A complete equational axiomatisation for prefix iteration

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A Complete Equational Axiomatisation for Prefix Iteration

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Abstract

Iteration is added to Minimal Process Algebra (MPA), which is a subset of BFA, that is equivalent to Milner's basic CCS. We present an equational axiomatisation for MPA, and prove that this axiomatisation is complete with respect to strong bisimulation equivalence. To obtain this result, we will set up a term rewriting system, based on the axioms, and show that bisimilar terms have the same normal form.

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1 Introduction

Kleene [Kle56] defined a binary operator * in the context of finite automata, where $E^*F$ denotes the iterate of E and F. Milner [Mil84] studied Kleene's star in the setting of (strong) bisimulation equivalence, and raised the question whether there exists a complete axiomatisation for it. Bergstra, Bethke & Ponse [BBP93] incorporated the binary Kleene star into Basic Process Algebra (BPA) [BK84], and they suggested three axioms for iteration. In [FZ93] it was proved that these three axioms, together with the five standard axioms of BFA, are a complete axiomatisation for BPA* with respect to bisimulation.

In this paper, we add the deadlock δ to the syntax. Sewell [Sew93] has proved that there does not exist a complete finite equational axiomatisation for BPA*. Therefore, we restrict the binary sequential composition $x \cdot y$ to the unary prefix sequential composition $a \cdot x$, to obtain Minimal Process Algebra MPA, equivalent to basic CCS [Mil80]. Moreover, we add (prefix) iteration $a^*x$ to the syntax, resulting in the algebra MPA. We propose two simple axioms for iteration, and prove that these two axioms together with four standard axioms from BPA, are a complete axiomatisation for MPA with respect to bisimulation.

The proof consists of producing a term rewriting system from the axioms, and showing that bisimilar normal forms are syntactically equal, modulo commutativity and associativity of the +. Hence, bisimilarity between terms is decidable.

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2 Minimal Process Algebra with Iteration

We assume an alphabet $A$ of atomic actions. The terms in our algebra $\text{MPA}_\delta^*(A)$ are built from a constant $\delta$, which represents deadlock, together with the binary alternative composition $x + y$, and the unary prefix sequential composition $a \cdot x$ and iteration $a^*x$, for $a \in A$. Table 1 presents an operational semantics for $\text{MPA}_\delta^*(A)$ in Plotkin style [Plo81]. The special symbol $\sqrt{\cdot}$ (pronounce ‘tick’) in this table represents successful termination.

<table>
<thead>
<tr>
<th>$x \xrightarrow{a} x'$</th>
<th>$x \xrightarrow{\sqrt{\cdot}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x + y \xrightarrow{a} x'$</td>
<td>$y + x \xrightarrow{a} \sqrt{\cdot}$</td>
</tr>
<tr>
<td>$a \cdot x \xrightarrow{a} x$</td>
<td>$a^*x \xrightarrow{a} a^*x$</td>
</tr>
<tr>
<td>$x \xrightarrow{b} \sqrt{\cdot}$</td>
<td>$a^*x \xrightarrow{b} \sqrt{\cdot}$</td>
</tr>
<tr>
<td>$a^*x \xrightarrow{b} \sqrt{\cdot}$</td>
<td>$x \xrightarrow{b} \sqrt{\cdot}$</td>
</tr>
</tbody>
</table>

Table 1: Action rules for $\text{MPA}_\delta^*(A)$

Our model for $\text{MPA}_\delta^*(A)$ consists of all the closed terms that can be constructed from the atomic actions and the three operators. That is, the BNF grammar for the collection of process terms is as follows, where $a \in A$:

$$p ::= \delta \mid p + p \mid a \cdot p \mid a^*p$$

In the sequel the operator $\cdot$ will often be omitted, so $ap$ denotes $a \cdot p$. As binding convention, $^*$ binds stronger than $\cdot$, which in turn binds stronger than $+$. Process terms are considered modulo (strong) bisimulation equivalence [Par81]. Intuitively, two process terms are bisimilar if they have the same branching structure.

**Definition 2.1** Two processes $p_0$ and $q_0$ are called bisimilar, denoted by $p_0 \leftrightarrow q_0$, if there exists a symmetric relation $R$ between processes such that:

1. $R(p_0, q_0)$,
2. if $p \xrightarrow{a} p'$ and $R(p, q)$, then there is a transition $q \xrightarrow{a} q'$ such that $R(p', q')$,
3. if $p \xrightarrow{a} \sqrt{\cdot}$ and $R(p, q)$, then $q \xrightarrow{a} \sqrt{\cdot}$.

Since the action rules in Table 1 are in path format [BV93], it follows that bisimulation equivalence is a congruence with respect to all the operators, i.e. if $p \leftrightarrow p'$ and $q \leftrightarrow q'$, then $p + q \leftrightarrow p' + q'$ and $ap \leftrightarrow ap'$ and $a^*p \leftrightarrow a^*p'$.

Table 2 contains an axiom system for $\text{MPA}_\delta^*(A)$, consisting of four axioms from $\text{BPA}_\delta(A)$ together with two axioms for iteration. In the sequel, $p = q$ will mean that this equality can be derived from these axioms. Our axiomatisation for $\text{MPA}_\delta^*(A)$ is sound with respect to
bisimulation equivalence, i.e. if \( p = q \) then \( p \cong q \). Since bisimulation is a congruence, this can be verified by checking soundness for each axiom separately. In this paper it is proved that the axiomatisation is complete with respect to bisimulation, i.e. if \( p \cong q \) then \( p = q \).

\[
\begin{array}{c}
A1 & x + y = y + x \\
A2 & (x + y) + z = x + (y + z) \\
A3 & x + x = x \\
A6 & x + \delta = x \\
MI1 & a \cdot a^*x + x = a^*x \\
MI2 & a^*(a^*x) = a^*x
\end{array}
\]

Table 2: Axioms for MPA\(\delta\)(A)

3 A Term Rewriting System

We want to define a term rewriting system (TRS) for process terms in that reduces bisimilar terms to the same normal form. However, it is not so easy to construct such a TRS for MPA\(\delta\)(A). Because then the terms \( a^*x + x \) and \( a^*x \) must reduce to the same normal form, and a rule \( a^*x \rightarrow a^*x + x \) does not terminate, so we need the rules:

\[
\begin{align*}
a^*(x + y) + x & \rightarrow a^*(x + y) \\
a^*x + x & \rightarrow a^*x
\end{align*}
\]

However, these rules are not yet sufficient, because they do not deal with the case \( a^*(b^*x) + x \leftrightarrow a^*(b^*x) \). Hence, for this case we must introduce extra rewrite rules. But then, they do not cover the case \( a^*(b^*(c^*x)) + x \leftrightarrow a^*(b^*(c^*x)) \), etc. So to obtain unique normal forms modulo bisimulation for MPA\(\delta\)(A), apparently we need an infinite number of rewrite rules.

To avoid this complication, we replace iteration by a new, equivalent operator \( a^\oplus_p \), representing the behaviour of \( a \cdot a^*p \). The construct \( a^\oplus x \) is called proper iteration. (Its standard notation would be \( a^+_x \), but we want to avoid ambiguous use of the \(+\).) The operational semantics and the axiomatisation for proper iteration are given in Table 3. They are obtained from the action rules and axioms for MPA\(\delta\)(A), using the obvious equivalence \( a^*x \leftrightarrow a^\oplus x + x \). The axiomatisation in Table 3 is complete for MPA\(\oplus\)(A) if and only if the axiomatisation in Table 2 is complete for MPA\(\delta\)(A).

3.1 The TRS for MPA\(\oplus\)(A)

From now on we consider process terms modulo commutativity and associativity of the \(+\).

Table 4 contains a the TRS for MPA\(\oplus\)(A). It is easy to see that all rules can be deduced from MPA\(\oplus\)(A). In each rule, the term at the left-hand side contains more symbols than the term at the right-hand side. Hence, the TRS is terminating.
\[
\begin{array}{c}
a^0x \xrightarrow{a} a^6x + x \\
\hline
\text{PMI1} \quad a(a^0x + x) = a^6x \\
\text{PMI2} \quad a^0(a^6x + x) = a^6x
\end{array}
\]

Table 3: Semantics and axioms for proper iteration

\[
\begin{array}{ll}
1. & x + x \longrightarrow x \\
2. & x + \delta \longrightarrow x \\
3. & a(a^0x + x) \longrightarrow a^6x \\
4. & a^0(a^6x + x) \longrightarrow a^6x \\
5. & a(a^0\delta) \longrightarrow a^6\delta \\
6. & a^0(a^6\delta) \longrightarrow a^6\delta
\end{array}
\]

Table 4: Rewrite rules for \(\text{MPA}_{g}^{B}(A)\)

In order to obtain unique normal forms our TRS must be \textit{weakly confluent}, i.e. if some term has reductions \(p'' \leftarrow p \longrightarrow p'\), then there must exist a term \(q\) such that \(p'' \longrightarrow q \longrightarrow p'\). Because then Newman’s Lemma says that the TRS yields unique normal forms. To obtain the property of weak confluence for our TRS, we have added the extra Rules 5,6. Weak confluence of our TRS can easily be verified by checking this property for all \textit{overlapping redexes}. However, there is no need to do so, because we will not use this property in the proof that bisimilar normal forms are syntactically equal modulo A1,2.

4 Normal Forms Decide Bisimulation Equivalence

In the previous section we have developed a TRS for \(\text{MPA}_{g}^{B}(A)\) that reduces terms to a normal form. Since all rewrite rules are sound with respect to bisimulation equivalence, it follows that each term is bisimilar with its normal forms. So in order to determine completeness of the axiomatisation for \(\text{MPA}_{g}^{B}(A)\) with respect to bisimulation, it is sufficient to prove that if two normal forms are bisimilar, then they are provably equal by the axioms A1,2.

Process terms are considered modulo A1,2. From now on, this equivalence is denoted by \(p \equiv q\), and we say that \(p\) and \(q\) are of the same form. Each process term \(p\) is a sum of terms that are of the form \(\delta\) or \(aq\) or \(a^0r\); these terms are called the \textit{summands} of \(p\).
Now we present the proof of our main theorem, which is in fact a simplified version of the completeness proof in \[FWZ93\], with some minor extra cases to deal with deadlock. In the proof we apply induction on the following weight function on terms.

\[
\begin{align*}
g(\delta) & = 0 \\
g(p + q) & = \max\{g(p), g(q)\} \\
g(ap) & = g(p) + 1 \\
g(a^s p) & = g(p) + 1
\end{align*}
\]

**Theorem 4.1** If two normal forms \(p\) and \(q\) are bisimilar, then they are of the same form.

**Proof.** We prove the theorem by induction on \(g(p) + g(q)\). The case \(g(p) + g(q) = 0\) is trivial, because then Rule 2 ensures that both \(p\) and \(q\) must be of the form \(\delta\). Now assume \(\delta\) that we have already proved the theorem for bisimilar normal forms \(p\) and \(q\) with \(g(p) + g(q) < n\). We prove it for \(g(p) + g(q) = n\).

1. Suppose that summands \(ar\) of \(p\) and \(as\) of \(q\) are bisimilar. Clearly, \(g(r) + g(s) < n\) and \(r \leftrightarrow s\). So the induction hypothesis yields \(r \cong s\).

2. Next, let summands \(ar\) of \(p\) and \(a^ss\) of \(q\) be bisimilar, so \(r \leftrightarrow a^ss + s\). We deduce a contradiction.
   
   If \(s \neq \delta\), then \(a^ss + s\) is a normal form, because we cannot apply Rules 1 or 2 to it. Moreover, \(g(r) + g(a^ss + s) < n\), so the induction hypothesis gives \(r \cong a^ss + s\). But then we can apply Rule 3 to \(ar\), so we have a contradiction.
   
   And if \(s = \delta\), then \(r \leftrightarrow a^ss\), so induction implies \(r \cong a^ss\). But then we can apply Rule 5 to \(ar\), which yields a contradiction again.

3. Finally, assume that summands \(a^sr\) and \(a^ss\) are bisimilar. We prove \(r \cong s\).
   
   Clearly \(a^sr + r \leftrightarrow a^ss + s\). If \(r\) and \(s\) do not contain summands that are bisimilar with \(a^ss\) and \(a^sr\) respectively, then \(r \leftrightarrow s\), so that induction yields \(r \cong s\), and we are done.
   
   So without loss of generality we may assume that \(r\) contains a summand that is bisimilar with \(a^ss\). According to point 2 this summand cannot be of the form \(at\), so apparently some summand \(a^su\) of \(r\) is bisimilar with \(a^ss\). Then induction yields \(u \cong s\). According to Rule 1, \(r\) can contain only one subterm of the form \(a^ss\). Its other summands must be bisimilar to summands of \(s\).
   
   Suppose that \(s\) contains a summand bisimilar to \(a^sr\). Then as before we see that this summand is of the form \(a^sr\), which indicates that \(s\) contains more symbols than \(r\). But on the other hand \(r\) has a summand \(a^ss\), so \(r\) contains more symbols than \(s\). This cannot be, so \(s\) and \(a^sr\) can have no behaviour in common.
   
   And suppose that \(s\) contains a summand bisimilar to the summand \(a^ss\) of \(r\). Then as before it follows that this summand is of the form \(a^ss\), which indicates that \(s\) contains more symbols than itself. Again, this cannot be.
   
   Hence, the summands of \(s\) unequal to \(\delta\) correspond with the summands of \(r\) unequal to \(a^ss\). We distinguish two possibilities:
REFERENCES

- $s \not\equiv \delta$. Then $r$ must be of the form $a^\ominus s + r'$, where $r' \Leftrightarrow s$. Induction implies $r' \equiv s$. But then we can apply Rule 4 to $a^\ominus r$, so we have a contradiction.

- $s \equiv \delta$. Then $r \equiv a^\ominus \delta$. But then we can apply Rule 6 to $a^\ominus r$, so once more we have a contradiction. □

Corollary 4.2 The axiomatisation $A1,2,3,6 + MI1,2$ for $MPA_\delta^\otimes(A)$ is complete with respect to bisimulation equivalence.

Proof. If two terms in $MPA_\delta^\otimes(A)$ are bisimilar, then according to Theorem 4.1 their normal forms are of the same form. Since all the rewrite rules can be deduced from $A1,2,3,6 + PMI1,2$, it follows that this is a complete axiom system for $MPA_\delta^\otimes(A)$. Then clearly $A1,2,3,6 + MI1,2$ is a complete axiomatisation for $MPA_\delta^\otimes(A)$. □

References


