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J.W. Cohen

Department of Operations Research, Statistics, and System Theory

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P.O. Box 94079, 1090 GB Amsterdam (NL)
Kruislaan 413, 1098 SJ Amsterdam (NL)
Telephone +31 20 592 9333
Telefax +31 20 592 4199

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J.W. Cohen
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

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Abstract

Present-day modelling of traffic in broadband communication requires the use of rather sophisticated stochastic processes. Although a large class of suitable stochastic processes is known in the literature, their rather complicated structure limits their use because of the laborious numerical evaluation involved. The present study concerns the so-called periodic Pollaczek processes. The characteristics of the arrival process, such as service time τ_n and interarrival time σ_n, are here periodic functions of n, i.e. the vector (τ_n, σ_n) and $(\tau_{n+N}, \sigma_{n+N})$ have the same distribution, N being the period. The sequence $\mathbf{w}_n, n = 1, 2, \ldots$, defined by

$$\mathbf{w}_{n+1} = [\mathbf{w}_n - \tau_n - \sigma_n]^+$$

is investigated; as such the queue under consideration is a direct generalisation of the classical Pollaczek $GI/G/1$ queue. It appears that the model is a quite flexible one, and moreover very accessible for numerical evaluation if the distributions of all the service times, or of all the interarrival times, have rational Laplace-Stieltjes transforms.

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1. Introduction

For the classical $GI/G/1$ waiting time model Pollaczek characterises the structure of the actual waiting time process $\mathbf{w}_n, n = 1, 2, \ldots$, by the relations

$$\begin{aligned} \mathbf{w}_{n+1} &= [\mathbf{w}_n + \rho_n]^+ \qquad n = 1, 2, \ldots, \\ \mathbf{w}_1 &= w_1 \geq 0, \end{aligned} \tag{1.1}$$

where $\mathbf{w}_n$ is the actual waiting time of the nth arriving customer, w_1 the initial waiting time and

$$\rho_n := \tau_n - \sigma_n; \tag{1.2}$$

with τ_n the service time of the nth arriving customer, σ_n the time between the nth and $(n+1)st$ arrival. The $\tau_n, n = 1, 2, \ldots$, and similarly, the $\sigma_n, n = 1, 2, \ldots$, are i.i.d. nonnegative stochastic variables, and further $\{\tau_n, n = 1, 2, \ldots\}$ and $\{\sigma_n, n = 1, 2, \ldots\}$ are independent families.

A generalisation of this classical model may be formulated as follows. let

$$\mathbf{x}_n, \; n = 1, 2, \ldots, \tag{1.3}$$

be a discrete time Markov chain with a countable, irreducible state space $\mathcal{S}$ and with stationary transition probabilities. Here $(p_{ij}), i, j \in \mathcal{S}$, shall denote the one-step transition probabilities; $\mathbf{x}_1 = x_1$ shall be the initial state of the $\mathbf{x}_n$-process.

Further, let $\rho(\gamma), j \in \mathcal{S}$, be a set of stochastic variables with distributions $r_j(\cdot), j \in \mathcal{S}$, and let

$$\rho_n \equiv \rho(\mathbf{x}_n), \quad n = 1, 2, \ldots, \tag{1.4}$$

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be stochastic variables defined on the $\mathbf{x}_n$-process, such that for all $n = 1, 2, \ldots$,

$$\Pr\{\rho_{n+1} < \rho, \mathbf{x}_{n+1} = j | \mathbf{x}_m = i_m, \rho_m = \rho(i_m), m = 1, \ldots, n\} = r_{i_n}(\rho) p_{i_n j}, \tag{1.5}$$

for all $i_n \in \mathcal{S}$.

Define the sequence $\mathbf{w}_n, n = 1, 2, \ldots$, by

$$\mathbf{w}_{n+1} = [\mathbf{w}_n + \rho_n]^+, \quad n = 1, 2, \ldots, \tag{1.6}$$

for given initial $\mathbf{w}_1 = w_1 \geq 0$.

The process $(\mathbf{x}_n, \mathbf{w}_n), n = 1, 2, \ldots$, as defined above will be called a Pollaczek waiting time process. Obviously if we take ρ_n to be the difference of two nonnegative stochastic variables $\tau(\mathbf{x}_n)$ and $\sigma(\mathbf{x}_n)$, so

$$\rho_n \equiv \tau(\mathbf{x}_n) - \sigma(\mathbf{x}_n), \tag{1.7}$$

then $\mathbf{w}_n$ may be interpreted as the actual waiting time of the nth arriving customer whose service time $\tau(\mathbf{x}_n)$ and next interarrival time $\sigma(\mathbf{x}_n)$ depend on the state of the $\mathbf{x}_n$-process.

The $(\mathbf{x}_n, \mathbf{w}_n)$-process has been introduced by ARJAS [2]. He presents a fairly detailed study, however, does not present results which are useful in actual performance studies. ASMUSSEN and THORISSON [3] study a similar model as Arjas, but with the $\mathbf{x}_n$-process replaced by a Markov process with a general state space. Their goal concerns the modelling of periodic queues. Indeed, the classical queueing models are not suited to model queueing situations with arrival processes, which lack a simple regenerative structure. The introduction of semi-Markov processes, see e.g. [7], [8], [17], has provided a greater flexibility in the modelling of the arrival, and also in that of the servicing process. The inherent numerical analysis is of course rather laborious, but generally within the limits of available computer facilities.

Queueing models with periodic arrival processes have been studied by several authors, see e.g. [3], [4], [5], [6]. The relevant studies mainly concentrate on the derivation of limit theorems and ergodicity conditions. Tangible results suited for numerical evaluation are hardly obtained. Actually, authors experience with the investigation of an $M/G/1$ model with a periodical arrival process with rate $\lambda(\cdot)$ and $\lambda(t + \mu) = \lambda(t)$ for all $t > 0, \mu$ being the period, is most disappointing indeed. Even for the very simple case with

$$\begin{aligned} \lambda(t) &= \lambda_1 \quad \text{for} \quad 0 \leq t \leq \tfrac{1}{2}\mu, \\ &= \lambda_2 \quad \;,, \quad \tfrac{1}{2}\mu < t < \mu; \end{aligned}$$

the analysis of the actual waiting time process is governed by a complicated integral equation, which seems hardly accessible for further analysis.

A more promising approach of periodical queueing processes is obtained whenever the characteristics of the queueing model are periodical functions of the number of arrivals. Such a model and actually a very general one is provided by the $(\mathbf{x}_n, \mathbf{w}_n)$-process introduced above, if we specify the state space $\mathcal{S}$ and the one-step transition probabilities as follows:

$$\mathcal{S} = \{1, 2, \ldots, N\}, \tag{1.8}$$

$$\begin{aligned} p_{ij} &= 1 \text{ for } j = i + 1,\ i = 1, \ldots,\ N - 1, \text{ and for } i = N,\ j = 1, \\ &= 0 \text{ otherwise.} \end{aligned}$$

This special process will be called a periodic Pollaczek process. In section 2 the structure of this process is described and some notations are introduced. The ergodicity conditions are introduced, and it is assumed that they apply. These conditions follow from a general study of LOYNES [9].

In section 3 the functional equations for the Laplace-Stieltjes transforms of the stationary distributions of the actual waiting times $\mathbf{w}_1, \ldots, \mathbf{w}_N$ are formulated for the case that the service times and the interarrival times do not all have lattice distributions. In section 4 the functional equations are derived for the case that all these distributions are lattice variables with state space the set of positive integers.

In section 5 we discuss shortly the case $M = 1$, i.e. the classical Pollaczek model, as the well-known results exposed here are needed in the subsequent sections.

The functional equations derived in section 3 and 4 formulate a Hilbert Boundary Value Problem for a set of functions. Such boundary value problems have been extensively studied, see e.g. [10], [11] [12]. The somewhat special type of coefficients in our functional equations, which is due to the fact that they consist of the L.S.-transforms of the service time and interarrival time distributions, makes it possible to construct quite explicit solutions if the L.S.-transforms of all the service time distributions are rational functions without a need to specify those of the interarrival time distributions. Similarly, if all the L.S.-transforms of the interarrival time distributions are rational.

In section 6 the periodical Pollaczek process is analysed for the case that all the service time distributions have a rational L.S.-transform, section 7 discusses tha case with the L.S.-tranforms of all the interarrival time distributions rational functions. Actually the results obtained in sections 6 and 7 are extensions of the $GI/K_n/1$ and the $K_m/G/1$ model, see [14].

The solutions obtained in sections 6 and 7 are quite explicit apart from the solution of a set of linear equations. The number of equations depends on the degrees of the polynomials in the denominators of the rational L.S.-transforms.

In section 8 the mixed case is studied. Here the L.S.-transforms of the service time distributions are rational for a subset of the indices $\{1, \ldots, N\}$, whereas the transforms of the interarrival time distributions are rational for the indices belonging to the complementary subset. However, for this case the results are less explixit.

In the final section 9 some models are discussed where the greater part of the service- and interarrival times are constant. Whenever the other service times and/or interarrival times have rational L.S.-transforms again explicit results can be obtained.

The analysis of the periodical Pollaczek process shows that this process is quite flexible in modelling queueing processes with a rather complicated traffic structure, in particular they seem to be useful to describe models with bursty traffic. Further algebraic and numerical research is needed to judge the usefulness of these processes in actual performance analysis.

2. The periodic Pollaczek $GI_N/G_N/1$ queueing model

With a stightly differend notation we reformulate the $(\mathbf{x}_n, \mathbf{w}_n)$-process with the property (1.8) of the preceding section.

For a fixed integer $N \geq$ let

$$\rho_n = (\rho_1^{(n)}, \ldots, \rho_N^{(n)}), \quad n = 1, 2, \ldots, \tag{2.1}$$

be a sequence of i.i.d. stochastic vectors, with the components of each vector independent stochastic variables.

The sequence of vectors $\mathbf{w}_n, n = 1, 2, \ldots,$

$$\mathbf{w}_n = (\mathbf{w}_1^{(n)}, \ldots, \mathbf{w}_N^{(n)}), \quad n = 1, 2, \ldots, \tag{2.2}$$

is recursively defined by: for $n = 1, 2, \ldots,$

$$\mathbf{w}_j^{(n)} = [\mathbf{w}_{j-1}^{(n)} + \rho_{j-1}^{(n)}]^+ \text{ for } j = 2, \ldots, N, \tag{2.3}$$

$$\mathbf{w}_1^{(n+1)} = [\mathbf{w}_N^{(n)} + \rho_N^{(n)}]^+,$$

with initial value

$$\mathbf{w}_1^{(1)} = w_1 \geq 0.$$

Concerning the $\rho_j^{(n)}$ it will be assumed that it is the difference of two independent and positive variables,

$$\rho_j^{(n)} = \tau_j^{(n)} - \sigma_j^{(n)}, \quad j = 1, \ldots, N; \quad n = 1, 2, \ldots. \tag{2.5}$$

The $\mathbf{w}_n$-process so defined will be called a *periodical* Pollaczek $GI_N/G_N/1$ queueing model with period N.

We introduce some further notations and assumptions.

By $\boldsymbol{\tau}_j, \boldsymbol{\sigma}_j$ and $\boldsymbol{\rho}_j, j = 1, \ldots, N$, we shall denote stochastic variables such that $\boldsymbol{\tau}_1, \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\tau}_N, \boldsymbol{\sigma}_N$, are all independent and for $n = 1, 2, \ldots$,

$$\boldsymbol{\tau}_j \sim \boldsymbol{\tau}_j^{(n)}, \; \boldsymbol{\sigma}_j \sim \boldsymbol{\sigma}_j^{(n)}, \; \boldsymbol{\rho}_j \sim \boldsymbol{\rho}_j^{(n)}, \; j = 1, \ldots, N. \tag{2.6}$$

It will be always assumed that: for $j = 1, \ldots, N$,

i. $\quad \beta_j := \mathrm{E}\{\boldsymbol{\tau}_j\} < \infty, \quad \alpha_j := \mathrm{E}\{\boldsymbol{\sigma}_j\} < \infty, \quad \gamma_j := \mathrm{E}\{\boldsymbol{\rho}_j\};$ (2.7)

ii. $\quad \gamma = \beta - \alpha < 0,$

where

$$\beta := \sum_{j=1}^{N} \beta_j, \quad \alpha := \sum_{j=1}^{N} \alpha_j, \quad \gamma := \sum_{j=1}^{N} \gamma_j.$$

Further we define for $j = 1, \ldots, N$,

$$\begin{aligned}
\alpha_j(\rho) &:= \mathrm{E}\{e^{-\rho \boldsymbol{\sigma}_j}\}, \quad \beta_j(\rho) := \mathrm{E}\{e^{-\rho \boldsymbol{\tau}_j}\}, && \mathrm{Re}\,\rho \geq 0, \\
\gamma_j(\rho) &:= \alpha_j(-\rho)\beta_j(\rho), && \mathrm{Re}\,\rho = 0, \\
\alpha(\rho) &:= \prod_{j=1}^{N} \alpha_j(\rho), \quad \beta(\rho) := \prod_{j=1}^{N} \beta_j(\rho), && \mathrm{Re}\,\rho \geq 0, \\
\gamma(\rho) &:= \alpha(-\rho)\beta(\rho), && \mathrm{Re}\,\rho = 0.
\end{aligned} \tag{2.8}$$

Whenever the $\alpha_j(\rho)$ and/or the $\beta_j(\rho)$ are rational functions of ρ the we write: for $j = 1, \ldots, N$,

$$\alpha_j(\rho) = \frac{a_{1j}(\rho)}{a_{2j}(\rho)}, \quad \beta_j(\rho) = \frac{b_{1j}(\rho)}{b_{2j}(\rho)}, \quad \mathrm{Re}\,\rho \geq 0, \tag{2.9}$$

with

i. $a_{2j}(\rho)$ *a polynomial of degree* m_j, (2.10)
$b_{2j}(\rho)$,, ,, ,, ,, n_j;

ii. $a_{1j}(\rho)$ *a polynomial of degree* $< m_j$,
$b_{1j}(\rho)$,, ,, ,, ,, $< n_j$;

iii. $a_{1j}(0) = a_{2j}(0) \neq 0$,
$b_{1j}(0) = b_{2j}(0) \neq 0$;

$$\text{iv.}\qquad a_k(\rho)=\prod_{j=1}^{N}a_{kj}(\rho)\quad b_k(\rho)=\prod_{j=1}^{N}b_{kj}(\rho),\quad k=1,2,\ \operatorname{Re}\rho\geq 0,$$

$$\mu:=\sum_{j=1}^{N}m_j,\qquad \nu:=\sum_{j=1}^{n}n_j;$$

and it is assumed that the various rational functions in (2.9) are irreducible, so that

$$\begin{array}{lllll}\alpha_j(\rho) & \textit{has} & m_j & \textit{poles} \quad \textit{in} & \operatorname{Re}\rho<0,\\ \beta_j(\rho) & ,, & n_j & ,, \quad ,, & \operatorname{Re}\rho<0;\end{array}\tag{2.11}$$

these poles counted according to their multiplicities.

REMARK 2.1. Unless stated otherwise it will always be assumed that $\boldsymbol{\tau}_j$ and $\boldsymbol{\sigma}_j$, $j=1,\ldots,N$, have absolutely continuous distributions, so that, cf. [16],

$$\begin{array}{lllll}|\alpha_j(\rho_0)|=1 & \textit{with} & \operatorname{Re}\rho_0=0 & \textit{implies} & \rho_0=0,\\ |\beta_j(\rho_0)|=1 & ,, & ,, & ,, & ,, .\end{array}\tag{2.12}$$

This restriction is for the greater part of the following analysis rather inessential, but the case with all $\boldsymbol{\tau}_j$ and all $\boldsymbol{\sigma}_j$ being lattice variables requires a slightly different approach. In such a case it is more convenient to work with the generating functions of the distributions of $\boldsymbol{\tau}_j$ and $\boldsymbol{\sigma}_j$, instead of using their Laplace-Stieltjes transforms, see section 4. □

REMARK 2.2. The nth arriving customer with $k=n \bmod N$ will be called a type "k"-customer. □

REMARK 2.3. Next to the vector sequence $\mathbf{w}_n$, $n=1,2,\ldots$, we introduce the vector sequence

$$\mathbf{i}_n=(\mathbf{i}_1^{(n)},\ldots,\mathbf{i}_N^{(n)}),\quad n=1,2,\ldots,$$

with

$$\mathbf{i}_j^{(n)}:=-[\mathbf{w}_j^{(n)}+\boldsymbol{\rho}_j^{(n)}]^-,\quad j=1,\ldots,N.\tag{2.13}$$

Obviously with probability one

$$\mathbf{i}_j^{(n)}\geq 0. \qquad □\tag{2.14}$$

3. DERIVATION OF THE FUNCTIONAL EQUATIONS

From the definition of the $\mathbf{w}_n$-process in the previous secton it follows that the sucessive epochs of the sequence

$$\mathbf{w}_1^{(n)},\ n=1,2,\ldots,\quad \textit{with}\ w_1=0,\tag{3.1}$$

at which $\mathbf{w}_1^{(n)}=0$ are regeneration points of the $\mathbf{w}_n$-process. Note, however, that if $\boldsymbol{\tau}_N>\boldsymbol{\sigma}_N$ with probability one then the sequence does not have such epochs. But the assumption (2.7)ii implies that at least on $h\in\{1,\ldots N\}$ exists for which the sequence $\mathbf{w}_h^{(n+h)}$, $n=1,2,\ldots$, does have epochs at which $\mathbf{w}_h^{(m)}=0$. Therefore the generality of the analysis is not restricted by assuming that $h=1$, because the arrival epoch of every type h-customer may be taken as the starting point of the arrival process. So we may and do assume that (3.1) possesses epochs at which $\mathbf{w}_1^{(n)}=0$.

Define

$$\mathbf{n}_1 := \min_{n=1,2,\ldots} \{n : \mathbf{w}_1^{(n+1)} = 0 | \mathbf{w}_1 = 0\}, \tag{3.2}$$

so that $\mathbf{n}_1 N$ is the number of customers served in a type 1-busy period, such a period being defined as the time interval between the successsive arrival epochs of two type 1-customers with zero waiting time. Note: $\mathbf{n}_1$ is also the number of type j-customers served in a type 1-busy period.

Obviously, the $\mathbf{w}_1^{(n)}$-process is a random walk on $[0, \infty)$ with the zero state reflecting. It is readily seen that its drift is negative since $\gamma < 0$, cf. (2.7)ii, so the $\mathbf{w}_1^{(n)}$-process is positive recurrent, cf. [9]. Its state space is not a subset of $[0, \infty)$ if at least one of the distributions of $\boldsymbol{\tau}_j$ or $\boldsymbol{\sigma}_j$ is absolutely continuous. It is readily seen that the same conclusions hold for each of the sequences

$$\mathbf{w}_j^{(n)},\ n = 1, 2, \ldots, \quad \textit{with}\ j = 1, 2, \ldots, N. \tag{3.3}$$

It follows that the $\mathbf{w}_j^{(n)}$-squences converge in distribution for $n \to \infty$ for every $j \in N$. Denote by $\mathbf{w}_j$ a stochastic variable with distribution the limiting distribution of the $\mathbf{w}_j^{(n)}$-sequence. It is readily shown that the sequence $\mathbf{i}_j^{(n)}, n = 1, 2, \ldots$, cf. (2.13), also converges in distribution for $n \to \infty$; denote by $\mathbf{i}_j$ a stochastic variable with distribution this limiting distribution. It then follows from (2.3), (2.6) and (2.13): for $j = 2, \ldots, N$,

$$\begin{aligned} &\mathbf{w}_j \sim [\mathbf{w}_{j-1} + \boldsymbol{\rho}_{j-1}]^+, \quad \mathbf{i}_{j-1} \sim -[\mathbf{w}_{j-1} + \boldsymbol{\rho}_{j-1}]^-, \\ &\mathbf{w}_1 \sim [\mathbf{w}_N + \boldsymbol{\rho}_N]^+, \qquad \mathbf{i}_1 \sim -[\mathbf{w}_N + \boldsymbol{\rho}_N]^-. \end{aligned} \tag{3.4}$$

From the identity: for real x

$$e^{-\rho x} + 1 = e^{-\rho[x]^+} + e^{-\rho[x]^-},$$

we obtain from (2.3) and (3.13)

$$\begin{aligned} &e^{-\rho \mathbf{w}_j^{(n+1)}} = e^{-\rho(\mathbf{w}_{j-1}^{(n)} + \boldsymbol{\rho}_{j-1}^{(n)})} + 1 - e^{\rho \mathbf{i}_{j-1}^{(n)}}, \quad j = 2, \ldots, N, \\ &e^{-\rho \mathbf{w}_1^{(n+1)}} = e^{-\rho(\mathbf{w}_N^{(n)} + \boldsymbol{\rho}_N^{(n)})} + 1 - e^{\rho \mathbf{i}_N^{(n)}}. \end{aligned} \tag{3.5}$$

Hence by taking expections in (3.5) and by noting that $\mathbf{w}_j^{(n)}$ and $\boldsymbol{\rho}_j^{(n)}$ are independent, cf. section 2, it follows, cf. (2.8): for $\mathrm{Re}\,\rho = 0$; $j = 2, \ldots, N$,

$$\begin{aligned} &\text{i.} \quad \mathrm{E}\{e^{-\rho \mathbf{w}_j}\} - \gamma_{j-1}(\rho)\mathrm{E}\{e^{-\rho \mathbf{w}_{j-1}}\} = -\rho \mathrm{E}\{\mathbf{i}_{j-1}\} \frac{1 - \mathrm{E}\{e^{\rho \mathbf{i}_{j-1}}\}}{-\rho \mathrm{E}\{\mathbf{i}_{j-1}\}}; \\ &\text{ii.} \quad \mathrm{E}\{e^{-\rho \mathbf{w}_1}\} - \gamma_N(\rho)\mathrm{E}\{e^{-\rho \mathbf{w}_N}\} = -\rho \mathrm{E}\{\mathbf{i}_N\} \frac{1 - \mathrm{E}\{e^{\rho \mathbf{i}_N}\}}{-\rho \mathrm{E}\{\mathbf{i}_N\}}. \end{aligned} \tag{3.6}$$

REMARK 3.1. Note that if at least one of the distributions of $\boldsymbol{\tau}_j$ or $\boldsymbol{\sigma}_j$, $j = 1, \ldots, N$, is absolutely continuous then all $\mathbf{w}_j$ and $\mathbf{i}_j$ are nonlattice variables. Further it is wellknown that

$$\frac{1 - \mathrm{E}\{e^{\rho \mathbf{i}_j}\}}{-\rho \mathrm{E}\{\mathbf{i}_j\}}, \tag{3.7}$$

is the Laplace-Stieltjes transform of an absolutely continuous distribution with support $(-\infty, 0)$; the ergodicity of the $\mathbf{w}_j^{(n)}$-sequence implies that $\mathrm{E}\{\mathbf{i}_j\}$ is finite. □

Put for $j = 1, \ldots, N$,

$$
\begin{aligned}
\Phi_j(\rho) &:= \mathrm{E}\{\mathrm{e}^{-\rho \mathbf{w}_j}\}, && \operatorname{Re}\rho \geq 0,\\
\Psi_j(\rho) &:= \frac{\mathrm{E}\{\mathbf{i}_j\}}{\alpha-\beta}\,\frac{1-\mathrm{E}\{\mathrm{e}^{\rho \mathbf{i}_j}\}}{-\rho \mathrm{E}\{\mathbf{i}_j\}}, \quad \text{if} \quad \mathrm{E}\{\mathbf{i}_j\} > 0, && \operatorname{Re}\rho \leq 0,\\
&:= 0 \qquad\qquad\qquad\qquad\qquad\quad = 0. &&
\end{aligned} \tag{3.8}
$$

Next introduce the vectors and matrices

$$\Phi(\rho) := (\Phi_1(\rho), \dots, \Phi_N(\rho)), \tag{3.9}$$

$$\Psi(\rho) := (\Psi_1(\rho), \dots, \Psi_N(\rho)),$$

$$\Gamma_{ij}(\rho) := \gamma_j(\rho)\delta_{ij}, \quad i, j \in \{1, \dots, N\},$$

δ_{ij} the Kronecker symbol,

$P = (p_{ij})$ with p_{ij} as in (2.8),

I the identity matrix.

Hence the relations in (3.6) may be rewritten as: for $\operatorname{Re}\rho = 0$,

$$\Phi(\rho)[I - \Gamma(\rho)P] = (\beta - \alpha)\rho\, \Psi(\rho)P. \tag{3.10}$$

It is readily verified that, cf. (2.8), for $\operatorname{Re}\rho = 0$,

$$D_N(\rho) := ||I - \Gamma(\rho)P|| = 1 - \gamma(\rho), \tag{3.11}$$

$$D_N(\rho)P[I - \Gamma(\rho)P]^{-1} = \tag{3.12}$$

$$
\begin{bmatrix}
\hat\gamma_2\hat\gamma_3\cdots\hat\gamma_N & 1 & \hat\gamma_2 & \cdots & \hat\gamma_2\hat\gamma_3\cdots\hat\gamma_{N-1}\\
\hat\gamma_3\cdots\hat\gamma_N & \hat\gamma_1\hat\gamma_3\cdots\hat\gamma_N & 1 & & \hat\gamma_3\cdots\hat\gamma_{N-1}\\
 & & \hat\gamma_1\hat\gamma_2\hat\gamma_4\cdots\hat\gamma_N & \cdots & \cdots\\
\cdots & \cdots & \cdots & & \cdots\\
\hat\gamma_N & \hat\gamma_1\hat\gamma_N & \cdots & & 1\\
1 & \hat\gamma_1 & \hat\gamma_1\hat\gamma_2 & \cdots & \hat\gamma_1\hat\gamma_2\cdots\hat\gamma_{N-1}
\end{bmatrix},
$$

where for brevity we have written

$$\hat\gamma_j \equiv \gamma_j(\rho).$$

It follows from (3.10) that : for $\operatorname{Re}\rho = 0$,

$$\Phi(\rho) = (\beta - \alpha)\rho\, \Psi(\rho)\, P[I - \Gamma(\rho)P]^{-1}, \tag{3.13}$$

note that $D_N(\rho)$ has in $\rho = 0$ a zero with multiplicity one, because

$$\frac{\mathrm{d}\gamma(\rho)}{\mathrm{d}\rho}\Big|_{\rho=0} = \beta - \alpha < 0. \tag{3.14}$$

From the definition of $\Phi_j(\rho)$ and $\Psi_j(\rho)$, $j = 1, \dots, N$, it follows that:

i. $\Phi_j(\rho)$ is regular for $\operatorname{Re}\rho > 0$, continuous for $\operatorname{Re}\rho \geq 0$, (3.15)

$\Phi_j(\rho) = 1$ for $\rho = 0$,

$\Phi_j(\rho) = O(1)$ for $|\rho| \to \infty$, $|\arg\,\rho| \leq \frac{1}{2}\pi$;

ii. $\Psi_j(\rho)$ is regular for $\mathrm{Re}\,\rho < 0$, continuous for $\mathrm{Re}\,\rho \leq 0$;

$\Psi_j(\rho) = \frac{\mathrm{E}\{\mathbf{i}_j\}}{\alpha - \beta}$ for $\rho = 0$,

$\Psi_j(\rho) = O(\frac{1}{|\rho|})$ for $|\rho| \to \infty$, $\frac{1}{2}\pi \leq \arg\,\rho \leq 1\frac{1}{2}\pi$.

The set of equations (3.13) for the unknown vectors $\Phi(\rho)$ and $\Psi(\rho)$, and similarly the equivalent set (3.10), formulate together with the conditions (3.15) a homogeneous Hilbert Boundary Value Problem for the $2N$ unknown components of $\Phi(\rho)$ and $\Psi(\rho)$, with the imaginary axis $\mathrm{Re}\,\rho = 0$ the line of discontinuity, cf. [10]. p. 384, see also [11], [12]. Actually, the functional equation (3.13) differs slightly from that discussed in [10] because the line of discontinuity is not a closed contour bounding a finite domain.

The construction of explicit solutions of these types of boundary value problem is generally hardly possible. However, for the three cases:

i. all $\beta_j(\rho)$ are rational functions,

ii. ,, $\alpha_j(\rho)$,, ,, ,,

iii. for a subset of N all $\beta_j(\rho)$ are rational
and for the complementary subset all $\alpha_j(\rho)$
are rational,

it is fairly simple to construct the explicit solutions.

In the context of Queueing Theory the Hilbert Boundary Value Problem formulated above have been studied in a slightly more general form by Miller [13]. In his approach it is shown that the matrix $I - \Gamma(\rho)P$ can be written as the product of two matrices, one with all its elements regular for $\mathrm{Re}\,\rho < 0$, the other with its elements regular for $\mathrm{Re}\,\rho > 0$. The existence of such a factorisation is also shown in [10], however, it leads to a rather formal solution of the problem. The cases, i and ii mentioned above will be analysed in the next section which quickly leads to results accessible for numerical evaluation.

REMARK 3.2. It is readily verified that for $\rho = 0$ the relation (3.10) is an identity, note that $\Gamma(0)$ is the identity matrix and $\Phi_j(0) = 1$, $j = 1, \ldots, N$. We next consider the relation (3.10) for $\mathrm{Re}\,\rho = 0$, $\rho \to 0$.

Because of (2.7) we have for $\mathrm{Re}\,\rho = 0, \rho \to 0$,

$$\begin{aligned} \Gamma_{ij}(\rho) &= 1 - \rho\gamma_j + o(\rho) \quad \text{for} \quad i = j, \\ &= 0 \qquad\qquad\qquad\quad ,, \quad i \neq j, \end{aligned} \tag{3.16}$$

so

$$\Gamma(\rho) = I + \rho\frac{\mathrm{d}}{\mathrm{d}\rho}\Gamma(\rho) + o(\rho)I. \tag{3.17}$$

Since P has an inverse we have from (3.10) for $\mathrm{Re}\,\rho = 0$,

$$\Phi(\rho)[P^{-1} - \Gamma(\rho)] = (\beta - \alpha)\rho\,\Psi(\rho),$$

and so by using (3.17) for $\mathrm{Re}\,\rho = 0$, $\rho \to 0$,

$$\Phi(\rho)[P^{-1} - I - \rho\frac{\mathrm{d}}{\mathrm{d}\rho}\Gamma(\rho) + o(\rho)I] = (\beta - \alpha)\rho\,\Psi(\rho).$$

Multiply (3.18) on the right by 1^T, the transposed of the unit vector $(1, \dots, 1)$, then since

$$[P^{-1} - I]1^T = (0, \dots, 0),$$

we obtain for $\mathrm{Re}\,\rho = 0$, $\rho \to 0$,

$$[-\Phi(\rho)\frac{\mathrm{d}}{\mathrm{d}\rho}\Gamma(\rho)\,|_{\rho=0} + o(1)I]1^T = (\beta - \alpha)\;\Psi(\rho)1^T, \tag{3.19}$$

Since, cf. (3.8) and (3.15),

$$\Phi(0) = (1, \dots, 1), \quad \Psi(0) = \frac{1}{\alpha - \beta}(\mathrm{E}\{\mathbf{i}_1\}, \dots, \mathrm{E}\{\mathbf{i}_N\}),$$

we have from (3.16) and (3.19) by letting $\rho \to 0$ that

$$\alpha - \beta = \sum_{j=1}^{N} \gamma_j = \sum_{j=1}^{N} \mathrm{E}\{\mathbf{i}_j\} = \sum_{j=1}^{N} -\mathrm{E}\{[\mathbf{w}_j + \boldsymbol{\rho}_j]^-\}. \tag{3.20}$$

The relation (3.20) shows that the average total idle time during a 1-cycle, i.e. the time between successive arrivals of type 1-customers is equal to $\alpha - \beta$; it is readily seen that the same result applies for a j-cycle; a j-cycle being defined analogously as a 1-cycle. □

4. The functional equations for the lattice case.

In the derivations of the preceding section we have used the Laplace-Stieltjes transforms of the various distributions involved. When, however, the state space of all $\boldsymbol{\tau}_j$ and $\boldsymbol{\sigma}_j$, $j = 1, \dots, N$, is the set of positive integers it is preferable to use generating functions instead of Laplace-Stieltjes transforms. In this section we shall derive the functional equation for the case of lattice variables. Exactly the same notation will be used but p shall stand for the variable of the various generating functions. So we have for $j = 1, \dots, N$,

$$\begin{aligned} \alpha_j(p) &:= \mathrm{E}\{p^{\boldsymbol{\sigma}_j}\}, & \beta_j(p) &:= \mathrm{E}\{p^{\boldsymbol{\tau}_j}\}, \quad |p| \le 1, \\ \gamma_j(p) &:= \alpha_j(p^{-1})\beta_j(p), & \gamma(p) &:= \prod_{j=1}^{N}\gamma_j(p), \quad |p| = 1, \end{aligned} \tag{4.1}$$

and if $\alpha_j(p)$ or $\beta_j(p)$ is a rational function of p then

$$\alpha_j(p) = \frac{a_{1j}(p)}{a_{2j}(p)}, \quad \beta_j(p) = \frac{b_{1j}(p)}{b_{2j}(p)}, \quad |p| \le 1, \tag{4.2}$$

with

									(4.3)
i.	$a_{2j}(p)$	*a*	*polynomial*	*in*	p	*with*	*degree*	m_j,	
	$b_{2j}(p)$	,,	,,	,,	p	,,	,,	n_j;	
ii.	$a_{1j}(p)$	,,	,,	,,	p	,,	,,	$< m_j$,	
	$b_{1j}(p)$	,,	,,	,,	p	,,	,,	$< n_j$;	
iii.	$a_{1j}(0) = a_{2j}(0) \neq 0$,								

$$b_{1j}(0) = b_{2j}(0) \neq 0;$$

iv. $a_{2j}(p)$ *has* m_j *poles in* $|p| > 1$,

$b_{2j}(p)$,, n_j ,, ,, $|p| > 1$.

It will always be assumed that $a_{1j}(p)$ and $a_{2j}(p)$ have no common factors, similarly for $b_{1j}(p)$ and $b_{2j}(p)$ and that the $\boldsymbol{\tau}_j$ and $\boldsymbol{\sigma}_j$ are aperiodic lattice variables, i.e.

$$\begin{aligned} &|\alpha_j(p)| = 1 \quad \textit{for} \quad a \quad p \quad \textit{with} \quad |p| = 1 \quad \textit{implies} \quad p = 1, \\ &|\beta_j(p)| = 1 \quad ,, \quad ,, \quad p \quad ,, \quad |p| = 1 \quad ,, \quad p = 1. \end{aligned} \tag{4.4}$$

Again it is assumed that the sequences $\mathbf{w}_j^{(n)}$, $n = 1, 2, \ldots$, converge in distribution for $n \to \infty$. The variables $\mathbf{w}_j$, $\mathbf{i}_j$, $j = 1, \ldots, N$, are introduced as in the previous section.
We start from the identity: for real y,

$$p^{[y]^+} + p^{[y]^-} = p^y + 1.$$

Hence

$$p^{\mathbf{w}_j} = p^{\mathbf{w}_{j-1} + \boldsymbol{\rho}_{j-1}} + 1 - p^{-\mathbf{i}_{j-1}}, \quad j = 2, \ldots, N, \tag{4.5}$$

$$p^{\mathbf{w}_1} = p^{\mathbf{w}_N + \boldsymbol{\rho}_N} + 1 - p^{-\mathbf{i}_N}.$$

So for $|p| = 1$, $j = 2, \ldots, N$,

$$\mathrm{E}\{p^{\mathbf{w}_j}\} - \gamma_{j-1}(p)\, \mathrm{E}\{p^{\mathbf{w}_{j-1}}\} = -(1 - \frac{1}{p})\mathrm{E}\{\mathbf{i}_{j-1}\} \frac{1 - \mathrm{E}\{p^{-\mathbf{i}_{j-1}}\}}{-(1 - \frac{1}{p})\mathrm{E}\{\mathbf{i}_{j-1}\}}, \tag{4.6}$$

$$\mathrm{E}\{p^{\mathbf{w}_1}\} - \gamma_N(p)\, \mathrm{E}\{p^{\mathbf{w}_N}\} = -(1 - \frac{1}{p})\mathrm{E}\{\mathbf{i}_N\} \frac{1 - \mathrm{E}\{p^{-\mathbf{i}_N}\}}{-(1 - \frac{1}{p})\mathrm{E}\{\mathbf{i}_N\}}.$$

Put for $j = 1, \ldots, N$,

$$\begin{aligned} \Phi_j(p) &:= \mathrm{E}\{p^{\mathbf{w}_j}\}, & |p| \le 1, \\ \Psi_j(p) &:= \frac{\mathrm{E}\{\mathbf{i}_j\}}{\alpha - \beta} \frac{1 - \mathrm{E}\{p^{\mathbf{i}_j}\}}{-(1 - \frac{1}{p})\mathrm{E}\{\mathbf{i}_j\}}, & |p| \ge 1, \end{aligned} \tag{4.7}$$

and introduce the vectors and matrices

$$\begin{aligned} \Phi(p) &:= (\Phi_1(p), \ldots, \Phi_N(p)), \\ \Psi(p) &:= (\Psi_1(p), \ldots, \Psi_N(p)), \\ \Gamma_{ij}(p) &:= \gamma_j(p)\delta_{ij}, \quad i, j \in \{1, \ldots, N\}. \end{aligned} \tag{4.8}$$

The relations (4.6) may now be rewritten as: for $|p| = 1$,

$$\Phi(p)[I - \Gamma(p)P] = (\beta - \alpha)(1 - p^{-1})\, \Psi(p)P.$$

From (4.9) we obtain as before: for $|p| = 1$, $p \neq 1$,

$$\Phi(p) = (\beta - \alpha)(1 - \tfrac{1}{p})\, P[I - \Gamma(p)P]^{-1}.$$

From the definition of $\Phi_j(p)$ and $\Psi_j(p)$, $j = 1, \dots, N$, it follows that

$$\text{i.}\quad \Phi_j(p) \text{ is regular for } |p| < 1, \text{ continuous for } |p| \leq 1, \tag{4.11}$$

$$\Phi_j(1) = 1,$$

$$\text{ii.}\quad \Psi_j(p) \text{ is regular for } |p| > 1, \text{ continuous for } |p| \geq 1,$$

$$\Psi_j(1) = \frac{\mathrm{E}\{\mathbf{i}_j\}}{\alpha - \beta},$$

$$\Psi_j(p) = O(1) \text{ for } |p| \to \infty.$$

The relation (4.9) for the vectors $\Phi(p)$ and $\Psi(p)$, and similarly the equivalent relation (4.10), formulates together with the condition (4.11) a homogeneous Hilbert Boundary Value Problem with the unit circle $|p| = 1$ the contour of discontinuity, cf. [10] p. 384.

5. Some remarks on the case $N = 1$

For the case $N = 1$ the relation (3.10) becomes: for $\mathrm{Re}\,\rho = 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}\frac{1-\gamma(\rho)}{(\beta-\alpha)\rho} = \frac{\mathrm{E}\{\mathbf{i}\}}{\alpha-\beta}\,\frac{1-\mathrm{E}\{\mathrm{e}^{\rho\mathbf{i}}\}}{-\rho\mathrm{E}\{\mathbf{i}\}}, \tag{5.1}$$

where we have written

$\mathbf{w}$ for $\mathbf{w}_1$ and $\mathbf{i}$ for $\mathbf{i}_1$.

By taking $\rho = 0$ in (5.1) it follows, cf. also (3.20),

$$\mathrm{E}\{\mathbf{i}\} = \alpha - \beta. \tag{5.2}$$

Let $\mathbf{v}$ be a positive stochastic variable with distribution

$$\begin{aligned} \Pr\{\mathbf{v} < v\} &= \frac{1}{\mathrm{E}\{\mathbf{i}\}}\int_0^v \Pr\{\mathbf{i} \geq u\}\mathrm{d}u, \quad && v > 0, \\ &= 0, && v \leq 0, \end{aligned} \tag{5.3}$$

then

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{v}}\} = \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{i}}\}}{\rho\mathrm{E}\{\mathbf{i}\}}, \quad \mathrm{Re}\,\rho \geq 0. \tag{5.4}$$

Hence, it follows from (5.1), (5.2) and (5.4), that: for $\mathrm{Re}\,\rho = 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}\frac{1-\gamma(\rho)}{(\beta-\alpha)\rho} = \mathrm{E}\{\mathrm{e}^{\rho\mathbf{v}}\}. \tag{5.5}$$

The present case is actually the classical $GI/G/1$ model with $\alpha(\rho)$ and $\beta(\rho)$ the Laplace-Stieltjes transforms of the interarrival time and of the service time distribution, with $\mathbf{w}$ a stochastic variable with distribution the stationary distribution of the actual waiting time process and $\mathbf{i}$ the idle time in a busy cycle, cf. [14], [15]. From the results of Queueing Theory of the $GI/G/1$ model, or from those of Fluctuation Theory, cf. [14], II.5 and I.6.6, it follows that there exists a unique pair of functions

$K_-(\rho)$ and $K_+(\rho)$ such that

i. $K_-(\rho)$ is regular for $\text{Re}\,\rho > 0$, continuous for $\text{Re}\,\rho \geq 0$, (5.6)

$K_-(0) = 1,\ K_-(\rho) = O(1)$ for $|\rho| \to \infty, \quad |\arg\ \rho| \leq \frac{\pi}{2}$,

$K_-(\rho)$ has no zeros in $\text{Re}\,\rho \geq 0$,

ii. $K_+(\rho)$ is regular for $\text{Re}\,\rho < 0$, continuous for $\text{Re}\,\rho \leq 0$,

$K_+(0) = 1,\ K_+(\rho) = O(-\log\ |\rho|)$ for $|\rho| \to \infty,\ \frac{1}{2}\pi \leq \arg\ \rho \leq \frac{3}{2}\pi$,

iii. $$\mathrm{e}^{K_+(\rho) - K_-(\rho)} = \frac{1 - \gamma(\rho)}{(\beta - \alpha)\rho}, \quad \text{Re}\,\rho = 0,$$

provided $\beta - \alpha < 0$, and

$$\sum_{j=1}^{N} (\tau_j - \sigma_j)$$

is not a lattice variable and nonzero with positive probability. For integral expressions expressing $K_+(\cdot)$ and $K_-(\cdot)$ as functionals of $\gamma(\cdot)$, see [14], p. 143 and [15].

Actually, the relation (5.6)iii describes the factorisation of

$$\frac{1 - \gamma(\rho)}{(\beta - \alpha)\rho}, \quad \text{Re}\,\rho = 0,$$

as the product of two functions with one the boundary value of a function regular in $\text{Re}\,\rho < 0$, the other regular in $\text{Re}\,\rho > 0$. A further result from $GI/G/1$ queueing theory is that:

$$\begin{aligned} \mathrm{E}\{\mathrm{e}^{-\rho \mathbf{w}}\} &= \mathrm{e}^{K_-(\rho)}, \quad \text{Re}\,\rho \geq 0, \\ \mathrm{E}\{\mathrm{e}^{\rho \mathbf{v}}\} &= \mathrm{e}^{K_+(\rho)}, \quad \text{Re}\,\rho \leq 0. \end{aligned} \tag{5.7}$$

6. THE CASE WITH ALL $\beta_j(\rho)$ RATIONAL

In this section we shall consider the case that all $\beta_j(\rho)$, $j = 1, \dots, N$, are rational functions of ρ, and no assumptions are made concerning the $\alpha_j(\rho)$, $j = 1, \dots, N$.

We start from the relation (3.13) written as: for $j = 1, 2, \dots, N$, and $\text{Re}\ \rho = 0$,

$$\Phi_j(\rho) = \frac{(\beta - \alpha)\rho}{1 - \gamma(\rho)} [\Psi(\rho)\ P[I - \Gamma(\rho)\ P]^{-1}]_j\ (1 - \gamma(\rho)), \tag{6.1}$$

$$\beta_j(\rho) = \frac{b_{1j}(\rho)}{b_{2j}(\rho)},\ \gamma_j(\rho) = \frac{b_{1j}(\rho)}{b_{2j}(\rho)}\ \alpha(-\rho),\ b_2(\rho) = \prod_{j=1}^{N} b_{2j}(\rho). \tag{6.2}$$

By using (5.6)iii we rewrite this relation as: for $j = 1, \dots, N$, $\text{Re}\,\rho = 0$,

$$\frac{b_2(\rho)}{b_{2j}(\rho)}\ \mathrm{e}^{-K_-(\rho)}\ \Phi_j(\rho) = \mathrm{e}^{-K_+(\rho)} \frac{b_2(\rho)}{b_{2j}(\rho)} [\Psi(\rho)\ P[I - \Gamma(\rho) P]^{-1}]_j\ (1 - \gamma(\rho)). \tag{6.3}$$

From (3.10), (3.15) and (5.6)i it is seen that the function in the lefthand side of (6.2) is the boundary value of a regular function in $\mathrm{Re}\,\rho > 0$ which for $|\rho| \to \infty$, $|\arg\ \rho| < \frac{1}{2}\pi$, behaves as $|\rho|^{\nu-n_j}$. By noting that in the elements of the jth column of the matrix in (3.12) the factor $\gamma_j(\rho)$ does not occur it is readily seen from (3.15) and (5.6)ii that the righthand side of (6.2) is the boundary value of a function regular in $\mathrm{Re}\,\rho < 0$ and that for $|\rho| \to \infty$, $\frac{1}{2}\pi <$ arg $\rho < \frac{3}{2}\pi$, it behaves as $|\rho|^{\nu-n_j}$. Consequently, these functions for $\mathrm{Re}\,\rho > 0$ and $\mathrm{Re}\ \rho < 0$, respectively are each other's analytic continuation. Hence by applying Liouville's theorem it follows that these functions are both a polynomial say, $P_j(\rho)$, of degree $\nu - n_j$, i.e. for $j = 1, \ldots, N$,

$$\Phi_j(\rho) = \frac{b_{2j}(\rho)}{b_2(\rho)}\ \mathrm{e}^{K_-(\rho)}\ P_j(\rho), \quad \mathrm{Re}\,\rho \geq 0, \tag{6.4}$$

$P_j(\rho)$ *a polynomial in* ρ *of degree* $\nu - n_j$, $\nu = \sum_{j=1}^{N} n_j$ *the degree of* $b_2(\rho)$.

Substitution of (6.4) into (3.6) leads to: for $j = 2, \ldots, N$, $\mathrm{Re}\,\rho = 0$,

$$\frac{b_{2j}(\rho)}{b_2(\rho)}\mathrm{e}^{K_-(\rho)} P_j(\rho) - \frac{b_{2j-1}(\rho)}{b_2(\rho)}\gamma_{j-1}(\rho)\mathrm{e}^{K_-(\rho)} P_{j-1}(\rho) = -\rho\ \Psi_{j-1}(\rho),$$

or by using (5.6)iii: for $j = 2, \ldots, N$, $\mathrm{Re}\,\rho = 0$,

$$[b_{2j}(\rho)P_j(\rho) - b_{2j-1}(\rho)\gamma_{j-1}(\rho)P_{j-1}(\rho)]\ \mathrm{e}^{K_+(\rho)} = \frac{1-\gamma(\rho)}{(\beta-\alpha)\rho} b_2(\rho)\rho\ \Psi_{j-1}(\rho). \tag{6.5}$$

The function in the lefthand side of (6.5) is the boundary value of a function regular for $\mathrm{Re}\,\rho < 0$, cf. (5.6)ii and (6.4), that in the righthand side is also such a function, cf. (3.9) and note that $\{1-\gamma(\rho)\}b_2(\rho)/\rho$ has no poles in $\mathrm{Re}\,\rho \leq 0$, since all $\beta_j(\rho)$ are rational. Hence by analytic continuation (6.5) holds for Re $\rho \leq 0$. It follows from the continuity of $\Psi_{j-1}(\rho)$ in $\mathrm{Re}\,\rho \leq 0$ that the zeros of

$$\frac{1-\gamma(\rho)}{(\beta-\alpha)\rho}\ b_2(\rho) \text{ in } \mathrm{Re}\,\rho \leq 0, \tag{6.6}$$

should be zeros of

$$b_{2j}(\rho)\ P_j(\rho) - b_{2j-1}(\rho)\gamma_{j-1}(\rho)\ P_{j-1}(\rho), \quad j = 2, \ldots, N; \tag{6.7}$$

note that $K_+(\rho)$ is continuous in $\mathrm{Re}\,\rho \leq 0$.

By applying Rouché's theorem it is readily shown, cf. [14], p. 323, that the function in (6.6) has for $\beta - \alpha < 0$ exactly ν zeros $\epsilon, \ldots \epsilon_\nu$, say, in $\mathrm{Re}\,\rho \leq 0$ and their real parts are all negative. Hence, if all these zeros have multiplicitly one, see remark 6.1 below, then: for $\rho = \epsilon_k$, $k = 1, \ldots, \nu$,

$$\text{i.} \quad b_{2j}(\rho)P_j(\rho) - b_{2j-1}(\rho)\gamma_{j-1}(\rho)P_{j-1}(\rho) = 0, \quad j = 2, \ldots, N, \tag{6.8}$$

$$\text{ii.} \quad b_{21}(\rho)P_1(\rho) - b_{2N}(\rho)\gamma_N(\rho)P_N(\rho) = 0;$$

the relation in (6.8)ii follows from the last realtion in (3.6) by using the same arguments as above.

From (3.15)i, (5.6)i and (6.4) we have

$$\frac{b_{2j}(0)}{b_2(0)}\ P_j(0) = 1, \quad j = 1, \ldots, N. \tag{6.9}$$

Because $P_j(\rho)$ is a polynomial of degree $\nu - n_j$, cf. (6.4), and so has $\nu - n_j + 1$ coefficients the determination of these polynomials requires the determination of

$$\sum_{j=1}^{N}(\nu - n_j + 1) = N + (N-1)\nu, \tag{6.10}$$

unknowns. Because $1-\gamma(\rho)$ is the main determinant of the set of equations (3.6) for the $\Phi_j(\rho)$, cf. (3.11), it follows that the relation (6.8)ii depends linearly on those in (6.8)i. So it is seen that the conditions (6.8)i and (6.9) represent $(N-1)\nu+N$ inhomogeneous linear equations for the $N+(N-1)\nu$, cf. (6.10), coefficients of the polynomials $P_j(\rho)$, $j=1,\dots,N$. Because the condition $\beta-\alpha<0$ implies that the $\mathbf{w}_n$-process has a unique stationary distribution, it follows that there exists a unique $\Phi(\rho)$, satisfying (3.15)i, and consequently, the system of linear, inhomogeneous equations (6.8)i and (6.9) has a unique solution if all zeros $\rho=\epsilon_k$, $k=1,\dots,\nu$, have multiplicity one.

REMARK 6.1. The sum of the multiplicities of the zeros of the function in (6.9) is always equal to n. If such a zero, say ϵ_1, has a multiplicity larger than one, then ϵ_1 should be also a zero with the same multiplicity of (6.7), and it is readily seen that again a system of linear equations for the coefficients of the $P_j(\cdot)$ is obtained which determines these coefficients. □

From the above it is seen that the polynomials $P_j(\rho)$, $j=1,\dots,N$, may be considered to have been determined. Hence, for the ultimate determination of the vectors $\Phi(\rho)$ and $\Psi(\rho)$ it remains to determine $K_+(\rho)$ and $K_-(\rho)$. From the preceding section it follows, cf. (5.7), that $\exp K(\rho)$, $\operatorname{Re}\rho\geq 0$ is the Laplace-Stieltjes transform of the stationary distribution of the actual waiting time of a $GI/K_n/1$ queue, since $\beta(\rho)=\prod_{j=1}^{N}\beta_j(\rho)$ is a rational function of ρ. That transform has been determined in [14], section II.5.10. From the results obtained there it follows that, cf. (II.5.190): for $\operatorname{Re}\rho\geq 0$,

$$e^{K_-(\rho)} = \frac{b_2(\rho)}{b_2(0)}\prod_{k=1}^{\nu}\frac{\epsilon_k}{\epsilon_k-\rho}. \tag{6.11}$$

Hence from (6.4), (6.9) and (6.11) it follows that: for $j=1,\dots,N$, $\operatorname{Re}\rho\geq 0$,

$$\Phi_j(\rho) = \frac{b_{2j}(\rho)P_j(\rho)}{b_{2j}(0)P_j(0)}\prod_{k=1}^{\nu}\frac{\epsilon_k}{\epsilon_k-\rho}, \qquad \operatorname{Re}\rho\geq 0, \tag{6.12}$$

if it is assumed that all ϵ_k have multiplicity one. Note that $b_{2j}(\rho)P_j(\rho)$ is a polynomial of degree ν.

So we have reached the following result.

For $\beta-\alpha<0$ the Laplace-Stieltjes transforms $\Phi_j(\rho)$, $j=1,\dots,N$; $\operatorname{Re}\rho\geq 0$, of the stationary distributions of the actual waiting times $\mathbf{w}_j$, $j=1,\dots,N$ (in an arrival-cycle) are given by

i. $$\Phi_j(\rho) = \frac{b_{2j}(\rho)P_j(\rho)}{b_{2j}(0)P_j(0)}\prod_{k=1}^{\nu}\frac{\epsilon_k}{\epsilon_k-\rho}, \qquad j=1,\dots,N; \quad \operatorname{Re}\rho\geq 0, \tag{6.13}$$

with

ii. ϵ_k, $k=1,\dots,\nu$, the zeros of

$$1-\prod_{j=1}^{N}\frac{b_{1j}(\rho)}{b_{2j}(\rho)}\alpha_j(-\rho) \text{ in } \operatorname{Re}\rho<0,$$

iii. $P_j(\rho)$, $j=1,\dots,N$, polynomials of degree $\nu-n_j$, determined by the conditions

$$b_{2j}(0)\,P_j(0) = b_2(0), \quad j=1,\dots,N,$$

and

for $\rho = \epsilon_k$, $k = 1, \ldots, \nu$,

$$b_{21}(\rho)P_1(\rho) - \gamma_N(\rho)b_{2N}(\rho)\, P_N(\rho) = 0,$$

$$b_{2j}(\rho)P_j(\rho) - \gamma_{j-1}(\rho)b_{2j-1}(\rho)\, P_{j-1}(\rho) = 0, \quad j = 2, \ldots, N.$$

7. The case with all $\alpha_j(\rho)$ rational

In this section we consider the case that all $\alpha_j(\rho)$, $j = 1, \ldots, N$, are rational functions of ρ, and no assumptions will be made concerning the $\beta_j(\rho)$, $j = 1, \ldots, N$.

We start from the relations (3.6) and by using (2.9) we have, cf. (3.8): for $\operatorname{Re}\rho = 0$, $j = 2, \ldots, N$,

$$\text{i)} \quad a_{2j-1}(-\rho)\Phi_j(\rho) - a_{2j-1}(-\rho)\gamma_{j-1}(\rho)\Phi_{j-1}(\rho) = (\beta - \alpha)\rho\Psi_{j-1}(\rho)a_{2j-1}(-\rho), \tag{7.1}$$

$$\text{ii)} \quad a_{2N}(-\rho)\Phi_1(\rho) - a_{2N}(-\rho)\gamma_N(\rho)\Phi_n(\rho) = (\beta - \alpha)\rho\Psi_N(\rho)a_{2N}(-\rho).$$

From (2.10) and (3.15) it is seen that the lefthand side of (7.1)i is the boundary value of a function regular in $\operatorname{Re}\rho > 0$, continuous in $\operatorname{Re}\rho \geq 0$, which for $|\rho| \to \infty$, $|\arg\rho| \leq \frac{1}{2}\pi$ behaves as $|\rho|^{m_J - 1}$. Similarly, it is seen that the righthand side in (7.1)i is the boundary value of a function regular in $\operatorname{Re}\rho < 0$, continuous in $\operatorname{Re}\rho \leq 0$ and which behaves as $|\rho|^{m_j - 1}$ for $|\rho| \to \infty$, $\frac{1}{2}\pi \leq \arg\rho \leq \frac{3}{2}\pi$. Hence these functions for $\operatorname{Re}\rho < 0$ and $\operatorname{Re}\rho > 0$ are each other's analytic continuation, and so by applying Liouville's theorem it follows that : for $\operatorname{Re}\rho \leq 0$, $j = 1, \ldots, N$,

$$(\beta - \alpha)\rho a_{2j}(-\rho)\Psi_j(\rho) = Q_j(\rho), \tag{7.2}$$

$Q_j(\rho)$ *a polynomial in* ρ *of degree* m_j, $Q_j(0) = 0$, cf. (3.8.)

For the present case we have, cf. (3.10), (3.11), (5.6)iii: for $\operatorname{Re}\rho = 0$,

$$\frac{D_N(\rho)}{(\beta - \alpha)\rho} = \frac{1 - \gamma(\rho)}{(\beta - \alpha)\rho} = \frac{a_2(-\rho) - a_2(-\rho)\gamma(\rho)}{(\beta - \alpha)\rho a_2(-\rho)} = \mathrm{e}^{K_+(\rho) - K_-(\rho)}. \tag{7.3}$$

Because $\beta - \alpha < 0$ it follows readily, cf. [14], p. 330, that for the present case $D_N(\rho)$ has μ zeros δ_k, $k = 1, \ldots, \mu$, in $\operatorname{Re}\rho \geq 0$, and $\delta_1 = 0$, $\operatorname{Re}\delta_k > 0$, $k = 2, \ldots, \mu$, a result easily obtained by applying Rouché's theorem.

Remark 7.1. For the sake of simplicity it is again assumed that these zeros all have multiplicity one, see also remark 6.1. □

From [14] p. 330, and from section 5 we obtain:

$$\mathrm{e}^{K_+(\rho)} = \frac{a_2(0)}{a_2(-\rho)} \prod_{k=2}^{\mu} \frac{\delta_k - \rho}{\delta_k}, \qquad \operatorname{Re}\rho \leq 0, \tag{7.4}$$

$$\mathrm{e}^{K_-(\rho)} = \frac{-a_2(0)(\alpha - \beta)\rho}{a_2(-\rho) - a_1(-\rho)b(\rho)} \prod_{k=2}^{\mu} \frac{\delta_k - \rho}{\delta_k}, \qquad \operatorname{Re}\rho \geq 0.$$

For $\operatorname{Re}\rho = 0$, we have, cf. (3.13),

$$\Phi(\rho) = \frac{(\beta - \alpha)\rho}{1 - \gamma(\rho)}\Psi(\rho)D_n(\rho)P[I - \Gamma(\rho)P]^{-1},$$

and so from (7.3): for $\operatorname{Re}\rho = 0$,

$$\Phi(\rho)e^{-K_-(\rho)} = \frac{e^{-K_+(\rho)}}{a_2(-\rho)}\Psi(\rho)a_2(-\rho)D_N(\rho)P[I-\Gamma(\rho)P]^{-1}. \tag{7.5}$$

Hence by using (7.2) and (7.3) we have: for $\operatorname{Re}\rho = 0$, $j = 1, \ldots, N$,

$$\Phi_j(\rho)e^{-K_-(\rho)} = \frac{\prod_{k=2}^{m}\delta_k}{a_2(0)}\frac{a_2(\rho)}{\prod_{k=2}^{m}(\delta_k-\rho)}\sum_{i=1}^{N}\frac{Q_i(\rho)}{\alpha_{2i}(\rho)}\frac{D_N(\rho)}{(\beta-\alpha)\rho}[P[I-\Gamma(\rho)P]^{-1}]_{ij}. \tag{7.6}$$

From the structure of the matrix in (3.12) and from the fact that the factor $a_2(-\rho)/a_{2i}(-\rho)$ occurs in the righthand side it is seen that this righthand side has a meromorphic continuation in $\operatorname{Re}\rho > 0$, since $Q_i(\rho)$ is a polynomial. The lefthand side of (7.6) has an analytic continuation in $\operatorname{Re}\rho > 0$, see (3.15)i and (5.6)i. Because these continuations are unique it follows that: for $j = 1, \ldots, N$,

$$\sum_{i=1}^{N}\frac{Q_i(\rho)}{\alpha_{2i}(-\rho)}\frac{D_n(\rho)}{(\beta-\alpha)\rho}[P[I-\Gamma(\rho)P]^{-1}]_{ij} = 0 \text{ for } \rho = \delta_k, \qquad k = 2, \ldots, \mu, \tag{7.7}$$

note that these δ_k, $k = 2, \ldots, \mu$, are the only poles of the mentioned meromorphic continuation.

Next we multiply (7.1) by $e^{-K_-(\rho)}$, then by using (7.2), (7.3) and (7.4) we obtain: for $\operatorname{Re}\rho = 0$, $j = 2, \ldots, N$,

i. $$\Phi_j(\rho)e^{-K_-(\rho)} = \frac{1}{\alpha_{2j-1}(-\rho)}[\alpha_{2j-1}(-\rho)\gamma_{j-1}(\rho)\Phi_{j-1}(\rho)e^{-K_-(\rho)} + Q_{j-1}(\rho)\frac{a_2(-\rho)}{a_2(0)}D_N(\rho)\prod_{k=2}^{\mu}\frac{\delta_k}{\delta_k-\rho}], \tag{7.8}$$

ii. $$\Phi_1(\rho)e^{-K_-(\rho)} = \frac{1}{\alpha_{2N}(-\rho)}[\alpha_{2N}(-\rho)\gamma_N(\rho)\Phi_N(\rho)e^{-K_-(\rho)} + Q_N(\rho)\frac{a_2(-\rho)}{a_2(0)}D_N(\rho)\prod_{k=2}^{\mu}\frac{\delta_k}{\delta_k-\rho}];$$

note that for $\rho = 0$ the relations (7.8) imply that

$$\Phi_1(0) = \Phi_2(0) = \ldots = \Phi_N(0). \tag{7.9}$$

To determine the polynomials $Q_j(\rho)$, $j = 1, \ldots, N$, note that $Q_j(\rho)$ contains m_j coefficients because $Q_j(0) = 0$, cf. (7.2); so in total $\mu = m_1 + \ldots + m_N$ coefficients have to be determined. Now consider (7.7) for $j = 1$, then these conditions lead to $\mu - 1$ equations for the μ unknown coefficients in the polynomials Q_j, $j = 1, \ldots, N$. Hence together with the norming condition $\Phi_1(0) = 1$, i.e. by using (7.6) for $j = 1$ and $\rho = 0$, we obtain a system of μ linear inhomogeneous equations for the μ unknowns. *Assume for the present that this system possesses a unique solution.* Then $\Phi_1(\rho)$, $\operatorname{Re}\rho \le 0$, is determined by (7.6). By taking successively $j = 2, \ldots, N$ in (7.8)i explicit expressions for $\Phi_j(\rho)$, $\operatorname{Re}\rho = 0$, are obtained. It also follows from (7.8)i, or equivalently from (7.1)i and (7.2), that for the expressions so obtained $\alpha_{2j-1}(-\rho)\Phi_j(\rho)$, $\operatorname{Re}\rho > 0$, $j = 2, \ldots, N$, has a unique analytic continuation in $\operatorname{Re}\rho > 0$. Consequently, it follows from (7.6) that the relations (7.8)i for $j = 2, \ldots, N$, are identically satisfied. Hence, by using (7.6) for $j = 2, \ldots, N$, it follows that the expressions for $\Phi_j(\rho)$, $j = 2, \ldots, N$, as constructed above, are regular for $\operatorname{Re}\rho > 0$ (note the structure of the matrix in (3.12)), which implies that the zeros of $\alpha_{2j-1}(-\rho)$ in $\operatorname{Re}\rho > 0$ are zeros of the sum between the square brackets in (7.8)i for every $j = 2, \ldots, N$.

REMARK 7.2. The analysis above is based, apart from the assumption introduced above, on the relations (7.8)i, for $j = 2, \ldots, N$, and on the relation (7.7) for $j = 1$. It is readily verified that this set of relations is equivalent with the system (7.1), cf. also (3.10), and also equivalent with (3.13), Hence the solution constructed above satisfies (7.1)ii, and similarly (7.8)ii. Note further, cf. also (7.9), that this solution leads to a unique determination of $\Psi_j(\rho)$, $\mathrm{Re}\,\rho \leq 0$, $j = 1, \ldots, N$, and that this solution possesses the properties (3.15). The verification of this is not difficult but requires some algebras. □

To complete our analysis it remains to justify the assumption introduced above concerning the uniqueness of the solution of the linear system of inhomogeneous equations. This justification is obtained as in the previous section. Viz. the condition $\beta - \alpha < 0$ implies that the $\boldsymbol{w}_n$-process has a unique stationary distribution and so there exits a unique $\Phi(\rho)$ satisfying (3.15)i, and consequently the system of equations (7.1)i for $j = 2, \ldots, N$, together with (7.6) for $j = 1$, should have a unique solution satisfying (3.15)i, and this is only possible if the system (7.7) for $j = 1$, together with $\Phi_1(0) = 1$ determines the μ coefficients of the $Q_j(\rho)$, $j = 1, \ldots, N$, uniquely.

8. On the mixed case

In section 6 the functional equations (3.10) and (3.13) have been analysed for the case with all $\beta_j(\rho)$, $j = 1, \ldots, N$, rational, whereas section 7 concerns the case with all $\alpha_j(\rho)$ rational. Obviously, the next variant is the case where

$$\begin{aligned} &\alpha_j(\rho), \quad j \in \mathcal{A}, \quad \text{are all rational} \\ &\beta_j(\rho), \quad j \in \mathcal{B}, \quad \text{,,} \quad \text{,,} \quad \text{,,} \end{aligned} \tag{8.1}$$

with

$$\mathcal{A} \cup \mathcal{B} = \{1, 2, \ldots, N\} \quad \text{and} \quad \mathcal{A} \cap \mathcal{B} = \emptyset.$$

It appears to be rather intricate to deduce a general algorithm for this case without a detailed description of the set $\mathcal{A}$. This is due to the structure of the relations (3.6), viz. in (3.6)i the cases that the indices $j-1$ and j both belong to the same set, i.e. both to $\mathcal{A}$ or both to $\mathcal{B}$, or belong to different sets have to be distinguished. This has to be done for every $j \in \mathcal{A} \cup \mathcal{B}$, and hence it generates a large number of variants to be considered. We, therefore, restrict the analysis to the case $N = 2$. This analysis provides indications for that of more complicated mixing, see also remark 8.1.

So we consider the case that $N = 2$ and that

$$\beta_1(\rho) \text{ and } \alpha_2(\rho) \text{ are both rational.} \tag{8.2}$$

From the relations (3.10) and (3.13) we have: for $\mathrm{Re}\,\rho = 0$,

$$\text{i.} \quad \Phi_1(\rho) - \gamma_2(\rho)\Phi_2(\rho) = (\beta - \alpha)\rho\Psi_2(\rho), \tag{8.3}$$

$$\text{ii.} \quad \Phi_2(\rho) = \frac{(\beta - \alpha)\rho}{D_N(\rho)}\{\Psi_1(\rho) + \gamma_1(\rho)\Psi_2(\rho)\}.$$

By using (8.2) and (5.6)iii we rewrite (8.3) as: for $\mathrm{Re}\,\rho = 0$,

$$\text{i.} \quad a_{22}(-\rho)\Phi_1(\rho) - a_{22}(-\rho)\gamma_2(\rho) = (\beta - \alpha)\rho a_{22}(-\rho)\Psi_2(\rho), \tag{8.4}$$

$$\text{ii.} \quad b_{22}(\rho)\Phi_2(\rho)\mathrm{e}^{-K_-(\rho)} = b_{21}(\rho)\Psi_1(\rho)\mathrm{e}^{-K_+(\rho)} + b_{21}(\rho)\gamma_1(\rho)\Psi_2(\rho)\mathrm{e}^{-K_+(\rho)}.$$

As in sections 6 and 7 it follows by analytic continuation and by using Liouville's theorem that:

$$\begin{aligned}
&\text{i.} && a_{22}(-\rho)\Phi_1(\rho) - a_{22}(-\rho)\gamma_2(\rho)\Phi_2(\rho) = Q_2(\rho), && \operatorname{Re}\rho \geq 0,\\
&\text{ii.} && (\beta-\alpha)\rho a_{22}(-\rho)\Psi_2(\rho) = Q_2(\rho), && \operatorname{Re}\rho \leq 0,\\
&\text{iii.} && b_{21}(\rho)\Phi_2(\rho)\mathrm{e}^{-K_-(\rho)} = P_2(\rho), && \operatorname{Re}\rho \geq 0,\\
&\text{iv.} && b_{21}(\rho)\Psi_1(\rho)\mathrm{e}^{-K_+(\rho)} + b_{21}(\rho)\gamma_1(\rho)\Psi_2(\rho)\mathrm{e}^{-K_+(\rho)} = P_2(\rho), && \operatorname{Re}\rho \leq 0,
\end{aligned} \tag{8.5}$$

with

$$\begin{aligned}
&\text{i.} && Q_2(\rho) && \text{a polynomial of degree } m_2,\\
& && Q_2(0) = 0,
\end{aligned} \tag{8.6}$$

$$\text{ii.} \quad P_2(\rho) \quad \text{a polynomial of degree } n_1.$$

Insert (8.5)iii into (8.5)i, this yields: for $\operatorname{Re}\rho \geq 0$,

$$a_{22}(-\rho)b_{21}(\rho)\Phi_1(\rho) = a_{22}(-\rho)\gamma_2(\rho)P_2(\rho)\mathrm{e}^{K_-(\rho)} + b_{21}(\rho)Q_2(\rho). \tag{8.7}$$

Next insert (8.5)ii into (8.5)iv, then: for $\operatorname{Re}\rho \leq 0$,

$$b_{21}(\rho)a_{22}(-\rho)(\beta-\alpha)\rho\Psi_1(\rho) = -b_{21}(\rho)\gamma_1(\rho)Q_2(\rho) + a_{22}(-\rho)(\beta-\alpha)\rho P_2(\rho)\mathrm{e}^{K_+(\rho)}. \tag{8.8}$$

Denote by ρ_{2k}, $k = 1,\dots,m_2$, the m_2 zeros of $\alpha_{22}(-\rho)$, counted according to their multiplicities, and, similarly, by ρ_{1h}, $h = 1,\dots,n_1$, the n_1 zeros of $\beta_{21}(\rho)$. For the sake of simplicity it is assumed that all zeros ρ_{2k} and ρ_{1h} have multiplicity one. From the definition of $\beta_1(\rho)$ and $\alpha_2(\rho)$ it follows that

$$\operatorname{Re}\rho_{2k} > 0,\ k = 1,\dots,m_2, \quad \text{and} \quad \operatorname{Re}\rho_{1h} < 0,\ h = 1,\dots,n_1. \tag{8.9}$$

Because $\Phi_1(\rho)$ should be regular for $\operatorname{Re}\rho > 0$, and $\Psi_1(\rho)$ should be regular for $\operatorname{Re}\rho < 0$, it follows from (8.7) and (8.8) that:

$$\begin{aligned}
&\text{i.} && \alpha_{21}(-\rho)\beta_2(\rho)P_2(\rho)\mathrm{e}^{K_-(\rho)} + b_{21}(\rho)Q_2(\rho) = 0 \quad \text{for} \quad \rho = \rho_{2k},\ k = 1,\dots,m_2,\\
&\text{ii.} && -b_{11}(\rho)\alpha_1(-\rho)Q_2(\rho) + \alpha_{22}(-\rho)(\beta-\alpha)\rho P_2(\rho)\mathrm{e}^{K_+(\rho)} = 0 \quad \text{for} \quad \rho = \rho_{1h},\ h = 1,\dots,n_1.
\end{aligned} \tag{8.10}$$

From (8.6) it is seen that the polynomial $Q_2(\rho)$ contains m_2 coefficients, and $P_2(\rho)$ has $n_1 + 1$ coefficients. The relations (8.10) lead to $n_1 + m_2$ linear equations for these coefficients, so that together with the norming condition

$$\Phi_2(0) = 1,$$

i.e. cf. (8.5)iii,

$$b_{21}(0) = P_2(0), \tag{8.11}$$

an inhomogeneous system of $n_1 + m_2 + 1$ linear equations for the $n_1 + m_2 + 1$ unknown coefficients is obtained. Once these coefficients are known the polynomials $Q_2(\rho)$ and $P_2(\rho)$ are known, and with these polynomials explicit expressions con be calculated for $\Phi_j(\rho)$, $\operatorname{Re}\rho \geq 0$ and $\Psi_j(\rho)$, $\operatorname{Re}\rho \leq 0$, $j = 1, 2$. As in the preceding sections it is argued that the system of $m_2 + n_1 + 1$ equations has a unique solution.

REMARK 8.1. The solution for the mixed case $N = 2$, constructed above contains the factors

$$\mathrm{e}^{K_+(\rho)},\ \mathrm{Re}\,\rho \leq 0 \quad \text{and} \quad \mathrm{e}^{K_-(\rho)},\ \mathrm{Re}\,\rho \geq 0,$$

and as such the above solution is still a rather formal solution. Actually, integral expressions do exist, for $K_+(\rho)$ and $K_-(\rho)$, cf. [15]. In particular such expressions are fairly simple if $\gamma(\rho) = \beta_1(\rho)\beta_2(\rho)\alpha_1(-\rho)\alpha_2(-\rho)$ is regular at $\rho = 0$. In general, however, these integrals are not so simple to evaluate; unless $\beta_1(\rho)\beta_2(\rho)$ or $\alpha_1(-\rho)\alpha_2(-\rho)$ are rational, but then we have again the case discussed in section 6 or 7, respectively. Obviously, for a mixed case, cf. (8.1), with $N > 2$, the same problem arises and thus makes it questionable whether it is of much use to proceed with a general investigation of the mixed cases for $N > 2$. □

9. The case with $\boldsymbol{\tau}_j$, $j = 2,\ldots,N$, and $\boldsymbol{\sigma}_j$, $j = 1,\ldots,N-1$, constants

In this section we consider the model of section 2 for the case that

$$\boldsymbol{\tau}_j = t_j,\ j = 2,\ldots,N, \qquad \boldsymbol{\sigma}_j = s_j,\ j = 1,\ldots,N-1,$$
$$t := \sum_{j=2}^{N} t_j, \qquad s := \sum_{j=1}^{N-1} s_j, \tag{9.1}$$

with the t_j and s_j nonnegative constants. So we have: for $\mathrm{Re}\rho = 0$,

$$\gamma_1(\rho) = \beta_1(\rho)\mathrm{e}^{s_1\rho}, \tag{9.2}$$
$$\gamma_j(\rho) = \mathrm{e}^{\rho(s_j - t_j)}, \qquad j = 2,\ldots,N-1,$$
$$\gamma_N(\rho) = \alpha_N(-\rho)\mathrm{e}^{-\rho t_N},$$
$$D_N(\rho) = 1 - \gamma(\rho) = 1 - \mathrm{e}^{\rho(s-t)}\beta_1(\rho)\alpha_N(-\rho),$$
$$= 1 - \mathrm{E}\{\mathrm{e}^{-\rho\boldsymbol{\tau}_1}\}\mathrm{E}\{\mathrm{e}^{\rho(\boldsymbol{\sigma}_N - (t-s))}\}.$$

Note that the condition $\beta - \alpha < 0$, cf. (2.7)ii, reads here

$$\beta_1 - \alpha_N + (t - s) < 0. \tag{9.3}$$

From (3.12) and (3.13) we obtain: for $\mathrm{Re}\,\rho = 0$,

$$\Phi_1(\rho) = \frac{(\beta - \alpha)\rho}{1 - \gamma(\rho)}[\Psi_N(\rho) + \alpha_N(-\rho)\mathrm{e}^{-\rho t_N}\Psi_{N-1}(\rho) + \tag{9.4}$$
$$\alpha_N(-\rho)\mathrm{e}^{-\rho(t_N + t_{N-1} - s_{N-1})}\Psi_{N-2}(\rho) + \ldots + \alpha_N(-\rho)\mathrm{e}^{-\rho(t-s+s_1)}\Psi_1(\rho)].$$

Put

$$T_N := \max[t_N, t_N + t_{N-1} - s_{N-1}, t_N + t_{N-1} + t_{N-2} - s_{N-1} - s_{N-2}, \ldots,$$
$$,\ldots, t - t_2 - (s - s_2), t - s + s_1]. \tag{9.5}$$

Denote by

$$(A_N, \infty) \text{ the support of } \boldsymbol{\sigma}_N, \tag{9.6}$$

so that $A_N \geq 0$ and for $\mathrm{Re}\,\rho \leq 0$,

$$\alpha_N(-\rho) = \mathrm{E}\{\mathrm{e}^{\rho\boldsymbol{\sigma}_N}\} = \mathrm{E}\{\mathrm{e}^{\rho(\boldsymbol{\sigma}_N - T_N)}\}\mathrm{e}^{\rho T_N}, \tag{9.7}$$

$$\alpha_N(-\rho) = e^{\rho T_N} \mathrm{E}\{e^{\rho(\sigma_N - T_N)}(\sigma_N > T_N)\} \quad \text{if} \quad A_N > T_N. \tag{9.8}$$

It will hence forth be assumed that

$$A_N > T_N, \tag{9.9}$$

so that with probability one

$$\sigma_N > T_N. \tag{9.10}$$

By using (5.6)iii with $D_N(\rho)$ as given in (9.2) we obtain from (9.4): for $\mathrm{Re}\,\rho = 0$,

$$\Phi_1(\rho)e^{-K_-(\rho)} = e^{-K_+(\rho)}[\Psi_N(\rho) + \mathrm{E}\{e^{\rho(\sigma_N - t_N)}\}\Psi_{N-1}(\rho) + \tag{9.11}$$
$$+\mathrm{E}\{e^{\rho(\sigma_N - (t_N + t_{N-1} - s_{N-1}))}\}\Psi_{N-2}(\rho) + \ldots + \mathrm{E}\{e^{\rho[\sigma_N - (t - s + s_1)]}\}\Psi_1(\rho)].$$

In (9.11) the lefthand side is the boundary value of a function regular in $\mathrm{Re}\,\rho > 0$, which behaves as $O(1)$ for $|\rho| \to \infty$, $|\arg\rho| \le \frac{1}{2}\pi$, cf. (3.15), (5.6); the righthand side is obviously the boundary value of a function regular in $\mathrm{Re}\,\rho < 0$, which also behaves as $O(1)$ for $|\rho| \to \infty$, $\frac{1}{2}\pi \le \arg\rho \le \frac{3}{2}\pi$, cf. (3.15), (5.6). Consequently, these functions are equal to a constant, so that by using the norming condition $\Phi_1(0) = 1$ we have:

$$\Phi_1(\rho) = e^{K_-(\rho)}, \qquad \mathrm{Re}\,\rho \ge 0, \tag{9.12}$$
$$\Psi_N(\rho) + \alpha_N(-\rho)e^{-\rho t_N}\Psi_{N-1}(\rho) + \ldots + \alpha_N(-\rho)e^{-\rho(t-s)}\Psi_1(\rho = e^{K_+(\rho)} \text{ for } \mathrm{Re}\,\rho \le 0.$$

From (3.10) we have: for $\mathrm{Re}\,\rho = 0$,

$$\text{i.} \quad \Phi_2(\rho) - (\beta - \alpha)\rho\Psi_1(\rho) = \beta_1(\rho)e^{\rho s_1}\Phi_1(\rho) = \beta_1(\rho)e^{\rho s_1}e^{K_-(\rho)}. \tag{9.13}$$
$$\text{ii.} \quad \Phi_{j+1}(\rho) - (\beta - \alpha)\rho\Psi_j(\rho) = e^{\rho(s_j - t_j)}\Phi_j(\rho), \quad j = 2, \ldots, N-1.$$

Because $\Phi_1(\rho)$, $\mathrm{Re}\,\rho \ge 0$, is known, cf. (9.12), it is seen that the relation (9.13)i formulates a standard Riemann Boundary Value Problem, cf. [18]; similarly for the equation (9.13)ii, when $\Phi_j(\rho)$ is known. To solve the equations (9.13) put

$$\text{i.} \quad H_1(\rho) := \frac{\rho}{2\pi i} \int\limits_{\mathrm{Re}\,\xi=0} \frac{\beta_1(\xi)e^{\xi s_1}}{(\xi - \rho)\xi} e^{K_-(\xi)} d\xi, \quad \text{for} \quad j = 1, \tag{9.14}$$

$$\text{ii.} \quad H_j(\rho) := \frac{\rho}{2\pi i} \int\limits_{\mathrm{Re}\,\xi=0} \frac{e^{\xi(s_j - t_j)}}{(\xi - \rho)\xi} \Phi_j(\xi) d\xi, \qquad j = 2, \ldots, N-1.$$

It is not difficult to show that these integrals are well defined for all ρ with $\mathrm{Re}\,\rho \ne 0$. Excluding for the sake of simplicity the case that τ_1 and σ_N are both lattice variables, cf. remark 2.1, it is readily verified that the integrand in (9.14)i satisfies for $\mathrm{Re}\,\rho = 0$ the Hölder condition, and so the integral in (9.14)i is well defined for $\mathrm{Re}\,\rho = 0$ as a singular Cauchy integral, cf. [10], [18]. It will be shown that this also applies for the integrals in (9.14)ii whenever $\mathrm{Re}\,\rho = 0$. Hence the following limits exist: for $j = 1, \ldots, N-1$, $\mathrm{Re}\,\rho = 0$,

$$H_j^+(\rho) := \lim_{\substack{\xi \to \rho \\ \mathrm{Re}\xi < 0}} H_j(\xi), \qquad H_j^-(\rho) := \lim_{\substack{\xi \to \rho \\ \mathrm{Re}\xi > 0}} H_j(\xi), \tag{9.15}$$

and the Plemelj-Sokhotski formulas hold, i.e. for $\mathrm{Re}\,\rho = 0$,

$$H_1^+(\rho) - H_1^- = \beta_1(\rho)e^{\rho s_1}e^{K_-(\rho)}, \tag{9.16}$$

$$H_j^+(\rho) - H_j^-(\rho) = \Phi_j(\rho)e^{\rho(s_j - t_j)}, \quad j = 2, \dots, N-1.$$

From (9.13)i and (9.16) we obtain: for $\mathrm{Re}\,\rho = 0$,

$$\Phi_2(\rho) + H_1^-(\rho) = (\beta - \alpha)\rho\Psi_1(\rho) + H_1^+(\rho). \tag{9.17}$$

The function in the lefthand side is the boundary value of a function regular in $\mathrm{Re}\,\rho > 0$, and from (3.15) and (9.14)i it is seen that it behaves as $O(1)$ for $|\rho| \to \infty$, $|\arg \rho| \le \frac{1}{2}\pi$; it also is seen that the function in the righthand side is the boundary value of a function regular in $\mathrm{Re}\,\rho < 0$, and that it is $O(1)$ for $|\rho| \to \infty$, $\frac{1}{2}\pi \le \arg \rho \le \frac{3}{2}\pi$. Hence by analytic continuation and by applying Liouville's theorem we obtain

$$\begin{aligned} &\Phi_2(\rho) + H_1(\rho) = C_2, && \mathrm{Re}\,\rho > 0, \\ &(\beta - \alpha)\rho\Psi_1(\rho) + H_1(\rho) = C_2, && \mathrm{Re}\,\rho < 0, \end{aligned} \tag{9.18}$$

with C_2 a constant.
By using again the *P.S*-formulas, cf. [18], for $\mathrm{Re}\,\rho = 0$, we have

$$H_1^+(\rho) = \frac{1}{2}\beta_1(\rho)e^{\rho s_1}e^{K_-(\rho)} + \frac{\rho}{2\pi i}\int\limits_{\mathrm{Re}\,\xi=0} \frac{\beta_1(\xi)e^{\xi s_1}}{(\xi - \rho)\xi}e^{K_-(\xi)}d\xi, \tag{9.19}$$

$$H_1^-(\rho) = -\frac{1}{2}\beta_1(\rho)e^{\rho s_1}e^{K_-(\rho)} + \frac{\rho}{2\pi i}\int\limits_{\mathrm{Re}\,\xi=0} \frac{\beta_1(\xi)e^{\xi s_1}}{(\xi - \rho)\xi}e^{K_-(\xi)}d\xi,$$

and so since, cf. (9.18), for $\mathrm{Re}\,\rho = 0$,

$$\Phi_2(\rho) + H_1^-(\rho) = C_2, \tag{9.20}$$

we obtain from (3.15) and (9.19) by taking here $\rho = 0$,

$$C_2 = 1 - \frac{1}{2} = \frac{1}{2}. \tag{9.21}$$

Hence $\Phi_2(\rho)$ is for $\mathrm{Re}\,\rho \ge 0$ completely determined by (9.14), (9.16), (9.18), (9.19) and (9.20). It may be readily shown from this result that $\Phi_2(\rho)$ satisfies the Hölder condition on $\mathrm{Re}\,\rho = 0$, and so the integral in (9.14)ii is well defined as a singular integral for $\mathrm{Re}\,\rho = 0$. Consequently, we can solve the functional equation in (9.13)ii for $j = 2$, since its righthand side is known. The solution is obtained along the same lines as that used to solve (9.13)i and it leads to the determination of $\Phi_2(\rho)$, $\mathrm{Re}\,\rho \ge 0$. Hence via recursion the functional equations (9.13)ii can be solved for $j = 2, \dots, N-1$. We shall not present here the detailed solution.

It is of interest to consider for the model discussed in this section, cf. (9.1) and (9.9), the case with

$$t_j = s_j, \qquad j = 2, \dots, N-1, \qquad t_N = s_1. \tag{9.22}$$

Hence the conditions (9.3) and (9.9) read

$$\beta_1 - \alpha_N < 0 \quad \text{and} \quad A_N > t_N, \tag{9.23}$$

and, cf. (9.2),

$$D_N(\rho) = 1 - \gamma(\rho) = 1 - \beta_1(\rho)\alpha_N(-\rho) = (\beta - \alpha)\rho e^{K_+(\rho) - K_-(\rho)}. \tag{9.24}$$

Further from (9.12) and (9.13)ii,

$$\text{i.} \quad \Phi_1(\rho) = e^{K_-(\rho)}, \qquad \mathrm{Re}\,\rho \geq 0, \tag{9.25}$$

$$\text{ii.} \quad \Phi_{j+1}(\rho) - \Phi_j(\rho) = (\beta - \alpha)\rho\Psi_j(\rho), \quad \mathrm{Re}\,\rho = 0, \quad j = 2, \ldots, N-1.$$

The functional equation (9.25)ii is again a boundary value problem, actually a very simple one, and its solution is readily seen to be given by: for $j = 2, \ldots, N$,

$$\begin{aligned} &\text{i.} && \Phi_j(\rho) = \Phi_2(\rho), && \mathrm{Re}\,\rho \geq 0, \\ &\text{ii.} && \rho\Psi_j(\rho) = 0, && \mathrm{Re}\,\rho \leq 0, \end{aligned} \tag{9.26}$$

with $\Phi_2(\rho)$ being determined as above, i.e. by (9.14)i, (9.18) and (9.21). The result (9.26) is not surprising because the service time of a type j-customer, $j = 2, \ldots, N-1$, and the next interarrival are equal.

The two models discussed above may appear to be some what artificial but they may serve as models for queueing processes with a burst-type traffic. Together with the models discussed in the preceding section it is seen that the periodic Pollaczek-process as introduced in the present study presents a flexible scheme for modelling queueing processes with a complicated traffic structure, cf. remark 9.2 below, and that it is fairly accessible to numerical evaluation.

REMARK 9.1. It is also of some interest to mention the model with

$$\tau_j = \sigma_j = \mu_j, \qquad j = 2, \ldots, N, \tag{9.27}$$

where the μ_j are constants. It then follows from (3.12) and (3.13): for $\mathrm{Re}\,\rho = 0$,

$$\Phi_1(\rho) = \frac{(\beta - \alpha)\rho}{1 - \gamma(\rho)} \sum_{j=1}^{N} \Psi_j(\rho), \tag{9.28}$$

$$\begin{aligned} \Phi_2(\rho) - \gamma_1(\rho)\Phi_1(\rho) &= (\beta - \alpha)\rho\Psi_1(\rho), \\ \Phi_{j+1}(\rho) - \Phi_j(\rho) &= (\beta - \alpha)\rho\Psi_j(\rho), \qquad j = 2, \ldots, N-1. \end{aligned}$$

with

$$\beta - \alpha = \beta_1 - \alpha_1, \qquad 1 - \gamma(\rho) = \beta_1(\rho)\alpha_1(-\rho), \qquad \mathrm{Re}\,\rho = 0.$$

As above it is readily deduced for the present case that $\Phi_1(\rho)$, $\mathrm{Re}\,\rho \geq 0$, is the $L.S$-transform of the stationary actual waiting time distribution of a GI/G/1 queue with $\beta_1(\rho)$ and $\alpha_2(\rho)$ the $L.S$-transforms of the service time and of the interarrival time distribution. The expression for $\Phi_2(\rho)$, $\mathrm{Re}\rho \geq 0$, can be obtained via the same arguments as used to derive (9.18), . . . , (9.21) and further it is seen that again

$$\Phi_j(\rho) = \Phi_2(\rho) \quad \text{for} \quad j = 2, \ldots, N.$$

It is readily seen that the present model is a periodic GI/G/1 queue where every first customer generates a fixed number of $N-1$ subsequent customers with constant service- and interarrival times, each service time being equal to the next interarrival time. This process may service as a simple model for bursty traffic. □

REMARK 9.2. To obtain some further insight in the modelling aspects of the periodic Pollaczek process consider the model of section 2 with all σ_j, $j = 1, \ldots, N$, constants, say $\sigma_j = s_j$ and put $s = s_1 + \ldots + s_N$. Consider for this process the embedded process $w_1^{(n)}$, $n = 1, 2, \ldots$, and compare it with the single server model D/G/1 with interarrival times equal to s and service time distribution that of the sum of $\tau_1, \ldots, \tau_N$. For this D/G/1 process the supply to the workload of the server occurs at the equally spaced arrival moments, where as in the $w_1^{(n)}$-process the supply to the workload is distributed, (discretely and unevenly) over each of the intervals of length s. Hence the supply of the workload for the $w_1^{(n)}$-process is a smoother process than that in the D/G/1 system. Consequently, the workload process in this periodic Pollaczek process is smoother, i.e. its jumps are less pronounced than those of the workload process of the D/G/1 system.

A further aspect of this periodic Pollaczek aspect in modelling traffic processes is the available freedom in choosing the distributions of the service times$\tau_1, \ldots, \tau_N$. This freedom may be used for modelling bursty traffic, for instance by choosing for one or more of the distributions of these variables distributions with a large $\beta_j^{(2)}/2\beta_j^2$. □

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