$$

Again, as $a < b$, $c < d$, we have by (4.15) that

$$\begin{aligned} &q^{ab} = (\lambda^u)^{-1} x_{ab}, \quad (q^{cd})^{-1} = ((\lambda^u)^{-1} x_{cd})^{-1} = \lambda^u x_{cd}^{-1} \\ &q^{ab}(q^{cd})^{-1} = (\lambda^u)^{-1} x_{ab}\lambda^u x_{cd}^{-1} \end{aligned} \tag{4.24}$$

so that (4.23) is identical with (4.7).

This finishes the proof of the theorem. (Though not necessary, given what has been shown about the rank of the various groups of relations involved, it is in fact now not difficult to show that inversely the groups of relations (4.7)-(4.9) imply the group (4.14), i.e. (4.17)-(4.20)).

4.25. COROLLARY. *Let $M_q^{n\times n}$ be the multiparameter quantum matrix algebra of section 3, i.e. $M_q^{n\times n} = d\langle t\rangle/I$ when $I$ is the ideal of the relations (4.6)-(4.9). Then $M_q^{n\times n}$ is a PBW algebra with the same Hilbert-Poincaré series as $k[t_1^1, \dots, t_n^n]$.*

PROOF. We already know that the dimension of the degree 2 part is exactly right viz. $n^2 + \binom{n^2}{2}$. The commutation relations are of the form

$$T_1 T_2 = R^{-1} T_2 T_1 R$$

Now $R$ satisfies the YBE, i.e.

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{4.26}$$

Now for the triple product $T_1T_2T_3$, $T_1 = T\otimes I\otimes I$, $T_2 = I\otimes T\otimes I$, $T_3 = I\otimes I\otimes T$, we have that

$$\begin{aligned} T_1(T_2T_3) &= T_1R_{23}^{-1}T_3T_2R_{23} = R_{23}^{-1}(T_1T_3)T_2R_{23} = R_{23}^{-1}R_{13}^{-1}T_3T_1R_{13}T_2R_{23} \\ &= R_{23}^{-1}R_{13}^{-1}T_3(T_1T_2)R_{13}R_{23} = R_{23}^{-1}R_{13}^{-1}T_3R_{12}^{-1}T_2T_1R_{12}R_{13}R_{23} \\ &= R_{23}^{-1}R_{13}^{-1}R_{12}^{-1}T_3T_2T_1R_{12}R_{13}R_{23} \end{aligned} \tag{4.27}$$

(Note that $R_{ij}T_k = T_kR_{ij}$ if $i \neq j \neq k \neq i$ because $R_{ij}$ only affects factors $i$ and $j$ where $T_k$ is the identity). We also have

$$\begin{aligned} (T_1T_2)T_3 &= R_{12}^{-1}T_2T_1R_{12}T_3 = R_{12}^{-1}T_2(T_1T_2)R_{12} = R_{12}^{-1}T_2R_{13}^{-1}T_3T_1R_{13}R_{12} \\ &= R_{12}^{-1}R_{13}^{-1}(T_2T_3)T_1R_{13}R_{12} = R_{12}^{-1}R_{13}^{-1}R_{23}^{-1}T_3T_2R_{23}T_1R_{13}R_{12} \\ &= R_{12}^{-1}R_{13}^{-1}R_{23}^{-1}T_3T_2T_1R_{23}R_{13}R_{12} \end{aligned} \tag{4.28}$$

The end products of (4.27) and (4.28) are the same proving the confluence conditions of the diamond lemma, [2], and the result follows. This argument: YBE $\Rightarrow$ confluence condition of diamond lemma has been observed before, [6].

4.29. COMMENTS ON THE OTHER SOLUTIONS OF THE YBE

The solution consisting of one block of type I gives, as is easily checked, no relations at all amoung the $t^i_j$.

The solutions consisting of one block of type II but with mixed $r^{aa}_{aa}$ give rise to a bialgebra $k\langle t\rangle/I(R)$ with nilpotent elements. Indeed if, say, $r^{aa}_{aa} = \lambda$, $r^{bb}_{bb} = \mu \neq \lambda$, then by (4.11)

$$\lambda t^a_b t^a_b = \mu t^a_b t^a_b \tag{4.30}$$

so that $(t^a_b)^2 = 0$. These are of course perfectly good solutions of the YBE and as such are of potential use in for example the business of constructing link invariants (cf section 5 below) but the bialgebras they define are not quantum groups in the (more or less) accepted sense of the word. (There is no consensus and some authors equate the concepts Hopf algebra and quantum group; I would be inclined to reserve the phrase quantum group for a Hopf algebra that is a PBW algebra and is a deformation of the function algebra of a linear algebraic group).

Let me also remark that in spite of nilpotents the bialgebras defined by a single type II block with mixed $r^{aa}_{aa}$ solution of the YBE are still pretty nice in the sense that its defining rewriting rules (commutation relations) are confluent (so that it is easy to write down a basis and a version of Gröbner basis theory probably applies).

4.30. QUANTUM GROUPS

Let again $R$ be a single type II block solution of the YBE with constant $r^{aa}_{aa}$ defining a multiparameter quantum matrix algebra $M_q = K\langle t\rangle/I(R)$ as described in section 3. As is shown in e.g. [1] there is a central element $d$ in $M_q$ (a quantum determinant) Such that the localization $M_q[d^{-1}]$ admits an antipode and thus becomes a Hopf algebra.

By the work of [6,12,13], cf also [4], the fact that $M_q$ comes from a solution of the YBE is also useful in establishing such facts.

5. YANG-BAXTER OPERATORS AND LINK INVARIANTS

For this section the Yang-Baxter equation takes the form

$$S_{12}S_{23}S_{12} = S_{23}S_{12}S_{23} \tag{5.1}$$

If $S = (s^{ab}_{cd})$, then in terms of the entries of $S$ this works out as

$$s^{ab}_{kl}s^{lc}_{mw}s^{km}_{uv} = s^{ai}_{uk}s^{bd}_{ij}s^{kj}_{uv} \tag{5.2}$$

There is a simple relation between (5.1) and the YBE (3.1): if $R = (r^{ab}_{cd})$ solves (3.1) then both

$$S = (s^{ab}_{cd}),\ s^{ab}_{cd} = r^{ab}_{dc}, \quad S' = (s'^{ab}_{cd}),\ s'^{ab}_{cd} = r^{ba}_{cd} \tag{5.3}$$

solve (5.1) (and vice versa). Let's check that for $S$. Putting (5.3) in the LHS of (5.2) gives

$$r^{ab}_{lk}r^{lc}_{wm}r^{km}_{vu} \tag{5.4}$$

which is the LHS of (3.2) with $uvw$ replaced by $wvu$; now put (5.3) in the RHS of (5.2) to find

$$r^{ai}_{ku}r^{bc}_{ji}r^{kj}_{vw} = r^{bc}_{ji}r^{ai}_{ku}r^{kj}_{wv} \tag{5.5}$$

which is the RHS of (3.2) also with $uvw$ replaced by $wvu$. The proof for $S'$ is as easy (except that now RHS and LHS switch).

5.6. DEFINITION ([22]). A Yang-Baxter operator consists of a quadruple $(S,\nu,\alpha,\beta)$ where $S$ is an $n^2\times n^2$ matrix satisfying the YBE in the form (5.1), $\mu$ is an $n\times n$ matrix, and $\alpha,\beta$ are invertible scalars which are related to $S$ by the conditions (5.7)-(5.9)

$$\nu\otimes\nu \text{ commutes with } S \tag{5.7}$$

$$Tr_2(S\circ(\nu\otimes\nu))=\alpha\beta\nu \tag{5.8}$$

$$Tr_2(S^{-1}\circ(\nu\otimes\nu))=\alpha^{-1}\beta\nu \tag{5.9}$$

Here if $M=(m^{ij}_{kl})$ is an $n^2\times n^2$ matrix (with the usual ordering $11,\ldots,1n;21,\ldots,2n;\ldots;n1,\ldots,nn$ of rows and columns), then $Tr_2(M)=N$ is the $n\times n$ matrix with entries

$$n^i_j=m^{i1}_{j1}+\ldots+m^{in}_{jn}, \tag{5.10}$$

i.e. if $M$ is written as an $n\times n$ matrix of $n\times n$ blocks then replace each block by its trace. If $\nu$ is invertible then (5.8) and (5.9) are equivalent to

$$Tr_2(S^{\pm1}\circ(I_n\otimes\nu))=\alpha^{\pm1}\beta I_n \tag{5.11}$$

(where $I_n$ is the $n\times n$ identity matrix).

Given a YB operator $(S,\nu,\alpha,\beta)$, Turaev's formula

$$T_S(\xi)=\alpha^{-w(\xi)}\beta^{-m}Tr(\rho_S(\xi)\circ\nu^{\otimes m}) \tag{5.12}$$

defines a link invariant. Here $\xi\in B_m$, the braid group a $m$ letters, $w(\xi)=\Sigma\epsilon_i$ if $\xi=\sigma^{\epsilon_1}_{i_1}\ldots\sigma^{\epsilon_r}_{i}$ where the $\sigma_i$ are the standard generators of $B_m$, and $\rho_S$ is the representation of the braid group (in $(K^n)^{\otimes m}$) defined by $S$, $\sigma_i\mapsto S_{ii+1}$; $T_S(\xi)$ is then independent of the particular braid that gives rise to a link $\xi$ by closure of the braid.

Now, given the solutions of the YBE described in section 3 it is natural to investigate whether these extend to Yang Baxter operators in the sense of Turaev (definition 5.6), and, if so, what the resulting link and knot invariants bring. Here I report some preliminary results only. Further work is in progress.

5.13. REMARKS. Both the constants $\alpha$ and $\beta$ can be normalized to 1. Indeed if $(S,\nu,\alpha,\beta)$ is a Yang-Baxter operator then $(\alpha^{-1}S,\beta^{-1}\nu,1,1)$ is another one. However, for the formulas below it is convenient to keep $\alpha$ (but $\beta$ will always be 1). As Turaev observes, if $\nu$ is diagonal, then (5.8) implies that $\bar S\bar\nu=\bar\alpha$ where $\bar S$ is the $n\times n$ matrix $\bar s^i_j=s^{ij}_{ij}$, $\bar\nu$ is the column vecxtor $(\mu_1,\ldots,\nu_n)^T$ and $\bar\alpha$ is the column vector $\alpha(1,1,\ldots,1)^T$. Thus, assuming $\nu$ is diagonal, it is unique if $\bar S$ is invertible, as it will be in the results reported below.

5.14. THEOREM. *Let $R$ be a solution of the YBE consisting of a single type II block, let $S=\tau R$ be its associated solution of (5.1). Then $S$ extends to a Yang-Baxter operator with the scalar $\alpha$ such that*

$$\alpha^2=(-1)^n\lambda^{k_\lambda-k_\mu+1}\mu^{k_\mu-k_\lambda+1} \tag{5.15}$$

*where $k_\lambda$ (resp. $k_\mu$) is the number of times that the solution $\lambda$ (resp. $\mu$) of $X^2=yX+z$ occurs as an $r^{ii}_{ii}$.*

PROOF. For the moment regard $R,R^{-1}$ and $S,S^{-1}$ as $n\times n$ matrices made up of blocks that are also $n\times n$ matrices. Observe that the diagonals of all the off-diagonal blocks are zero. Take $\nu=$

diag$(\nu_1, \ldots, \nu_n)$, the diagonal $n \times n$ matrix with diagonal entries $\nu_1, \ldots, \nu_n$. Because $\nu$ is diagonal and $s^{ab}_{cd} = 0$ unless $\{a,b\} = \{c,d\}$, (5.7) holds. It also follows (cf (5.10)) that the conditions (5.8), (5.9) only involve the diagonal blocks of $S$ and $S^{-1}$. As is easily checked the inverse $R^{-1}$ of $R$ is equal to

$$\begin{aligned} (R^{-1})^{ab}_{ba} &= \lambda^{-1} + \mu^{-1} && \text{if } \ a < b \\ (R^{-1})^{ab}_{ab} &= z^{-1}x_{ba} && \text{if } \ a < b \\ (R^{-1})^{ab}_{ab} &= x^{-1}_{ab} && \text{if } \ a > b \\ (R^{-1})^{aa}_{aa} &= (R^{aa}_{aa})^{-1} && (= \lambda^{-1}(\text{resp. } \mu^{-1})) \end{aligned} \tag{5.16}$$

Indeed if $a < b$, $\{a,b\} \neq \{c,d\}$

$$(RR^{-1})^{ab}_{cd} = R^{ab}_{ij}(R^{-1})^{ij}_{cd} = R^{ab}_{ab}(R^{-1})^{ab}_{cd} + R^{ab}_{ba}(R^{-1})^{ba}_{cd} = 0$$

Further, if $a < b$, $a = c$, $b = d$

$$(RR^{-1})^{ab}_{ab} = R^{ab}_{ab}(R^{-1})^{ab}_{ab} + R^{ab}_{ba}(R^{-1})^{ba}_{ab} = zx^{-1}_{ba}z^{-1}x_{ba} + 0 = 1$$

and if $a < b$, a=d, $b = c$

$$(RR^{-1})^{ab}_{ba} = R^{ab}_{ab}(R^{-1})^{ab}_{ba} + R^{ab}_{ba}(R^{-1})^{ba}_{ba} = zx^{-1}_{ba}(\lambda^{-1} + \mu^{-1}) + (\lambda + \mu)x^{-1}_{ba} = 0$$

because $z = -\lambda\mu$.

The other cases $a = b$, $a > b$ are even easier to check.

Switching $\lambda$ and $\mu$ if necessary we can assume that $r^{11}_{11} = \lambda$. Let the pattern of $\lambda$'s and $\mu$'s be the following (writing $\rho_i = r^{ii}_{ii}$ for convenience)

$$\rho_1 = \ldots = \rho_{d_1} = \lambda;\ \rho_{d_1+1} = \ldots = \rho_{d_1+d_2} = \mu;\ \rho_{d_1+d_2+1} = \ldots = \rho_{d_1+d_2+d_3} = \lambda;\ \ldots$$

Let $r$ be the number of switches $(d_1, d_1+1), \ldots, (d_r, d_r+1)$, so that $\rho_n = \lambda$ if $r$ even and $\rho_n = \mu$ if $r$ is odd. It is now easy to see that the equations (5.8), (5.9) (with $\beta = 1$) amound to the following, (where the equations resulting from (5.8) are written on the left and those from (5.9) on the right. (Here, as in the above, to follow the calculations, it is useful to keep the first example of (3.45) in front of one).

$$
\begin{array}{ll}
\lambda\nu_1 = \alpha & \lambda^{-1}\nu_1 + (\lambda^{-1}+\mu^{-1})(\nu_2 + \ldots + \nu_n) = \alpha^{-1} \\
\lambda\nu_2 + (\mu+\lambda)\nu_1 = \alpha & \lambda^{-1}\nu_2 + (\lambda^{-1}+\mu^{-1})(\nu_3 + \ldots + \nu_n) = \alpha^{-1} \\
\vdots & \vdots \\
\lambda\nu_{d_1-1} + (\mu+\lambda)(\nu_1 + \ldots + \nu_{d_1-2}) = \alpha & \lambda^{-1}\nu_{d_1-1} + (\lambda^{-1}+\mu^{-1})(\nu_{d_1} + \ldots + \nu_n) = \alpha^{-1} \\
\lambda\nu_{d_1} + (\mu+\lambda)(\mu_1 + \ldots + \mu_{d_1-1}) = \alpha & \lambda^{-1}\nu_{d_1} + (\lambda^{-1}+\mu^{-1})(\nu_{d_1+1} + \ldots + \nu_n) = \alpha^{-1} \\
\mu\nu_{d_1+1} + (\mu+\lambda)(\nu_1 + \ldots + \nu_{d_1}) = \alpha & \mu^{-1}\nu_{d_1+1} + (\lambda^{-1}+\mu^{-1})(\nu_{d_1+2} + \ldots + \nu_n) = \alpha^{-1} \\
\mu\nu_{d_1+2} + (\mu+\lambda)(\nu_1 + \ldots + \nu_{d_1+1}) = \alpha & \mu^{-1}\nu_{d_1+2} + (\lambda^{-1}+\mu^{-1})(\nu_{d_1+3} + \ldots + \nu_n) = \alpha^{-1} \\
\vdots & \vdots \\
\mu\nu_{d_1+d_2-1} + (\mu+\lambda)(\nu_1 + \ldots + \nu_{d_1+d_2-2}) = \alpha & \mu^{-1}\nu_{d_1+d_2-1} + (\lambda^{-1}+\mu^{-1})(\nu_{d_1+d_2} + \ldots + \nu_n) = \alpha^{-1} \\
\mu\nu_{d_1+d_2} + (\mu+\lambda)(\nu_1 + \ldots + \nu_{d_1+d_2-1}) = \alpha & \mu^{-1}\nu_{d_1+d_2} + (\lambda^{-1}+\mu^{-1})(\nu_{d_1+d_2+1} + \ldots + \nu_n) = \alpha^{-1} \\
\mu\nu_{d_1+d_2+1} + (\mu+\lambda)(\nu_1 + \ldots + \nu_{d_1+d_2}) = \alpha & \mu^{-1}\nu_{d_1+d_2+1} + (\lambda^{-1}+\mu^{-1})(\nu_{d_1+d_2+2} + \ldots + \nu_n) = \alpha^{-1} \\
\mu\nu_{d_1+d_2+2} + (\mu+\lambda)(\nu_1 + \ldots + \nu_{d_1+d_2+1}) = \alpha & \mu^{-1}\nu_{d_1+d_2+2} + (\lambda^{-1}+\mu^{-1})(\nu_{d_1+d_2+3} + \ldots + \nu_n) = \alpha^{-1} \\
\vdots & \vdots \\
\kappa\nu_n + (\mu+\lambda)(\nu_1 + \ldots + \nu_{n-1} = \alpha & \kappa^{-1}\nu_n = \alpha^{-1}
\end{array}
$$

where $\kappa = \lambda$ (resp. $\mu$) depending on whether $r$ is even (resp. odd). Now observe that substituting the $(i+1)$-th from the $i$-th equation both left and right results in the same relation between $\nu_{i+1}$ and $\nu_i$ viz. either $\nu_{i+1} = -\lambda^{-1}\mu\nu_i$, or $\nu_{i+1} = -\mu^{-1}\lambda\nu_i$, or $\nu_{i+1} = -\nu_i$. This results in the following recipe for the $\nu$'s

$$
\nu_1 = \lambda^{-1}\alpha
$$

$$
\nu_i = \begin{cases} (-\lambda^{-1}\mu)\nu_{i-1} & \text{if} \quad \rho_i = \lambda = \rho_{i-1} \\ -\nu_{i-1} & \text{if} \quad \rho_i = \lambda,\ \rho_{i-1} = \mu \\ (-\mu^{-1}\lambda)\nu_{i-1} & \text{if} \quad \rho_i = \mu = \rho_{i-1} \\ -\nu_{i-1} & \text{if} \quad \rho_i = \mu,\ \rho_{i-1} = \lambda \end{cases} \tag{5.17}
$$

$$
\nu_n = \begin{cases} \lambda\alpha^{-1} & \text{if} \quad r \text{ is even} \\ \mu\alpha^{-1} & \text{if} \quad r \text{ is odd} \end{cases}
$$

It follows that, depending on the number, $r$, of switches from $\lambda$ to $\mu$ or vice versa

$$
\begin{array}{lll}
\text{if } r \text{ is even} & \nu_n = (-1)^{n-1}\lambda^{k_\mu - k_\lambda + 1}\nu_1, & \nu_1\lambda^{-1}\alpha,\ \nu_n = \lambda\alpha^{-1} \\
\text{if } r \text{ is odd} & \nu_n = (-1)^{n-1}\lambda^{k_\mu - k_\lambda}\mu^{k_\lambda - k_\mu}\nu_1, & \nu_1 = \lambda^{-1}\alpha,\ \nu_n = \mu\alpha^{-1}
\end{array} \tag{5.18}
$$

where $k_\lambda$ is the number of $i$'s for which $\rho_i = \lambda$ and $k_\mu$ the number of $i$'s for which $\rho_i = \mu$, $k_\lambda + k_\mu = n$. In both cases it follows that

$$
\alpha^2 = (-1)^{n-1}\lambda^{k_\lambda - k_\mu + 1}\mu^{k_\mu - k_\lambda + 1} \tag{5.19}
$$

and for both $\alpha$'s solving (5.19) (taking if necessary a quadratic extension of $K$) (5.17) then specifies $\nu_1, \ldots, \nu_n$ such that (5.8), (5.9) are satisfied (with $\beta = 1$). This concludes the proof of the theorem

5.20. REMARK. Both choices for $\alpha$ in (5.19) give up to sign the same link invariant, cf [22], 3.3.

5.21. COROLLARY. *Let $R$ be any solution of the YBE as described by theorem 3.44 and $S = \tau R$ the corresponding solution of (5.1). Then $S$ extends to a Yang Baxter operator $(S, \nu, \alpha, 1)$ if any only if for all blocks*

$$\alpha^2 = (-1)^{n_i-1}\lambda_i^{k_{\lambda_i}-k_{\mu_i}+1}\mu_i^{k_{\mu_i}-k_{\lambda_i}+1} \quad \text{for a block of type II of size } n_i \tag{5.22}$$

$$\alpha^2 = \lambda_i^2 \quad \text{for a block of type I} \tag{5.23}$$

$$\alpha^2 = \lambda_i^2 \quad \text{for a block of size 1} \tag{5.24}$$

PROOF. Take $\nu$ diagonal. From the form of $S$ (and $S^{-1}$ which has the same form) one easily sees that (5.8) and (5.9) only involve the separate blocks and the $\nu$'s with corresponding indices. It is trivial to check (5.23), (5.24) for blocks of type I and size 1. Finally (5.7) holds because $s_{cd}^{ab} = 0$ unless $\{a,b\} = \{c,d\}$ and $\nu$ is diagonal.

The next result is perhaps a disappointment. With $\binom{n}{2}$ extra variables in an $n^2 \times n^2$ single type II block solution of (5.1) it might be hoped (even expected) that these will give some extra information when employed to define link invariants via Turaev's formula (5.12). This is not the case, and using both solutions $\lambda$ and $\mu$ of $X^2 = yX + z$ (instead of just 1) for the $\rho_a = s_{aa}^{aa}$ also gives nothing new.

5.25. PROPOSITION. Let $S$ be a single type II block solution of (5.1). Let $\mu$ occur $m$ times as a $\rho_a$, $m \leq \frac{1}{2}n$. Then the link invariant $T_S$ defined by $S$ by formula (5.12) using the extended YB operator $(S,\nu,\alpha,1)$ defined by theorem (5.14) is the same as the one defined by the single type II block solution $S_1$ of size $(n-2m)^2 \times (n-2m)^2$, $x_{ij} = 1 = z$ for all $i,j$, same $y$ as $S$ (i.e. it is one of the "classical" $A_s$ invariants of Turaev).

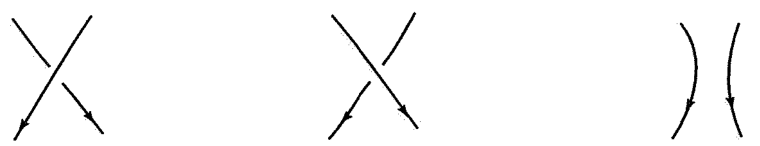
PROOF. It follows immediately from (5.12) that $(S,\nu,\alpha,\beta)$ and $(\rho S,\nu,\rho\alpha,\beta)$ define the same link invariant. We can therefore assume $z = 1$, i.e. $\lambda = q$, $\mu = -q^{-1}$. Then, by (5.15), $\alpha = \pm q^{n-2m}$. A simple check now shows that $S$ satisfies the relation

$$S - S^{-1} = (q - q^{-1})I_{n^2} \tag{5.26}$$

and this also satisfied by $S_1$. It follows that the link invariants $T$ and $T_1$ defined by $S$ and $S_1$ (or $-S_1$ which does not matter by 5.20) both satisfy, [22], the same skein relation.

$$q^{n-2m}T_S(L_+) - q^{2m-n}T_S(L_-) = (q - q^{-1})T_S(L_0) \tag{5.27}$$

where $L_+$, $L_-$, and $L_0$ are three oriented links which are identical except for one crossing where they look respectively like



+ − 0

By repeated changing of + crossings to − crossings any link can be turned into an unlink. Thus the value of $T_S$ is uniquely determined by the skein relation (5.27) and its values on $k$-component unlinks. The latter are equal to $(\nu_1 + \ldots + \nu_n)^k$. Finally one checks that $(\nu_1 + \ldots + \nu_n) = (\bar{\nu}_1 + \ldots + \bar{\nu}_{n-2m})$ where $(S_1,\bar{\nu},\alpha,1)$ is the YB operator belonging to $S_1$. This is (with induction) seen as follows. If $d_i$ is the shortest run of $\lambda$'s or $\mu$'s, then if $i = 1$, the pattern $d_2 - d_1, d_3, \ldots, d_{r+1}$ gives the same trace value of $\nu$ as the original (because $\nu_{d_1+1} = -\nu_{d_1}$, $\nu_{d_1+i} = -\nu_{d_1-i+1}$, $i = 1,\ldots,d_1$) and similarly if $i > 1$, the pattern $d_1,\ldots,d_{i-1}$, $d_{i+1},\ldots,d_{r+1}$ gives the same trace value of $\nu$ as the original. This proves the proposition.

5.28. REMARK. This result (proposition 5.25), illustrates the previous remark (cf (3.46)) that the $\binom{n}{2}$ extra diagonal parameters in the general on type II block solution of the YBE, i.e. the $x_{ij}$ and $z$, play in some sense a trivial role, while there is but one essential parameter, viz. $y$ (or $q$). On the other hand the corresponding quantum groups, the general $\binom{n}{2}+1$ parameter one, and the classical 1 parameter one are not isomorphic.

5.29. INVARIANTS FROM DIAGONAL SOLUTIONS
On the other hand, perhaps surprisingly, the diagonal solutions of the YBE can give rise to nontrivial knot invariants. Take for example the $n=2$, 2 blocks of size 1 solution:

$$R=\begin{pmatrix} x_{11} & & & \\ & zx_{21}^{-1} & & \\ & & x_{21} & \\ & & & x_{22} \end{pmatrix}, \quad R^{-1}=\begin{pmatrix} x_{11}^{-1} & & & \\ & z^{-1}x_{21}^{-1} & & \\ & & x_{21}^{-1} & \\ & & & x_{22}^{-1} \end{pmatrix} \tag{5.30}$$

with corresponding solutions of (5.1)

$$S=\begin{pmatrix} x_{11} & & & \\ & 0 & x_{21} & \\ & zx_{21}^{-1} & 0 & \\ & & & x_{22} \end{pmatrix}, \quad S^{-1}=\begin{pmatrix} x_{11}^{-1} & & & \\ & 0 & z^{-1}x_{21} & \\ & x_{21}^{-1} & 0 & \\ & & & x_{22}^{-1} \end{pmatrix} \tag{5.31}$$

This $S$, for $x_{11}=x_{22}$, extends to a Yang Baxter operator $(S,\nu,\alpha,\beta)$ with $\nu=I_2$, if $\alpha=x_{11}=x_{22}$, $\beta=1$ and gives rise to a link invariant that takes the following values on the following links


$L_0$

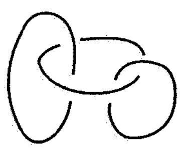

$L_1$

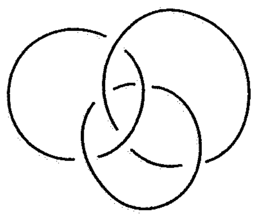

$L_2$

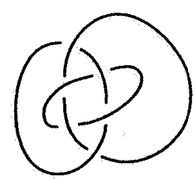

$L_3$


$L_4$ (trefoil)


$L_5$


$L_6$


$L_7$


$L_8$

$$\begin{gathered} T(L_0)=2,\ T(L_1)=4,\ T(L_2)=2+2\gamma,\ T(L_3)=2+2\gamma^2,\ T(L_4)=2 \\ T(L_5)=2+2\gamma,\ T(L_6)=2(1+\gamma)^2,\ T(L_7)=2+6\gamma^2,\ T(L_8)=6+2\gamma^2 \end{gathered} \tag{5.32}$$

Here $\gamma=r_{12}^{12}r_{21}^{21}=z$. Thus, this invariant counts components, can detects various ways in which components are linked but does not distinguish between e.g. trefoil and unknot ($L_0$, and $L_4$; cf also $L_2$ and $L_5$). The two size 1 blocks themselves give only the trivial invariant, thus this example shows conclusively that putting two blocks nontrivially together can definitely give nontrivial extra information.

5.33. REMARK. The representations of the raid group on $k$ strings $B_k$ defined by $S$ and $S_1$ in proposition (5.25) are different (even if $m = 0$), but this difference does not show up in the trace formula (5.12). This can also be seen directly in cases where there is no relation like (5.26), which is important in dealing with solutions $S$ which do not consist of a single block. Indeed:

5.34. THEOREM. *$R$ be an invertible $n^2 \times n^2$ matrix with diagonal entries $x_{ij}$ and possibly nonzero diagonal entries $q_{ij} = r_{ij}^{ij}$, $i < j$, and no other nonzero entries. Let $S = \tau R$. Let $w = \sigma_{i_1}^{\epsilon_1} \ldots \sigma_{i_m}^{\epsilon_m}$, $\epsilon_i \in \{1, -1\}$ be an element of the braid group $B_k$ of braids on $k$ strings. Let $S_i = I_n^{\otimes i-1} \otimes S \otimes I_n^{\otimes k-i-1}$ and let $S_w = S_{i_1}^{\epsilon_1} \ldots S_{i_m}^{\epsilon_m}$. Then the diagonal elements of $S_w$ are Laurent polynomials in the $q_{ij}$, the $x_{ii}$, and the products $x_{ij}x_{ji} = z_{ij}$*

PROOF. The only off-diagonal elements of $S$ are of the form

$$s_{ji}^{ij} = r_{ij}^{ij} = x_{ij}, \quad s_{ij}^{ji} = r_{ji}^{ji} = x_{ji} = x_{ij}^{-1} z_{ij} \tag{5.35}$$

The off-diagonal elements of $R^{-1}$ are equal to $-q_{ij} x_{ji}^{-1} x_{ij}^{-1}$, $i < j$. It follows that the diagonal elements of $S^{-1} = R^{-1}\tau$ are of the form

$$x_{ii}^{-1}, \quad -q_{ij}(x_{ij}x_{ij})^{-1} = -q_{ij} z_{ij}^{-1} \tag{5.36}$$

and that the off-diagonal elements of $S^{-1}$ are of the form

$$(S^{-1})_{ji}^{ij} = x_{ji}^{-1} = z_{ij}^{-1} x_{ij}, \quad (S^{-1})_{ij}^{ji} = x_{ij}^{-1} \tag{5.37}$$

Now consider a diagonal element of $S_w$. Such an element is a sum of products of the form

$$t_{i_1(2)\ldots i_n(2)}^{i_1(1)\ldots i_n(1)} t_{i_1(3)\ldots i_n(3)}^{i_1(2)\ldots i_n(2)} \cdots t_{i_1(m)\ldots i_n(m)}^{i_1(m-1)\ldots i_n(m-1)} \tag{5.38}$$

with $i_l(m) = i_l(1)$, $l = 1, \ldots, n$, and $r_{i_1(l+1)\ldots\ldots i_n(l+1)}^{i_1(l)\ldots i_n(l)}$ an element of $S_{i_l}^{\epsilon_l}$. Because of (5.35)-(5.37) each product (5.38) is zero unless all the permutations

$$\begin{pmatrix} i_1(l)\ldots i_n(l) \\ i_1(l+1)\ldots i_n(l+1) \end{pmatrix}$$

are of the form identity or $\tau_k$ where $\tau_k$ is the transposition $(k\ k+1)$ that interchanges the $k$-th and $(k+1)$-th entries and leaves all others in place. The identity permutations produce diagonal entries from $S_{i_l}$ or $S_{i_l}^{-1}$ and by (5.35)-(5.37) these are of the desired form. The remaining permutations in (5.38) form a word $\omega$ in the $\tau_1, \ldots, \tau_{n-1}$ that is equal to the identity in the permutation group $\Pi_n$ on $n$-letters. The relations between the generators $\tau_1, \ldots, \tau_{n-1}$ of $\Pi_n$ are the following

$$\begin{aligned} &\tau_k^2 = 1 \\ &\tau_k \tau_{k+1} \tau_k \tau_{k+1}^{-1} \tau_k^{-1} \tau_{k+1}^{-1} = 1 \\ &\tau_k \tau_l \tau_k^{-1} \tau_l^{-1} = 1 \quad \text{if} \quad |k-l| \geq 2 \end{aligned} \tag{5.39}$$

It follows that somewhere in the word $\omega$ one of the three left hand sides of (5.39) occurs and by induction (in the length of $\omega$) it follows that if suffices to check that in all three cases the corresponding factors in (5.38) combine to give a monomial of the desired form. Observe that $S$ and $S^{-1}$ have the same off-diagonal entries except for a factor $z_{ij}$. Thus replacing each $S_l^{-1}$ with $S_l$ only changes things by monomials in the $z_{ij}$ and we may assume that all $\epsilon_l$ are 1.

In the first case we obtain a product

$$t^{\alpha_1 ab\alpha_2}_{\alpha_1 ba\alpha_2} t^{\alpha_1 ba\alpha_2}_{\alpha_1 ab\alpha_2}$$

which is equal to $x_{ab}x_{ba} = z_{ab}$. Here and below the $\alpha_i$ stand for strings of indices that remain unchanged.

In the case of the second type of relation of (5.39) we obtain a product

$$t^{\alpha_1 abc\alpha_2}_{\alpha_1 bac\alpha_2} t^{\alpha_1 bac\alpha_2}_{\alpha_1 bca\alpha_2} t^{\alpha_1 bca\alpha_2}_{\alpha_1 cba\alpha_2} t^{\alpha_1 cba\alpha_2}_{\alpha_1 cab\alpha_2} t^{\alpha_1 cab\alpha_2}_{\alpha_1 acb\alpha_2} t^{\alpha_1 acb\alpha_2}_{\alpha_1 abc\alpha_2}$$

which is equal to $x_{ab}x_{ac}x_{bc}x_{ba}x_{ca}x_{cb} = z_{ab}z_{bc}z_{ac}$.

Finally in the case of the third type of relation of (5.39) we obtain a product

$$t^{\alpha_1 ab\alpha_2 cd\alpha_3}_{\alpha_1 ba\alpha_2 cd\alpha_3} t^{\alpha_1 ba\alpha_2 cd\alpha_3}_{\alpha_1 ba\alpha_2 dc\alpha_3} t^{\alpha_1 ba\alpha_2 dc\alpha_3}_{\alpha_1 ab\alpha_2 dc\alpha_3} t^{\alpha_1 ab\alpha_2 dc\alpha_3}_{\alpha_1 ab\alpha_2 cd\alpha_3}$$

which is equal to $x_{ab}x_{cd}x_{ba}x_{dc} = z_{ab}z_{cd}$. This concludes the proof.

5.40. COROLLARY. *Let $R$ be any one of the solutions of the YBE described in theorem 3.44 and suppose conditions (5.22)-(5.24) of corollary (5.21) hold (so that there is an YB operator $(\tau R, \nu, \alpha, \beta)$). Then the corresponding link invariant is a Laurent polynomial in the $\lambda_i, z_i, z_{ij}$.*

PROOF. If there are no blocks of type I present this is an immediate corollary of theorem 5.34. The presence of a block of type I changes very little (essentially on extra scalar multiple of the identity block in $S$ and the result remains true.

5.41. NEW INVARIANTS FROM MIXED SOLUTIONS

We already know from 5.29 that putting together several blocks (in a nontrivial way can give real extra information. In the case of $n = 4$ and 2 (different) type II blocks of size 2 the resulting link invariant will be a Laurent polynomial in $\lambda_1, \lambda_2, z_1, z_2, z_{12}$. One of the $z$'s, say $z_1$, can be normalized away (or absorbed into $\alpha$ which is the same thing) so that the result is a Laurent polynomial in four variables (with one nontrivial relation given by 5.22 between them and there does not seem to be any obvious way to write this polynomial in terms of known "classical" ones. In particular there is in general (e.g. for $\lambda_1 \neq \lambda_2$) no relation like (5.26). Just what this polynomial and all the other ones arising from theorem 3.44 via corollary 5.21 bring in terms of new invariants remains to be explored.

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APPENDIX 1. *Direct proof that the ideal generated by the elements (2.3), (2.4) is a Hopf algebra ideal in $k\langle t\rangle$.*

Let $I$ be the ideal in $k\langle t\rangle$ generated by the elements (2.3), (2.4). Under the comultiplication of $k\langle t\rangle$ we have

$$t^a_r t^b_r - q^{ab} t^b_r t^a_r \mapsto t^a_{i_1} t^b_{i_1} \otimes t^{i_1}_r t^{i_2}_r - q^{ab} t^b_{j_1} t^a_{j_2} \otimes t^{j_1}_r t^{j_2}_r \tag{A1.1}$$

First consider the terms on the right of (A1.1) with $i_1 = i_2$ and $j_1 = j_2$. These balance in pairs:

$$t^a_i t^b_i \otimes t^i_r t^i_r - q^{ab} t^a_i \otimes t^i_r t^i_r = (t^a_i t^b_i - q^{ab} t^b_i t^q_i) \otimes t^i_r t^i_r \in I \otimes k\langle t\rangle \tag{A1.2}$$

The remaining terms on the *RHS* of (A1.1) are treated in groups of four ($i \neq j$).

$$\begin{aligned}
&t^a_i t^b_j \otimes t^i_r t^j_r - q^{ab} t^b_i t^a_j \otimes t^i_r t^j_r + t^a_j t^b_i \otimes t^j_r t^i_r - q^{ab} t^b_j t^q_i \otimes t^j_r t^i_r \\
&\equiv (t^a_i t^b_j - q^{ab} t^b_i t^a_j + (q^{ij})^{-1} t^a_j t^b_i - (q^{ab})(q^{ij})^{-1} t^b_j t^q_i) \otimes t^i_r t^j_r \\
&\equiv 0 \qquad \bmod (I \otimes k\langle t\rangle + k\langle t\rangle \otimes I)
\end{aligned} \tag{A1.3}$$

(where the first congruence is in fact $\bmod(k\langle t\rangle \otimes I)$ and the second $\bmod\ I \otimes (k\langle t\rangle)$).
The elements (2.4) are twice as complicated to treat. Under the comulliplication (2.4) goes to

$$\begin{aligned}
&t^a_{i_1} t^b_{i_2} \otimes t^{i_1}_r t^{i_2}_s - q^{ab} t^b_{j_1} t^a_{j_2} \otimes t^{j_1}_r t^{j_2}_s \\
&\quad + (q^{rs})^{-1} t^a_{k_1} t^b_{k_2} \otimes t^{k_1}_s t^{k_2}_r - (q^{ab})(q^{rs})^{-1} t^b_{l_1} t^a_{l_2} \otimes t^{l_1}_s t^{l_2}_r
\end{aligned} \tag{A1.4}$$

The terms with $i_1 = i_2$ balance with those with $j_1 = j_2$ for the same value ($i_1 = i_2 = j_1 = j_2$):

$$t^a_i t^b_i \otimes t^i_r t^i_s - q^{ab} t^b_i t^a_i \otimes t^i_r t^i_s = (t^a_i t^b_i - q^{ab} t^b_i t^a_i) \otimes t^i_r t^i_s \in I \otimes k\langle t\rangle.$$

Similarly the terms with $k_1 = k_2$ balance with those of $l_1 = l_2$ for the same value.
Recall that if $a = b$ the element (2.4) is zero. So $a \neq b$ in (A1.4). The remaining terms of (A1.4) are dealt with in groups of 8 as follows:

$$\begin{aligned}
&t^a_i t^b_j \otimes t^i_r t^j_s + t^a_j t^b_i \otimes t^j_r t^i_s - q^{ab} t^b_i t^a_j \otimes t^i_r t^j_s - q^{ab} t^b_j t^a_i \otimes t^j_r t^i_s \\
&+ (q^{rs})^{-1} t^a_i t^b_j \otimes t^i_s t^j_r + (q^{rs})^{-1} t^a_j t^b_i \otimes t^j_s t^i_r - (q^{ab})(q^{rs})^{-1} t^b_i t^a_j \otimes t^i_s t^j_r - (q^{ab})(q^{rs})^{-1} t^b_j t^a_i \otimes t^j_s t^i_r \\
&= (t^a_i t^b_j - q^{ab} t^b_i t^a_j + (q^{ij})^{-1} t^a_j t^b_i - (q^{ab})(q^{ij})^{-1} t^b_j t^a_i) \otimes t^i_r t^j_s \\
&- (q^{ij})^{-1} t^a_j t^b_i \otimes (t^i_r t^j_s - q^{ij} t^j_r t^i_s + (q^{rs})^{-1} t^i_s t^j_r - q^{ij} (q^{rs})^{-1} t^j_s t^i_r) \\
&+ (q^{ab})(q^{ij})^{-1} t^b_j t^a_i \otimes (t^i_r t^j_s - q^{ij} t^j_r t^i_s + (q^{rs})^{-1} t^i_s t^j_r - q^{ij} (q^{rs})^{-1} t^j_s t^i_r) \\
&+ (t^a_i t^b_j - q^{ab} t^b_i t^a_j + (q^{ij})^{-1} t^a_j t^b_i - (q^{ab})(q^{ij})^{-1} t^b_j t^a_i) \otimes (q^{rs})^{-1} t^i_s t^j_r
\end{aligned}$$

which is in $I \otimes k\langle t\rangle + k\langle t\rangle \otimes I$. Above the *RHS* differs from the *LHS* only in regrouping and the insertion of the four terms $(q^{ij})^{-1} t^a_j t^b_i \otimes t^i_r t^j_s$, $q^{ab} (q^{ij})^{-1} t^b_j t^a_i \otimes t^i_r t^j_s$, $(q^{ij})^{-1} (q^{rs})^{-1} t^a_j b^b_i \otimes t^i_s t^j_r$, $q^{ab} (q^{ij})^{-1} (q^{rs})^{-1} t^b_j t^a_i \otimes t^i_s t^j_r$, each both with a plus and a minus sign.
This proves that $I_L$ is a bialgebra ideal. The proof for $I_R$ is completely analogous.

APPENDIX 2. *Derivation of the R-equations (R1)-(R7) of subsection 3.6 and proof that these are all equations.*

The general equation is (cf (3.2))

$$r^{ab}_{k_1k_2}r^{k_1c}_{uk_3}r^{k_2k_3}_{vw} = r^{bc}_{l_1l_2}r^{al_2}_{l_3w}r^{l_3l_1}_{uv} \tag{A.2.1}$$

By lemma (3.4) we know that under the condition

$$r^{ab}_{cd} = 0 \quad \text{unless} \quad \{a,b\} = \{c,d\} \tag{A2.2}$$

both sides of (A2.1) are zero unless $\{a,b,c\} = \{u,v,w\}$

CASE 1. $a = b = c = u = v = w$.
Then the $LHS$ of (A2.1) is nonzero iff $k_1 = k_2 = k_3 = a$ and then is equal to $(r^{aa}_{aa})^3$. Similarly the $RHS$ of (A2.1) is nonzero iff $l_1 = l_2 = l_3 = a$ and then is also equal to $(r^{aa}_{aa})^3$. No extra equation results from this case

CASE 2. $a \neq b \neq c \neq a$.
There are six subcases to be considerd, namely how the $u, v, w$ match up with the $a, b, c$

*Subcase 2.1.* $u = a,\ v = b,\ w = c$.
For a nonzero term on the $LHS$ we need $k_1 = a,\ k_2 = b$ or $k_1 = b,\ k_2 = a$
2.1.1. $k_1 = a,\ k_2 = b \Rightarrow k_3 = c$ (because $u = a = k_1$) giving a term $r^{ab}_{ab}r^{ac}_{ac}r^{bc}_{bc}$.
2.1.2. $k_1 = b,\ k_2 = a \Rightarrow$ zero because $r^{bc}_{ak_3} = 0$ for all $k_3$.
For a nonzero term on the $RHS$ we need $l_1 = b,\ l_2 = c$ or $l_1 = c,\ l_2 = b$
2.1.3. $l_1 = b,\ l_2 = c \Rightarrow l_3 = a$ (because $w = c = l_2$) giving a term $r^{bc}_{bc}r^{ac}_{ac}r^{ab}_{ab}$
2.1.4. $l_1 = c,\ l_2 = b \Rightarrow$ zero because $r^{ab}_{l_3c} = 0$ for all $l_3$.
Thus always $LHS = RHS$ in this subcase and no extra equation results.

*Subcase 2.2.* $u = a,\ v = c,\ w = b$.
For a nonzero term on the $LHS$ we need $k_1 = a,\ k_2 = b$ or $k_1 = b,\ k_2 = a$
2.2.1. $k_1 = a,\ k_2 = b \Rightarrow k_3 = c$ (because $u = a = k_1$) giving a term $r^{ab}_{ab}r^{ac}_{ac}r^{bc}_{cb}$
2.2.2. $k_1 = b,\ k_2 = a \Rightarrow$ zero because $r^{bc}_{ak_3} = 0$ for all $k_3$.
For a nonzero term on the $RHS$ we need $l_1 = b,\ l_2 = c$ or $l_1 = c,\ l_2 = b$.
2.2.3. $l_1 = b,\ l_2 = c \Rightarrow$ zero because $r^{ac}_{l_3b} = 0$ for all $l_3$.
2.2.4. $l_1 = c,\ l_2 = b \Rightarrow l_3 = a$ (because $l_2 = w = b$) giving a term $r^{bc}_{cb}r^{ab}_{ab}r^{ac}_{ac}$.
Thus always $LHS = RHS$ in this subcase and no extra equation results.

*Subcase 2.3.* $u = b,\ v = a,\ w = c$.
For a nonzero term on the left hand side we need $k_1 = a,\ k_2 = b$ or $k_1 = b,\ k_2 = a$
2.3.1. $k_1 = a,\ k_2 = b \Rightarrow$ zero because $r^{ac}_{bk_3} = 0$ for all $k_3$.
2.3.2. $k_1 = b,\ k_2 = a \Rightarrow k_3 = c$ (because $k_1 = b = u$) giving a term $r^{ab}_{ba}r^{bc}_{bc}r^{ac}_{ac}$.
For a nonzero term on the $RHS$ we need $l_1 = b,\ l_2 = c$ or $l_1 = c,\ l_2 = b$.
2.3.3. $l_1 = b,\ l_2 = c \Rightarrow l_3 = a$ (because $l_2 = c = w$) giving a term $re^{bc}_{bc}r^{ac}_{ac}r^{ab}_{ba}$.
2.3.4. $l_1 = c,\ l_2 = b \Rightarrow$ zero because $r^{ab}_{l_3c} = 0$ for all $l_3$.
Thus always $LHS = RHS$ in this subcase and no extra equation results

*Subcase 2.4.* $u = b,\ v = c,\ w = a$.
For a nonzero term on the $LHS$ we need $k_1 = a,\ k_2 = b$ or $k_1 = b,\ k_2 = a$.
t 2.4.1. $k_1 = a,\ k_2 = b \Rightarrow$ zero because $r^{ac}_{bk_3} = 0$ for all $k_3$
2.4.2. $k_1 = b,\ k_2 = a \Rightarrow k_3 = c$ (because $u = k_1 = b$) giving a term $r^{ab}_{ba}r^{bc}_{bc}r^{ac}_{ca}$.

For a nonzero term on the $RHS$ we need $l_1 = b,\ l_2 = c$ or $l_2 = c,\ l_1 = b$.
2.4.3. $l_1 = b,\ l_2 = c \Rightarrow l_3 = c$ (because $w = a,\ l_2 = c$) giving a term $r^{bc}_{bc} r^{ac}_{ca} r^{cb}_{bc}$.
2.4.4. $l_1 = c,\ l_2 = b \Rightarrow l_3 = b$ (because $w = a,\ l_2 = b$) giving a term $r^{bc}_{cb} r^{ab}_{ba} r^{bc}_{bc}$.
Thus $LHS = RHS$ in this subcase holds iff

$$r^{bc}_{bc}(r^{ab}_{ba} r^{ac}_{ca}) = r^{bc}_{bc}(r^{ac}_{ca} r^{cb}_{bc} + r^{bc}_{cb} r^{ab}_{ba}) \tag{R1}$$

*Subcase 2.5.* $u = c,\ v = a,\ w = b$.
For a nonzero term on the $LHS$ we need $k_1 = a,\ k_2 = b$ or $k_1 = b,\ k_2 = a$.
2.5.1. $k_1 = a,\ k_2 = b \Rightarrow k_3 = a$ (because $k_1 = a,\ u = c$) giving a term $r^{ab}_{ab} r^{ac}_{ca} r^{ba}_{ab}$.
2.5.2. $k_1 = b,\ k_2 = a \Rightarrow k_3 = b$ (because $k_1 = b,\ u = c$) giving a term $r^{ab}_{ba} r^{bc}_{cb} r^{ab}_{ab}$.
For a nonzero term on the $RHS$ we need $l_1 = b,\ l_2 = c$ or $l_1 = c,\ l_2 = b$.
2.5.3. $l_1 = b,\ l_2 = c \Rightarrow$ zero because $r^{ac}_{l_3 b} = 0$ for all $l_3$.
2.5.4. $l_1 = c,\ l_2 = b \Rightarrow l_3 = a$ (because $w = b = l_2$) giving a term $r^{bc}_{cb} r^{ab}_{ab} r^{ac}_{ca}$.
Thus $LHS = RHS$ in this subcase holds iff

$$r^{ab}_{ab}(r^{ac}_{ca} r^{ba}_{ab} + r^{ab}_{ba} r^{bc}_{cb}) = r^{ab}_{ab}(r^{bc}_{cb} r^{ac}_{ca}) \tag{R2}$$

*Subcase 2.6.* $u = c,\ v = b,\ w = a$.
For a nonzero term on the $LHS$ we need $k_1 = a,\ k_2 = b$ or $k_1 = b,\ k_2 = a$.
2.6.1. $k_1 = a,\ k_2 = b \Rightarrow k_3 = a$ (because $u = c,\ k_1 = a$) giving a term $r^{ab}_{ab} r^{ac}_{ca} r^{ba}_{ba}$.
2.6.2. $k_1 = b,\ k_2 = a \Rightarrow k_3 = b$ (because $u = c,\ k_1 = b$) giving a term $r^{ab}_{ba} r^{bc}_{cb} r^{ab}_{ba}$.
For a nonzero term on the $RHS$ we need $l_1 = b,\ l_2 = c$ or $l_1 = c,\ l_2 = b$.
2.6.3. $l_1 = b,\ l_2 = c \Rightarrow l_3 = c$ (because $w = a,\ l_2 = c$) giving a term $r^{bc}_{bc} r^{ac}_{ca} r^{cb}_{cb}$.
2.6.4. $l_1 = c,\ l_2 = b \Rightarrow l_3 = b$ (because $w = a,\ l_2 = b$) giving a term $r^{bc}_{cb} r^{ab}_{ba} r^{bc}_{cb}$.
Thus $LHS = RHS$ in this subcase holds iff

$$r^{ab}_{ab} r^{ba}_{ba} r^{ac}_{ca} + r^{ab}_{ba} r^{avb}_{ba} r^{bc}_{cb} = r^{bc}_{bc} r^{cb}_{cb} r^{ac}_{ca} + r^{ab}_{ba} r^{ba}_{cb} r^{bc}_{cb} \tag{R3}$$

CASE 3. $a = b \neq c$.
Again there are a number of subcases to consider depending on how the $u, v, w$ match up with the $a, b, c$. The six possibilities a priori coincide in pairs giving 3 subcases.

*Subcase 3.1.* $u = v = a = b,\ w = c$.
For a nonzero term on the $LHS$ we need $k_1 = a,\ k_2 = a$.
3.1.1. $k_1 = k_2 = a \Rightarrow k_3 = c$ (Because $k_1 = a = u$) giving a term $r^{aa}_{aa} r^{ac}_{ac} r^{ac}_{ac}$.
For a nonzero term on the $RHS$ we need $l_1 = a,\ l_2 = c$ or $l_1 = c,\ l_2 = a$
3.1.2. $l_1 = a,\ l_2 = c \Rightarrow l_3 = a$ (Because $w = c = l_2$) giving a term $r^{ac}_{ac} r^{ac}_{ac} r^{aa}_{aa}$.
3.1.3. $l_1 = c,\ l_2 = a \Rightarrow$ zero because $r^{aa}_{l_3 c} = 0$ for all $l_3$.
Thus always $LHS = RHS$ in this subcase and no extra equation results.

*Subcase 3.2.* $u = w = a = b,\ v = c$.
For a nonzero term on the $LHS$ we need $k_1 = k_2 = a$
3.2.1. $k_1 = k_2 = a \Rightarrow k_3 = c$ (because $u = a = k_1$) giving a term $r^{aa}_{aa} r^{ac}_{ac} r^{ac}_{ca}$.
For a nonzero term on the $RHS$ we need $l_1 = a,\ l_2 = c$ or $l_1 = c,\ l_2 = a$
3.2.2. $l_1 = a,\ l_2 = c \Rightarrow l_3 = c$ (because $l_2 = c,\ w = a$) giving a term $r^{ac}_{ac} r^{ac}_{ca} r^{ca}_{ac}$.
3.2.3. $l_1 = c,\ l_2 = a \Rightarrow l_3 = a$ (because $a = l_2 = w$) giving a term $r^{ac}_{ca} r^{aa}_{aa} r^{ac}_{ac}$.
Thus $LHS = RHS$ in this subcase iff

$$r^{ac}_{ac} r^{ac}_{ca} r^{ca}_{ac} = 0 \tag{R4}$$

*Subcase 3.3.* $u = c,\ v = w = a = b$.
For a nonzero term on the $LHS$ we need $k_1 = k_2 = a$
3.3.1. $k_1 = a = k_2 \Rightarrow k_3 = a$ (because $u = c,\ k_1 = a$) giving a term $r^{aa}_{aa} r^{ac}_{ca} r^{aa}_{aa}$.
For a nonzero term on the $RHS$ we need $l_1 = a,\ l_2 = c$ or $l_1 = c,\ l_2 = a$
3.3.2. $l_1 = a,\ l_2 = c \Rightarrow l_3 = c$ (because $w = a,\ l_2 = c$) giving a term $r^{ac}_{ac} r^{ac}_{ca} r^{ca}_{ca}$.
3.3.3. $l_1 = c,\ l_2 = a \Rightarrow l_3 = a$ (because $w = a,\ l_2 = a$) giving a term $r^{ac}_{ca} r^{aa}_{aa} r^{ac}_{ca}$.
Thus $LHS = RHS$ in this subcase iff

$$r^{aa}_{aa} r^{aa}_{aa} r^{ac}_{ca} = r^{aa}_{aa} r^{ac}_{ca} r^{ac}_{ca} + r^{ac}_{ac} r^{ca}_{ca} r^{ac}_{ca} \tag{R5}$$

CASE 4. $a \neq b = c$.
As in case 3, there are three subcases to consider

*Subcase 4.1.* $u = a,\ v = w = b = c$.
For a nonzero term on the $LHS$ we need $k_1 = a,\ k_2 = b$ or $k_1 = b,\ k_2 = a$
4.1.1. $k_1 = a,\ k_2 = b \Rightarrow k_3 = b$ (because $u = a$) giving a term $r^{ab}_{ab} r^{ab}_{ab} r^{bb}_{bb}$.
4.1.2. $k_1 = b,\ k_2 = a \Rightarrow$ zero because $r^{bb}_{ak_3} = 0$ for all $k_3$.
For a nonzero term on the $RHS$ we need $l_1 = l_2 = b$
4.1.3. $l_1 = l_2 = b \Rightarrow l_3 = a$ (because $l_2 = b = w$) giving a term $r^{bb}_{bb} r^{ab}_{ab} r^{ab}_{ab}$.
Thus always $LHS = RHS$ in this subcase and no extra equation results

*Subcase 4.2.* $u = b = w = c,\ v = a$.
For a nonzero term on the $LHS$ we need $k_1 = a,\ k_2 = b$ or $k_1 = b,\ k_2 = a$
4.2.1. $k_1 = a,\ k_2 = b \Rightarrow k_3 = a$ (because $u = b,\ k_1 = a$) giving a term $r^{ab}_{ab} r^{ab}_{ba} r^{ba}_{ab}$.
4.2.2. $k_1 = b,\ k_2 = a \Rightarrow k_3 = b$ (because $u = b = k_1$) giving a term $r^{ab}_{ba} r^{bb}_{bb} r^{ab}_{ab}$.
For a nonzero term on the $RHS$ we need $l_1 = l_2 = b$
4.2.3. $l_1 = l_2 = b \Rightarrow l_3 = a$ (because $w = b = l_2$) giving a term $r^{bb}_{bb} r^{ab}_{ab} r^{ab}_{ba}$.
Thus $LHS = RHS$ in this subcase iff

$$r^{ab}_{ab} r^{ab}_{ba} r^{ba}_{ab} = 0$$

giving (R4) for the second time

*Subcase 4.3.* $u = v = b = c,\ w = a$.
For a nonzero on the $LHS$ we need $k_1 = a,\ k_2 = b$ or $k_1 = b,\ k_2 = a$
4.3.1. $k_1 = a,\ k_2 = b \Rightarrow k_3 = a$ (because $u = b,\ k_1 = a$) giving a term $r^{ab}_{ab} r^{ab}_{ba} r^{ba}_{ba}$.
4.3.2. $k_1 = b,\ k_2 = a \Rightarrow k_3 = a$ (because $u = b,\ k_1 = b$) giving a term $r^{ab}_{ba} r^{bb}_{bb} r^{ab}_{ba}$.
For a nonzero term on the $RHS$ we need $l_1 = l_2 = b$
4.3.3. $l_1 = b,\ l_2 = b \Rightarrow l_3 = b$ (because $w = a,\ l_2 = b$) giving a term $r^{bb}_{bb} r^{ab}_{ba} r^{bb}_{bb}$.
Thus $RHS = LHS$ in this subcase iff

$$r^{bb}_{bb} r^{bb}_{bb} r^{ab}_{ba} = r^{bb}_{bb} r^{ab}_{ba} r^{ab}_{ba} + r^{ab}_{ab} r^{ba}_{ba} r^{ab}_{ba} \tag{R6}$$

Note that this is not the same equation as (R5) (also after changing $b$ to $a$, $a$ to $c$)

CASE 5. $a = c \neq b$ As in case 3 and 4, there are three subcases to consider.

*Subcase 5.1.* $u = w = a = c,\ v = b$.
For a nonzero term in the $LHS$ we need $k_1 = a, k_2 = b$ or $k_1 = b,\ k_2 = a$
5.1.1. $k_1 = a,\ k_2 = b \Rightarrow k_3 = a$ (because $u = k_1 = a$) giving a term $r^{ab}_{ab} r^{aa}_{aa} r^{ba}_{ba}$.
5.1.2. $k_1 = b,\ k_2 = a \Rightarrow k_3 = b$ (because $k_1 = b,\ u = a$) giving a term $r^{ab}_{ba} r^{ba}_{ab} r^{ab}_{ba}$.
For a nonzero term on the $RHS$ we need $l_1 = b,\ l_2 = a$ or $l_1 = a,\ l_2 = b$

5.1.3. $l_1 = b,\ l_2 = a \Rightarrow l_3 = a$ (because $w = a = l_2$) giving a term $r^{ba}_{ba} r^{aa}_{aa} r^{ba}_{ab}$.
5.1.4. $l_1 = a,\ l_2 = b \Rightarrow l_3 = b$ (because $w = a,\ l_2 = b$) giving a term $r^{ba}_{ab} r^{ab}_{ba} r^{ba}_{ab}$.
Thus $LHS = RHS$ in this subcase iff

$$r^{ab}_{ba} r^{ba}_{ab} r^{ab}_{ba} = r^{ab}_{ba} r^{ba}_{ab} r^{ba}_{ab} \tag{R7}$$

*Subcase 5.2.* $u = v = a = c,\ w = b$.
For a nonzero term on the $LHS$ we need $k_1 = a,\ k_2 = b$ or $k_1 = b, k_2 = a$
5.2.1. $k_1 = a,\ k_2 = b \Rightarrow k_3 = a$ (because $u = k_1 = a$) giving a term $r^{ab}_{ab} r^{aa}_{aa} r^{ba}_{ab}$.
5.2.2. $k_1 = b,\ k_2 = a \Rightarrow k_3 = b$ (because $u = a,\ k_1 = b$) giving a term $r^{ab}_{ba} r^{ba}_{ab} r^{ab}_{ab}$.
For a nonzero term on the $RHS$ we need $l_1 = b,\ l_2 = a$ or $l_1 = a,\ l_2 = b$
5.2.3. $l_1 = b,\ l_2 = a \Rightarrow$ zero because $r^{aa}_{l_3 b} = 0$ for all $l_3$
5.2.4. $l_1 = a,\ l_2 = b \Rightarrow l_3 = a$ (because $w = b = l_2$) giving a term $r^{ba}_{ab} r^{ab}_{ab} r^{aa}_{aa}$.
Thus $LHS = RHS$ in this subcase iff

$$r^{ab}_{ab} r^{ab}_{ba} r^{ba}_{ab} = 0$$

giving (R4) for the third time.

*Subcase 5.3.* $u = b,\ v = w = a = c$.
For a nonzero term on the $LHS$ we need $k_1 = a,\ k_2 = b$ or $k_1 = b,\ k_2 = a$
5.3.1. $k_1 = a,\ k_2 = b \Rightarrow$ zero because $r^{aa}_{bk_3} = 0$ for all $k_3$
5.3.2. $k_1 = b,\ k_2 = a \Rightarrow k_3 = a$ (because $u = b = k_1$) giving a term $r^{ab}_{ba} r^{ba}_{ba} r^{aa}_{aa}$.
For a nonzero term on the $RHS$ we need $l_1 = b,\ l_2 = a$ or $l_1 = a,\ l_2 = b$
5.3.3. $l_1 = b,\ l_2 = a \Rightarrow l_3 = a$ (because $w = a = l_2$) giving a term $r^{ba}_{ba} r^{aa}_{aa} r^{ab}_{ba}$.
5.3.4. $l_1 = a,\ l_2 = b \Rightarrow l_3 = b$ (because $w = a,\ l_2 = b$) giving a term $r^{ba}_{ab} r^{ab}_{ba} r^{ba}_{ba}$.
Thus $LHS = RHS$ in this subcase iff

$$r^{ba}_{ba} r^{ab}_{ba} r^{ba}_{ab} = 0$$

giving (R4) for the fourth time.