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A structural characterisation of product form stochastic Petri nets

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Abstract

Product form results for the equilibrium distribution of stochastic Petri nets are available in the literature. These results are based on assumptions for the Markov chain describing the Petri net, and not on the structure of the Petri net. The structure of the Petri net is one of the most important parts in the analysis of Petri nets, and many results on this structure are available in the literature. Hence, it seems natural to characterise the product form property on a structural level. This paper provides such a characterisation: it gives a necessary and sufficient condition for the existence of a solution for the traffic equations (the basic equations allowing product form), completely in terms of the T-invariants of the Petri net.
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1. Introduction

Performance is an important issue in the design and implementation of real life systems such as computer systems, telecommunication networks, and flexible manufacturing systems. In many theoretical and practical studies of performance models involving stochastic effects, the statistical distribution of items over places is of great interest since most of the performance measures such as throughput and utilization can be derived from this distribution. If we are interested in quantitative results we can use approximation and simulation techniques. Analytical results, however, yield vital insight into the qualitative behaviour of the system. In particular, qualitative results related to the structure of the system are of great importance.

For queueing networks an important analytical result is the *product form equilibrium distribution* for the number of customers at the stations. Product form distributions were found by Jackson [17], and are nowadays known for a fairly wide class of queueing models (e.g., Baskett *et al.* [2], Boucherie and van Dijk [4], Henderson and Taylor [15], Serfozo [23]). The obvious advantage of these product form distributions is their simplicity which make them easy to use for computational issues as well as for theoretical reflections on performance models involving congestion as a consequence of queueing.

Recently, product form results were found for the marking process of stochastic Petri nets by Lazar and Robertazzi [18]. Although these results were shown for a very special class of stochastic Petri nets consisting only of linear task sequences, the notion of competition over resources incorporated in these models cannot be included in queueing networks without the introduction of state-dependent routing. Still, product form results very similar to those obtained by Jackson [17] were found. Since these first product form results various extensions have been found. In a number of papers, Henderson *et al.* [13], [14], [16] derive product form results for stochastic Petri nets similar to those obtained for batch routing queueing networks (Boucherie and van Dijk [4], Henderson and Taylor [15]). Frosch [11], [12] derived product form results for closed synchronized systems of stochastic sequential processes, a class of Petri nets in which state machines are synchronized via buffers.

The product form results for stochastic Petri nets are based on the assumption that a positive solution exists for a linear set of equations similar to the traffic equations for queueing networks. However, a characterisation of this assumption based on the structure of the Petri net is not available in the literature. This paper provides such a characterisation. We show that a necessary and sufficient condition for the existence of a positive solution for the traffic equations is that all transitions of the Petri net are covered by closed support T-invariants. A T-invariant is a closed support T-invariant if the firing sequence is a linear chain of transitions, that is a closed support T-invariant closely resembles the 'task sequences' used by Lazar and Robertazzi [18] to prove their product form result. As will be shown via examples, the class of Petri nets used in the present paper is substantially larger than the class of Lazar and Robertazzi.

Product form results for stochastic Petri nets of a completely different type are derived by Boucherie [3]. There the equilibrium distribution for a stochastic Petri net containing several subnets linked via buffer places is shown to be a product over the subnets under some conditions. Also, closed form expressions for the equilibrium distribution of stochastic Petri nets are derived by Florin and Natkin [9]. In that paper the equilibrium distribution of a stochastic Petri net with finite reachability set is shown to be a sum of product form distributions. The number of product form distributions in this sum is related to the number of distinct markings of the Petri net, a number that is usually substantially smaller than the cardinality of the reachability set. We do not consider these types of closed form equilibrium distributions in this paper.

In section 2 we present the basic Petri net notation. In section 3 we present the structural characterisation of the Petri net allowing us to provide necessary and sufficient conditions for the existence of a solution for the traffic equations. We will also give some known product form theorems based on the existence of such a solution. This allows us to illustrate the results by means of some simple examples in section 4.

2. Model

This section presents the basic definitions of stochastic Petri nets. For additional results and definitions, see the recent survey of Murata [21]. The specific assumptions and definitions needed to obtain product forms for stochastic Petri nets will be given in section 3.

Definition 2.1 (Marked stochastic Petri net) *A marked stochastic Petri net is a 6-tuple*

$$\mathcal{SPN} = (P, T, I, O, R, \mathbf{m}_0),$$

where $P = \{p_1, \ldots, p_N\}$ *is a finite set of places;* $T = \{t_1, \ldots, t_M\}$ *is a finite set of transitions;* $P \cap T = \emptyset$ *and* $P \cup T \neq \emptyset$; $I, O : P \times T \to \mathbb{N}_0$ *are the input and output functions identifying the relation between the places and the transitions;* $R = (r(t_1), \ldots, r(t_M))$ *is a set of firing rates drawn from exponential distributions; and* $\mathbf{m}_0$ *is the initial marking.*

A *marking* $\mathbf{m} = (\mathbf{m}(n),\ n = 1, \ldots, N)$ of a Petri net is a vector in $\mathbb{N}_0^N$, where $\mathbf{m}(n)$ represents the number of tokens at place p_n, $n = 1, \ldots, N$.

Distributions associated with different transitions are independent, and each transition of the Petri net is due to exactly one transition $t \in T$ that fires. The *execution policy* of the stochastic Petri net is the *race model with age memory* (cf. Ajmone Marsan *et al.* [1]).

From $I(\cdot,\cdot)$ and $O(\cdot,\cdot)$ we obtain the vectors $\mathbf{I}(t) = (I_1(t), \ldots, I_N(t))$, and $\mathbf{O}(t) = (O_1(t), \ldots, O_N(t))$, where $I_i(t) = I(p_i, t)$, and $O_i(t) = O(p_i, t)$. The vectors $\mathbf{I}(t)$, and $\mathbf{O}(t)$ are called *input, and output bags* of transition $t \in T$, representing the number of tokens needed at the places to fire transition t, and the number of tokens released to the places after firing of transition t. Furthermore, define the sets of places corresponding to input and output bags of transitions as ${}^{\bullet}t = \{p \in P | I(p,t) > 0\}$, the set of places giving input to transition t, $t^{\bullet} = \{p \in P | O(p,t) > 0\}$, the set of places receiving output from transition t. If transition t is *enabled* in marking $\mathbf{m}$ and fires, then the next state of the Petri net is $\mathbf{m}' = \mathbf{m} - \mathbf{I}(t) + \mathbf{O}(t)$. Symbolically this will be denoted as $\mathbf{m}[t > \mathbf{m}'$. A necessary and sufficient condition for t to be enabled is that $\mathbf{m}(n) \geq I_n(t)$, $n = 1, \ldots, N$.

A finite sequence of transitions $\sigma = t_{\sigma_1} t_{\sigma_2} \cdots t_{\sigma_k}$ is a finite *firing sequence* of the Petri net if there exists a sequence of markings $\mathbf{m}_{\sigma_1}, \ldots, \mathbf{m}_{\sigma_k}$ for which $\mathbf{m}_{\sigma_i}[t_{\sigma_i} > \mathbf{m}_{\sigma_{i+1}}$,

$i = 1, \ldots, k-1$. In this case marking $\mathbf{m}_{\sigma_k}$ is *reachable* from marking $\mathbf{m}_{\sigma_1}$ by firing σ, denoted as $\mathbf{m}_{\sigma_1}[\sigma > \mathbf{m}_{\sigma_k}$. The *reachability set* $\mathcal{M}(\mathbf{m}_0)$ is a subset of $\mathbb{N}_0^N$ and gives all possible markings of the Petri net with initial marking $\mathbf{m}_0$.

The *incidence matrix* is the $N \times M$ matrix A with entries $A(i,t) = O_i(t) - I_i(t)$ describing the change in the number of tokens in place p_i if transition t fires, $i = 1, \ldots, N$, $t \in T$. A vector $\bar{\sigma}$ is the *firing count vector* of the firing sequence σ if $\bar{\sigma}(t)$ equals the number of times transition t occurs in the firing sequence σ. If $\mathbf{m}_0[\sigma > \mathbf{m}$, then $\mathbf{m} = \mathbf{m}_0 + A\bar{\sigma}$, an equation referred to as the *state equation* for the Petri net.

A vector $\mathbf{x} \in \mathbb{N}_0^M$ is a *T-invariant* if $\mathbf{x} \neq 0$, and $A\mathbf{x} = 0$. From the state equation we obtain that a T-invariant corresponds to a firing sequence that brings a marking back to itself (Murata [21]). The *support* of a T-invariant $\mathbf{x}$ is the set of transitions corresponding to non-zero entries of $\mathbf{x}$, and is denoted by $\|\mathbf{x}\|$, i.e. $\|\mathbf{x}\| = \{t \in T | \mathbf{x}(t) > 0\}$. A T-invariant $\mathbf{x}$ is a *minimal T-invariant* if there is no other T-invariant $\mathbf{x}'$ such that $\mathbf{x}'(m) \leq \mathbf{x}(m)$ for all m. A support is minimal if no proper nonempty subset of the support is also a support of a T-invariant. From Memmi and Roucairol [19] we obtain that there is a unique minimal T-invariant corresponding to a minimal support (*minimal support T-invariant*), and any T-invariant can be written as a linear combination of minimal support T-invariants. A vector $\mathbf{y} \in \mathbb{N}_0^N$ is a *P-invariant* (sometimes called S-invariant) if $\mathbf{y} \neq 0$, and $\mathbf{y}A = 0$. P-invariants correspond to conservation of tokens in subsets of places. For example, the set of places of a Petri net corresponding to a closed Jackson network is a P-invariant. Definitions of and results for minimal support etc. are analogous to those for T-invariants.

The stochastic process describing the evolution of the Petri net is a continuous-time Markov chain with state space isomorphic to the reachability set, that is with state space $\mathcal{M}(\mathbf{m}_0)$ (Molloy [20]). The transition rates of this Markov chain are denoted by $Q = (q(\mathbf{m}, \mathbf{m}'),\ \mathbf{m}, \mathbf{m}' \in \mathcal{M}(\mathbf{m}_0))$. A collection of positive numbers, $m = (m(\mathbf{m}),\ \mathbf{m} \in \mathcal{M}(\mathbf{m}_0))$, is called an *invariant measure* if it satisfies the *global balance equations*,

$$\sum_{\mathbf{m}' \in \mathcal{M}(\mathbf{m}_0)} \{m(\mathbf{m})q(\mathbf{m}, \mathbf{m}') - m(\mathbf{m}')q(\mathbf{m}', \mathbf{m})\} = 0, \quad \mathbf{m} \in \mathcal{M}(\mathbf{m}_0).$$

When m is a proper distribution over $\mathcal{M}(\mathbf{m}_0)$ it will be called an *equilibrium distribution*, and will be denoted by $\pi = (\pi(\mathbf{m}),\ \mathbf{m} \in \mathcal{M}(\mathbf{m}_0))$.

As the Markov chain is chosen such that it describes the evolution of the stochastic Petri net under consideration, irreducibility and positive recurrence properties necessary to obtain a unique equilibrium distribution for the Markov chain should be characterised directly from the Petri net structure. A Petri net is *live* if, no matter what marking has been reached from $\mathbf{m}_0$, it is possible to ultimately fire any transition of the net by progressing through some further sequence. For unicity of the equilibrium distribution we must add the assumption that the Petri net is (strongly) connected. An extensive discussion of liveness, and related concepts is given in Murata [21].

3. Product form results

Without loss of generality, we may assume that the firing rate associated with transition $t \in T$ with input bag $\mathbf{I}(t)$ and output bag $\mathbf{O}(t)$ can be written as $r(t) = \mu(t)p(\mathbf{I}(t), \mathbf{O}(t))$, a form chosen in accordance with the literature on product form results (e.g., Jackson [17], Baskett *et al.* [2]).

Assume that the stochastic Petri net can be represented by a stable and regular, continuous-time Markov chain $\mathbf{X} = \{X(t),\ t \geq 0\}$ at state space $\mathcal{M}(\mathbf{m}_0)$. Then the transition rates of $\mathbf{X}$ are

$$q(\mathbf{I}(t), \mathbf{O}(t); \mathbf{m} - \mathbf{I}(t)) = \mu(t)p(\mathbf{I}(t), \mathbf{O}(t)), \tag{3.1}$$

for all $t \in T$, $\mathbf{m} \in \mathcal{M}(\mathbf{m}_0)$ such that $\mathbf{m} - \mathbf{I}(t) \in \mathbb{N}_0^N$. Here $q(\mathbf{I}(t), \mathbf{O}(t); \mathbf{m} - \mathbf{I}(t))$ is the transition rate associated with transition t bringing $\mathbf{m}$ to $\mathbf{m} - \mathbf{I}(t) + \mathbf{O}(t)$. The total transition rate from $\mathbf{m}$ to $\mathbf{m}' = \mathbf{m} - \mathbf{I}(t) + \mathbf{O}(t)$ is $q(\mathbf{m}, \mathbf{m}') = \sum_{\{\mathbf{n},\ t\in T:\mathbf{n}+\mathbf{I}(t)=\mathbf{m},\ \mathbf{n}+\mathbf{O}(t)=\mathbf{m}'\}} q(\mathbf{I}(t), \mathbf{O}(t); \mathbf{n})$.

Let $\mathbf{x}^1, \ldots, \mathbf{x}^h$ denote the minimal support T-invariants found from the incidence matrix. The following definition and assumption are essential to the analysis presented in this paper. Closedness of T-invariants was first defined by Donatelli and Sereno [8] as a unifying principle to obtain product form distributions for stochastic Petri nets. A necessary condition for a product form equilibrium distribution similar to closedness is presented in Henderson *et al.* [13], Corollary 1.

Definition 3.1 (Closed set) *For $\mathcal{T} \subset T$ define $\mathcal{R}(\mathcal{T})$, the set of input and output bags for the transitions in $\mathcal{T}$, as*

$$\mathcal{R}(\mathcal{T}) = \bigcup_{t\in\mathcal{T}} \{\mathbf{I}(t) \cup \mathbf{O}(t)\}.$$

$\mathcal{T}$ is a closed set if for any $\mathbf{g} \in \mathcal{R}(\mathcal{T})$ there exist $t,\ t' \in \mathcal{T}$ such that $\mathbf{g} = \mathbf{I}(t)$, as well as $\mathbf{g} = \mathbf{O}(t')$, that is if each output bag is also an input bag for a transition in $\mathcal{T}$.

Assumption 3.2 (Minimal closed support T-invariants) *Assume that all transitions $t \in T$ are covered by minimal closed support T-invariants, that is assume that for all $t \in T$ there exists an $i \in \{1, \ldots, h\}$ such that $t \in \|\mathbf{x}^i\|$ and $\|\mathbf{x}^i\|$ is a closed set.*

Observe that the essential part of the assumption is that all transitions are contained in a *closed* support. The assumption that all transitions are covered by minimal support T-invariants (closed or not closed) is a natural assumption if we are interested in the equilibrium or stationary distribution of a stochastic Petri net. If this assumption is not satisfied, then there exists a transition, say t_0, that is enabled in a reachable marking $\mathbf{m}$, and $t_0 \notin \bigcup_{i=1}^h \|\mathbf{x}^i\|$ (if t_0 is never enabled, then we can delete t_0 from T). Let t_0 fire in marking $\mathbf{m}$. Then there exists no firing sequence from $\mathbf{m} - \mathbf{I}(t_0) + \mathbf{O}(t_0)$

back to $\mathbf{m}$ (otherwise t_0 would be contained in a T-invariant). Thus $\mathbf{m}$ is a transient state and does not appear in the equilibrium description of the stochastic Petri net. As a consequence, both $\mathbf{m}$ and t_0 can be deleted from the equilibrium description of the Petri net.

The structural characterisation of product form results for stochastic Petri nets is completely based on Assumption 3.2. We now proceed with a characterisation of minimal *closed* support T-invariants. This shows the relation between minimal closed support T-invariants and 'task sequences' (corresponding to a number of tasks that must be executed consecutively) as introduced by Lazar and Robertazzi [18]. This turns out to be a key-notion when product form equilibrium distributions are desired.

Theorem 3.3 *Assume that* $\mathbf{x}$ *is a minimal closed support* T*-invariant. Then the firing sequence of* $\mathbf{x}$ *is 'linear', that is for each* $t \in \|\mathbf{x}\|$ *there is a unique* $t' \in \|\mathbf{x}\|$ *such that* $\mathbf{O}(t) = \mathbf{I}(t')$*. As a consequence* $x_i \leq 1$, $i = 1, \ldots, M$*. Conversely, if the firing sequence of a* T*-invariant* $\mathbf{x}$ *is linear, then* $\mathbf{x}$ *is a closed support* T*-invariant.*

Proof Let $t \in \|\mathbf{x}\|$. The existence of $t' \in \|\mathbf{x}\|$ such that $\mathbf{O}(t) = \mathbf{I}(t')$ follows from the closedness of $\|\mathbf{x}\|$. To proof the unicity, let $t \in \|\mathbf{x}\|$, and $t', t'' \in \|\mathbf{x}\|$ such that $\mathbf{O}(t) = \mathbf{I}(t') = \mathbf{I}(t'')$. Without loss of generality, assume that $\mathbf{O}(t') \neq \mathbf{O}(t'')$ (otherwise $t' = t''$). As a consequence there exists a place p such that $p \in \{(t')^\bullet \cup (t'')^\bullet\} \setminus \{(t')^\bullet \cap (t'')^\bullet\}$. Without loss of generality, assume that $p \in (t')^\bullet$.

From the closedness of $\|\mathbf{x}\|$ we obtain that there exist two *distinct* transitions, say $t_1' \in \|\mathbf{x}\|$, and $t_1'' \in \|\mathbf{x}\|$ such that $\mathbf{O}(t') = \mathbf{I}(t_1')$, and $\mathbf{O}(t'') = \mathbf{I}(t_1'')$, and we must have one of the following three situations:

- **(a)** $\mathbf{O}(t_1') = \mathbf{O}(t_1'')$, but this implies that there exist two firing sequences within $\|\mathbf{x}\|$ that can fire independently from $\mathbf{I}(t)$ to $\mathbf{O}(t_1')$, in contrast with the assumption that $\mathbf{x}$ is a minimal T-invariant.
- **(b)** $\exists\, p' \in \{(t_1')^\bullet \cup (t_1'')^\bullet\} \setminus \{(t_1')^\bullet \cap (t_1'')^\bullet\}$, and $p' \in (t_1')^\bullet$. This is the situation observed when we considered t' and t'' and is either followed by situation (a), (b), or (c).
- **(c)** as (b), but now $p' \in (t_1'')^\bullet$. It is obvious that this is followed by (a), (b), or (c) too.

Finally, since $\mathbf{x}$ is a T-invariant, it must be that the firing sequences starting with t' and t'', say $t't_1' \cdots t_{\alpha'}'$ and $t''t_1'' \cdots t_{\alpha''}''$, are such that $\mathbf{O}(t_{\alpha'}') = \mathbf{O}(t_{\alpha''}'')$ for some α', α'', that is situation (a) must occur finally, which contradicts the assumption that $\mathbf{x}$ is a minimal T-invariant, because we have created two firing sequences that can independently be fired from $\mathbf{I}(t)$ to $\mathbf{O}(t_{\alpha'}')$. This establishes unicity.

Unicity implies that each transition $t \in \mathbf{x}$ can occur at most once in the firing sequence associated with $\mathbf{x}$, i.e. that $x_i \leq 1$, $i = 1, \ldots, M$.

If the firing sequence of a T-invariant $\mathbf{x}$ is linear, then for each $t \in \|\mathbf{x}\|$ there exist $s, s' \in \|\mathbf{x}\|$ such that $\mathbf{O}(s) = \mathbf{I}(t)$, $\mathbf{O}(t) = \mathbf{I}(s')$ implying that $\mathbf{x}$ has closed support. □

The important property of closed support T-invariants with respect to product form results is that the residual marking of tokens that remain at the places during one complete firing of the T-invariant is the same for all transitions, that is the firing sequence can be represented by the sequence of markings $\mathbf{m} = \mathbf{n}+I(t_{i_1}) \to \mathbf{n}+I(t_{i_2}) \to \cdots \to \mathbf{n}+I(t_{i_k}) \to \mathbf{n}+I(t_{i_1})$, with $\mathbf{n} \equiv \mathbf{m}-\mathbf{I}(t_{i_1})$ the residual marking. This observation is the basis of the classification of the transitions into *equivalence classes* as presented below. This classification is based on a classification presented in Frosch [10], Frosch and Natarajan [11] for cyclic state machines. In the case of cyclic state machines the input bag of a transition basically contains only one place, whereas the generalisation to closed support T-invariants incorporates more general input bags. The classification will then be used to construct a solution to the traffic equations, a set of linear equations defined by analogy with the traffic equations for queueing networks.

Definition 3.4 (Traffic equations) *For $t \in T$, an invariant measure, $y = (y(\mathbf{I}(t))$, $t \in T)$, for the traffic equations is a mapping $y : \mathbb{N}_0^N \to \mathbb{R}^+$ that satisfies the traffic equations for all $t \in T$ (recall the definition of the transition rates (3.1))*

$$\sum_{t' \in T} \{y(\mathbf{I}(t))\mu(t)p(\mathbf{I}(t), \mathbf{I}(t')) - y(\mathbf{I}(t'))\mu(t')p(\mathbf{I}(t'), \mathbf{I}(t))\} = 0. \tag{3.2}$$

Remark 3.5 (Traffic equations) The definition of the traffic equations relies heavily on the assumption that all transitions are covered by closed support T-invariants. Otherwise $p(\mathbf{I}(t), \mathbf{I}(t'))$ may be zero for all $t' \in T$ since without the assumption of closedness $\mathbf{O}(t)$ need not be an input bag for some transition t'. In fact, from Assumption 3.2 we obtain that for each t there exists a t' such that $\mathbf{O}(t) = \mathbf{I}(t')$, and the first summation in the traffic equations is equivalent to $\sum_{O(t) \in \mathbb{N}_0^N} y(\mathbf{I}(t))\mu(t)p(\mathbf{I}(t), \mathbf{O}(t))$. Obviously, the second summation is equivalent to $\sum_{I(t') \in \mathbb{N}_0^N : \mathbf{O}(t') = \mathbf{I}(t)} y(\mathbf{I}(t'))\mu(t')p(\mathbf{I}(t'), \mathbf{O}(t'))$, which shows that under Assumption 3.2 the traffic equations do not exclude any transitions depositing or consuming $\mathbf{I}(t)$. In particular, Assumption 3.2 implies that the traffic equations are equivalent to the global balance equations for the Markov chain with transition rates (3.1), a result used below to prove that Assumption 3.2 is necessary and sufficient for the existence of a solution for the traffic equations. □

We will now show that Assumption 3.2 is necessary and sufficient for the existence of an invariant measure for the traffic equations (3.2). Before proving this result we first characterise the minimal support T-invariants that are connected as (3.2) decomposes into disjoint sets of equations, one set of equations for each equivalence class of connected T-invariants.

Assume that the minimal support T-invariants $\mathbf{x}^1, \ldots, \mathbf{x}^h$ are numbered such that $ClT \stackrel{\text{def}}{=} \{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$ is the set of minimal closed support T-invariants ($k \leq h$).

Definition 3.6 (Common input bag relation) *Let* $\mathbf{x}$, $\mathbf{x}' \in ClT$. *We say that* $\mathbf{x}$, $\mathbf{x}'$ *are in common input bag relation (notation:* $\mathbf{x}\ CI\ \mathbf{x}'$*) if there exist* $t \in \|\mathbf{x}\|$, $t' \in \|\mathbf{x}'\|$ *such that* $\mathbf{I}(t) = \mathbf{I}(t')$. *The relation* CI^* *is the transitive closure of* CI.

The transitive closure of a relation is defined as follows: if $\mathbf{x}$, $\mathbf{x}'$, $\mathbf{x}'' \in ClT$, and $\mathbf{x}\ CI\ \mathbf{x}'$, $\mathbf{x}'\ CI\ \mathbf{x}''$, then we define $\mathbf{x}\ CI^*\ \mathbf{x}'$, $\mathbf{x}'\ CI^*\ \mathbf{x}''$, *and* $\mathbf{x}\ CI^*\ \mathbf{x}''$. This reflects the property that we can go from $\mathbf{x}$ to $\mathbf{x}''$ via $\mathbf{x}'$. This makes the common input bag relation CI^* an equivalence relation on ClT.

The common input bag relation characterises the irreducible sets of the Markov chain $\mathbf{Y} = (Y(t),\ t \geq 0)$ at finite state space $S = \{\mathbf{I}(t),\ t \in T\}$ with transition rates $q(\mathbf{I}(t), \mathbf{I}(t')) = \mu(t)p(\mathbf{I}(t), \mathbf{I}(t'))$. This Markov chain $\mathbf{Y}$ corresponds to the routing chain as defined in Henderson *et al.* [13], [16]. Let $CI(\mathbf{x})$ be the equivalence class of $\mathbf{x} \in ClT$, that is $CI(\mathbf{x}) = \{\mathbf{x}' | \mathbf{x}\ CI^*\ \mathbf{x}'\}$. The equivalence classes partition ClT: each $\mathbf{x} \in ClT$ belongs to exactly one equivalence class.

Let $\mathbf{x} \in ClT$ with equivalence class $CI(\mathbf{x})$. Define $S(\mathbf{x}) \subset S$, the input bags corresponding to $CI(\mathbf{x})$, as

$$S(\mathbf{x}) = \{\mathbf{I}(t) | \exists\ \mathbf{x}' \in CI(\mathbf{x}) \text{ such that } x'_t > 0\}.$$

The following theorem shows that the partition of ClT into equivalence classes $\{CI(\mathbf{x})\}_{\mathbf{x} \in ClT}$ induces a partition $\{S(\mathbf{x})\}_{\mathbf{x} \in ClT}$ of S into irreducible sets of the Markov chain $\mathbf{Y}$ if and only if Assumption 3.2 is satisfied.

Theorem 3.7 (Structural characterisation) *Assumption 3.2 is necessary and sufficient for the existence of an invariant measure for the traffic equations (3.2).*

Proof Observe that the state-independent traffic equations (3.2) are the global balance equations of $\mathbf{Y}$ at state space S. Therefore it is sufficient to prove that Assumption 3.2 is necessary and sufficient for the partition of S into irreducible sets $\{S(\mathbf{x})\}_{\mathbf{x} \in ClT}$.

Let $\mathbf{x}, \mathbf{x}' \in ClT$. If $\mathbf{x}' \in CI(\mathbf{x})$ then $S(\mathbf{x}') = S(\mathbf{x})$, since $CI(\mathbf{x}) = CI(\mathbf{x}')$. If $S(\mathbf{x}') \cap S(\mathbf{x}) \neq \emptyset$, then $\exists\ t \in T$ such that $\mathbf{I}(t) \in S(\mathbf{x}') \cap S(\mathbf{x})$ implying that $\exists\ \mathbf{x}'' \in CI(\mathbf{x})$ for which $\exists\ s \in T$ such that $x''_s > 0$ and $\mathbf{I}(s) = \mathbf{I}(t)$, and $\exists\ \mathbf{x}''' \in CI(\mathbf{x}')$ for which $\exists\ s'$ such that $x'''_{s'} > 0$ and $\mathbf{I}(s') = \mathbf{I}(t)$. Thus $CI(\mathbf{x}'') = CI(\mathbf{x}''')$ implying $CI(\mathbf{x}) = CI(\mathbf{x}')$, in turn implying that $S(\mathbf{x}') = S(\mathbf{x})$. This shows that $S(\mathbf{x}') = S(\mathbf{x})$ if $CI(\mathbf{x}') = CI(\mathbf{x})$, and $S(\mathbf{x}') \cap S(\mathbf{x}) = \emptyset$ if $CI(\mathbf{x}') \cap CI(\mathbf{x}) = \emptyset$.

Assumption 3.2 implies that for all $t \in T$, $\exists\ \mathbf{x} \in ClT$ such that $t \in \|\mathbf{x}\|$, i.e. $\exists\ S(\mathbf{x})$ such that $\mathbf{I}(t) \in S(\mathbf{x})$. As a consequence $\{S(\mathbf{x})\}_{\mathbf{x} \in ClT}$ forms a partition of S.

Let $\mathbf{I}(t), \mathbf{I}(t') \in S(\mathbf{x})$. Then $\exists\ \mathbf{x}', \mathbf{x}'' \in CI(\mathbf{x})$ for which $\exists\ s, s' \in T$ such that $x'_s > 0$ and $x''_{s'} > 0$, and $\mathbf{I}(s) = \mathbf{I}(t)$ and $\mathbf{I}(s') = \mathbf{I}(t)$, but also $\mathbf{x}' CI^* \mathbf{x}''$. Thus $\exists\ \sigma$, firing-sequence, such that $\mathbf{I}(t)[\sigma > \mathbf{I}(t')$. Let $\mathbf{I}(t) \in S(\mathbf{x})$, $\mathbf{I}(t') \in S(\mathbf{x}')$, $S(\mathbf{x}) \cap S(\mathbf{x}') = \emptyset$. Assume $\exists\ \sigma$, firing sequence, such that $\mathbf{I}(t)[\sigma > \mathbf{I}(t')$ then $\mathbf{x}' \in CI(\mathbf{x})$ implying that

$S(\mathbf{x}) = S(\mathbf{x}')$. As a consequence $\{S(\mathbf{x})\}_{\mathbf{x} \in ClT}$ forms a partition of S into irreducible sets. The Perron-Frobenius theorem (cf. Seneta [22]) implies that a positive solution exists to the marking independent traffic equations.

Conversely, assume that an invariant measure exists to the marking independent traffic equations. This immediately implies that for all $t \in T \ \exists \ t' \in T$ such that $\mathbf{O}(t) = \mathbf{I}(t')$. Furthermore, the existence of this invariant measure implies that S is partitioned in irreducible sets. Let V_i, $i = 1, \ldots, v$, denote the irreducible sets of $\mathbf{Y}$. Let $t \in T$ and i_0 such that $\mathbf{I}(t) \in V_{i_0}$. Since V_{i_0} is an irreducible set we have that for all $\mathbf{v} \in V_{i_0} \ \exists \ \sigma, \sigma'$ such that $\mathbf{I}(t)[\sigma > \mathbf{v}$, and $\mathbf{v}[\sigma' > \mathbf{I}(t)$. Thus $\tilde{\sigma} = \sigma\sigma'$ is a closed support T-invariant. Similarly, from the irreducibility we may conclude that all T-invariants contained in V_{i_0} have closed support. From Memmi and Roucairol [19] we obtain that each support of an invariant can be decomposed into a union of minimal supports which implies that t is covered by a minimal closed support T-invariant. □

Remark 3.8 (Structural characterisation) In the literature, one usually assumes that a solution for the traffic equations exists, and necessary conditions are derived from this assumption (e.g., Henderson *et al.* [13]). Theorem 3.7 provides a *necessary and sufficient* structural condition for the existence of a solution of the traffic equations, only. We will now illustrate the difference between Assumption 3.2 and the conditions of Henderson *et al.* [13], [16] that are *necessary* for the existence of a solution for the traffic equations. This also shows that Assumption 3.2 is a *new condition* for the characterisation of product form results.

Henderson *et al.* [13] introduce the following necessary condition for the existence of a solution for the traffic equations (Corollary 1): *for all* $\mathbf{g} \in \mathcal{R}(T) = \bigcup_{t \in T} \{\mathbf{I}(t) \cup \mathbf{O}(t)\}$ *there exist* $t, s \in T$ *such that* $\mathbf{g} = \mathbf{I}(t)$, $\mathbf{g} = \mathbf{O}(s)$, *that is* $\mathcal{R}(T)$ *is a closed set.* Obviously, Assumption 3.2 implies this condition, since Assumption 3.2 not only assumes that such $t, s \in T$ exist, but also that t, s are elements of the support of a single minimal closed support T-invariant. The reversed statement is not true, as is shown in the following example taken from Coleman [6], where the example is given to illustrate that the condition of Corollary 1 from Henderson *et al.* [13] is not sufficient for the existence of a solution for the traffic equations.

Consider the Petri net depicted in Figure 1. From the incidence matrix

$$A = \begin{pmatrix} -1 & 0 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 & -1 \end{pmatrix}$$

we obtain that this net has 3 minimal support T-invariants: $\mathbf{x}^1 = (10010)$, $\mathbf{x}^2 = (00101)$, $\mathbf{x}^3 = (12001)$, of which $\mathbf{x}^1$ and $\mathbf{x}^2$ have closed support, but $\mathbf{x}^3$ does not have closed support. (This can be seen from Theorem 3.3, or from the definition of

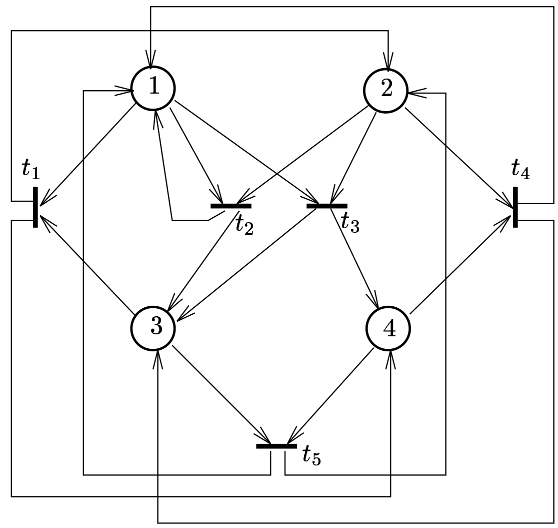

Figure 1: Petri net violating Assumption 3.2

closed sets.) Since transition t_2 is contained in $\|\mathbf{x}^3\|$ only, t_2 cannot be covered by a minimal closed support T-invariant, which contradicts Assumption 3.2. In contrast, the condition of Corollary 1 from Henderson *et al.* [13] is satisfied, also for transition t_2.

The state space of the routing chain is

$$S = \{\mathbf{I}(t_1), \mathbf{I}(t_2), \mathbf{I}(t_4), \mathbf{I}(t_5)\}, \quad (\mathbf{I}(t_2) = \mathbf{I}(t_3)),$$

and the solution of the traffic equations (3.2) is (up to a multiplicative constant)

$$y(\mathbf{I}(t_1)) = 1/\mu_1,\ y(\mathbf{I}(t_4)) = 1/\mu_4,\ y(\mathbf{I}(t_2)) = 0,\ y(\mathbf{I}(t_3)) = 0,\ y(\mathbf{I}(t_5)) = 0,$$

which shows that the condition of Corollary 1 from Henderson *et al.* [13] is not sufficient for the existence of a *positive* solution of the traffic equations. □

We are now able to present a first product form theorem for stochastic Petri nets. This theorem is formulated by analogy with similar results for batch routing queueing networks, and shows the similarity between stochastic Petri nets and batch routing queueing networks at the Markovian level.

Theorem 3.9 *Assume that an invariant measure y exists to the marking independent traffic equations (3.2), and a function $\pi_y : \mathcal{M}(\mathbf{m}_0) \to \mathbb{R}^+$ such that for all $\mathbf{n} + I(t) \in \mathcal{M}(\mathbf{m}_0)$, $t, s \in T$ with $p(\mathbf{I}(t), \mathbf{I}(s)) > 0$,*

$$\frac{\pi_y(\mathbf{n} + \mathbf{I}(t))}{\pi_y(\mathbf{n} + \mathbf{I}(s))} = \frac{y(\mathbf{I}(t))}{y(\mathbf{I}(s))}. \tag{3.3}$$

Then $\pi_y(\mathbf{m})$, $\mathbf{m} \in \mathcal{M}(\mathbf{m}_0)$, *is an invariant measure of the Markov chain* $\mathbf{Y}$ *describing the stochastic Petri net. If* $B^{-1} = \sum_{\mathbf{m}\in\mathcal{M}(\mathbf{m}_0)} \pi_y(\mathbf{m}) < \infty$, *then* $\pi(\mathbf{m}) = B\pi_y(\mathbf{m})$, $\mathbf{m} \in \mathcal{M}(\mathbf{m}_0)$, *is an equilibrium distribution of the Markov chain describing the stochastic Petri net.*

The proof of Theorem 3.9 can be found in the literature (Boucherie and van Dijk [4], Henderson and Taylor [16]). The key-idea of Theorem 3.9 is that the marking independent solution $y(\cdot)$ of the traffic equations is translated into a marking dependent solution with the same properties. This is reflected in Condition (3.3). This establishes the *product form* nature of the equilibrium distribution.

Note that Condition (3.3) is a condition on y and *not* on the structure of the Petri net. Furthermore, as is shown in section 4.2, if a solution $y(\cdot)$ of the traffic equations is found, a function $\pi_y(\cdot)$ satisfying (3.3) cannot always be found without additional assumptions on the Petri net. We will now provide a structural characterisation of the Petri net guaranteeing (3.3). The rank condition is taken from Coleman *et al.* [7]. The result of this characterisation is that condition (3.3) is satisfied with a function π_y that is a product over the places of the Petri net.

Theorem 3.10 *Assume that all transitions are covered by minimal closed support T-invariants. Then, with* y *the invariant measure for the traffic equations,* π_y *satisfying (3.3) has the form*

$$\pi_y(\mathbf{m}) = \prod_{i=1}^{N} c_i(y)^{\mathbf{m}(i)} \tag{3.4}$$

if and only if

$$Rank(A) = Rank([A|\mathbf{C}(y)]), \tag{3.5}$$

where $[A|\mathbf{C}(y)]$ *is the matrix* A *augmented with the row* $\mathbf{C}(y)$, *defined as*

$$\mathbf{C}(y)_j = \log\left[y(\mathbf{I}(t_j))/y(\mathbf{O}(t_j))\right], \quad j = 1, \ldots, M.$$

In this case the N*-vector* $\mathbf{c}(y) = (\log c_i(y),\ i = 1, \ldots, N)$ *satisfies the matrix equation*

$$\mathbf{c}(y)A + \mathbf{C}(y) = 0. \tag{3.6}$$

Observe that the solution y for the state-independent traffic equations is defined up to multiplicative factors at the irreducible sets of the routing chain $\mathbf{Y}$ at state space S only. This cannot give rise to problems in the above theorem, since we only use the ratios $y(\mathbf{I}(t))/y(\mathbf{I}(s))$, where $\mathbf{I}(t)$ and $\mathbf{I}(s)$ are in the same irreducible set of $\mathbf{Y}$, in the definition of $\mathbf{C}(y)$. This quotient is unique at each irreducible set, and therefore $\mathbf{C}(y)$ is uniquely determined.

Theorem 3.10 and its proof are taken from Coleman *et al.* [7]. This theorem characterises product forms for stochastic Petri nets based on the incidence matrix. The product form (3.4) is of the Jackson-type since it is a product over the places similar to the result of Jackson [17]. Note that the Petri nets are substantially more complex than Jackson networks.

Observe that Theorem 3.10 states that a product form solution (3.4) exists if and only if the invariant measure $y(\cdot)$ for the traffic equations is such that $\mathbf{C}(y)$ is orthogonal to the right null space of A containing all T-invariants. The product form distribution (3.4) contains one term for each token in the Petri net. Therefore, the only dependence between tokens lies in the normalising constant.

Remark 3.11 (Generalisations) The results of this section can immediately be generalised to also include marking dependent firing rates

$$q(\mathbf{I}(t), \mathbf{O}(t); \mathbf{m} - \mathbf{I}(t)) = \mu(t)\frac{\psi(\mathbf{m} - \mathbf{I}(t))}{\phi(\mathbf{m})}p(\mathbf{I}(t), \mathbf{O}(t)),$$

where $\psi(\mathbf{m} - \mathbf{I}(t))/\phi(\mathbf{m})$ is the marking dependent firing rate. This does not affect the analysis as can be seen from the literature on batch routing queueing networks (cf. Boucherie and van Dijk [4]: $\mathbf{I}(t)$, $\mathbf{O}(t)$ and correspond to the batches of departing and arriving customers, $\mu(t)\psi(\mathbf{m} - \mathbf{I}(t))/\phi(\mathbf{m})$ is the service rate, and $p(\mathbf{I}(t), \mathbf{O}(t))$ is the routing probability for the customers in the batch). The equilibrium distribution becomes

$$\pi(\mathbf{m}) = B\phi(\mathbf{m})\pi_y(\mathbf{m}).$$

The inclusion of a marking dependent part in the firing rates allows for more general Petri nets. The structural analysis based on $p(\mathbf{I}(t), \mathbf{O}(t))$ is not affected, but some marking dependent properties can be modelled using ψ. Furthermore, $p(\mathbf{I}(t), \mathbf{O}(t))$ can be generalised to a marking dependent function $p(\mathbf{I}(t), \mathbf{O}(t); \mathbf{m} - \mathbf{I}(t))$, which allows us to introduce inhibitor arcs in the Petri net formalism. The Petri nets obtained via these two generalisations cannot be completely characterised at the structural level: some of the transitions that are enabled in the net with firing rates $\mu(t)p(\mathbf{I}(t), \mathbf{O}(t))$ can be excluded in a marking dependent way. Some results in this direction can be found in Boucherie and Sereno [5]. □

4. Examples

In this section we present some examples illustrating the structural characterisation presented above. First, in example 4.1 we present the product form results obtained by Lazar and Robertazzi [18]. In example 4.2 we present some examples of Petri nets

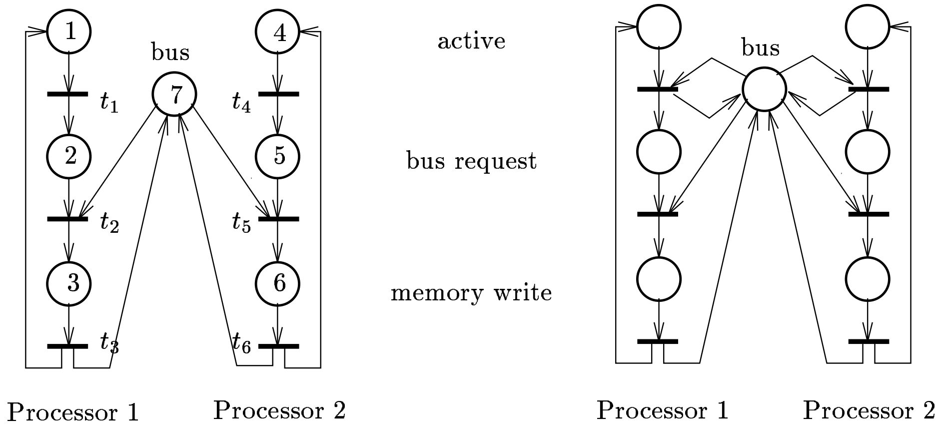

Figure 2: a: original Petri net model b: modified Petri net model

that are covered by closed support T-invariants, but with different behaviour: a net that always has a product form equilibrium distribution, a net that sometimes has such a distribution, and a net that does not have an equilibrium distribution at all. This shows that closed support T-invariants can be rather complex, and illustrates the theoretical results of section 3.

4.1 The dual processor system
The Petri nets discussed by Lazar and Robertazzi [18] are of the form presented here. We will illustrate the framework of Lazar and Robertazzi with an example.

Consider the dual processor system. It consists of two processors sharing a single memory. The processors may refer to the shared memory through a bus. A processor is allowed to work only if the bus is available (!), hence conflicts between the processors occur as only one of the processors may utilize the bus. The assumption that the processors are allowed to work only if the bus is available is necessary to obtain a product form equilibrium distribution, and is reflected in the assumption of Robertazzi and Lazar that a task sequence is only allowed to proceed if there is a non-zero probability that it can return to its current state without the need for a state change in other task sequences. The practical consequence of this assumption is that is arcs are added in the Petri net of Figure 2a representing the dual processor system without modifications. This results in the Petri net of Figure 2b representing the dual processor system in which processors can work only when the bus is available.

The Petri nets of Figure 2 have two T-invariants $\mathbf{x}^1 = (111000)$, $\mathbf{x}^2 = (000111)$. As a consequence of the extra arcs which are added because we have assumed that a processor is allowed to work only if the bus is available, both T-invariants for the Petri net of Figure 2b are minimal closed support T-invariants. Note that the Petri net without the extra arcs has the same two minimal support T-invariants. This is an immediate consequence of the fact that the extra arcs do not contribute to the incidence matrix A, which shows that Assumption 3.2 cannot be verified on the basis of the incidence matrix A only, but needs to be verified directly from the input and output functions $I(\cdot,\cdot)$ and $O(\cdot,\cdot)$.

The transition rates of the Petri net are of the form (3.1):

$$q(\mathbf{I}(t_i), \mathbf{O}(t_i); \mathbf{m} - \mathbf{I}(t_i)) = \mu(t_i),$$

for $i = 1, \ldots, 6$, such that $\mathbf{m} - \mathbf{I}(t_i) \in \mathbb{N}_0^N$. In Lazar and Robertazzi [18] initially one token is present at places 1, 4 and 7. The equilibrium distribution is

$$\pi(\mathbf{m}) = B \prod_{i=1}^{6} (1/\mu(t_i))^{m_i}, \quad \mathbf{m} \in \mathcal{M}(\mathbf{m}_0),$$

where the reachability set $\mathcal{M}(\mathbf{m}_0)$ is

$$\begin{aligned}\mathcal{M}(\mathbf{m}_0) = \mathcal{M}(1001001) = \{&(1001001), (0101001), (0011000), (1000101),\\ &(0100101), (0010100), (1000101), (1000101)\}.\end{aligned}$$

From Theorem 3.9 we obtain that except for the normalisation constant B, the equilibrium distribution has the same form if the assumption of safeness (at most one token in each place) made by Lazar and Robertazzi [18] is removed. The only difference is the reachability set $\mathcal{M}(\mathbf{m}_0)$. This result shows the power of the use of T-invariants in the analysis of Petri nets: the form of the equilibrium distribution is completely determined by the T-invariants, regardless of the shape of the reachability set.

4.2 Closed support T-invariants

This example considers three stochastic Petri nets that are covered by closed support T-invariants, but with completely different behaviour. The Petri net of Figure 3a has a product form equilibrium distribution, the net of Figure 3b has a product form equilibrium distribution for a specific choice of the firing rates (related to conflicting T-invariants), and the net of Figure 3c may not possess an equilibrium distribution (due to a possibly unbounded number of tokens).

Consider the Petri net depicted in Figure 3a. From the incidence matrix

$$A = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 2 & 1 & -2 & 2 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$

we obtain that this net has two minimal support T-invariants $\mathbf{x}^1 = (10100)$, $\mathbf{x}^2 = (01111)$, which are both minimal closed support T-invariants, and two minimal support

P-invariants $\mathbf{y}^1 = (11011)$, $\mathbf{y}^2 = (20112)$. Since the T-invariants share $\mathbf{I}(t_1)$ they are in common input bag relation, which implies that the routing chain has one irreducible set: $S = \{\mathbf{I}(t_1), \mathbf{I}(t_3), \mathbf{I}(t_4), \mathbf{I}(t_5)\}$ $(\mathbf{I}(t_1) = \mathbf{I}(t_2))$.

Denote $\mu(t_{12}) = \mu(t_1) + \mu(t_2)$, $b = \mu(t_2)/\mu(t_{12})$, the probability that transition t_2 fires before transition t_1 when transitions t_1 and t_2 are enabled. The solution of the traffic equations is (up to normalisation)

$$y(\mathbf{I}(t_1))\mu(t_{12}) = y(\mathbf{I}(t_3))\mu(t_3) = 1, \quad y(\mathbf{I}(t_4))\mu(t_4) = y(\mathbf{I}(t_5))\mu(t_5) = b.$$

The solution π_y to (3.3) is not immediately obvious from these relations, therefore we apply Theorem 3.10 to derive this solution. The vector $\mathbf{C}(y)$ can be obtained from the solution of the traffic equations:

$$\mathbf{C}(f) = \left(\log\left[\tfrac{\mu(t_3)}{\mu(t_{12})}\right], \log\left[\tfrac{\mu(t_5)}{b\mu(t_{12})}\right], \log\left[\tfrac{\mu(t_{12})}{\mu(t_3)}\right], \log\left[\tfrac{b\mu(t_3)}{\mu(t_4)}\right], \log\left[\tfrac{\mu(t_4)}{\mu(t_5)}\right]\right).$$

It can easily be verified that $Rank(A) = Rank(A|\mathbf{C}(y))$ without any conditions on the firing rates. The solution $\mathbf{c}(y)$ of the system of equations (3.6) is (we have set $c_1(y) = c_3(y) = 1$ as normalisation)

$$c_1(y) = 1,\ c_2(y) = \frac{\mu(t_{12})}{\mu(t_3)},\ c_3(y) = 1,\ c_4(y) = \frac{b\mu(t_{12})}{\mu(t_5)},\ c_5(y) = \frac{b\mu(t_{12})}{\mu(t_4)}$$

is a solution to (3.6), and the equilibrium distribution is (cf. Coleman *et al.* [7])

$$\pi_y(\mathbf{m}) = \left(\frac{\mu(t_{12})}{\mu(t_3)}\right)^{m(2)} \left(\frac{b\mu(t_{12})}{\mu(t_5)}\right)^{m(4)} \left(\frac{b\mu(t_{12})}{\mu(t_4)}\right)^{m(5)}$$

is an invariant measure for the Petri net at reachability set

$$\mathcal{M}(\mathbf{m}_0) = \{\mathbf{m} : \mathbf{y}^1 \bullet (\mathbf{m} - \mathbf{m}_0) = 0,\ \mathbf{y}^2 \bullet (\mathbf{m} - \mathbf{m}_0) = 0\},$$

where $\bullet$ denotes the inner product of the two vectors.

Consider the Petri net depicted in Figure 3b. This Petri net has incidence matrix

$$A = \begin{pmatrix} -1 & 1 & -2 & 2 \\ 1 & -1 & 2 & -2 \end{pmatrix}.$$

Observe that each transition is covered by the minimal closed support T-invariants $\mathbf{x}^1 = (1100)$, $\mathbf{x}^2 = (0011)$, but that $\mathbf{x}^3 = (2001)$, and $\mathbf{x}^4 = (0210)$, are also minimal support T-invariants that do not have closed support.

The routing chain has two irreducible sets $S(\mathbf{x}^1) = \{\mathbf{I}(t_1), \mathbf{I}(t_2)\}$, and $S(\mathbf{x}^2) = \{\mathbf{I}(t_3), \mathbf{I}(t_4)\}$. Theorem 3.9 implies that the traffic equations have a positive solution. This solution is

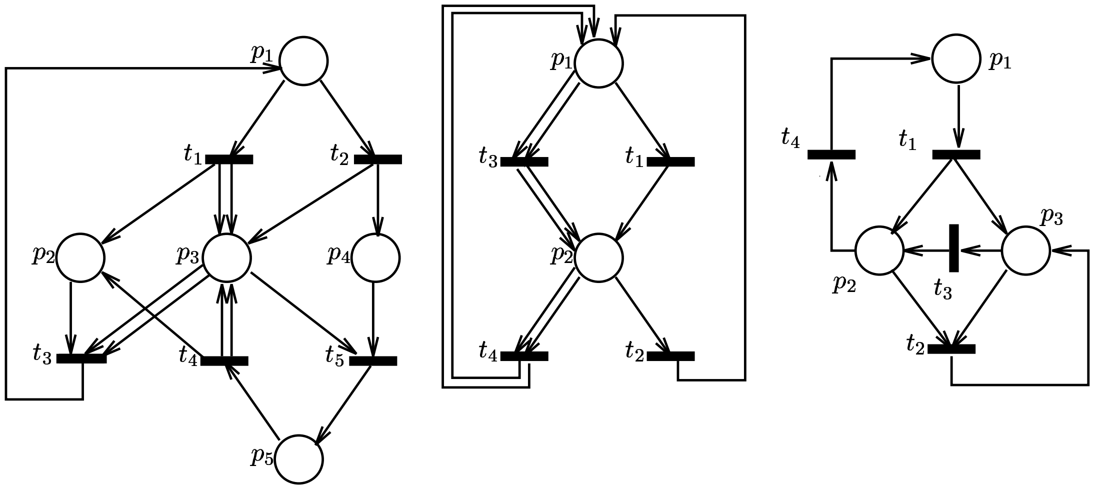

Figure 3: a. Figure 3: b. Figure 3: c.

$$\frac{y^1(I(t_2))}{y^1(I(t_1))} = \frac{\mu(t_1)}{\mu(t_2)}, \quad \frac{y^2(I(t_4))}{y^2(I(t_3))} = \frac{\mu(t_3)}{\mu(t_4)},$$

with corresponding vector $\mathbf{C}(y)$

$$\mathbf{C}(y) = \left(\log\left[\frac{\mu(t_2)}{\mu(t_1)}\right], \log\left[\frac{\mu(t_1)}{\mu(t_2)}\right], \log\left[\frac{\mu(t_4)}{\mu(t_3)}\right], \log\left[\frac{\mu(t_3)}{\mu(t_4)}\right]\right).$$

The matrix $[A|\mathbf{C}(y)]$ is

$$[A|\mathbf{C}(y)] = \begin{pmatrix} -1 & 1 & -2 & 2 \\ 1 & -1 & 2 & -2 \\ C_1 & C_2 & C_3 & C_4 \end{pmatrix},$$

and $Rank([A|\mathbf{C}(y)]) = Rank(A) = 1$ if and only if $C_1 + C_2 = 0$, $2C_1 - C_3 = 0$, $2C_1 + C_4 = 0$, that is if and only if

$$\left(\frac{\mu(t_2)}{\mu(t_1)}\right)^2 = \frac{\mu(t_4)}{\mu(t_3)}. \tag{4.1}$$

If this is the case, the Petri net has an equilibrium distribution

$$\pi(\mathbf{m}) = B\left(\frac{\mu(t_2)}{\mu(t_1)}\right)^{\mathbf{m}(1)},$$

at reachability set

$$\mathcal{M}(\mathbf{m}_0) = \{\mathbf{m} : \mathbf{m}(1) + \mathbf{m}(2) = \mathbf{m}_0(1) + \mathbf{m}_0(2)\}.$$

This example provides an interpretation and explanation of the rank condition (3.5) of Theorem 3.10. As can be seen from Figure 3b, for two tokens to move from place 1 to place 2 we have two possibilities. In the first case (via t_1) the tokens jump one after the other, in the second case (via t_3) the tokens jump simultaneously. The probability flow for these two possibilities must be the same. This is reflected in the condition (4.1) on the firing rates: two transitions with rate $\mu(t_1)$ must be proportional to one transition at rate $\mu(t_3)$.

Finally, consider the Petri net of Figure 3c. The Petri net has one T-invariant $\mathbf{x} = (1111)$ covering all transitions, and $\mathbf{x}$ has closed support. From Theorem 3.7 we obtain that the traffic equations have a positive solution. This solution is (up to a multiplicative constant)

$$y(\mathbf{I}(t_1)) = 1/\mu(t_1), \quad y(\mathbf{I}(t_2)) = 1/\mu(t_2), \quad y(\mathbf{I}(t_3)) = 1/\mu(t_3), \quad y(\mathbf{I}(t_4)) = 1/\mu(t_4),$$

and the Petri net has an invariant measure

$$m(\mathbf{m}) = \left(\frac{\mu(t_3)\mu(t_4)}{\mu(t_1)\mu(t_2)}\right)^{\mathbf{m}(1)} \left(\frac{\mu(t_3)}{\mu(t_2)}\right)^{\mathbf{m}(2)} \left(\frac{\mu(t_4)}{\mu(t_2)}\right)^{\mathbf{m}(3)}.$$

From Figure 3c we can see that the number of tokens in the net is unbounded (repetitive firing of transitions t_1 and t_4 increases the number of tokens by 1), but that for every marking a firing sequence to $\mathbf{m}_0 = (100)$ exists. If $\mu(t_3)\mu(t_4) < \mu(t_1)\mu(t_2)$, $\mu(t_3) < \mu(t_2)$, $\mu(t_4) < \mu(t_2)$ the Petri net has an equilibrium distribution

$$\pi(\mathbf{m}) = Bm(\mathbf{m}), \quad \mathbf{m} \in \mathcal{M}(\mathbf{m}_0) = \mathbb{N}_0^3 \setminus \{0\}.$$

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