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# Structural Identifiability of Dynamic Systems Operating under Feedback with Application to Economic Systems

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**Abstract**

In economics and econometrics, economic phenomena such as part of a national economy or the the behavior of a firm are modelled as economic systems. The behavior of these phenomena takes place under feedback thus the inputs or instruments depend directly on the current and possibly past values of the observations. Economic modeling must therefore take account of the fact that economic systems are operating under feedback.

In this paper the modeling of dynamic systems operating under feedback is posed as a problem of system theory. Attention is limited to linear dynamic systems that are structured by, say, economic laws. Structural identifiability of the parametrization of a class of dynamic systems is defined as injectiveness of the map from the parameter vector to the observations. The problem posed is whether a structured deterministic linear dynamic system operating under feedback is structurally identifiable. Necessary and sufficient conditions for structurally identifiability are presented. For several economic systems structural identifiability is investigated.

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## 1 Introduction

The purpose of this paper is to report on structural identifiability of linear dynamic systems operating under feedback or in closed-loop. The systems considered are models of economic phenomena.

The motivation for this paper is system identification of economic phenomena. System identification is the subject in which the construction of mathematical models from observed phenomena is studied. In economics one studies models of macro economic or micro

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economic phenomena such as part of a national economy or of a firm. Such a mathematical model will be called an *economic system.* One distinguishes in such a system inputs or instruments, outputs, and states. It is a well known fact that economic systems operate in closed-loop or under feedback. This means that the inputs or the instruments depend on the state of the system or on its output. Therefore in system identification of economic phenomena account must be made of the fact that an economic system operates under feedback. As far as the authors know this identification under feedback isssue has not been studied in depth in the economics and econometrics literature. Yet it is a major limitation on modeling of economic phenomena. Hence the motivation of the problem.

The dynamic systems considered in this paper are economic systems. The authors favor the study of economic systems that are derived from economic laws. This should be seen in contrast with modeling in econometrics. There linear relations are assumed between outputs, lagged values of outputs, and instrument variables of an economic model. Attention will further be restricted to linear dynamic systems. Because the linear dynamic systems considered are structured by economic laws, the class of structured linear dynamic systems will be used.

In a structured linear dynamic system the parameters of the system depend on a parameter vector. The parametrization consists of the set of parameter vectors and the map that associates such a vector with the parameters of the system. Intuitively, a parametrization is said to be *structurally identifiable* if the map from the parameter vector to the observations of inputs and outputs is injective. A formal definition of structural identifiability is presented in section 4. Structural identifiability implies that it is in principal possible to uniquely determine the parameter vector form the observations. In econometrics another concept of identifiability is used. This concept refers to the convergence of a sequence of parameter estimates. In system identification attention should be given to both structural identifiability of a parametrization and convergence of parameter estimates. The latter concept will not be treated in this paper.

The problem of the paper is to characterize structural identifiability of linear dynamic systems operating under feedback. Two cases are studied. In the first case the control is based on state feedback and in the second one on dynamic output feedback. Because the motivation comes from economic modeling the control law will be assumed known and fixed.

Identification and identifiability of linear dynamic systems operating under feedback has been studied by several authors. The assumptions differ from paper to paper. Most papers treat this problem for stochastic systems in matrix polynomial form. The control law may be assumed known or unknown. There may or may not be an excitation signal entering the closed-loop system. The novelty of the problem studied in this paper is that structured linear dynamic systems in state space form are considered with a fixed known control law. In addition, in this paper attention is given to structural identifiability from the initial condition response.

The results of the paper are necessary and sufficient conditions for structural identifiability of structured linear dynamic systems operating under feedback. Several structured economic systems are tested for structural identifiability. Some economic systems are structural identifiable, one system is not.

A description of the content by section follows. A problem formulation is presented in section 2. Dynamic systems are formulated in section 3. Structural identifiability of linear dynamic systems is defined in section 4. Conditions for structural identifiability of linear dynamic systems operating under state feedback and dynamic output feedback are derived in the sections 5 and 6. Conclusions and open problems are mentioned in section 7.

## 2 Problem formulation

. In this section the problem is motivated and formulated.

Consider an economic organization whose behavior is to modelled. Examples are a national economy, an economic sector, or a firm. The behavior may be modelled by economic variables and by relations between these variables. An economic system is then a system of difference or differential equations that describe the behavior of these variables. The authors prefer to use a state space description of such a system as used in system theory. For an open economy with fixed output, see example 5.6, the state space description is given by

$$x(t+1) = A(p)x(t) + B(p)u(t), \quad x(t_0) = x_0, \tag{1}$$

where $x_1(t) = e(t)$ represents the exchange rate, $x_2(t) = y(t)$ represents income, $p = (\alpha, \beta, \gamma, \delta, \epsilon) \in P \subset R^5$ are the parameters of the model, and

$$A(p) = \begin{pmatrix} 0 & \beta \\ \delta & \gamma \end{pmatrix}, \quad B(p) = \begin{pmatrix} \alpha & 0 \\ 0 & \epsilon \end{pmatrix}. \tag{2}$$

See example 5.6 for a detailed description and derivation of the model. One calls the set $P$ and the map $p \mapsto (A(p), B(p))$ the parametrization of this system.

Suppose that one observes the behavior of the economy over a finite horizon, say $T = \{t_0, t_0+1, \ldots, t_1\}$. The observations consist of $\{x(t), u(t), t \in T\}$. The problem of system identification is to construct a model from these observations that is realistic and not overly complex. A major question within the system identification problem is identifiability. The question is whether the parametrization of the model class is such that from the observations one can uniquely determine the parameter vector. In the example sketched above this may be phrased as that, given the input $u$, the map

$$p \mapsto (A(p), B(p), x_0(p)) \mapsto \{x(t), u(t), t \in T\}$$

is injective. A precise definition of identifiability is presented in section 4.

Economic organizations almost always operate under feedback. Thus the inputs or instrument variables depend on current and past values of the observations. This fact must be accounted for in the investigation of identifiability of an economic system.

Consider a feedback law of the form

$$u(t) = Fx(t) \tag{3}$$

where $F \in R^{m \times n}$ is assumed to be known. The closed-loop dynamic system consisting of the open-loop system (1) and the feedback law (3) is then given by

$$x(t+1) = [A(p) + B(p)F]x(t), \quad x(t_0) = x_0. \tag{4}$$

Identifiability of the dynamic system (1) operating under the feedback law (3) may then be phrased as injectiveness of the map

$$p \mapsto (A(p) + B(p)F, x_0(p)) \mapsto \{x(t), u(t), t \in T\}.$$

Under certain conditions one may be able to determine the system matrix $A(p) + B(p)F$. In this case the identifiability question then is whether for $p, q \in P$ the equation

$$A(p) + B(p)F = A(q) + B(q)F$$

implies that $p = q$. Note that in this problem the matrix $F$ is fixed and its value assumed known.

The problem of the paper is then to formulate structural identifiability of dynamic systems operating under feedback, to derive necessary and sufficient conditions for structural identifiability, and to investigate whether several well known economic systems operating under feedback are structural identifiable. Besides identifiability under state feedback also identifiability under dynamic output feedback is investigated.

## 3 Dynamic systems

The reader is assumed to be familiar with the concept of a dynamic system as used in system theory. For references on system theory see [6, 26]. A dynamic system is a mathematical model for a phenomenon such as part of a national economy, a firm, a robot arm, an electric circuit, etc. A dynamic system consists of states, inputs, and outputs and relations between these variables.

The notation is fairly standard. Let $Z_+ = \{1, 2, \ldots\}$, $N = \{0, 1, 2, \ldots\}$, $R$ the set of the real numbers, $R^n$ the n-dimensional vector space over $R$, and $R^{n \times n}$ the set of the $n \times n$ matrices with elements in $R$.

**Definition 3.1** *A* time-invariant discrete-time linear dynamic system *in* state space form *is a dynamic system in which the state, input, and output space are respectively $X = R^n$, $U = R^m$, $Y = R^k$, the time index set is $T = \{t_0, t_0 + 1, \ldots t_1\}$ and the state, input, and output function are related by*

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \quad x(t_0) = x_0, & (5) \\ y(t) &= Cx(t) + Du(t), & (6) \end{aligned}$$

*where $A \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{k \times n}$, $D \in R^{k \times m}$. Denote the system by $\sigma$ and the class of linear dynamic systems by $L\Sigma$. The indices $n, m, k$ and the matrices $A, B, C, D$ are called the* parameters *of this system. Denote the parameters by*

$$p = (A, B, C, D) \in L\Sigma P(n, m, k), \quad L\Sigma P(n, m, k) = R^{n \times n} \times R^{n \times m} \times R^{k \times n} \times R^{k \times m}.$$

*A time-invariant continuous-time finite-dimensional linear system in state space form is a dynamical system defined on the same sets as defined above and by the equations*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0, & (7) \\ y(t) &= Cx(t) + Du(t). & (8) \end{aligned}$$

The adjectives discrete-time and time-invariant will often be omitted when they are clear from the context.

Modeling of an economic phenomenon by laws and relations of economics leads to a dynamic system that is structured. In subsection 5.2 examples are presented of a structured dynamic system. A formal definition of this class of systems follows.

**Definition 3.2** *A* structured linear dynamic system *is a time-invariant finite-dimensional linear dynamic system in state space form together with a* parameter set $P \subset R^r$ *for some* $r \in N$ *and maps*

$$A : P \rightarrow R^{n \times n}, B : P \rightarrow R^{n \times m}, C : P \rightarrow R^{k \times n}, D : P \rightarrow R^{k \times m},$$

*such that* $(A(p), B(p), C(p), D(p)) \in L\Sigma P(n, m, k)$. *The dynamic system will be represented by the equations*

$$x(t+1) = A(p)x(t) + B(p)u(t), \quad x(t_0) = x_0, \tag{9}$$

$$y(t) = C(p)x(t) + D(p)u(t). \tag{10}$$

*Denote by* $SL\Sigma P(n, m, k)$ *the set of parameters of this structured system or*

$$SL\Sigma P(n, m, k) = \{(A(p), B(p), C(p), D(p)) \in L\Sigma P(n, m, k) | p \in P\},$$

*and by* $f : P \rightarrow SL\Sigma P(n, m, k)$ *the* parametrization map

$$p \mapsto (A(p), B(p), C(p), D(p)).$$

*The concept of a continuous-time structured linear dynamic system is defined similarly.*

**Definition 3.3** *Consider a subset* $L\Sigma_1$ *of the set of linear dynamic systems* $L\Sigma$, *and denote the corresponding sets of parameters by* $L\Sigma P_1$ *and* $L\Sigma P$. *A* parametrization *of the set* $L\Sigma P_1$ *is a tuple* $(P, f)$ *such that* $P \subset R^r$ *is a set,* $f : P \rightarrow L\Sigma P_1$ *is a map, and the map* $f$ *is surjective. One says that* $P$ *is the* parameter set *and the map* $f$ *the* parametrization map.

Subsection 5.2 contains several examples of structured linear dynamic systems with their parametrization.

The key property of a parametrization, say $(P, f)$ of $L\Sigma P_1$, is that it is surjective. Thus every element of $L\Sigma P_1$ is the image of an element in $P$ under $f$.

The relation between inputs and outputs of a linear dynamic system in state space form is given by

$$y(t) = \sum_{s=t_0}^{t-1} CA^{t-1-s}Bu(s) + Du(t) + CA^{t-t_0}x_0 = \sum_{s=t_0}^{t} W(t-s)u(s) + CA^{t-t_0}x_0. \tag{11}$$

The function $W$ is called the *impulse response function* of this system. Denote by $\mathbf{W}$ the set of impulse response functions $W : T \rightarrow R^{k \times m}$ in which the indices $k, m$ are in general not indicated. Define the map that associates the parameter of a linear dynamic system with the impulse response function $g : L\Sigma P(n, m, k) \rightarrow \mathbf{W}$

$$g((A, B, C, D))(t) = \begin{cases} CA^{t-1}B, & t > 0, \\ D, & t = 0. \end{cases} \tag{12}$$

# 4 Structural identifiability

A key concept in modeling is that of identifiability. Yet, in much of the literature there is confusion on its definition. In this paper a definition of identifiability will be used that is derived from system theory.

In econometrics the concept of identifiability is often defined as the property that the parameter estimates converge to the true parameter. For this it is assumed that the observed data are generated by a system in the model class whose parameter value is then called the true parameter. In system theory and system identification a distinction is made between the identifiability of the parametrization of the model class and the convergence properties of a sequence of parameter estimates. Convergence analysis of parameter estimation algorithms is relevant to system identification. However, identifiability as a property of a parametrization of a model class must also be studied.

Intuitively, identifiability of the parametrization of a class of dynamic systems is the property that from the observations of inputs and outputs one can uniquely determine the parameter value. This intuitive concept is formalized below for a structured linear dynamic system.

The concept of *structural identifiability* has been introduced by R. Bellman and K.J. Aström [2]. Since then structural identifiability of linear and nonlinear structured systems has been studied by many authors. For an entry into the literature see the book [27] and the collection of papers [28]. An early paper on the subject is [10]. Below the concept of structural identifiability is defined. New to the literature is the distinction between structural identifiability from the impulse response function and from the initial condition response. For economic systems this distinction is relevant.

Consider a structured linear dynamic system of the form

$$x(t+1) = A(p)x(t) + B(p)u(t), \quad x(t_0) = x_0, \tag{13}$$

$$y(t) = C(p)x(t) + D(p)u(t). \tag{14}$$

The relation between the output, inputs, and the initial condition is then

$$y(t) = \sum_{s=t_0}^{t-1} C(p)A(p)^{t-1-s}B(p)u(s) + D(p)u(t) + C(p)A(p)^{t-t_0}x_0 \tag{15}$$

$$= \sum_{s=t_0}^{t} W_p(t-s)u(t) + C(p)A(p)^{t-t_0}x_0 \tag{16}$$

where $W_p : T \rightarrow R^{k\times m}$

$$W_p(t) = \begin{cases} C(p)A(p)^{t-1}B(p), & t > 0, \\ D(p), & t = 0. \end{cases} \tag{17}$$

There are now two ways to proceed that will be termed *structural identifiability from the impulse response function* and *structural identifiability from the initial condition response.* These cases are next discussed in the indicated order.

## 4.1 Structural identifiability from the impulse response function

It is argued that under certain conditions one may limit attention to structural identifiability from the impulse response.

**Assumption 4.1** *Conditions required for the use of structural identifiability of a structured linear system from the impulse response function.*

1. *The dynamic system is stable.*
2. *The time horizon is long relative to the dynamics of the system.*
3. *The possibility exists to experiment freely with the input function.*

For linear dynamic systems derived from economic models, the assumptions 2 and 3 are generally not satisfied. One could also say that economists are not allowed to experiment with the instruments of a national economy. The director of a firm can experiment with his instruments but the aims of the firm will in general deter him from experimentation. In this paper structural identifiability from the impulse response function is studied because it is a general concept that is indirectly also useful for economic systems.

Because of the assumptions 4.1.1 and 4.1.2 the effect of the initial condition may be neglected. The resulting map from input to output is then

$$y(t) = \sum_{s=t_0}^{t} W_p(t-s)u(t).$$

Assume one is able to make many experiments with different inputs. Then from the observations of inputs and outputs one is able to obtain the impulse response function. Therefore knowledge of inputs and outputs of such a system is equivalent, under the above assumptions, to knowledge of the impulse response function. Hence identifiability of the parametrization of a class of systems may be phrased in terms of the impulse response function.

**Definition 4.2** *Consider a structured linear dynamic system in state space form*

$$\begin{aligned} x(t+1) &= A(p)x(t) + B(p)u(t), \quad x(t_0) = x_0, \\ y(t) &= C(p)x(t) + D(p)u(t), \end{aligned}$$

*with parametrization* $(P, f)$ $f : P \to L\Sigma P_1(n, m, k)$. *Let* $g : L\Sigma P_1(n, m, k) \to \mathbf{W}$ *be as defined in (12). The parametrization* $(P, f)$ *is said to be* structural identifiable from the impulse response *if the map*

$$g \circ f : P \to \mathbf{W}, \quad p \mapsto W_p,$$

*see (17), is injective.*

Note that in general the map $g \circ f$ is not surjective, the image of $P$ under $g \circ f$ is a proper subset of $\mathbf{W}$. Structural identifiability of a continuous-time linear dynamic system is defined analogously.

## 4.2 Structural identifiability from the initial condition response

The investigation of structural identifiability of economic systems requires also the study of identifiability given the response of the system to an initial condition.

Consider a structured linear dynamic system without input function, say with representation

$$x(t+1) = A(p)x(t), \quad x(t_0) = x_0, \tag{18}$$
$$y(t) = C(p)x(t). \tag{19}$$

The output as function of the initial condition is given by $y(t) = C(p)A(p)^{t-t_0}x_0$. Note that now $x_0 \in X = R^n$ is not observed. In the following $x_0$ is considered to be a function of the parameter vector $p \in P$ and denoted by $x_0(p)$. In this case let

$$SL\Sigma P(n,k) = \{(A(p), C(p), x_0(p)) \in R^{n\times n} \times R^{k\times n} \times R^n | p \in P\}$$

and $f : P \to SL\Sigma P(n,k)$ the parametrization map $p \mapsto (A(p), C(p), x_0(p))$. Let $g : SL\Sigma P(n,k) \to Y^T$ and $g \circ f : P \to Y^T$

$$g((A, C, x_0)) = \{y(t), t \in T\}, \quad y(t) = CA^{t-t_0}x_0, \tag{20}$$
$$(g \circ f)(p) = \{y_p(t), t \in T\}, \quad y_p(t) = C(p)A(p)^{t-t_0}x_0(p). \tag{21}$$

**Definition 4.3** *Consider a structured linear dynamic system in state space form without input function*

$$x(t+1) = A(p)x(t), \quad x(t_0) = x_0(p), \tag{22}$$
$$y(t) = C(p)x(t), \tag{23}$$

*with parametrization* $(P, f)$. *This parametrization is said to be* structurally identifiable from the initial condition response *if the map*

$$g \circ f : P \to Y^T, p \mapsto \{y_p(t), t \in T\},$$

*see (21),is injective.*

## 4.3 Conditions for structural identifiability

Conditions for structural identifiability are based on realization theory. For time-invariant finite-dimensional linear systems the realization problem has been solved. Hence equivalent conditions for structural identifiability may be formulated. For other classes of dynamic systems the realization problem is still open so conditions for structural identifiability may not be known.

In the following terminology of algebraic geometry is used, see [5],and [17, X, §3]. A property is said to be *generic* on $P \subset R^r$ if it holds for all $p \in P$ outside an algebraic variety or an algebraic set. An *algebraic set* is defined by a finite set of polynomials according to

$$\{p \in P | f_1(p) = 0, \ldots, f_k(p) = 0\},$$

where $f_i : P \to R$, $i = 1, \ldots, k$ are polynomials. An *algebraic variety* is an algebraic set in which the polynomials are derived from linear relations, possibly from a determinant.

**Definition 4.4** *Consider the structured linear dynamic system*

$$\begin{aligned} x(t+1) &= A(p)x(t) + B(p)u(t), \quad x(t_0) = x_0, && (24) \\ y(t) &= C(p)x(t) + D(p)u(t), && (25) \end{aligned}$$

*with parametrization* $(P, f)$, $f : P \to L\Sigma P_1(n, m, k)$.

**a** *This system is called* structurally reachable *if for all* $p \in P$ *outside an algebraic variety*

$$rank \begin{pmatrix} B(p) & A(p)B(p) & \ldots & A(p)^{n-1}B(p) \end{pmatrix} = n.$$

*In this case one says that* $(A(.), B(.))$ *is a* structured reachable pair.

**b** *This system is called* structurally observable *if for all* $p \in P$ *outside an algebraic variety*

$$rank \begin{pmatrix} C(p) \\ C(p)A(p) \\ \vdots \\ C(p)A(p)^{n-1} \end{pmatrix} = n.$$

*In this case one says that* $(A(.), C(.))$ *is a* structured observable pair.

**c** *This system is said to be* structurally minimal *if for all* $p \in P$ *outside an algebraic variety it is a minimal realization of its impulse response function.*

**Theorem 4.5** *Consider the structured linear dynamic system*

$$\begin{aligned} x(t+1) &= A(p)x(t) + B(p)u(t), \quad x(t_0) = x_0, && (26) \\ y(t) &= C(p)x(t) + D(p)u(t), && (27) \end{aligned}$$

*with parametrization* $(P, f)$, $f : P \to L\Sigma P_1(n, m, k)$.

**a** *This structured linear dynamic system is structurally minimal iff it is structurally reachable and structurally observable.*

**b** *Assume that the structured linear dynamic system is structurally reachable and structurally observable. Then this parametrization is structurally identifiable from the impulse response iff for all* $p, q \in P$ *outside an algebraic variety and* $T \in R^{n \times n}$ *nonsingular, the equations*

$$A(p) = TA(q)T^{-1}, B(p) = TB(q), C(p) = C(q)T^{-1}, D(p) = D(q), \qquad (28)$$

*imply that* $p = q$. *If the system is structurally identifiable then, for all* $q \in P$ *outside an algebraic variety, the system of equations (28) for the pair* $(p, T) \in P \times R^{n \times n}$, *with* $T$ *nonsingular, has the unique solution* $(p, T) = (q, I)$.

**Proof** This result follows directly from the main result of realization theory for time-invariant finite-dimensional linear systems. For references see [6, Chapter 9] [14, 15], and [26, section 5.5]. □

The result of theorem 4.5 is not stated in [27, 28]. Surprisingly enough the authors have not found this result in the literature. For continuous-time structured linear dynamic systems the conditions are the same as for the discrete-time case.

Conditions for structural reachability are available in the literature, see [8, 18, 21, 22, 27]. Conditions for structural reachability and for condition 4.5.b may be checked by a symbolic manipulation program, see [28].

Next the characterization is stated of structural identifiability from the initial condition response.

**Theorem 4.6** *Consider the structured linear dynamic system without input*

$$\begin{aligned} x(t+1) &= A(p)x(t), \quad x(t_0) = x_0, && (29)\\ y(t) &= C(p)x(t), && (30) \end{aligned}$$

*with the parametrization* $(P, f)$

$$\begin{aligned} f: &\mapsto SL\Sigma P_1(n,k), && (31)\\ SL\Sigma P_1(n,k) &= \{(A(p), C(p), x_0(p)) \in R^{n \times n} \times R^{k \times n} \times R^n | p \in P\}. && (32) \end{aligned}$$

*Assume that the structured linear dynamic system is:*

1. *structurally observable, or, for all* $p \in P$ *outside an algebraic variety,* $(A(p), C(p))$ *is an observable pair;*

2. *the system is structurally reachable from the initial condition response, or, for all* $p \in P$ *outside an algebraic variety,* $(A(p), x_0(p))$ *is a reachable pair.*

*Under these conditions, the parametrization* $(P, f)$ *of this class of structured linear dynamic systems is structurally identifiable from the initial condition response iff for all* $p, q \in P$ *outside an algebraic variety and* $T \in R^{n \times n}$ *nonsingular, the equations*

$$A(p) = T^{-1}A(q)T, C(p) = C(q)T, x_0(p) = T^{-1}x_0(q), \qquad (33)$$

*imply that* $p = q$.

For continuous-time structured linear dynamic systems the conditions are the same. Assumption 2 of theorem 4.6, that for all $p \in P$ outside an algebraic variety $(A(p), x_0(p))$ is a reachable pair, is necessary. If it does not hold then for a set $P_1 \subset P$ that is not an algebraic variety, the initial condition does not excite all modes of the system. For example, consider

$$\begin{aligned} x(t+1) &= \begin{pmatrix} A_{11}(p) & A_{12}(p) \\ 0 & A_{22}(p) \end{pmatrix} x(t), \quad x_0(p) = \begin{pmatrix} x_{01}(p) \\ 0 \end{pmatrix}, && (34)\\ y(t) &= x(t). && (35) \end{aligned}$$

In this case the initial condition response does not contain information on the parameters in the matrices $A_{12}(p)$ and $A_{22}(p)$. It is then advisable to reduce the state space representation such that the condition of structurable reachability is satisfied.

**Proof** $\Rightarrow$. Suppose that there exists $p, q \in P$, $p \neq q$, not in algebraic variety, and $T \in R^{n \times n}$ nonsingular such that

$$A(p) = T^{-1}A(q)T, C(p) = C(q)T, x_0(p) = T^{-1}x_0(q).$$

Then the initial condition responses associated with $p, q \in P$ are the same contradicting structural identifiability.
$\Leftarrow$. Let $p, q \in P$ be such that $(g \circ f)(p) = (g \circ f)(q)$ or

$$C(p)A(p)^{t-t_0}x_0(p) = C(q)A(q)^{t-t_0}x_0(q), \quad t \in T. \tag{36}$$

Consider

$$W_p(t) = C(p)A(p)^{t-t_0}x_0(p),$$

as an impuls response function. Excluding an algebraic variety, it follows from (36), the assumptions, and realization theory, that there exists a $T \in R^{n \times n}$ nonsingular such that

$$C(p) = C(q)T,\ A(p) = T^{-1}A(q)T, x_0(p) = T^{-1}x_0(q). \tag{37}$$

From (37) and the condition follows that $p = q$. Hence the system is structurally identifiable.
□

# 5 Structural identifiability of a structured linear dynamic system operating under state feedback

In this section structural identifiability is treated for structured linear dynamic systems operating under state feedback.

The issues of identification and identifiability conditions for dynamic systems operating under feedback or in closed-loop have been discussed by many authors. An early survey paper on the issue of identification under feedback is that by I. Gustavsson, L. Ljung, and T. Söderström [12]. Related papers by the same authors are [20, 23, 24]. A book that briefly discusses this issue is [19]. Conditions for identifiability under feedback when the controller is unknown are presented in [1]. These papers limit attention to stochastic systems in matrix polynomial form driven by white noise. The control law may or may not be known. The necessary and sufficient conditions for identifiability are phrased in terms of bounds on indices. Structural identifiability of deterministic linear dynamic systems operating under a fixed known control law apparently has not been treated.

## 5.1 Conditions for structural identifiability in case of state feedback

Consider a structured linear dynamic system consisting of a state recursion only, say described by

$$x(t+1) = A(p)x(t) + B(p)u(t), \quad x(t_0) = x_0. \tag{38}$$

Suppose that the state function $x$ is observed.

Consider a control problem for this dynamic system. Control objectives are usually stability of the closed-loop system, good transient response, and robustness against unmodelled disturbances and against model uncertainty. Feedback of the state function is almost always used to achieve these objectives. Consider a time-invariant feedback law $F \in R^{m \times n}$ and the input function $u(t) = Fx(t)$. The combination of the linear system (38) and the feedback control law $u(t) = Fx(t)$ is given by

$$x(t+1) = [A(p) + B(p)F]x(t), \quad x(t_0) = x_0, \tag{39}$$

and will be called the *closed-loop system* associated with these objects. The system (38) is then said to operate under state feedback or in closed-loop.

**Problem 5.1** *Consider the structured linear system (38) with parametrization $(P, f)$ $f : P \to SL\Sigma P(n, m)$,*

$$SL\Sigma P(n,m) = \{(A(p), B(p)) \in R^{n \times n} \times R^{n \times m} | p \in P\}.$$

*Consider the control based on state feedback $F \in R^{m \times n}$ and the input function $u(t) = Fx(t)$. The resulting closed-loop system is given by (39). Denote the closed-loop system by $CSL\Sigma$, its parameter set by*

$$CSL\Sigma P(n) = \{(A(p) + B(p)F) \in R^{n \times n} | p \in P\},$$

*and its parametrization by $(P, f_1)$, $f_1 : P \to CSL\Sigma P(n)$, $f_1(p) = A(p) + B(p)F$. Is this parametrization structurally identifiable from the impulse response function?*

**Assumption 5.2** *1. The open-loop structured linear dynamic system (38) is structurally identifiable from the impulse response function.*

*2. The feedback law $u(t) = Fx(t)$ is fixed and known.*

Assumption 5.2.1 may be checked from the specification of the class of structured linear dynamic systems. In problem 5.1 only one state feedback law is used. This choice is motivated by modeling in economics. A national economy or a firm is under constant online control. The assumption that the control law is linear is debatable. However, a linear control law is consistent with the class of linear dynamic systems selected. The assumption that the control law is constant, is realistic only for a short time horizon. A change of government or a change of directorship for the national bank may lead to a change in the control law. Thus for a short time horizon the assumption of a linear control law may be realistic in economic modeling. It is assumed that the value of the feedback matrix $F$ is known. This assumption is consistent with economic modeling. If the value of the feedback matrix is not known then a totally different realization problem is obtained, see [13].

Consider the problem of structural identifiability from the initial condition response for the structured linear dynamic system (38) with the feedback law $F \in R^{m \times n}$. The closed-loop system is given by (39). Note that the state is observed. If $(A(p) + B(p)F, x_0)$ is a structured reachable pair then one can uniquely determine the matrix $A(p)+B(p)F$ from the state trajectory $\{x(t), t \in T\}$. The analysis of structural identifiability from the initial condition response proceeds then as for structural identifiability from the impulse response, see below. If $(A(p)+B(p)F, x_0)$ is not a structured pair, then in general the system is not structurally identifiable from the initial condition response and a reduction of the state space representation must be made.

**Proposition 5.3** *Consider problem 5.1. The parametrization $(P, f_1)$ of the closed-loop system (39) is structurally identifiable from the impulse response function iff for all $q \in P$ outside an algebraic variety*

$$A(p)+B(p)F = A(q)+B(q)F, \tag{40}$$

*for $p \in P$ implies $p = q$.*

**Proof** This follows directly from theorem 4.6. □

The consequences of not accounting for structural identifiability may be dramatic. Consider a system identification experiment. Suppose that the model class consists of a structured linear dynamic system as in (38). Suppose further that the phenomenon may be modelled by a system in the model class, say with parameter value $p_0 \in P$. Suppose that the actual phenomenon is controlled in closed-loop by a state feedback law $u(t) = F_1 x(t)$ for some $F_1 \in R^{m \times n}$. Any identification procedure will then produce a parameter estimate $p_1 \in P$ in the set

$$P_1(p_0, F_1) = \{p \in P | A(p_0) + B(p_0)F = A(p) + B(p)F\}.$$

The closed-loop system is given by

$$\begin{aligned} x(t+1) &= [A(p_1)+B(p_1)F_1]x(t) = [A(p_0)+B(p_0)F_1]x(t), \\ &= A(p_1)x(t) + B(p_1)u(t), \quad x(t_0) = x_0. \end{aligned}$$

Suppose now that the state feedback law is changed, say to $u(t) = F_2 x(t)$ for $F_2 \in R^{m \times n}$. The real closed-loop system is then given by

$$x(t+1) = [A(p_0)+B(p_0)F_2]x(t), \quad x(t_0) = x_0.$$

However, the modeller imagines the closed-loop system to be

$$x(t+1) = [A(p_1)+B(p_1)F_2]x(t), \quad x(t_0) = x_0.$$

In general $A(p_0)+B(p_0)F_2 \neq A(p_1)+B(p_1)F_2$. Predictions of the state function based on the incorrect parameter estimate may therefore be way of the mark or lead to an unstable system as the following example shows.

**Example 5.4** Consider the structured linear dynamic system of first order

$$x(t+1) = a(p)x(t) + b(p)u(t), \quad x(t_0) = x_0, \tag{41}$$

with $P = R^2$, $p \in P$, $a(p) = p_1$, $b(p) = p_2$, and the control law $u(t) = fx(t)$. The closed-loop system is given by

$$x(t+1) = [a(p) + b(p)f]x(t), \quad x(t_0) = x_0. \tag{42}$$

The question whether structural identifiability from the impulse response function holds leads to the investigation of the equation

$$a(p) + b(p)f = a(q) + b(q)f \ \Leftrightarrow\ p_1 + p_2 f = q_1 + q_2 f, \tag{43}$$

given $q \in P$ for $p \in P$. In general, for $f \neq 0$, one does not obtain $p = q$. In fact

$$P_1(q) = \{p \in P | a(p) + b(p)f = a(q) + b(q)f\} = \{p \in P | p_1 + p_2 f = q_1 + q_2 f\}$$

is an affine set.

Let,

$$p_0 = \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix}, \quad x(t+1) = 0.5x(t) + 0.3u(t), \quad x(t_0) = x_0.$$

Let $f_1 = 1$. Then $a(p_0) + b(p_0)f_1 = 0.8$ and

$$P_1(p_0, f_1) = \{p \in P | a(p) + B(p)f_1 = a(p_0) + b(p_0)f_1\} = \{p \in P | p_1 + p_2 = 0.8\}.$$

Suppose that the modeller selects

$$p = \begin{pmatrix} 0.9 \\ -0.1 \end{pmatrix} \in P_1(p_0, f_1).$$

Based on this estimate a control engineer or economist may select the feedback law $u(t) = f_2 x(t)$ with $f_2 = 2$. Then the real system satisfies $a(p_0) + b(p_0)f_2 = 0.5 + 0.3 \times 2 = 1.1$ and hence is unstable, while the control engineer or economist expects the system to behave like $a(p) + b(p)f_2 = 0.9 - 0.1 \times 2 = 0.7$.

### 5.2 Structural identifiability of economic systems operating under state feedback

The purpose of this subsection is to investigate structural identifiability of several economic systems. By an economic system we mean a set of differential or difference equations which describe the evolution of an economic phenomenon over time. Below we introduce a few examples of simple macroeconomic models taken from a standard textbook, see [4].

The authors favor the study of economic systems that are based on economic laws. This should be seen in contrast with modeling in econometrics where the output variables are assumed to depend linearly on lagged outputs and lagged instruments. Laws of macro economics or micro economics lead to dynamical systems that are structured. References that contain dynamic systems of economic models are [3, 4, 7, 16]. It is assumed that the economic system includes a specification of inputs and outputs. Thus no consideration is given to the question of distinguishing inputs and outputs in an observation vector nor of causality. In the literature on economics causality has been investigated, for example by W. Granger [11].

**Example 5.5** The first example is the Solow model of economic growth [25]. Let $Y$ denote output, $K$ capital, $N$ labour force, $C$ consumption, $I$ investment, $G$ government spending, $S$ savings, $T$ taxes, $D$ goverment debt, $r$ interest rate, and $\delta$ depreciation rate. Consider the following relations between these variables

$$\begin{aligned} Y(t) &= C(t) + I(t) + G(t) = C(t) + S(t) + T(t), && (44) \\ I(t) &= \dot{K}(t) - \delta K(t), && (45) \\ \dot{D}(t) &= rD(t) + G(t) - T(t), && (46) \\ Y(t) &= F(K(t), N(t)). && (47) \end{aligned}$$

Here F(K,N) is a production function of the economy. The first equation postulates that output is partially consumed in private sector, partially by the government, and partially invested. The second equation states that the amount of capital changes due to investment and to depreciation of the existing capital stock.

In what follows we assume that the production function is homogeneous of degree one (or has constant return to scale) and the population grows at rate $n$. We also assume here that all the population is included into the labour force. This is not so realistic but we do not want to consider problems of employment within the model.

We denote by lower case letters the per capita (equal to per worker) values of the variables. Then system (44,45,46,47) takes the form

$$\begin{aligned} f(k) &= c + i + g, && (48) \\ i &= \dot{k} - \delta k - nk, && (49) \\ \dot{d} &= rd + g - \tau, && (50) \end{aligned}$$

where $f(k) = F(K/N, 1)$. We assume that $f(k)$ is strictly concave, $f(0) = 0$, $f'(0) = \infty$, $f'(\infty) = 0$. Under these conditions maximization of profits in production leads to the following expression for the interest rate (see, for example,[4])

$$r = f'(k) - \delta. \qquad (51)$$

Now system (48,49,50,51) can be represented in the following form

$$\begin{aligned} \dot{k} &= f(k) - g - c + (n + \delta)k, && (52) \\ \dot{d} &= (f'(k) - \delta)d + g - \tau. && (53) \end{aligned}$$

Suppose that there is some central planner who makes a decision on consumption (i.e. which part of the income is to be consumed), on government spendings and on taxes, so that c, g, and t are the instruments and can be choosen proportionally to some other variables. For example, consumption $c = sf(k)$ (a constant part of the income is consumed), $\tau = \tau_0 + \xi d$ (amount of taxes is proportional to the existing government debt), $g = \chi f(k)$.

Let's denote by $k^*$ and $d^*$ the steady state values of capital and debt. Linearising equations (52,53) around steady state gives

$$\begin{aligned} \dot{k} &= +(n + \delta)(k - k^*) - \Delta c + f'(k^*)(k - k^*) - \Delta g, && (54) \\ \dot{d} &= f''(k^*)(k - k^*)d^* + (f'(k^*) - \delta)(d - d^*) + \Delta g - \Delta \tau. && (55) \end{aligned}$$

The resulting system can be represented as the following state space system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (56)$$
$$y(t) = x(t), \quad (57)$$

where

$$x_1(t) = k(t) - k^*, x_2(t) = d(t) - d^*, u_1 = \Delta c(t), u_2(t) = \Delta g(t), u_3(t) = \Delta\tau(t),$$

$$A = \begin{pmatrix} +(n+\delta) + f'(k^*) & 0 \\ f''(k^*)d^* & (f'(k^*) - \delta) \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}. \quad (58)$$

Consider the economic system (56,57). This closed-loop economic system is structurally identifiable from the impulse response if the open loop system is structurally identifiable from the impulse response. The argument of a proof of this statement goes as follows. Because the $B$ matrix does not depend on the parameter vector, the equation $A(p) + BF = A(q) + BF$ implies $A(p) = A(q)$. From assumption 5.2.1 follows that $p = q$. Hence the closed-loop economic system is structural identifiable from the impulse response.

**Example 5.6** As the second example we consider a version of an open economy model developed by Mundell-Fleming and Dornbusch [9]. It is assumed that the country is small enough and takes all foreign variables as given and uneffected by its own actions. The exchange rate $E$ is assumed to be flexible and equal to a price of domestic currency in terms of foreign currency. Let i be a domestic rate of interest and i' be a foreign rate of interest. Then by arbitrage domestic bonds and foreign bonds must pay the same rate of return. Thus

$$i = i' + \frac{1}{E}\frac{dE}{dt}.$$

Demand for domestic goods depends on the income, real exchange rate, and the index of fiscal policy, so it can be represented in the form

$$Y = A(Y, \frac{EP'}{P}, F).$$

where Y is income, P and P' are price level in the domestic and foreign country respectively, F is the index of fiscal policy. Money market is described by so-called LM equation

$$\frac{M}{P} = L(Y, i),$$

where M is the amount of money, $i$ is the nominal interest rate. Usually this equation is used in the inverse form

$$i = i(\frac{M}{P}, Y),$$

where i is assumed to be linear function of its arguments with $\partial i/\partial(M/P) < 0$ and $\partial i/\partial Y > 0$. We will present two versions of this model. One of them assumes that prices are fixed and output adjusts slowly to spendings and the second one assumes that output is fixed and prices adjust. In addition we will consider a combined model.

To obtain the first equation of the model we combine the equations and get

$$\frac{1}{E}\frac{dE}{dt} = i(\frac{M}{P}, Y) - i'.$$

We can write this equation in terms of logarithms of M,P,Y,E which we denote by the corresponding lower case letters. Thus we obtain

$$\dot{e} = \alpha(m - p) + \beta y.$$

All the variables representing the economy abroad are assumed to be equal to one so that logarithms are equal to zero.

The second equation of the model takes the form

$$\dot{Y} = \phi(A(Y, EP'/P, F) - Y), \tag{59}$$

which expresses that output adjusts to movements in spendings. We can write this equation in terms of logarithms as follows

$$\dot{y} = \gamma y + \delta(e - p) + \epsilon f,$$

where we assume that $\log P' = 0$. Thus for a fixed price level $p$ we obtain the following model

$$\dot{e} = \alpha m + \beta y, \tag{60}$$

$$\dot{y} = \gamma y + \delta e + \epsilon f. \tag{61}$$

Here m and f are instruments of the government which represent the amount of money and government spendings. We assume e and y to be state variables and m and f to be controls. The system has the following state space representation

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{62}$$

$$y(t) = x(t), \tag{63}$$

where $x_1(t) = e(t)$, $x_2(t) = y(t)$, $u_1(t) = m(t)$, $u_2(t) = f(t)$,

$$A = \begin{pmatrix} 0 & \beta \\ \delta & \gamma \end{pmatrix}, \quad B = \begin{pmatrix} \alpha & 0 \\ 0 & \epsilon \end{pmatrix}.$$

In the sequel a particular control law is needed. It is assumed that a particular control law is chosen from the family specified by the equations

$$m(t) = \zeta y(t) + \eta e(t), f(t) = \nu y(t). \tag{64}$$

This may be written as $u(t) = Fx(t)$ with

$$F = \begin{pmatrix} \eta & \zeta \\ 0 & \nu \end{pmatrix}. \tag{65}$$

For all feedback laws $F \in R^{m \times n}$ outside an algebraic variety the closed-loop economic system

$$\dot{x}(t) = [A(p) + B(p)F]x(t), \quad x(t_0) = x_0,$$

is not structurally identifiable from the impulse response.

Even for the particular feedback law (65) the closed-loop system is not structurally identifiable from the impuls response.

It is shown that the closed-loop system is not structurally identifiable. Let

$$F = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}. \tag{66}$$

The relation

$$A(p) + B(p)F = A(q) + B(q)F \tag{67}$$

is equivalent with

$$\alpha_1 f_{11} = \alpha_2 f_{11}, \quad \alpha_1 f_{12} + \beta_1 = \alpha_2 f_{12} + \beta_2, \tag{68}$$

$$\delta_1 + \epsilon_1 f_{21} = \delta_2 + \epsilon_2 f_{21}, \quad \gamma_1 + \epsilon_1 f_{22} = \gamma_2 + \epsilon_2 f_{22}. \tag{69}$$

Because by assumption the possible values of $F$ exclude an algebraic variety, $f_{ij} \neq 0$ for all $i, j = 1, 2$, hence from (68) follows that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. It is clear that there exist $p \neq q$ such that (67) is satisfied.

Consider the particular feedback law (64). Thus $f_{21}$ in (69) is equal to zero. Then from (69) we have that $\delta_1 = \delta_2$. However, we still cannot uniquely determine $\gamma$ and $\epsilon$ from (69). This means that the system is not structurally identifiable from the impuls response with this particular feedback law.

**Example 5.7** If in example 5.6 we allow for flexibility both in prices and output then we obtain the model of the form

$$\dot{e} = \alpha(m - p) + \beta y, \tag{70}$$

$$\dot{y} = \gamma y + \delta(e - p) + \epsilon f, \tag{71}$$

$$\dot{p} = \phi y + \psi(e - p) + \xi f. \tag{72}$$

This system has the following state space representation $x_1 = e$, $x_2 = y$, $x_3 = p$, $u_1 = m$, $u_2 = f$,

$$A = \begin{pmatrix} 0 & \beta & -\alpha \\ \delta & \gamma & -\delta \\ \psi & \phi & -\psi \end{pmatrix}, \quad B = \begin{pmatrix} \alpha & 0 \\ 0 & \epsilon \\ 0 & \xi \end{pmatrix}.$$

To be definite, we assume that a particular control law is chosen from the family specified by the equations

$$m(t) = \zeta y(t) + \eta e(t) + \mu p(t), f(t) = \nu y(t). \tag{73}$$

This may be written as $u(t) = Fx(t)$ with

$$F = \begin{pmatrix} \eta & \zeta & \mu \\ 0 & \nu & 0 \end{pmatrix}. \tag{74}$$

The first equation shows that monetary policy $m$ can be expressed as a linear combination of output, exchange rate and prices. This choice can be justified by the fact that monetary policy has strong effect on all the three state variables. It seems to be natural that money supply influence the price level and interest rate. On the other hand changes in interest rate cause capital inflow or outflow which in its turn influence the exchange rate. However, this has effect on price of imports providing exports with a competitive advantage or disadvantage. Changes in amount of exports lead to changes in the level of income. Thus monetary policy not only influence prices and exchange rate but the level of output as well.

Equation (73) says that the government purchases a constant proportion of output which is the usual assumption in the economic literature. Generaly speaking we could assume that $f$ depends on both $p$ and $y$ in (64). However, as one can see from the further analysis this will not lead to any new conclusions on the identifiability of the model.

The structured linear dynamic system defined above with the feedback law (74) may be shown to be structurally identifiable under state feedback.

# 6 Structural identifiability of a structured linear dynamic system operating under dynamic output feedback

In this section the problem of structural identifiability is treated for a structured linear dynamic system operating under dynamic output feedback. Thus the state of the dynamic system is not observed but an output function is. In control theory it is often argued that for a dynamic system of which only the output is observed the attainment of the control objectives requires the use of dynamic output feedback. The solution of the linear-quadratic-Gaussian (LQG) optimal stochastic control problem also shows this. For control of a linear dynamic system attention in dynamic output feedback is often limited to a controller consisting of a linear dynamic system and of a linear control law operating on the state of this controller. Such a controller has not been studied extensively in the economics and econometrics literature. Note however, that a controller based on an ARMAX representation as used in econometrics, is often a dynamic output feedback controller.

## 6.1 Conditions for structural identifiability in case of dynamic output feedback

Consider a structured linear dynamic system of the form

$$x(t+1) = A(p)x(t) + B(p)u(t), \quad x(t_0) = x_0, \tag{75}$$

$$y(t) = C(p)x(t) + D(p)u(t). \tag{76}$$

Suppose that the output of this system can be observed. Suppose further that a *dynamic output feedback law* or *dynamic controller* is used consisting of another discrete-time time-invariant linear dynamic system

$$z(t+1) = Fz(t) + Gy(t), \quad z(t_0) = z_0, \tag{77}$$

$$u(t) = Hz(t), \tag{78}$$

where $Z = R^{n_z}$, $F \in R^{n_z \times n_z}$, $G \in R^{n_z \times k}$, and $H \in R^{m \times n_z}$. The combination of the structured linear system (75,76) and the dynamic output feedback (77,78) is then called the associated *closed-loop system.* It is given by the equations

$$\begin{pmatrix} x(t+1) \\ z(t+1) \end{pmatrix} = \begin{pmatrix} A(p) & B(p)H \\ GC(p) & F+GD(p)H \end{pmatrix} \begin{pmatrix} x(t) \\ z(t) \end{pmatrix}, \begin{pmatrix} x(t_0) \\ z(t_0) \end{pmatrix} = \begin{pmatrix} x_0 \\ z_0 \end{pmatrix}, \quad (79)$$

$$\begin{pmatrix} y(t) \\ u(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} C(p) & D(p)H \\ 0 & H \\ 0 & I \end{pmatrix} \begin{pmatrix} x(t) \\ z(t) \end{pmatrix}. \quad (80)$$

The output of this system consists of $y, u, z$ because all three functions can be observed. The state of the controller can be observed because the controller has been build by the control engineer.

**Problem 6.1** *Consider the structured linear dynamic system (75,76) with parametrization* $(P, f)$ $f : P \to SL\Sigma P$,

$$\begin{aligned} & SL\Sigma P(n,m,k) \\ = & \{((A(p), B(p), C(p), D(p)) \in R^{n\times n} \times R^{n\times m} \times R^{k\times n} \times R^{k\times m} | p \in P\}. \end{aligned}$$

*Consider the control law (77,78) based on dynamic output feedback. The resulting closed-loop system is given by (79,80). Denote this system by*

$$\bar{x}(t) = \bar{A}(p)\bar{x}(t), \bar{x}(t_0) = x_0, \quad (81)$$

$$\bar{y}(t) = \bar{C}(p)\bar{x}(t). \quad (82)$$

*Denote the set of closed-loop systems by* $CSL\Sigma$ *and* $CSL\Sigma P(n+n_z, k+m+n_z)$, *and its parametrization by* $(P, f_1)$,

$$\begin{aligned} f_1 & \to CSL\Sigma P(n+n_z, k+m+n_z) \\ & \subset R^{(n+n_z)\times(n+n_z)} \times R^{(k+m+n_z)\times(n+n_z)} \times R^{n+n_z}, \end{aligned}$$

$$f_1(p) = \left( \begin{pmatrix} A(p) & B(p)H \\ GC(p) & F+GD(p)H \end{pmatrix}, \begin{pmatrix} C(p) & D(p)H \\ 0 & H \\ 0 & I \end{pmatrix}, \begin{pmatrix} x_0(p) \\ z_0 \end{pmatrix} \right).$$

*Is this parametrization of the closed-loop structurally identifiable from the initial condition response?*

As in the previous section, attention is limited to a fixed control law. This limitation is motivated by economic modeling. In 6.1 the problem is posed of structural identifiability from the initial condition response. Note that the state process $x$ is not available as observation. The requested structural identifiability property therefore differs from that studied in section 5. The input function is regarded as an observation component. Since also the state of the controller is an observation component, the input could be deleted. It is left in to emphasize the structure of the problem.

Problem 6.1 will investigated under several assumptions.

**Assumption 6.2** *1. The open-loop structured linear dynamic system (75,76) is structurally reachable, structurally observable, and structurally identifiable from the impulse response function.*

*2. The dynamic control law (77,78) is assumed known, fixed, and a minimal realization of its impulse response function.*

*3. The closed-loop structured system (79,80) is structurally observable.*

Assumption 1 of 6.2 must be imposed because otherwise the closed-loop system cannot be structurally identifiable. The assumption can be checked from the specification of the class of systems. Assumption 2 must also be imposed because otherwise the closed-loop system cannot be structurally identifiable. Because the controller is assumed known the assumption can easily be checked. It is well known that the interconnection of two minimal linear dynamic systems in closed-loop need not be minimal. If the closed-loop system is not minimal then there is a cancellation of dynamics. This is in some sense a rare event. The control objective of robustness of the closed-loop system for unmodelled dynamics implies that cancellation of dynamics should be avoided. Therefore assumption 3 is based on sound control theoretic arguments. What remains of the indeterminedness of the closed-loop system is the invariance due to the state space transformation. This indeterminedness is investigated below.

**Lemma 6.3** *Consider problem 6.1 and the assumptions 6.2. Assume that the for almost all $p \in P$ $(\bar{A}(p), \bar{x}_0(p))$ is a reachable pair. The closed-loop system is structurally identifiable from the initial condition response iff for $p, q \in P$ outside an algebraic variety and $T \in R^{(n+n_z)\times(n+n_z)}$ nonsingular of the form*

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & I \end{pmatrix},$$

*with $T_{11} \in R^{n\times n}$, $T_{12} \in R^{n\times n_z}$, $T_{11}$ nonsingular, the equations*

$$C(p) = C(q)T_{11}, \quad (83)$$
$$D(p)H = C(q)T_{12} + D(q)H, \quad (84)$$
$$T_{11}A(p) + T_{12}GC(q) = A(q)T_{11}, \quad (85)$$
$$GD(p)H = GD(q)H + GC(q)T_{12}, \quad (86)$$
$$T_{11}B(p)H + T_{12}F + T_{12}GD(p)H = A(q)T_{12} + B(q)H, \quad (87)$$
$$T_{11}x_0(p) + T_{12}z_0 = x_0(q), \quad (88)$$

*imply that $p = q$.*

**Proof** From 4.6 follows that the closed-loop system (79,80) is structurally identifiable from the initial condition response iff the condition of that theorem holds. It follows from assumption 6.2 that the system (79,80) is structurally observable. Because by assumption for all $p \in P$ outside an algebraic variety $(\bar{A}(p), \bar{x}_0(p))$ is a structured reachable pair, the assumptions of 4.6 holds. Assume first that the system is structurally identifiable from the initial condition response. Let $q \in P$, and consider the equations for $p \in P$ and $T$ nonsingular. Partition

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in R^{(n+n_z)\times(n+n_z)}$$

in which $T_{12} \in R^{n\times n_z}$ and the other matrices are conformally. The equation $\bar{C}(p) = \bar{C}(q)T$ is equivalent with the relations

$$\begin{aligned} T_{21} &= 0, T_{22} = I, && (89) \\ C(p) &= C(q)T_{11}, && (90) \\ D(p)H &= C(q)T_{12} + D(q)H. && (91) \end{aligned}$$

The equation $T\bar{A}(p) = \bar{A}(q)T$ is equivalent with the relations (85,86,87) and

$$GC(q)T_{11} = GC(p).$$

The latter relation follows from (90). Finally $T\bar{x}_0(p) = \bar{x}_0(q)$ is equivalent to (88).

Conversely, assume that the characterization of the lemma holds. The conditions of the lemma then yield by the above arguments that for almost all $q \in P$ the equations

$$T\bar{A}(p) = \bar{A}(q)T, \bar{C}(p) = \bar{C}(q)T, T\bar{x}_0(p) = \bar{x}_0(q).$$

for $p \in P$ and $T \in R^{(n+n_z)\times(n+n_z)}$ imply that $p = q$. From theorem 4.6 then follows that the system is structurally identifiable from the initial condition response. □

**Proposition 6.4** *Consider the structured linear dynamic system*

$$\begin{aligned} x(t+1) &= A(p)x(t) + B(p)u(t), \quad x(t_0) = x_0, && (92) \\ y(t) &= x(t), && (93) \end{aligned}$$

*with parametrization $(P, f)$, the control law (77,78), and the assumption 6.2. Note that the state $x$ is observed. Assume that $n_z \geq m$ and $rank(H) = m$. Assume in addition that $(\bar{A}(.), \bar{x}_0)$ is generically a reachable pair. Then the parametrization $(P, f_1)$ of the closed-loop system*

$$\begin{pmatrix} x(t+1) \\ z(t+1) \end{pmatrix} = \begin{pmatrix} A(p) & B(p)H \\ G & F \end{pmatrix} \begin{pmatrix} x(t) \\ z(t) \end{pmatrix}, \quad \begin{pmatrix} x(t_0) \\ z(t_0) \end{pmatrix} = \begin{pmatrix} x_0 \\ z_0 \end{pmatrix}, \qquad (94)$$

$$\begin{pmatrix} y(t) \\ u(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & H \\ 0 & I \end{pmatrix} \begin{pmatrix} x(t) \\ z(t) \end{pmatrix}, \qquad (95)$$

*is structurally identifiable from the initial condition response.*

**Proof** Proposition 6.3 is applied. From (83) and $C = I$ follows that $T_{11} = I$. From (84) and $C = I$ follows that $T_{12} = 0$. With this it follows from (85) that $A(p) = A(q)$, from (87) that $B(p)H = B(q)H$, and from (88) that $x_0(p) = x_0(q)$. If $n_z \geq m$ and $rank(H) = m$, then $B(p)H = B(q)H$ implies that $B(p) = B(q)$. From assumption 6.2 follows that the structured linear system is structurally identifiable from the impulse response function, hence

$$A(p) = A(q), \; B(p) = B(q), \; C(p) = I = C(q), \; D(p) = 0 = D(q),$$

imply that $p = q$. From proposition 6.3 then follows that the closed-loop system is structurally identifiable. □

The conclusion of proposition 6.4 is that although a structured linear system may not be structurally identifiable under state feedback, by using dynamic output feedback satisfying the conditions of 6.4 the closed-loop system becomes structurally identifiable. The important conditions are that $rank(H) = m$ or that the controller has sufficient high dynamics, and that $(\bar{A}(.), \bar{x}_0)$ is a reachable pair or that $\bar{x}_0$ excites the system sufficiently well.

Instead of a dynamic output feedback for the control system (75,76) one may also consider a static output feedback controller, say $u(t) = Fy(t)$, for some $F \in R^{m \times k}$. The closed-loop control system is then given by the recursion

$$x(t+1) = [A(p) + B(p)FC(p)]x(t), \quad x(t_0) = x_0.$$

The reader should now be able to characterize structural identifiability from the initial condition response for this system.

## 6.2 Structural identifiability of economic systems operating under dynamic output feedback

**Example 6.5** Consider example 5.6 the Dornbusch model for an open economy with a fixed price level. The dynamic system is given by

$$x(t+1) = A(p)x(t) + B(p)u(t), \quad x_0(t_0) = x_0, \tag{96}$$

$$y(t) = x(t), \tag{97}$$

$$A(p) = \begin{pmatrix} 0 & \beta \\ \delta & \gamma \end{pmatrix}, \quad B(p) = \begin{pmatrix} \alpha & 0 \\ 0 & \epsilon \end{pmatrix}, \tag{98}$$

$$p = (\alpha, \beta, \gamma, \delta, \epsilon) \in R^5. \tag{99}$$

As argued in example 5.6 this system is not structurally identifiable from the impulse response under state feedback. Consider the dynamic output feedback

$$z(t+1) = Fz(t) + Gy(t), \quad z(t_0) = z_0, \tag{100}$$

$$u(t) = Hz(t), \tag{101}$$

with $n_z = 2$ and $rank(H) = 2$.

Assume that the dynamic control law (100,101) is known, fixed, and a minimal realization of its impulse response function. Assume also that the closed-loop system consisting of (96,97) and (100,101) is structurally minimal. Assume that the pair $(\bar{A}(p), \bar{x}(p))$ is structurally reachable. It will be argued that under these conditions the closed-loop system is structurally identifiable from the initial condition response. Because

$$B(p) = \begin{pmatrix} \alpha & 0 \\ 0 & \epsilon \end{pmatrix},$$

the system is structurally reachable. Because $C = I$ it is structurally observable. A short calculation establishes that it is structurally identifiable from the impulse response. Then condition 1 of assumption 6.2 is satisfied. It follows then from proposition 6.4 that the closed-loop system is structurally identifiable from the initial condition response.

## 7 Conclusion and open problems

The motivation of this paper is modeling in economics. That economic systems operate in closed-loop must be taken in account in modeling of economic phenomena. The problem considered in the paper is whether the parameters of an economic system operating in closed-loop can be determined uniquely from the observations. This problem is formulated as a structural identifiability problem of a structured linear dynamic system operating in closed-loop. Necessary and sufficient conditions for structural identifiability are presented. For several economic systems structural identifiability is investigated. Some economic systems are structurally identifiable and one is not.

An open problem is structural identifiability of a structured linear system operating under feedback with exogeneous input. Other open problems are structural identifiability of other classes of structured dynamic systems operating under feedback, such as positive linear dynamic systems and nonlinear dynamic systems. Economic systems may be positive linear dynamic systems, see [3], or be nonlinear dynamic systems. Structural identifiability of nonlinear dynamic systems has been investigated to some extent, see [27, 28].

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## References

[1] B.D.O. Anderson and M. Gevers. Identifiability of linear stochastic systems operating under feedback. *Automatica*, 18:195–213, 1982.

[2] R. Bellman and K.J. Aström. On structural identifiability. *Math. Biosciences*, 7:329–339, 1970.

[3] A. Berman, M. Neumann, and R.J. Stern. *Nonnegative matrices in dynamic systems.* John Wiley & Sons, New York, 1989.

[4] O.J. Blanchard and S. Fischer. *Lectures on macroeconomics.* MIT Press, Cambridge, MA, 1989.

[5] J. Bochnak, M. Coste, and M.-F. Roy. *Géometrie algébrique réelle.* Springer-Verlag, Berlin, 1987.

[6] F.M. Callier and C.A. Desoer. *Linear system theory.* Springer-Verlag, New York, 1991.

[7] G.C. Chow. *Analysis and control of dynamic economic systems.* John Wiley & Sons, New York, 1975.

[8] E.J. Davison. Connectability and structural controllability of composite systems. *Automatica*, 13:109–123, 1977.

[9] R. Dornbusch. Expectations and exchange rate dynamics. *J. of Political Economics*, 84:1161–1176, 1976.

[10] K. Glover and J.C. Willems. Parametrizations of linear dynamical systems: Canonical forms and identifiability. *IEEE Trans. Automatic Control*, 19:640–645, 1974.

[11] C.W.J. Granger. Testing for causality - a personal viewpoint. *J. Econ. Dyn. Control*, 2:329–352, 1980.

[12] I. Gustavsson, L. Ljung, and T. Söderström. Identification of processes in closed-loop - identifiability and accuracy aspects. *Automatica*, 13:59–75, 1977.

[13] C. Heij. *Deterministic identification of dynamical systems*, volume 127 of *Lecture Notes in Control and Information Sciences.* Springer-Verlag, Berlin, 1989.

[14] R.E. Kalman. Mathematical description of linear dynamical systems. *SIAM J. Control*, 1:152–192, 1963.

[15] R.E. Kalman, P.L.Falb, and M.A. Arbib. *Topics in mathematical systems theory.* McGraw-Hill Book Co., New York, 1969.

[16] D. Kendrick. *Stochastic control for economic models.* McGraw-Hill Book Co., New York, 1981.

[17] S. Lang. *Algebra.* Addison-Wesley Publ. Co., Reading, MA, 1971.

[18] Ching-Tai Lin. Structural controllability. *IEEE Trans. Automatic Control*, 19:201–208, 1974.

[19] L. Ljung. *System identification: Theory for the user.* Prentice-Hall, Inc., Englewood Cliffs, NJ, 1987.

[20] L. Ljung, I. Gustavsson, and T. Söderström. Identification of linear multivariable systems operating under feedback. *IEEE Trans. Automatic Control*, 19:836–840, 1974.

[21] K. Murota. Combinatorial dynamical system theory: General framework and controllability criteria. *Discrete Applied Math.*, 22:241–265, 1988/89.

[22] R.W. Shields and J.B. Pearson. Structural controllability of multiinput linear systems. *IEEE Trans. Automatic Control*, 21:203–212, 1976.

[23] T. Söderström, I. Gustavsson, and L. Ljung. Identifiability conditions for linear systems operating in closed-loop. *Int. J. Control*, 21:243–255, 1975.

[24] T. Söderström, L. Ljung, and I. Gustavsson. Identifiability conditions for linear multivariable systems operating under feedback. *IEEE Trans. Automatic Control*, 21:837–840, 1976.

[25] R. Solow. A contribution to the theory of economic growth. *Quarterly J. of Econometrics*, 70:65–94, 1956.

[26] E.D. Sontag. *Mathematical control theory: Deterministic finite dimensional systems.* Springer-Verlag, New York, 1990.

[27] E. Walter. *Identifiability of state space models*, volume 46 of *Lecture Notes in Biomathematics.* Springer-Verlag, Berlin, 1982.

[28] E. Walter, editor. *Identifiability of parametric models.* Pergamon Press, Oxford, 1987.