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Abstract

This paper introduces a new form of local balance and the corresponding product form results. It is shown that these new product form results allow capacity constraints at the stations of a queueing network without reversibility assumptions and without special blocking protocols. In particular, exact product form results for heavily loaded queueing networks are obtained.

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1. Introduction

Queueing networks are widely used in computer performance evaluation, telecommunications and manufacturing. In particular, product form queueing networks allow us to obtain accessible expressions for performance measures of interest. A major restriction of product form queueing networks is that they do not allow the introduction of capacity constraints, unless very restrictive assumptions are made on the routing process of the customers.

Product forms for queueing networks are based on the traffic equations. It is the structure of the traffic equations that introduces the restrictive assumptions necessary to obtain product form results when upper boundaries for the number of customers at stations are introduced. In contrast, lower boundaries for the number of customers at the queues can be chosen almost arbitrarily without destroying the product form result. This is also a consequence of the form of the traffic equations.

The traffic equations are the equilibrium equations for a queueing network containing only one customer. As such these traffic equations are the Kolmogorov forward equations for the Markov chain describing the behaviour of a single customer in the queueing network. For this Markov chain one might also consider the Kolmogorov backward equations. A new type of traffic equations, closely related to the backward equations will be referred to as the backward traffic equations, and are the basis for the analysis and results presented in this paper.

This paper shows that product forms can be derived based on the backward traffic equations. These product forms are similar to the product forms obtained for 'standard' queueing networks such as Jackson networks [7], Gordon and Newell networks [5], BCMP networks [1], and Kelly-Whittle networks [8], [10]. The major difference between product forms based on the backward traffic equations and product forms based on the 'standard' traffic equations is that upper boundaries for the number of customers at the stations can be introduced without restrictive conditions on the routing process when the backward traffic equations are used, whereas lower boundaries introduce restrictions on the routing process. In particular, this allows us to obtain product form results for heavily loaded queueing networks without restrictions, such as blocking protocols.

In section 2 we discuss the backward traffic equations and the corresponding form of local balance, that is used to prove the product form result. In section 3 we investigate the structure of the backward traffic equations in detail. In particular, blocking protocols are investigated. Furthermore, section 3 presents the generalisation to more general traffic equations. This generalisation is straightforward, and therefore postponed to this last section so as not to distract the reader from the ideas behind the backward traffic equations and corresponding product form results. Finally, product forms are discussed for heavily loaded queueing networks. The paper is concluded with a discussion of the results in section 4.

2. Model and product form result

Consider a continuous-time queueing network consisting of N queues, or stations labelled $1, \ldots, N$. Assume that the queueing network can be represented by a stable, regular, continuous-time Markov chain $\mathbf{X} = \{X(t),\ t \geq 0\}$ at state space $S \subseteq \mathbb{N}_0^N$. A state $\mathbf{n} = (n_1, \ldots, n_N) \in S$ is a vector with components $n_i \in \mathbb{N}_0$ denoting the number of customers at queue i, $i = 1, \ldots, N$.

Let $q(\mathbf{n}, \mathbf{n}')$ denote the transition rate from state $\mathbf{n}$ to state $\mathbf{n}'$. As transitions correspond to customers routing among the queues, we have $q(\mathbf{n}, \mathbf{n}') = 0$ unless $\mathbf{n}' = \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j$, $i, j = 0, \ldots, N$, where $\mathbf{e}_i$ denotes the i-th unit vector, and $\mathbf{e}_0 = 0$. A transition $\mathbf{n} \rightarrow \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j$ corresponds to a customer routing from station i to station j, and for notational convenience station 0 represents the outside.

Assume that $\mathbf{X}$ is irreducible at S and that $\mathbf{X}$ possesses a stationary or equilibrium distribution $\pi = (\pi(\mathbf{n}),\ \mathbf{n} \in S)$ at S. Then π is the unique solution of the global balance equations (Whittle [10])

$$\sum_{\mathbf{n}' \in S} \{\pi(\mathbf{n})q(\mathbf{n}, \mathbf{n}') - \pi(\mathbf{n}')q(\mathbf{n}', \mathbf{n})\} = 0, \quad \mathbf{n} \in S.$$

An immediate consequence of the transition structure of $\mathbf{X}$ is that the global balance equations can be written as

$$\sum_{i,j=0}^{N} \{\pi(\mathbf{n})q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) - \pi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})\} = 0, \quad \mathbf{n} \in S. \tag{2.1}$$

Product form results for queueing networks are based on local balance, balancing for each queue separately the flow out of a state due to the *departure* of customers and the flow into that state due to the *arrival* of customers. A distribution π satisfies local balance if for $i = 0, \ldots, N$

$$\sum_{j=0}^{N} \{\pi(\mathbf{n})q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) - \pi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})\} = 0, \quad \mathbf{n} \in S. \tag{2.2}$$

A distribution π that satisfies (2.2) is the equilibrium distribution because (2.2) is the decomposition of (2.1) into a set of equations for each i separately.

Similar to the decomposition of (2.1) into a set of equations for each i, (2.1) can also be decomposed into a set of equations for each j. This gives a form of local balance that we will refer to as *backward local balance* equating for each queue separately the flow out of a state due to the *arrival* of customers and the flow into that state due to the *departure* of customers from that queue. A distribution π^b satisfies backward local balance if for $j = 0, \ldots, N$

$$\sum_{i=0}^{N} \left\{\pi^b(\mathbf{n})q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) - \pi^b(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})\right\} = 0, \quad \mathbf{n} \in S. \tag{2.3}$$

At first glance, (2.2) and (2.3) seem to express a similar form of local balance. However, note that in general $\pi \neq \pi^b$, and that a solution π to (2.2) may exist while a solution π^b to (2.3) does not exist and vice versa. This is an immediate consequence of the restrictive nature of the assumption that a process satisfies (backward) local balance.

Below we present sufficient conditions for the existence of a solution to the backward local balance equations. These conditions are similar to the conditions sufficient to find a solution to the local balance equations. Note that the solution can be the same only if $\mathbf{X}$ is reversible.

Assume that the transition rates have the form

$$q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) = \frac{\phi(\mathbf{n} + \mathbf{e}_j)}{\psi(\mathbf{n})} p_{ij}(\mathbf{n} + \mathbf{e}_j), \tag{2.4}$$

where ϕ, ψ, and p_{ij} are arbitrary functions such that $\psi(\mathbf{n}) > 0$ for all $\mathbf{n} \in S$. We have the following

Theorem 2.1 *Assume that a positive solution $\{d_i\}_{i=1}^N$ exists for the backward state dependent traffic equations for $j = 0, \ldots, N$,*

$$\sum_{i=0}^{N} \{d_j p_{ij}(\mathbf{n}+\mathbf{e}_j) - d_i p_{ji}(\mathbf{n}+\mathbf{e}_j)\} = 0, \quad d_0 = 1. \tag{2.5}$$

Then $\mathbf{X}$ has a unique equilibrium distribution

$$\pi(\mathbf{n}) = B\psi(\mathbf{n}) \prod_{k=1}^{N} \left(\frac{1}{d_k}\right)^{n_k}, \quad \mathbf{n} \in S, \tag{2.6}$$

where $B^{-1} = \sum_{\mathbf{n}\in S} \psi(\mathbf{n}) \prod_{k=1}^{N} \left(\frac{1}{d_k}\right)^{n_k} < \infty$, and π satisfies backward local balance (2.3).

Proof Insertion of (2.4) and (2.6) in the backward local balance equations gives

$$\begin{aligned}
&\sum_{i=0}^{N} \{\pi(\mathbf{n})q(\mathbf{n}, \mathbf{n}-\mathbf{e}_i+\mathbf{e}_j) - \pi(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)q(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j, \mathbf{n})\} \\
&= B\phi(\mathbf{n}+\mathbf{e}_j) \prod_{k=1}^{N} \left(\frac{1}{d_k}\right)^{n_k+\delta_{kj}} \sum_{i=0}^{N} \{d_j p_{ij}(\mathbf{n}+\mathbf{e}_j) - d_i p_{ji}(\mathbf{n}+\mathbf{e}_j)\},
\end{aligned}$$

and (2.5) completes the proof. □

Remark 2.2

1. The structure of the transition rates (2.4) is similar to the structure used in the literature on product form queueing networks (Boucherie and van Dijk [3], Serfozo [9])

$$q(\mathbf{n}, \mathbf{n}-\mathbf{e}_i+\mathbf{e}_j) = \frac{\gamma(\mathbf{n}-\mathbf{e}_i)}{\psi(\mathbf{n})} p_{ij}(\mathbf{n}-\mathbf{e}_i). \tag{2.7}$$

   The main difference is the dependence on $\mathbf{n}+\mathbf{e}_j$ in (2.4) and on $\mathbf{n}-\mathbf{e}_i$ in (2.7). This difference is related to the structure of the local balance equations. From (2.3) and the proof of Theorem 2.1 we see that $\phi(\mathbf{n}+\mathbf{e}_j)$ appears as a constant for these local balance equations, whereas from (2.2) we similarly see that $\gamma(\mathbf{n}-\mathbf{e}_i)$ is a constant for this set of local balance equations. This is the explanation of the choice (2.4) for the transition rates.

2. The backward traffic equations have a structure related to the backward equations for the single customer process. This is best illustrated when we consider the state-independent routing version of (2.5):

$$\sum_{i=0}^{N} \{p_{ij} d_j - p_{ji} d_i\} = 0, \quad d_0 = 1. \tag{2.8}$$

The standard traffic equations for a queueing network with routing probabilities p_{ij} are

$$\sum_{j=0}^{N} \{c_i p_{ij} - c_j p_{ji}\} = 0, \quad c_0 = 1. \tag{2.9}$$

The difference between (2.8) and (2.9) is that d_i appears on the other side of the routing probabilities. Note that (2.8) does not correspond to the backward equations of the single customer process. The backward equations are $\sum_{i=0}^{N} \{p_{ji} d_i - p_{ji} d_j\} = 0$. Therefore, in contrast with the results for product form queueing networks, where the existence of a solution to the traffic equations (2.9) follows from the Perron-Frobenius theorem, for the backward process a solution to (2.8) is not guaranteed. In section 3.1 we will present special cases in which a solution to (2.8) exists, and in section 3.2 we present blocking results that are such that the solution for (2.8) is a solution for (2.5) too.

3. As a consequence of the form of the transition rates (2.4), and the traffic equations (2.5), the equilibrium distribution has a product form similar to the standard product form for queueing networks. The ideas behind the product form (2.6) are different: it is derived on the basis of the backward local balance equations. As is discussed below, this allows us to introduce capacity constraints at the stations without additional conditions on the transition structure. In particular, backward local balance allows us to derive product form results for heavily loaded networks without constraints on the transition rates such as blocking protocols.

3. Examples and extensions

In contrast with standard product form queueing networks, where a positive solution for the traffic equations (2.9) is guaranteed by the Perron-Frobenius theorem, the backward traffic equations (2.8) will not always have a positive solution. Section 3.1 presents some special cases in which a solution for the backward traffic equations is guaranteed. In section 3.2 we will investigate the behaviour of the state-dependent traffic equations at the boundary of the state space, and in section 3.3 we generalise the result of Theorem 2.1 to state-dependent solutions of the traffic equations. Finally, section 3.4 generalises the product form results of Gordon and Newell [6] for heavily loaded cyclic queueing networks to arbitrary heavily loaded queueing networks.

3.1 State-independent routing
Consider the state-independent backward traffic equations (2.8), a special case of (2.5) in which the state is removed from the routing function. A solution for (2.8) will in general not satisfy (2.5) because of conditions imposed by the boundary of the state space S, but in most applications a solution of (2.8) is the basis of a solution for (2.5). This will be discussed in section 3.2.

If $P = (p_{ij})$ is doubly stochastic, that is

$$\sum_{j=0}^{N} p_{ij} = 1, \quad \text{and} \quad \sum_{i=0}^{N} p_{ij} = 1,$$

then the backward traffic equations are equivalent to the backward equations for the single customer Markov chain. Thus if this Markov chain is irreducible the Perron-Frobenius theorem guarantees a unique solution $d_i = 1$ for all i. This is not a very interesting case.

It is not assumed that the coefficients p_{ij} figuring in the backward traffic equations are routing probabilities, that is $\sum_{j=0}^{N} p_{ij} = h_i \neq 1$ is possible. The following choices, each with $\bar{p}_{ij}$ the routing probabilities for a single customer Markov chain, exploit the possibility that $h_i \neq 1$ is allowed. In these examples the routing function is chosen such that the Perron-Frobenius theorem for the single customer process can be used to guarantee a unique solution for (2.8).

Let $\{\bar{c}_i\}_{i=1}^{N}$ be the unique solution to $\sum_{j=0}^{N}\{\bar{c}_i\bar{p}_{ij} - \bar{c}_j\bar{p}_{ji}\} = 0$, $\bar{c}_0 = 1$, then

$$\begin{array}{llll} \text{if} & p_{ij} = \bar{c}_i\bar{c}_j\bar{p}_{ij} & \text{we have} & d_i = 1/\bar{c}_i, \\ \text{if} & p_{ij} = \bar{c}_j^2\bar{p}_{ji} & \text{we have} & d_i = 1/\bar{c}_i, \\ \text{if} & p_{ij} = (\bar{c}_i/\bar{c}_j)\bar{p}_{ij} & \text{we have} & d_i = \bar{c}_i, \\ \text{if} & p_{ij} = \bar{p}_{ji} & \text{we have} & d_i = \bar{c}_i, \end{array}$$

and in each of these cases $\{d_i\}_{i=1}^{N}$ is unique. The first two cases correspond to the Markov chain $\mathbf{X}$ describing customers, whereas the last two cases correspond to $\mathbf{X}$ describing vacancies. A discussion on this interpretation can be found in Boucherie [2].

Finally, if the routing probabilities $\bar{p}_{ij}$ are such that a solution for the detailed balance equations $\bar{c}_i\bar{p}_{ij} = \bar{c}_j\bar{p}_{ji}$ exists, then for $p_{ij} = \bar{p}_{ij}$ we have that $d_i = 1/\bar{c}_i$ satisfies $p_{ij}d_j = p_{ji}d_i$.

The above examples make use of the Perron-Frobenius theorem for a related Markov chain with routing probabilities $\bar{p}_{ij}$. In general we have to solve (2.8) to see whether or not a positive solution exists.

To provide some insight in the solutions presented above, assume that the solutions for (2.8) presented above satisfy (2.5) too. Consider the cases $p^1_{ij} = \bar{c}_i\bar{c}_j\bar{p}_{ij}$ and $p^2_{ij} = (\bar{c}_i/\bar{c}_j)\bar{p}_{ij}$ discussed above. The transition rates (2.4) corresponding to these cases are

$$q^1(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) = \frac{\phi(\mathbf{n} + \mathbf{e}_j)}{\psi(\mathbf{n})} \bar{c}_i \bar{c}_j \bar{p}_{ij}, \quad q^2(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) = \frac{\phi(\mathbf{n} + \mathbf{e}_j)}{\psi(\mathbf{n})} (\bar{c}_i / \bar{c}_j) \bar{p}_{ij},$$

and $\mathbf{X}$ has a unique equilibrium distributions

$$\pi^1(\mathbf{n}) = B\psi(\mathbf{n}) \prod_{k=1}^{N} \bar{c}_k^{n_k}, \quad \text{and} \quad \pi^2(\mathbf{n}) = B\psi(\mathbf{n}) \prod_{k=1}^{N} (1/\bar{c}_k)^{n_k}.$$

Observe that π^1 satisfies standard local balance (2.2) for the process with transition rates (2.7), and that π^1 therefore is the equilibrium distribution for the process with transition rates (2.7) too. In contrast, we cannot select functions ϕ, ψ, and γ such that π^2 satisfies standard local balance (2.2). Therefore, the product form solutions for the backward local balance equations (2.3) found in Theorem 2.1 are in general *different* from the product form results based on standard local balance (2.2).

3.2 Blocking protocols (for open queueing networks)

In this section we discuss the behaviour of $\mathbf{X}$ at boundaries. In particular, blocking protocols are presented that preserve the product form equilibrium distribution. As will be shown, due to the structure of the local balance equations, upper limit blocking corresponding to capacity constraints can be introduced without restrictions, but lower bounds require some assumptions on the p_{ij}.

For simplicity, assume that $p_{ij}(\mathbf{n} + e_j)$ is state-independent, except for states near the boundary of S, that is for all $\mathbf{n}$, j such that $\mathbf{n} + e_j \in S$

$$p_{ij}(\mathbf{n} + e_j) = p_{ij} b_{ij}(\mathbf{n} + e_j),$$

where $b_{ij}(\mathbf{n} + e_j) = 1$ away from the boundary in a way presented below, and p_{ij} is such that a positive solution exists for (2.8).

If (2.8) admits a solution for $p_{ij}d_j - p_{ji}d_i = 0$, then for all i, j, $\mathbf{n} + e_j$ we may set $b_{ij}(\mathbf{n} + e_j) = 1$. This result is similar to the result for reversible processes based on local balance (2.2).

Consider the case in which a 'reversible' solution for (2.8) does not exist. In figure 1 for states in the interior of S global balance and its decomposition into local balance (2.3) is depicted for a 2 queue system. For comparison, the decomposition into local balance (2.2) is depicted too.

For a product form equilibrium distribution to be a solution of (2.2) it is known that special blocking protocols must be used to change certain transitions near the boundary of S. For example, the *stop protocol* also referred to as *communication blocking* stops all queues except for a saturated queue to preserve product form (cf. Van Dijk [4]). A similar protocol is discussed below for (2.3).

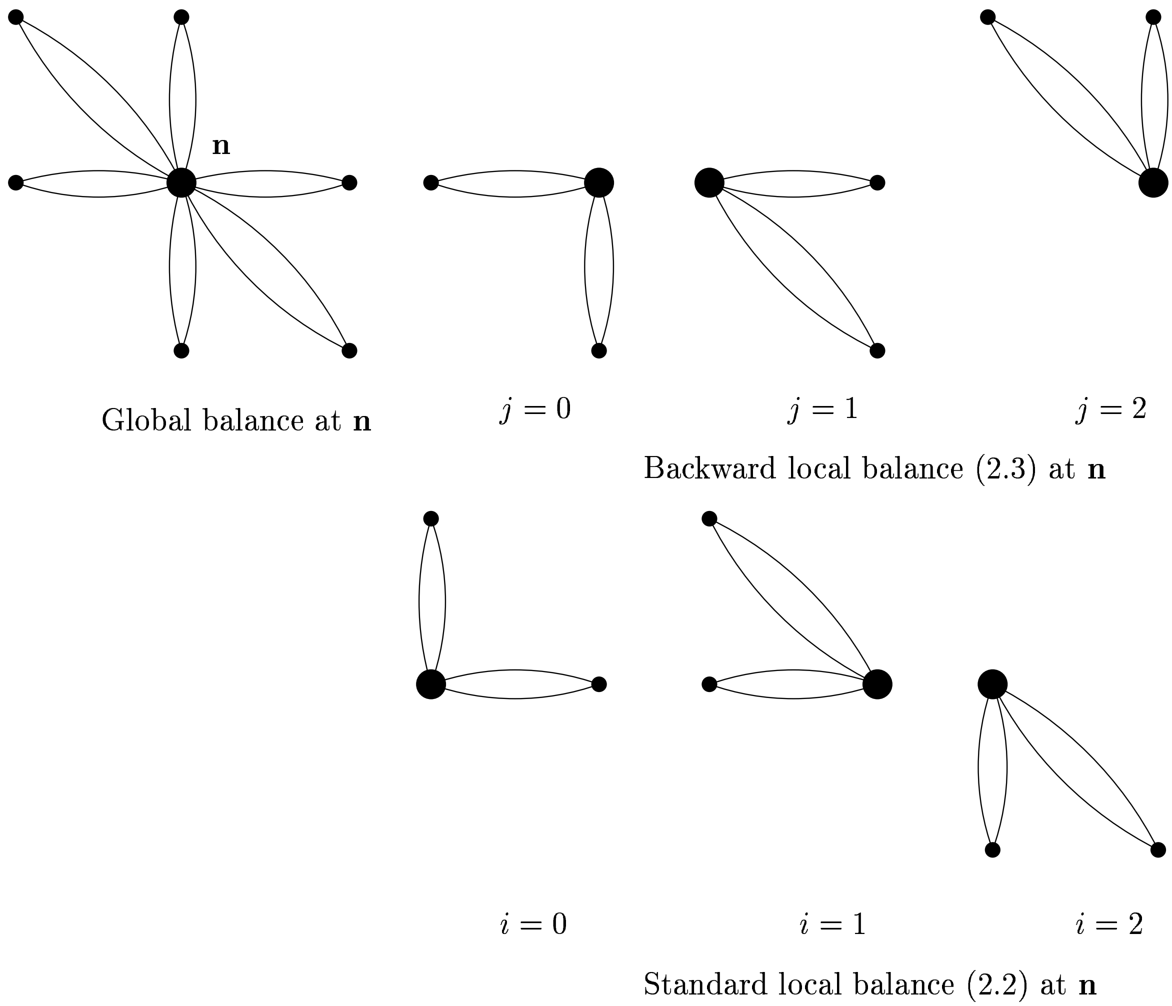

Figure 1: Decomposition of global balance

Consider the two queue system of figure 1. When we introduce a capacity constraint, say $n_2 \leq N_2$, this results in an upper bound for the state space. As a consequence, for $\mathbf{n}$ such that $n_2 = N_2$ transitions $\mathbf{n} \leftrightarrow \mathbf{n} + e_2$ and $\mathbf{n} \leftrightarrow \mathbf{n} - e_1 + e_2$ cannot occur. This corresponds to stopping of local balance for $j = 2$, but local balance for $j = 0$ and $j = 1$ is not affected by this bound. Therefore, if $\{d_i\}_{i=1}^2$ is a solution for (2.8), then $\{d_i\}_{i=1}^2$ is a solution for (2.3) at the upper bound too. The same holds true for a queueing network of N queues:

> *If $\{d_i\}_{i=1}^N$ is a solution for (2.8) then $\{d_i\}_{i=1}^N$ is also a solution for (2.3) at an upper bound without any restrictions on the transition rates.*

When we introduce the same upper bound $n_2 \leq N_2$ for local balance (2.2), from figure 1 we see that local balance for $i = 2$ is not affected, but that the upwards transitions for $i = 0$, and $i = 1$ cannot occur. Therefore, for the solution $\{c_i\}_{i=1}^2$ for (2.9) to be a solution for (2.2) we must *stop* local balance for $i = 0$ and $i = 1$ at the boundary $n_2 = N_2$.

> *This implies that local balance (2.3) is more suited for queueing networks with capacity constraints.*

What we have gained at the upper bounds by using (2.3) instead of (2.2) is lost at the lower bounds. For queueing networks for which the equilibrium distribution satisfies (2.2) it is well-known that a lower bound corresponding to zero customers at the queues can be introduced without any restrictions on the process. This lower bound is a natural restriction on the network. For queueing networks for which the equilibrium distribution is found as a solution for (2.3) we have to introduce a blocking protocol to preserve the solution to (2.8). This can immediately be seen from figure 1. If $n_2 = 0$ then transitions $\mathbf{n} \leftrightarrow \mathbf{n} - e_2$ and $\mathbf{n} \leftrightarrow \mathbf{n} + e_1 - e_2$ can no longer occur. Therefore local balance for $j = 0$ and $j = 1$ must be stopped, whereas local balance for $j = 2$ is not affected. This blocking protocol at the boundary $n_2 = 0$ can easily be formulated for queueing networks with N stations and states:

> *If a queue is empty only those transitions in which a customer moves to the empty queue are allowed.*

The blocking protocol corresponds to a service system in which when a queue is empty only a smart customer that selects the empty queue for its next service will be served, but can also be interpreted as a protocol that applies a load balancing procedure, balancing the load over the queues of the network.

From the above discussion we obtain that we have two blocking protocols that preserve product form. The first protocol (communication blocking) changes the behaviour of the process at upper boundaries, and the second protocol (smart customer blocking) changes the behaviour of the process at lower boundaries. When a process stays close to a lower boundary with high probability (light traffic) the communication blocking protocol may be used, and we may use smart customer blocking when the process stays away from its lower boundaries with high probability (heavy traffic). In section 3.4 below we consider the case in which the process stays away from its lower boundaries with probability 1.

3.3 General traffic equations

The result of section 2 can easily be generalised to general state-dependent solutions for the backward state dependent traffic equations (2.5). This gives the analog of the general state dependent traffic equations for queueing networks based on (2.2), as can for example be seen from Theorem 5.1 on p. 137 of Van Dijk [4]. The proof of the theorem is similar to the proof of Theorem 2.1 and is therefore omitted.

Theorem 3.1 *Assume that a positive function $H(\mathbf{n})$ exists that satisfies*

$$\sum_{i=0}^{N} \{H(\mathbf{n})p_{ij}(\mathbf{n}+\mathbf{e}_j) - H(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)p_{ji}(\mathbf{n}+\mathbf{e}_j)\} = 0, \tag{3.1}$$

for all $\mathbf{n} \in S$, $j = 0, \dots, N$ for which $\phi(\mathbf{n}+\mathbf{e}_j) > 0$. Then $\mathbf{X}$ has a unique equilibrium distribution

$$\pi(\mathbf{n}) = B\psi(\mathbf{n})H(\mathbf{n}), \quad \mathbf{n} \in S,$$

where $B^{-1} = \sum_{\mathbf{n}\in S} \psi(\mathbf{n})H(\mathbf{n}) < \infty$, and π satisfies backward local balance (2.3).

The above theorem generalises Theorem 2.1 in two ways. Firstly, the solution for the traffic equations is a state-dependent function. Secondly, (3.1) is required only if $\phi(\mathbf{n}+\mathbf{e}_j) > 0$. Observe that the above theorem establishes a decomposition of the equilibrium distribution in a service part $\psi(\cdot)$ and a routing part $H(\cdot)$, and that these parts are linked via the normalisation constant B only.

3.4 Heavily loaded networks

When we introduce a capacity constraint at each of the stations of a closed queueing network, in general product forms cannot be derived. As will be shown below, in lightly and heavily loaded systems product forms can still be obtained.

Let $S = \{\mathbf{n} : \ 0 \le n_i \le M_i < \infty\}$, that is assume that at most M_i customers are allowed at queue i, $i = 1, \dots, N$. Let M be the number of customers in the network and assume that all stations are single server queues. The transition rates for this network are

$$q(\mathbf{n}, \mathbf{n}-\mathbf{e}_i+\mathbf{e}_j) = \mu_i p_{ij} \mathbf{1}(n_i > 0, \ n_j < M_j).$$

If $M \leq \min_i\{M_i\}$, then blocking of customers does not occur, and the equilibrium distribution is of product form:

$$\pi(\mathbf{n}) = B \prod_{i=1}^{N} (c_i/\mu_i)^{n_i}, \quad \mathbf{n} \in S, \quad (M \leq \min_i\{M_i\}),$$

where $\{c_i\}_{i=1}^N$ is a solution of the standard traffic equations (2.9), a result that can be concluded from the standard local balance equations (2.2).

Now assume that $M \geq \sum_{i=1}^N M_i - \min_i\{M_i\}$, that is assume that the network is heavily loaded such that no queue can empty. From section 3.2 we obtain that the backward local balance equations do not impose restrictions on the process. Let $\{d_i\}_{i=1}^N$ be a solution of the backward traffic equations (2.8) for the process with routing function $\mu_i p_{ij}$:

$$\sum_{i=1}^{N} \{\mu_i p_{ij} d_j - \mu_j p_{ji} d_i\} = 0, \quad j = 1, \ldots, N. \tag{3.2}$$

Then

$$\pi(\mathbf{n}) = B \prod_{i=1}^{N} (1/d_i)^{n_i}, \quad \mathbf{n} \in S, \quad (M \geq \sum_{i=1}^{N} M_i - \min_i\{M_i\}),$$

is the equilibrium distribution at S, which can be concluded from Theorem 2.1 and backward local balance (2.3).

Observe that $\pi(\mathbf{n}) = B \prod_{i=1}^N (1/d_i)^{n_i}$ does *not* satisfy (2.2). Due to blocking of customers at the stations a product form cannot be obtained from standard product form theory, but can easily be concluded from Theorem 2.1.

When the network is cyclic we have $p_{i,i+1} = 1$, and $d_i = 1/\mu_{i-1}$, $i = 2, \ldots, N$, $d_1 = 1/\mu_N$, is the solution of (3.2). The equilibrium distribution is $\pi(\mathbf{n}) = B \prod_{i=1}^N \mu_i^{n_{i+1}}$, where $n_{N+1} \stackrel{\text{def}}{=} n_1$. This result is obtained by Gordon and Newell [6] via job-hole duality arguments. Therefore, Theorem 2.1 generalises the results of Gordon and Newell [6] to queueing networks with arbitrary routing.

4. Discussion

This paper has presented a new form of local balance for queueing networks and the corresponding product form results. These product form results have a structure closely related to standard product form results. A major difference is the behaviour at boundaries of the state space. As is shown, the product form queueing networks introduced in this paper allow us to introduce capacity constraints at the queues without conditions on the transition rates. Smart customer blocking is introduced to preserve local balance at lower boundaries of the state space.

It is shown that product form results for heavily loaded queueing networks can be obtained from backward local balance. These results extend the product form results obtained by Gordon and Newell [6] for heavily loaded cyclic networks to networks with arbitrary topology under heavy load.

The product form results introduced here may allow adequate approximations for queueing networks in heavy traffic, as can be seen from the smart customer blocking protocol at lower boundaries and the results related to job-hole duality. Additional research is necessary to investigate this possibility in detail. Furthermore, product form results for queueing networks and approximation schemes are available based on standard local balance (2.2). Similar results seem to be possible based on backward local balance (2.3).

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