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Axioms for $\aleph_0$-categorical orderings

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Axioms for $\aleph_0$-categorical Orderings

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Abstract

Every $\aleph_0$-categorical linear ordering is finitely axiomatizable. This follows trivially from the characterization of $\aleph_0$-categorical orderings in [Ros69]. But this proof does not show what the axioms are. Here it is shown how to construct an axiom for any $\aleph_0$-categorical ordering. To this end the notion of a *handle* for an $\aleph_0$-categorical ordering A is introduced. A handle is a formula that allows us to decide if two points from another $\aleph_0$-categorical ordering B can be extended to a convex subordering $A' \subseteq B$ that is isomorphic to A.

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1. INTRODUCTION

We will present a recipe for the construction of an axiom for any $\aleph_0$-categorical ordering. An $\aleph_0$-categorical ordering is a countable linear ordering $\mathfrak{A}$ such that all countable linear orderings $\mathfrak{B}$ that are logically equivalent to $\mathfrak{A}$, are also isomorphic to $\mathfrak{A}$. Here logically equivalent means that every closed first-order formula holds in one of the orderings if and only if it holds in the other. Logical equivalence (also known as elementary equivalence or first-order equivalence) is a model theoretic concept; model theory, as defined in [CK91], forms the general framework of this text.

In [Ros69] Rosenstein shows that all $\aleph_0$-categorical orderings can be generated inductively. From this characterization it follows that there are only countably many isomorphism classes of $\aleph_0$-categorical orderings. Another corollary is that the $\aleph_0$-categorical orderings are finitely axiomatizable. The only problem with this corollary is that it does not provide the axioms.

Suppose we were given a $\aleph_0$-categorical ordering and we were asked to produce an axiom for it. One way to go about this is of course to try every closed formula in the first-order

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Suppose we were given a $\aleph_0$-categorical ordering and we were asked to produce an axiom for it. One way to go about this is of course to try every closed formula in the first-order language of linear orderings and verify if it is an axiom for the given $\aleph_0$-categorical ordering or not. However, suppose we know the position of the ordering in Rosensteins characterization, can we construct an axiom for the ordering along the same lines as the construction of the ordering itself.

Yes, we can. It is not a trivial procedure, however. The main problem is that it turns out to be necessary to define convex suborderings that do not have endpoints, neither in the subordering, nor in the enclosing ordering. We will see that this problem can be solved by using interior points of the subordering. Formulæ that specify that there is a convex subordering of some kind containing some elements are called handles for that kind of subordering. The fact that 'smaller' $\aleph_0$-categorical orderings are constructed 'first' (have a lower rank) in the characterization allows us to form convex suborderings that have a lower rank than the enclosing ordering. Using a slightly generalized form of relativizing formulæ (restricting the scope of all quantifiers of a closed formula not to a formula with *exactly* one parameter but to a formula with *at least* one parameter), we can define these suborderings and specify there relative positions. The result is can be used for both an axiom for the enclosing ordering and a handle for it that can be used to make formulæ for even larger $\aleph_0$-categorical orderings.

The structure of this article is as follows. First we will see that, without loss of generality, we can restrict ourselves to the power set of the rational numbers instead of taking all countable linear orderings into account. Second Rosensteins characterization of the $\aleph_0$-categorical orderings will be given. Then the notion of relativizing formulas with extra parameters will be introduced. Finally the axioms and handles for all $\aleph_0$-categorical orderings will be constructed.

2. The power set of the rationals

Because this article is about countable linear orderings and every countable linear ordering is isomorphic to some subset of $\mathbb{Q}$ with the induced order relation, we will speak of subsets of $\mathbb{Q}$ as orderings and leave the induced order relation implicit. So instead of discussing all countable linear orderings we restrict ourselves to the power set of $\mathbb{Q}$. We denote the collection of all subsets of $\mathbb{Q}$ that are isomorphic to a countable linear ordering $\mathfrak{A}$ by $\overline{\mathfrak{A}}$. For $A \subseteq \mathbb{Q}$ we will write $\overline{A}$ instead of $\overline{\langle A, < \rangle}$.

To compare this notation to the terminology used in [Ros82] we have a look at the ordering $\langle \mathbb{N}, < \rangle$. Assume that the definition of $\mathbb{N}$ and $\mathbb{Q}$ is such that $\omega \neq \mathbb{N}$ and of course $\mathbb{N} \subset \mathbb{Q}$. In that case the *order type* $\boldsymbol{\omega}$ of ω is the class of all orderings that are isomorphic to $\langle \omega, \in \rangle$. Thus $\langle A, < \rangle \in \boldsymbol{\omega}$ for every $A \in \overline{\omega}$. We also see that $\mathbb{N} \in \overline{\omega}$ and so $\overline{\mathbb{N}} = \overline{\omega}$. From now on we will use equivalence classes $\overline{A}$ where Rosenstein would use order types.

To define when a linear ordering is $\aleph_0$-categorical we have to specify when two linear orderings are elementary equivalent.

Definition 1 *If $\mathfrak{A}$ and $\mathfrak{B}$ are linear orderings then $\mathfrak{A}$ and $\mathfrak{B}$ are* elementary equivalent *if and only if for all closed formulæ φ*

$$\mathfrak{A} \models \varphi \qquad \textit{if and only if} \qquad \mathfrak{B} \models \varphi$$

Notation: $\mathfrak{A} \equiv \mathfrak{B}$.

Definition 2 *A countable linear ordering* $\mathfrak{A}$ *is* $\aleph_0$-categorical *if and only if for every countable linear ordering* $\mathfrak{B}$ *such that* $\mathfrak{B} \equiv \mathfrak{A}$ *holds* $\mathfrak{B} \cong \mathfrak{A}$.

It would be nice if we could take a subset I of $\mathbb{Q}$, associate each element i of I with its own subset A_i of $\mathbb{Q}$ and then substitute A_i for every element $i \in I$. Because $\langle \mathbb{Q} \times \mathbb{Q}, <_L \rangle$, with $<_L$ the lexicographical ordering on $\mathbb{Q} \times \mathbb{Q}$, is a countable linear ordering, we can find a function $F: \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ such that for all $p, q, r, s \in \mathbb{Q}$ holds $\langle p, q \rangle <_L \langle r, s \rangle$ if and only if $F(p, q) < F(r, s)$. With this function F we can now define the desired substitution operation and some other useful operations on orderings.

Definition 3 *If* $I \subseteq \mathbb{Q}$ *and for all* $\{ i \in I \mid A_i \} \subset \wp(\mathbb{Q})$, *then*

$$\sum_{i \in I} A_i := \{ F(i, x) \mid i \in I \wedge x \in A_i \}$$

If $A_0, \ldots, A_n \subseteq \mathbb{Q}$, *then*

$$\sum \langle A_0, \ldots, A_n \rangle := \sum_{i \in \{0, \ldots, n\}} A_i$$

If $A, B \subseteq \mathbb{Q}$, *then*

$$A \otimes B := \sum_{i \in B} A_i \quad (A_i = A \text{ for all } i \in B)$$

In the following definition we will use that the $<$-relation on the rationals can be extended to a partial ordering $\lessdot$ on the power set of the rationals by defining that for all $A, B \subseteq \mathbb{Q}$

$$A \lessdot B \qquad \text{if and only if} \qquad a < b \quad (\text{for all } a \in A, b \in B)$$

Note that this relation is only a proper partial order when restricted to $\wp(\mathbb{Q}) \setminus \{\varnothing\}$, because $\varnothing \lessdot A \lessdot \varnothing$ holds for every $A \subseteq \mathbb{Q}$, which contradicts the antisymmetry of partial orderings.

The term *convex subset* occured several times already. Here follows the formal definition.

Definition 4 *A subset* C *of a set* $A \subseteq \mathbb{Q}$ *is a* convex *subset of* A *if and only if there are* $L, R \subseteq A$ *such that* $A = L \cup C \cup R$ *and* $L \lessdot C \lessdot R$.

In this definition L, R and C are allowed to be empty.

In order to characterize an element of a subset of $\mathbb{Q}$ in terms of its immediate neighborhood we define the following

Definition 5 *For an element* a *of a set* $A \subseteq \mathbb{Q}$

$$A^{<a} := \{ b \in A \mid b < a \}$$
$$A^{>a} := \{ b \in A \mid a < b \}$$

If $A^{<a} = \varnothing$ we say that a is the *left endpoint* of A. Likewise, if $A^{>a} = \varnothing$ then a is the *right endpoint* of A. If $A^{<a} \neq \varnothing$ and has a right endpoint then a has a *gap at its left*, otherwise it is a *limit point from the left* in A. If $A^{>a} \neq \varnothing$ and has a left endpoint then a has a *gap at its right*, otherwise it is a *limit point from the right* in A. If a has a gap at both sides it is *isolated* in A and if a is a limit point from both sides it is a *condensation* point in A.

3. Rosensteins characterization

For Rosensteins characterization of $\aleph_0$-categorical orderings we need an additional operator on orderings.

Definition 6 *Let $p_0, p_1, p_2, \ldots$ the sequence of the prime numbers, then*

$$Q_n := \left\{ \tfrac{t}{p_n^m} \,|\, t, m \in \mathbb{N}^+ \right\} \setminus \mathbb{N}$$

Let $n \in \mathbb{N}$ and $A_0, \ldots, A_n \subseteq \mathbb{Q}$, then

$$\sigma\langle A_0, \ldots, A_n \rangle := \bigcup_{i \in \{0,\ldots,n\}} A_i \otimes Q_i$$

This operator σ is called the *shuffle operator.* A simple non-trivial example of the shuffle operator is $\sigma\langle\{0,1\}\rangle$. This produces a subset of $\mathbb{Q}$ where every element has a gap on one side and is a limit point from the other side.

Note that if more than one subset of $\mathbb{Q}$ is shuffled, then infinitely many copies of each are densely distributed among each other. (To get a feeling for this, one could try to work out what $\sigma\langle\{0,1\}, \mathbb{Q}\rangle$ is like, *e.g.*, by defining a subset of $\mathbb{Q}$ that is isomorphic to it.)

We can now define inductively a special collection $\mathcal{M}$ of subsets of $\mathbb{Q}$. Rosenstein showed that this collection coincides with the collection of $\aleph_0$-categorical subsets of $\mathbb{Q}$. (Actually Rosenstein didn't restrict himself to subsets of $\mathbb{Q}$, but used *order types* instead.)

Definition 7 *For $n \in \mathbb{N}$ we define $\mathcal{M}_n \subseteq \wp(\mathbb{Q})$ and using these we define a class $\mathcal{M}$ of countable linear orderings as follows.*

$$\begin{aligned} \mathcal{M}_0 &:= \{\{0\}\} \\ \mathcal{M}_{n+1} &:= \{\textstyle\sum\langle A_0, \ldots, A_n\rangle \,|\, n \in \mathbb{N} \wedge A_0, \ldots, A_n \in \mathcal{M}_n\} \cup \\ & \{\sigma\langle A_0, \ldots, A_n\rangle \,|\, n \in \mathbb{N} \wedge A_0, \ldots, A_n \in \mathcal{M}_n\} \\ \mathcal{M} &:= \bigcup \left\{ \overline{A} \,|\, \exists n \in \mathbb{N}[A \in \mathcal{M}_n] \right\} \end{aligned}$$

For $A \in \mathcal{M}$ we define $r_{\mathcal{M}}(A)$ as the smallest $n \in \mathbb{N}$ such that there is a $B \in \mathcal{M}_n$ with $B \cong A$.

This number $r_{\mathcal{M}}(A)$ is called the *rank* of A. Our goal is to construct an axiom for every member of $\mathcal{M}$. An axiom for say $A \in \mathcal{M}$ is constructed under the assumption that for every $B \in \mathcal{M}$ that has a lower rank than A we have already found not only an axiom φ_B, but also a handle ψ_B.

Actually there is one set that is a $\aleph_0$-categorical ordering but is not an element of $\mathcal{M}$. This is the empty set. It is not too difficult, however, to find an axiom for the empty ordering, *e.g.*

$$\neg\exists x_0[x_0 = x_0]$$

To facilitate the construction and application of these formulæ an an extended notion of relativizing formulæ will be introduced now.

4. Relativizing formulæ

For the construction of the formulæ in the next section of this article we need to be able to relativize a formula with respect to another. That is, we want to be able to restrict the scope of the quantifiers of one formula to a subset of the domain specified by another. Furthermore we need several parameters for *both* formulæ, whereas normally a formula is only relativized with respect to a formula with only one parameter. In other words we want to be able to restrict the quantifiers of a formula to a *parameterized* definable subset of the domain.

Definition 8 *Let $\psi = \psi(x_0, \ldots, x_n)$ such that x_n is a free variable of ψ or $n = 0$. We define the* relativization *φ^ψ of a formula φ with respect to ψ as follows.*

1. $x_i^\psi := x_{i+n+1}$
2. $(x_i = x_j)^\psi := (x_i^\psi = x_j^\psi)$
3. $(x_i \sqsubset x_j)^\psi := (x_i^\psi \sqsubset x_j^\psi)$
4. $(\varphi_0 \wedge \varphi_1)^\psi := (\varphi_0^\psi \wedge \varphi_1^\psi)$
5. $(\neg\varphi)^\psi := \neg(\varphi^\psi)$
6. $(\exists x_i[\varphi])^\psi := \exists x_{i+n+1}[\varphi^\psi \wedge \forall x_n[x_n = x_{i+n+1} \rightarrow \psi]]$

The effect of rule 1 is that the parameters of φ are shifted as to avoid collision with the parameters of ψ. Rule 6 specifies that a formula with a quantifier is modified to restrict the scope of this quantifier to the 'range' of ψ. The other rules only point out that relativization is linear in all other ways to make a formula from terms or smaller formulæ.

To illustrate the idea of the range of ψ as a parameterized definable set we look at the special case that the parameters for φ are in the range of ψ. That is, if we have a sequence $a_0, \ldots, a_{n+m} \in A$ such that $A \models \psi[a_0, \ldots, a_{n-1}, a_i]$ for every $i \in \{n+1, \ldots, n+m\}$. In this case

$$A \models \varphi^\psi[a_0, \ldots, a_{n+m}]$$

if and only if

$$\{\, a \in A \mid A \models \psi[a_0, \ldots, a_{n-1}, a] \,\} \models \varphi[a_{n+1}, \ldots, a_{n+m}]$$

Note that x_n is *not* a free variable of φ^ψ.

5. Construction of axioms

By now we have the necessary tools to construct the promised formulæ. The extended relativization will help us build bigger formulæ from smaller ones and the concept of a handle enables us to 'grab' a result of a shuffle operation using internal points of it where we would have liked to use real numbers to cut out a convex subset of an ordering.

We will view elements of $\mathcal{M}$ as constructed from other elements of $\mathcal{M}$ that have a lower rank. Looking at the operators $\sum$ and σ we note that the constituting orderings $A_0, \ldots, A_n$

have one or infinitely many counterparts in the new ordering A. These counterparts are *convex* subsets of A that are elementary equivalent, and because they are $\aleph_0$-categorical even isomorphic, to one of $A_0, \dots, A_n$. Conversely: every element $a \in A$ can be extended to such a convex subset.

Theorem 1 *For every $A \in \mathcal{M}$ there is a handle, that is, a formula ψ_A such that for every countable linear ordering B and all pairs $b_0, b_1 \in B$ holds*

$$B \models \psi_A[b_0, b_1]$$

if and only if there exists a convex subset $A' \subseteq B$ such that $b_0, b_1 \in A'$ and $A' \equiv A$.

Furthermore there is a formula α_A such that for all countable linear orderings B holds $B \models \alpha_A$ if and only if $B \equiv A$.

Proof: Let $A \in \mathcal{M}$ such that for every B with rank B less than rank A the required formulæ can be constructed.

If $r_{\mathcal{M}}(A) = 0$ then let $\psi_A := (x_0 = x_1)$ and $\alpha_A := \exists x_0 \forall x_1 [x_0 = x_1]$

If $r_{\mathcal{M}}(A) \neq 0$ then there are two possibilities: A is the finite sum of some elements of $\mathcal{M}$ of lower rank, or A is the shuffling of some elements of $\mathcal{M}$ of lower rank.

Let $A_0, \dots, A_n \in \mathcal{M}$ such that $A = \sum \langle A_0, \dots, A_n \rangle$ or $A = \sigma\langle A_0, \dots, A_n \rangle$. The induction hypothesis gives us for every $m \in \{0, \dots, n\}$ a handle $\psi_{A_m} = \psi_{A_m}(x_0, x_1)$ for A_m and a formula α_{A_m} that axiomatizes A_m.

We define for every $m \in \{0, \dots, n\}$ and every $i, j \in \mathbb{N}$

$$\psi_{m,i,j} := \forall x_{k+1} \forall x_{k+2} [(x_i = x_{k+1} \wedge x_j = x_{k+2}) \rightarrow \psi_{A_m}^{x_k = x_k}]$$

Where k is the maximum of i and j. This is a trick to use x_i and x_j instead of x_0 and x_1. The formula ψ_{A_m} is only relativized to a tautology to shift the parameters out of the way. Below we will see more interesting applications of the extended relativization.

Suppose $A = \sigma\langle A_0, \dots, A_n \rangle$. Let

$$\varepsilon_{i,j} := \psi_{0,i,j} \vee \dots \vee \psi_{n,i,j}$$

Now $A \models \varepsilon_{i,j}[a_0, \dots, a_k]$ if and only if the set $\{a_i, a_j\}$ can be extended to a convex subset that is elementary equivalent to an A_m with $m \in \{0, \dots, n\}$. If $A \models \varepsilon_{i,j}[a_0, \dots, a_k]$ we say that a_i is *near* to a_j. Note that for every $a \in A$ holds that a is near to a. In order to specify that nearness is a transitive relation on A we define

$$\vartheta := (\varepsilon_{0,1} \wedge \varepsilon_{1,2}) \rightarrow \varepsilon_{0,2}$$
$$\vartheta' := (\forall x_0 \forall x_1 \forall x_2 [\vartheta])^{x_2 \sqsubseteq x_4 \wedge x_4 \sqsubseteq x_3}$$

Formula ϑ' says that nearness is a transitive relation between two points specified by x_2 and x_3. This is a proper use of the extended relativization. The quantifiers over x_0, x_1 and x_2, as well as the implicit quantifiers in ϑ are restricted to the points between x_2 and x_3. Furthermore, the indices of these variables as well as the variables in ϑ are increased by five to make room for the variables of the formula $x_2 \sqsubseteq x_4 \wedge x_4 \sqsubseteq x_3$. Note that x_2 and x_3 are the only free variables of ϑ'.

We abbreviate $x_i \sqsubset x_j \wedge \neg\varepsilon_{i,j}$ as $x_i \ll x_j$. In words: x_i is much less than x_j.

In order to define a handle for A it we will not use the two given points directly, but rather two points that bracket these points from a proper distance. For this purpose we define the following formula:

$$\xi := x_2 \ll x_0 \wedge x_2 \ll x_1 \wedge x_0 \ll x_3 \wedge x_1 \ll x_3$$

So ξ says that x_2 is much less than both x_0 and x_1, and both x_0 and x_1 are much less than x_3. We will use this formula ξ to restrict the quantifiers of the handle.

Let now for $m \in \{0, \dots, n\}$

$$\begin{aligned}
\gamma_m &:= \exists x_2[x_0 \ll x_2 \wedge x_2 \ll x_1 \wedge \psi_{m,2,2}] \\
\gamma &:= x_0 \ll x_1 \rightarrow (\gamma_0 \wedge \cdots \wedge \gamma_n) \\
\gamma' &:= (\forall x_0 \forall x_1[\gamma])^{x_2 \sqsubseteq x_4 \wedge x_4 \sqsubseteq x_3}
\end{aligned}$$

Formula γ_m says that there is a convex subset properly between x_0 and x_1 that is elementary equivalent to A_m. Formula γ says that if x_0 is much less than x_1 then there is for every $m \in \{0, \dots, n\}$ a convex subset properly between x_0 and x_1 that is elementary equivalent to A_m. Formula γ' takes this a step further and says that there is such a subset between any x_0 and x_1 in the interval $[x_2, x_3]$ with $x_0 \ll x_1$. In other words: γ' says that the 'suborderings' $A_0, \dots, A_n$ lie *dense* between x_2 and x_3.

We define

$$\varphi := (\forall x_0[\alpha_{A_0}^{\varepsilon_{0,1}} \vee \cdots \vee \alpha_{A_n}^{\varepsilon_{0,1}}])^{x_2 \sqsubseteq x_4 \wedge x_4 \sqsubseteq x_3}$$

In words: for every point x_0 between x_2 and x_3, the subset of all points near to x_0 is elementary equivalent to one of $A_0, \dots, A_n$.

To conclude the construction for the case that A is the shuffle of some elements of $\mathcal{M}$ of lower rank we define the following formulæ

$$\begin{aligned}
\beta &:= \xi \wedge \gamma' \wedge \varphi \wedge \vartheta' \\
\psi_A &:= \exists x_2 \exists x_3[\beta] \\
\alpha_A &:= \forall x_0 \forall x_1[\psi_A]
\end{aligned}$$

The formula β says that

1. The points x_0 and x_1 lie properly between x_2 and x_3;
2. suborderings $A_0, \dots, A_n$ lie dense between x_2 and x_3;
3. For any x_4 between x_2 and x_3, the set of points near to x_4 is elementary equivalent to one of $A_0, \dots, A_n$;
4. the relation *near to* is transitive on the interval $[x_2, x_3]$.

With Fraïssés method of extending finite partial isomorphisms it is quite straightforward to show that α_A axiomatizes A. Furthermore if B is a countable linear ordering and $b_0, b_1 \in B$

are elements of B such that $B \models \psi_A[b_0, b_1]$ then choose b_2 and b_3 such that $B \models \beta[b_0, b_1, b_2, b_3]$. If A' is the subset of all elements of B that are much greater than b_2 and much less than b_3, then $A' \equiv A$. It follows that ψ_A is a handle for A.

When A is a finite sum it is much easier than than the case where A is the result of the shuffle operator. Suppose that $A = \sum \langle A_0, \dots, A_n \rangle$. If there is an index $m \in \{0, \dots, n\}$ such that $A_m \cong \sum \langle B_0, \dots, B_k \rangle$ we can rewrite A as

$$\sum \langle A_0, \dots, A_{m-1}, B_0, \dots, B_k, A_{m+1}, \dots, A_n \rangle$$

so we can assume that for every $m \in \{0, \dots, n\}$ either $A_m \cong \{0\}$ or $A_m \cong \sigma\langle B_0, \dots, B_k \rangle$. Furthermore if there is an $m \in \{0, \dots, n\}$ such that $A_m \equiv A_{m+1} \not\cong \{0\}$, then A is isomorphic to

$$\sum \langle A_0, \dots, A_{m-1}, A_{m+1}, \dots, A_n \rangle$$

so we can also assume that if $A_m \equiv A_{m+1}$ then $A_m \cong A_{m+1} \cong \{0\}$. We define

$$\begin{aligned} \beta &:= x_0 \sqsubset x_1 \wedge \cdots \wedge x_{n-1} \sqsubset x_n \wedge \\ &\quad\ \forall x_{n+1}[\psi_{0,0,n+1} \vee \cdots \vee \psi_{n,n,n+1}] \\ \alpha_A &:= \exists x_0 \cdots \exists x_n[\beta] \end{aligned}$$

The formula β says that the points $x_0, \dots, x_n$ are in the right order and that for every point x_{n+1} there is an $m \in \{0, \dots, n\}$ such that there is a convex subset in A that contains x_m and x_{n+1} and that is elementary equivalent with A_m.

In order to conclude the case for $A = \sum \langle A_0, \dots, A_n \rangle$ we define the following formulæ:

$$\begin{aligned} \delta &:= (x_2 \sqsubseteq x_4 \vee \psi_{0,2,4}) \wedge (x_4 \sqsubseteq x_3 \vee \psi_{n,4,3}) \\ \psi_A &:= \exists x_2 \exists x_3[x_2 \sqsubseteq x_0 \wedge x_2 \sqsubseteq x_1 \wedge x_0 \sqsubseteq x_3 \wedge x_1 \sqsubseteq x_3 \wedge \alpha_A^\delta] \end{aligned}$$

The formula δ says that x_4 is in or near to the interval $[x_2, x_3]$. Formula ψ_A says that there is an interval $[x_2, x_3]$ that contains x_0 and x_1 such that the points in or near $[x_2, x_3]$ form a convex subset elementary equivalent to A. It follows that ψ_A is a handle for A.

So when $A = \sigma\langle A_0, \dots, A_n \rangle$ as well as when $A = \sum \langle A_0, \dots, A_n \rangle$ we can construct an axiom as well as a handle for A.

Thus, using induction on the rank of A, we can say that we are able to construct an axiom – as well as a handle – for every element of $\mathcal{M}$.

■

Finally a hint for the reader who believes that φ_A holds in A for $A \in \mathcal{M}$, but wants to verify that if there is a $B \subseteq \mathbb{Q}$ such that φ_A holds in B then B has to be isomorphic to A. On any $\aleph_0$-categorical subset C of $\mathbb{Q}$ we can define an equivalence relation by: $p \sim_C q$ if and only if there exists a convex subset C' of C, such that $p, q \in C'$ and $r_{\mathcal{M}}(C') < r_{\mathcal{M}}(C)$. Then it is possible to prove that if $A \in \mathcal{M}$ and $B \subseteq \mathbb{Q}$ such that $B \models \varphi_A$, then it is possible to find an isomorphism $f\colon A/{\sim_A} \to B/{\sim_B}$ such that for all $[p] \in A/{\sim_A}$ holds $[p] \cong f([p])$.

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