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Abstract

This report presents the proof of a conjecture made in earlier work. The conjecture was that the detection of the medial axis defined by a p-q-r–metric requires only local operations (restricted to a 16 point neighborhood) on the distance transformed image. This is related to the relative positions the centers of spheres defined by the p-q-r–metric can have, if one sphere is included in the other. The conjecture is proved by considering the extension of chamfer metrics to $\mathbb{R}^2$ and a general algorithm is described.

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1. Introduction

In a previous report [3] I described an algorithm for the computation of the medial axis of a discrete set, defined as the locus of centers of maximal spheres defined by the p-q–r–chamfer metric d. The algorithm is based on inspection of the internal distance transform ρ_X of a set X. I have been able to show that a point x is a medial axis point of X if there is no point y in a neighborhood N_x of x such that $\rho_X(y) \geq \rho_X(x) + d(x, y)$. In the algorithm I have been able to prove for the general p-q-r–chamfer metric, the environment N_x depends on the value of $\rho_X(x)$. A closer inspection for a large number of p-q-r–chamfer metrics showed that inspecting a 16 point environment is sufficient, regardless of the value of $\rho_X(x)$. Yet I was not able to prove this in general for all p-q-r–metrics.

In this report, I present a theorem, which implies that a 16 point neighborhood is sufficiently large for all points in every p-q-r–metric. This theorem is proved by considering the extension of p-q-r–chamfer metrics to the continuous plane.

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2. Discrete and Continuous Chamfer Metrics

This section describes the p-q-r–metric for the continuous plane $\mathbb{R}^2$ and the discrete grid $\mathbb{Z}^2$. We also present some properties of the spheres defined by these metrics, and of polygons in general. In most cases, proofs are omitted because they are either straightforward or can be found in previous work [2, 3].

The definition of the p-q-r–chamfer metric uses sixteen so-called prime vectors, which can be divided into three groups. These vectors are:

1. Four vectors $(\pm 1, 0)$ and $(0, \pm 1)$ which will be denoted as $v_{pi}, i = 1, \ldots, 4$.
2. Four vectors $(\pm 1, \pm 1)$ which will be denoted as $v_{qi}, i = 1, \ldots, 4$.
3. Eight vectors of the form $(\pm 1, \pm 2)$ and $(\pm 2, \pm 1)$, which will be denoted $v_{ri}, i = 1, \ldots, 8$.

The set of all sixteen prime vectors will be denoted V. Let $l : V \to \mathbb{N}$ be a function which assigns a positive integer weight to each prime vector such that $l(v_{pi}) = p$, $l(v_{qi}) = q$ and $l(v_{ri}) = r$. It will be assumed that the sixteen vectors $v/l(v)$ are the corners of a convex polygon (i.e. $r < p + q$, $p < r/2$ and $q < 2r/3$) and that the greatest common divisors of p and r and of q and r are 1 [2]. A common choice is to take $(p, q, r) = (5, 7, 11)$.

2.1 Definition *Let V be the set of prime vectors described above and let l be the weight functions satisfying the conditions mentioned above. The p-q-r–chamfer metric on $\mathbb{R}^2$ is defined by*

$$d(x, y) = \inf \left\{ \sum_{v \in V} |n_v| l(v) \mid \sum_{v \in V} n_v v = y - x, n_v \in \mathbb{R} \right\}$$

It is not difficult to see that the p-q-r–chamfer metric is indeed a metric, and that the "spheres" defined by this metric are polygons with 16 corners.

2.2 Lemma *The restriction of the p-q-r–chamfer metric defined above to the grid $\mathbb{Z}^2$ yields the classical p-q-r–chamfer metric on $\mathbb{Z}^2$* [1, 2].

Note that the range of d, when restricted to the grid, does not contain all integers, but only those which can be written in the form $pn + rm$ or $qn + rm$, where n and m are nonnegative integers. The set of all such integers will be denoted as D. D contains all but a finite number of positive integers.

Spheres in the continuous plane and the discrete grid are defined as follows.

2.3 Definition *For each $r \in \mathbb{R}, r \geq 0$ and $x \in \mathbb{R}^2$, the continuous sphere $S(x, r)$ is defined by*

$$S(x, r) = \{y \in \mathbb{R}^2 \mid d(x, y) \leq r\}.$$

2.4 Definition *For each $x \in \mathbb{Z}^2$ and $r \in D$, the discrete sphere $B(x, r)$ is defined by*

$$B(x, r) = \{y \in \mathbb{Z}^2 \mid d(x, y) \leq r\}.$$

Continuous and discrete spheres are related as follows. Suppose $x \in \mathbb{Z}^2$ and $r \in \mathbb{R}, r \geq 0$. Then

$$S(x, r) \cap \mathbb{Z}^2 = B(x, r^-), \tag{2.1}$$

where r^- is the number $\max\{s \in D \mid s \leq r\}$. This implies that, for $x \in \mathbb{Z}^2$ and $r \in D$,

$$B(x, r) = S(x, r) \cap \mathbb{Z}^2. \tag{2.2}$$

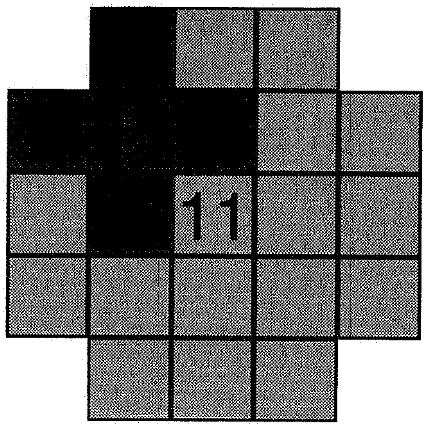

Figure 2.1: An example showing that inclusion of discrete spheres can not be derived directly from the distance transform.

For continuous spheres, the equivalence $S(x,r) \subseteq S(y,s) \Leftrightarrow s \geq r + d(x,y)$ holds. From figure 2.1, it can be seen that this is not true for discrete spheres. The figure uses the 5-7-11–metric. The smaller sphere of radius 5 is contained in the larger sphere of radius 11, but the distance between their centers is 7, while the difference of their radii is only 6.

2.5 Definition *The relation $\sqsubseteq$ between subsets A and B of $\mathbb{R}^2$ is defined by*

$$A \sqsubseteq B \text{ iff } A \cap \mathbb{Z}^2 \subseteq B \cap \mathbb{Z}^2.$$

Obviously, $A \subseteq B$ implies $A \sqsubseteq B$, but the reverse is in general not true.

The following lemmas describe the behaviour of convex sets under scalings. The scaling $M(c,\lambda)(X)$ of a subset X of $\mathbb{R}^2$ with center $c \in \mathbb{R}^2$ and magnification factor $\lambda \in \mathbb{R}$ is defined as the set $\{\lambda(x-c)+c \mid x \in X\}$.

2.6 Lemma *Let X be a convex subset of $\mathbb{R}^2$ and let $x \in X$, $\lambda > 1$. Then $X \subseteq M(x,\lambda)(X)$.*

2.7 Lemma *Let P be a convex polygon. Suppose that the carriers of two of its sides AB and CD intersect in a point Z, and that B is closer to Z than A and C is closer to Z than D. Suppose λ is a scaling factor such that $B' = \lambda(B-Z)+Z$ lies between B and A and $C' = \lambda(C-Z)+Z$ lies between C and D. Then the only part of P which is not contained in $M(Z,\lambda)(P)$ is the polygon bounded by the boundaries of the polygons between B and C and between B' and C', and the segments BB' and CC'.*

This lemma is illustrated in figure 2.2. It can be proven by considering, for all lines l through Z, the intersections of l with P and $M(Z,\lambda)(P)$. These intersections are either both equal to $\emptyset$ or partially overlapping segments of l.

3. Restriction to 16 Point Neighborhoods

In a previous report [3] I obtained the following result.

3.1 Theorem *Let X be a bounded subset of $\mathbb{Z}^2$. A point $x \in X$ is a medial axis point if there is no point y with $B(x,\rho(x)) \subseteq B(y,\rho(y))$ and one of the following holds:*
(1) $y - x = (a,0)$ or $(0,a)$, with $|a| \leq \rho(x)/q$.
(2) $y - x = (a,a)$ or $(a,-a)$, with $|a| \leq \rho(x)/p$.
(3) $y - x = (\pm1,\pm2)$ or $(\pm2,\pm1)$.

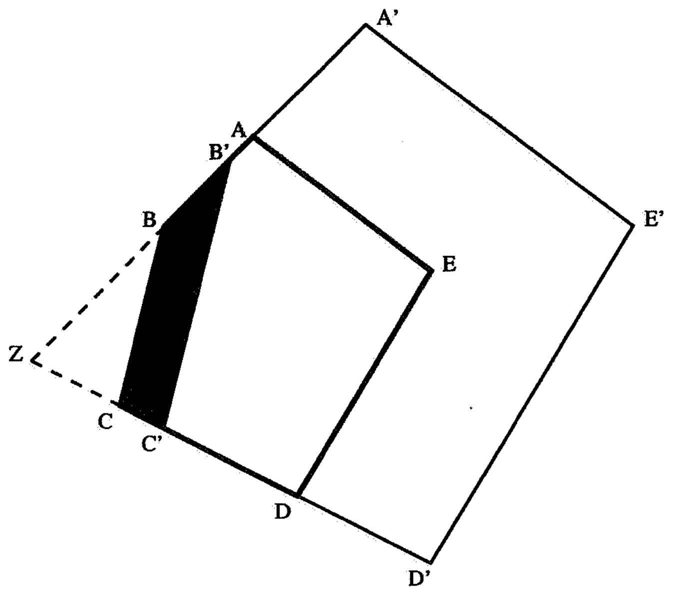

Figure 2.2: Illustration of lemma 2.7. The only part of the small polygon ABCDE which is not contained in the magnified polygon A'B'C'D'E' is the shaded area.

From this result, is was deduced that the medial axis defined by the p-q-r–metric can be computed by inspection of a star shaped neighborhood of each point. I was not been able to show whether this inspection could be restricted to a 16 point neighborhood, because I was not able to solve the folowing problem.

3.2 Question *Let y be a point of the form $(-a, 0)$ or $(-a, -a)$, with $a \in \mathbb{N}$, $a > 1$. Suppose that $B((0,0), r) \subseteq B(y, s)$ for some $r, s \in D$. Is there an eight-connected neighbor y' of $(0,0)$ such that*

$$B((0,0), r) \subseteq B(y', r) \subseteq B(y, s)$$

for some $r' \in D$?

In this report it is shown that this question can be answered affirmatively. Therefore, we prove the following theorem.

3.3 Theorem *Let $r, s \in D$, $a \in \mathbb{N}$ with $a > 1$.*
(1) If $B((0,0), r) \subseteq B((-a, 0), s)$, then there is an $r' \in D$ such that

$$B((0,0), r) \subseteq B((-1, 0), r') \subseteq B((-a, 0), s).$$

(2) If $B((0,0), r) \subseteq B((-a, -a), s)$, then there is an $r' \in D$ such that

$$B((0,0), r) \subseteq B((-1, -1), r') \subseteq B((-a, -a), s).$$

PROOF. We will prove only the first part of the theorem, as the second part can be proved in a similar way. It can be assumed without loss of generality that s is the smallest value in D for which $B((0,0), r) \subseteq B((-a, 0), s)$ holds. The theorem is shown by considering continuous spheres in stead of discrete ones. We will prove that there is an $r' \in \mathbb{R}$ such that

$$S((0,0), r) \sqsubseteq S((-1, 0), r') \sqsubseteq S((-a, 0), s). \tag{3.1}$$

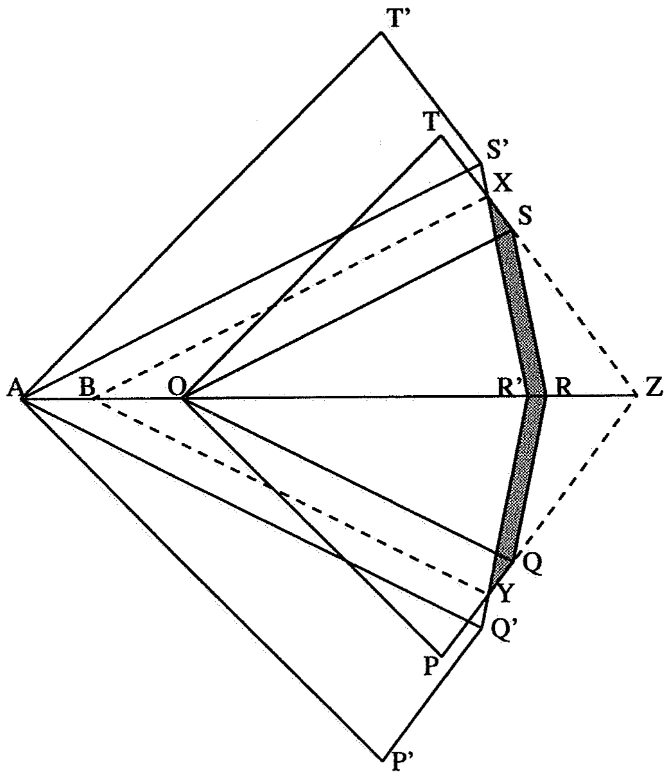

Figure 3.1: The relative positions of two spheres. See the text for an explanation.

The theorem follows immediately from this relation by applying (2.1) and definition 2.5.

Consider the octagons $S((0,0),r)$ and $S((-a,0),s)$. Parts of these polygons are depicted in figure 3.1. The center of the small sphere is the origin O. Some of its corners, P,Q,R,S and T, are marked. The center of the larger sphere is $A=(-a,0)$; some of its corners, P',Q',R',S' and T', are marked.

There must be a grid point on the segment RS or on the segment ST. If S' lies below the line ST, both the segments RS and ST lie outside the larger sphere, and so does at least one grid point. This would violate the inclusion relation. Therefore, S' lies above the line ST.

If S' lies above the line RS, the smaller sphere would be included in the interior of the larger one, and it would be possible to find a value $s' < s$ for which $S((0,0),r) \subseteq S((0,-a),s')$ holds. This would imply $B((0,0),r) \subseteq B((0,-a),s'^{-})$, violating the assumtion that s is the smallest value for which this inclusion holds. Therefore, S' lies below the line RS.

The boundaries of the spheres intersect in the points X and Y. The projection of these points in the $(2,1)-$ respectively the $(2,-1)-$directions is the point B. Two degenerate cases can occur. The points R and R' can coincide, or the point S' can lie on the segment ST. In these situations, X will be chosen to be S or S', respectively.

Note that the relation $S((0,0),r) \subseteq S((-a,0),s)$ does not hold in general, but only if R and R' coincide. On the other hand, since $S((0,0),r) \sqsubseteq S((-a,0),s)$, the shaded area $YQRSXR'$ does not contain any points from $\mathbb{Z}^2$.

The proof is as follows. We construct a family of spheres $S((-t,0),r(t))$ such that $r(0)=r$, $r(a)=s$ and

$$S((-t,0),r(t)) \sqsubseteq S((-t',0),r(t')) \tag{3.2}$$

if $t \leq t'$. This implies

$$S((0,0),r) \sqsubseteq S((-t,0),r(t)) \sqsubseteq S((-a,0),s) \tag{3.3}$$

for any $t \in [0,a]$. Then (3.1) follows directly by taking $t = 1$ in (3.3).

The family $S((-t,0),r(t))$ is found by transforming $S((0,0),r)$ gradually into $S((-a,0),s)$. Intuitively, the family can be described as the sequence of spheres which is obtained by gradually shifting and enlarging $S((0,0),r)$ in such a way that S moves along the segment SX to X, and then along the segment XS' to S', while the center of the sphere moves from $(0,0)$ to $(-a,0)$.

The family consists of two parts. The first part is obtained by applying the transformation $M(Z,\lambda)$ to $S((0,0),r)$ with λ ranging from 1 to $|R'X|/|RS|$. This yields the spheres $S((-t,0),r(t))$ with t between 0 and b.

The second part of the family is obtained by applying the transformation $M(R',\lambda)$ to the sphere $S((-b,0),r(b))$, with λ ranging from 1 to $|R'S'|/|R'X|$. This yields the spheres $S((-t,0),r(t))$ with t between b and a.

It remains to be shown that the family thus obtained satisfies (3.2). For the first part of the family, this follows from lemma 2.7, combined with the fact that the shaded area in figure 3.1 contains no grid points. For the second part, increasingness with respect to $\subseteq$ follows immediately from lemma 2.6. Therefore, increasingness with respect to $\sqsubseteq$ certainly holds. ∎

4. The Improved Algorithm

Theorem 3.3 implies that it is sufficient to consider 16 point neighborhoods in the detection of medial axis point. In this section, the operations which are to be performed in such a neighborhood will be described, and the resulting algorithm will be presented.

Let $x \in X$ be a point of a set $X \subset \mathbb{Z}^2$. In order to determine whether x is a medial axis point, it must be checked for each neighbor y of X in a sixteen point environment, whether

$$B(x,\rho(x)) \subseteq B(y,\rho(y)) \tag{4.1}$$

holds. There are three cases to be discerned: y is a direct (4-connected) neighbor of x, y is an indirect (8-connected but not 4-connected) neighbor of x, or y is at a knights move from x.

In the last case, we have shown [3] that (4.1) is true if and only if $\rho(y) \geq \rho(x)+r$. Now suppose that y is a direct neighbor of x. It can be assumed without loss of generality that $x = (0,0)$ and $y = (-1,0)$. Then (4.1) is true if and only if $\rho((-1,0)) \geq \rho((0,0)) + \Delta\rho$, where $\Delta\rho$ is given by

$$\Delta\rho = max\{d((-1,0),z) - \rho((0,0)) \mid z \in B((0,0),\rho((0,0)))\} \tag{4.2}$$

Consider again figure 3.1, and suppose that A is the point $(-1,0)$. The point z which maximizes $d((-1,0),z)$ in (4.2) is a grid point on the boundary of the larger sphere which is also included in the smaller sphere. If S' lies strictly above the line ST, z lies on the segment $R'X$, but if S' lies on the segment ST, then z can also lie on the segment XT.

In the first case, $d((0,0),z) = \max\{s \in p\mathbb{N}+r\mathbb{N} \mid s \leq \rho((0,0))\}$, so $d((-1,0),z) = p+\max\{s \in p\mathbb{N}+r\mathbb{N} \mid s \leq \rho((0,0))\}$. In the second case, $d((-1,0),z) = d((0,0),z)+r-q = \rho((0,0))+r-q$. Using the notation $s_{(pr)} = \max\{s' \in p\mathbb{N}+r\mathbb{N} \mid s' \leq s\}$, we find

$$\Delta\rho = \max(r-q, \rho(x)_{(pr)} - \rho(x) + p). \tag{4.3}$$

For all but a finite number of values for $\rho(x)$, $\rho(x) \in p\mathbb{N}+r\mathbb{N}$, implying $\rho(x)_{(pr)} = \rho(x)$ and $\Delta\rho = p$. Therefore, a table of the values of $\Delta\rho$ can be precomputed for those values of $\rho(x)$ which are not in $p\mathbb{N}+r\mathbb{N}$, while for other values $\Delta\rho = p$ can be used.

A expression similar to (4.3) can be found for the case where y is an indirect neighbor of x. Note that for p-q-r–metrics with small values of p, q and r, we have $p+q-r=1$, such that $\Delta\rho$ will always be equal to p or $p-1$.

These observations lead to the following algorithm.

4.1 Algorithm *Let X be a bounded subset of $\mathbb{Z}^2$ and let d be a p-q-r–metric.*

(1) Compute the internal distance transform ρ of X.

(2) For each $x \in X$:

- *If a direct neighbor y of x satisfies $\rho(y) \geq \max(\rho(x) + r - q, \rho(x)_{(pr)} + p)$, mark x as a non-medial axis point;*
- *If an indirect neighbor y of x satisfies $\rho(y) \geq \max(\rho(x) + r - p, \rho(x)_{(qr)} + q)$, mark x as a non-medial axis point;*
- *If a knights move neighbor y of x satisfies $\rho(y) \geq \rho(x) + r$, mark x as a non-medial axis point;*
- *otherwise mark x as a medial axis point.*

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