Centrum voor Wiskunde en Informatica

REPORT RAPPORT

An approximate Riemann solver for magnetohydrodynamics (that works in more than one dimension)

K.G. Powell

Department of Numerical Mathematics

Report NM-R9407 March 1994

CWI is the National Research Institute for Mathematics and Computer Science. CWI is part of the Stichting Mathematisch Centrum (SMC), the Dutch foundation for promotion of mathematics and computer science and their applications.
SMC is sponsored by the Netherlands Organization for Scientific Research (NWO). CWI is a member of ERCIM, the European Research Consortium for Informatics and Mathematics.

P.O. Box 94079, 1090 GB Amsterdam (NL)
Kruislaan 413, 1098 SJ Amsterdam (NL)
Telephone +31 20 592 9333
Telefax +31 20 592 4199

An Approximate Riemann Solver for Magnetohydrodynamics
(That Works in More than One Dimension)

Kenneth G. Powell
Department of Aerospace Engineering
The University of Michigan
Ann Arbor, MI 48109-2118, USA

Abstract

An approximate Riemann solver is developed for the governing equations of ideal magnetohydrodynamics (MHD). The Riemann solver has an eight-wave structure, where seven of the waves are those used in previous work on upwind schemes for MHD, and the eighth wave is related to the divergence of the magnetic field. The structure of the eighth wave is not immediately obvious from the governing equations as they are usually written, but arises from a modification of the equations that is presented in this paper. The addition of the eighth wave allows multi-dimensional MHD problems to be solved without the use of staggered grids or a projection scheme, one or the other of which was necessary in previous work on upwind schemes for MHD. A test problem made up of a shock tube with rotated initial conditions is solved to show that the two-dimensional code yields answers consistent with the one-dimensional methods developed previously.

AMS Mathematics Subject Classification (1991): 35L65, 65M20, 76W05, 85-08, 85A30.
Keywords and Phrases: magnetohydrodynamics; MHD; approximate Riemann solver.
Note: This report will be published elsewhere. The research reported was performed during visits to ICASE and CWI.

1 Introduction

The governing equations of ideal magnetohydrodynamics (MHD) describe the physics of a conducting fluid in which the following assumptions hold:

$$\begin{aligned} \frac{\lambda}{L} &\ll 1 \\ \frac{\epsilon}{\tau\sigma} &\ll 1 \\ \left(\frac{V}{c}\right)^2 &\ll 1 \\ \frac{\mu}{\rho V L} &\ll 1 \end{aligned} \tag{1}$$

where ρ, V, τ and L are, respectively, characteristic density, speed, time and length scales for the problem, c is the speed of light, and ϵ and σ represent the dielectric constant and conductivity of the

Report NM-R9407
ISSN 0169-0388
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

fluid. These equations, written in conservation-law form, are

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho\mathbf{u} \\ \mathbf{B} \\ E \end{pmatrix} + \nabla\cdot\begin{pmatrix} \rho\mathbf{u} \\ \rho\mathbf{u}\mathbf{u} + \mathbf{I}\left(p + \frac{\mathbf{B}\cdot\mathbf{B}}{2}\right) - \mathbf{B}\mathbf{B} \\ \mathbf{u}\mathbf{B} - \mathbf{B}\mathbf{u} \\ \left(E + p + \frac{\mathbf{B}\cdot\mathbf{B}}{2}\right)\mathbf{u} - \mathbf{B}\left(\mathbf{u}\cdot\mathbf{B}\right) \end{pmatrix} = 0 \tag{2}$$

where $\mathbf{I}$ is a 3×3 identity matrix, ρ is the density, $\mathbf{u}$ is the velocity, p is the pressure, $\mathbf{B}$ is the magnetic field, and E is the energy, defined as

$$E = \frac{p}{\gamma - 1} + \rho\frac{\mathbf{u}\cdot\mathbf{u}}{2} + \frac{\mathbf{B}\cdot\mathbf{B}}{2} \ . \tag{3}$$

Solutions of these equations can yield insight into a number of problems governed by fluid-dynamic and electromagnetic effects.

Much of the past work in solving these equations has been based on Rusanov and Lax-Wendroff techniques. Only recently have authors begun to work on upwind schemes for solving these equations. In particular, Brio and Wu [2], Zachary and Colella [8], and Dai and Woodward [7] have done some of the early development of Riemann-solver-based schemes for the MHD equations. Their work has been based not on the system of eight conservation laws as written in Equation 2, but instead on the closely related system that comes from assuming B_x = constant and dropping the evolution equation for B_x. This yields a 7×7 system. The reason for their use of this modified system arises from the fact that one of the equations governing the magnetic field is

$$\nabla \cdot \mathbf{B} = 0 \ , \tag{4}$$

which, in one dimension, becomes the constraint B_x = constant.

The eigenvalues and eigenvectors of this 7×7 system are well known (see, for example, the book by Jeffrey and Taniuti [3]); they correspond to:

- one entropy wave traveling with speed u;
- two Alfvén waves traveling with speed $u \pm c_a$ where

  $$c_a = \frac{B_x}{\sqrt{\rho}}$$

  is the Alfvén speed;
- four magneto-acoustic waves, two "fast" and two "slow", traveling with speed $u \pm c_f$ and $u \pm c_s$ respectively, where

  $$c_{f,s}^2 = \frac{1}{2}\left(\frac{\gamma p + \mathbf{B}\cdot\mathbf{B}}{\rho} \pm \sqrt{\left(\frac{\gamma p + \mathbf{B}\cdot\mathbf{B}}{\rho}\right)^2 - 4\frac{\gamma p B_x^2}{\rho^2}}\right) \ .$$

An (x, t) diagram of the wave interactions at a cell interface is shown in Figure 1.

Given these seven eigenvalues and corresponding right and left eigenvectors, it is possible to develop a linear approximate Riemann solver ala Roe [2, 8], or a more nonlinear approximate Riemann solver [7]. Once some questions as to how to scale the left and right eigenvectors of the system are answered (for a very nice description of the problems of scaling in the MHD eigensystem, and an elegant solution to these problems, see the paper by Roe and Balsara [5]), a robust solver for one-dimensional unsteady problems in MHD can be developed.

Building a code capable of solving two- or three-dimensional problems from the one-dimensional Riemann solver building block is not, unfortunately, as straightforward as in the case of the Euler equations. In the one-dimensional problem, no evolution equation is necessary for the component of $\mathbf{B}$

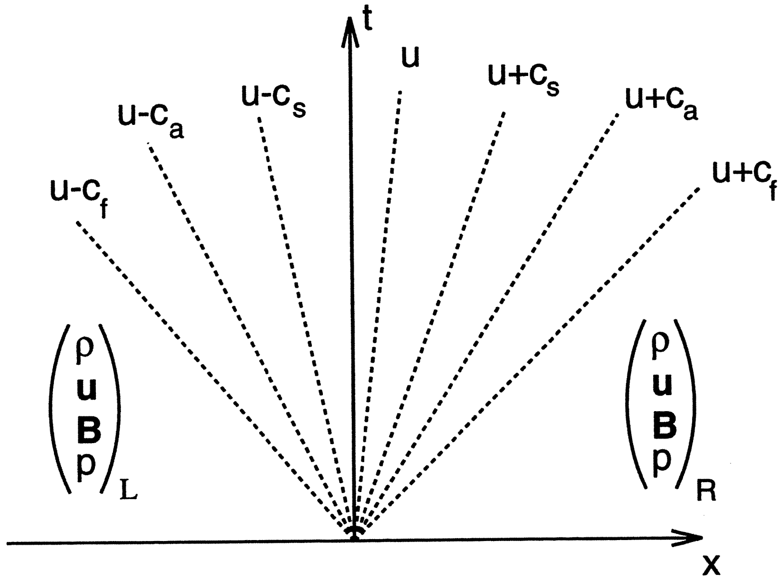

Figure 1: Waves in the One-Dimensional MHD Riemann Problem

normal to a cell interface, because the condition of Equation 4 implies that $B_{n_L} = B_{n_R}$. However, in a two-dimensional problem, this is no longer true. In two dimensions, the discrete constraint corresponding to Equation 4 is

$$\sum_{faces} B_n \, ds = 0 \ , \tag{5}$$

and so a jump in B_n *is* allowed across a face; it simply must be balanced by the jumps across the other faces of the cell. Thus, a separate procedure for updating this portion of the magnetic field must be implemented, and must be implemented in such a way as to satisfy the constraint implied in Equation 5. It should be noted that the $\nabla \cdot \mathbf{B}$ constraint is a headache not just for *upwind* schemes for MHD, but for solution of MHD problems in more than one dimension by *any* method. Typically, one of three approaches is taken to satisfy this constraint:

- a projection scheme, in which a Poisson equation must be solved to subtract off the portion of the magnetic field that leads to a non-zero divergence;
- non-collocated variables (e.g. a staggered-grid approach), so that the constraint is met identically;
- a vector-potential description of the magnetic field, so that the constraint is met identically.

A very different approach is taken in the work presented here. Instead of solving a seven-wave Riemann problem, with an added procedure to update the remaining B-field component which assures that Equation 4 is satisfied, an eight-wave Riemann solver, in which *all* of the magnetic field components are updated, is developed and tested.

2 Derivation of the Eight-Wave Riemann Solver

Given the primitive variables

$$\mathbf{W} = (\rho, u, v, w, B_x, B_y, B_z, p) \ , \tag{6}$$

Equation 2 may be rewritten in quasilinear form as

$$\frac{\partial \mathbf{W}}{\partial t} + A_p \frac{\partial \mathbf{W}}{\partial x} + B_p \frac{\partial \mathbf{W}}{\partial y} + C_p \frac{\partial \mathbf{W}}{\partial z} = 0 \ , \tag{7}$$

where, for example

$$A_p = \begin{bmatrix} u & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u & 0 & 0 & -\frac{B_x}{\rho} & \frac{B_y}{\rho} & \frac{B_z}{\rho} & \frac{1}{\rho} \\ 0 & 0 & u & 0 & -\frac{B_y}{\rho} & -\frac{B_x}{\rho} & 0 & 0 \\ 0 & 0 & 0 & u & -\frac{B_z}{\rho} & 0 & -\frac{B_x}{\rho} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_y & -B_x & 0 & -v & u & 0 & 0 \\ 0 & B_z & 0 & -B_x & -w & 0 & u & 0 \\ 0 & \gamma p & 0 & 0 & (\gamma-1)\mathbf{u}\cdot\mathbf{B} & 0 & 0 & u \end{bmatrix}. \tag{8}$$

The Riemann solver would normally be based on the eigensystem of A_p, but it is evident that this matrix is singular — the fifth row of the matrix is zero, leading to a zero eigenvalue. This zero eigenvalue is clearly non-physical — the eigenvalues should appear either singly as the $x-$component of the flow speed, u, or in pairs symmetric about u. It also does not bode well numerically — the mode corresponding to this eigenvalue will be undamped.

The approach taken here is to look for a way in which to modify the governing equations so as to make A_p non-singular. The criteria that should be met by the modified matrix A'_p are:

- The eigenvalues and eigenvectors corresponding to the seven waves in the one-dimensional ($B_x =$ constant) Riemann solver remain unchanged;
- The eigenvalue corresponding to the new eighth wave is equal to u (the only physical choice for a single eigenvalue);
- The left and right eigenvectors corresponding to the new eight wave "make sense";
- In the case $B_x =$ constant, the eight-wave Riemann problem reduces to the seven-wave Riemann problem.

With these criteria in mind, it becomes possible to find a modified version of A_p, given some patience and some facility with Maple's symbolic manipulation capabilities. The modified matrix that meets the above criteria is

$$A'_p = \begin{bmatrix} u & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u & 0 & 0 & 0 & \frac{B_y}{\rho} & \frac{B_z}{\rho} & \frac{1}{\rho} \\ 0 & 0 & u & 0 & 0 & -\frac{B_x}{\rho} & 0 & 0 \\ 0 & 0 & 0 & u & 0 & 0 & -\frac{B_x}{\rho} & 0 \\ 0 & 0 & 0 & 0 & u & 0 & 0 & 0 \\ 0 & B_y & -B_x & 0 & 0 & u & 0 & 0 \\ 0 & B_z & 0 & -B_x & 0 & 0 & u & 0 \\ 0 & \gamma p & 0 & 0 & 0 & 0 & 0 & u \end{bmatrix}. \tag{9}$$

The eigensystem of this matrix is composed of the following eight waves, with their corresponding eigenvalues λ, left eigenvectors $\tilde{\ell}$ and right eigenvectors $\tilde{r}$:

One Entropy Wave

$$\lambda_e = u$$

$$\tilde{\ell}_e = \left(1, 0, 0, 0, 0, 0, 0, -\frac{1}{a^2}\right)$$

$$\tilde{r}_e = (1,0,0,0,0,0,0,0)^{\mathrm{T}} \; ; \tag{10}$$

Two Alfvén Waves

$$\begin{aligned}
\lambda_a &= u \pm \frac{B_x}{\sqrt{\rho}} \\
\tilde{\ell}_a &= \left(0,0,-B_z,B_y,0,\pm\frac{B_z}{\sqrt{\rho}},\mp\frac{B_y}{\sqrt{\rho}},0\right) \\
\tilde{r}_a &= (0,0,-B_z,B_y,0,\pm\sqrt{\rho}B_z,\mp\sqrt{\rho}B_y,0)^{\mathrm{T}} \; ;
\end{aligned} \tag{11}$$

Four Magneto-acoustic Waves

$$\begin{aligned}
\lambda_{f,s} &= u \pm c_{f,s} \\
\tilde{l}_{f,s} &= \left(0,\pm\rho c_{f,s},\mp\frac{B_xB_y\rho c_{f,s}}{\rho c_{f,s}^2-B_x^2},\mp\frac{B_xB_z\rho c_{f,s}}{\rho c_{f,s}^2-B_x^2},0,\frac{B_y\rho c_{f,s}^2}{\rho c_{f,s}^2-B_x^2},\frac{B_z\rho c_{f,s}^2}{\rho c_{f,s}^2-B_x^2},1\right) \\
\tilde{r}_{f,s} &= \left(\rho,\pm c_{f,s},\mp\frac{B_xB_y c_{f,s}}{\rho c_{f,s}^2-B_x^2},\mp\frac{B_xB_z c_{f,s}}{\rho c_{f,s}^2-B_x^2},0,\frac{B_y\rho c_{f,s}^2}{\rho c_{f,s}^2-B_x^2},\frac{B_z\rho c_{f,s}^2}{\rho c_{f,s}^2-B_x^2},\gamma p\right)^{\mathrm{T}} \; ;
\end{aligned} \tag{12}$$

One "Divergence" Wave

$$\begin{aligned}
\lambda_d &= u \\
\tilde{\ell}_d &= (0,0,0,0,1,0,0,0) \\
\tilde{r}_d &= (0,0,0,0,1,0,0,0)^{\mathrm{T}} \; .
\end{aligned} \tag{13}$$

It is important to note that the first seven waves yield the same eigenvectors and eigenvalues as the seven-wave Riemann problem, with the additional information that none of them carries a change in B_x (the fifth entry of each right eigenvector is zero), and none of the wave strengths is proportional to a jump in B_x (the fifth entry of each left eigenvector is zero). The new eighth wave travels with the $x-$component of the flow speed (its eigenvalue is u), and it carries a jump in B_x (the only non-zero entry in the left eigenvector is the entry corresponding to B_x), and affects only the $x-$component of the magnetic field (the only non-zero entry in the right eigenvector is the entry corresponding to B_x).

It is clear that the eigensystem of this modified matrix has all of the desired properties. In the case $B_x =$ constant, the strength of the eighth wave is zero, and the model reverts to that of the seven-wave problem. The new wave simply gives a rational procedure for dealing with non-zero jumps in B_x across the cell faces, which will in general occur when problems in two or three dimensions are being solved. The question remains, however, of what the modification of the matrix A_p (and the corresponding changes to B_p and C_p) has done to the system of conservation laws.

This can be seen by collecting the source terms due to the modifications to A_p, B_p and C_p and transforming to conserved variables. The new equation set, which has the eight-wave eigensystem described above, is

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho\mathbf{u}\\ \mathbf{B}\\ E\end{pmatrix} + \nabla\cdot\begin{pmatrix}\rho\mathbf{u}\\ \rho\mathbf{u}\mathbf{u}+\mathbf{I}\left(p+\frac{\mathbf{B}\cdot\mathbf{B}}{2}\right)-\mathbf{B}n\mathbf{B}\\ \mathbf{u}\mathbf{B}-\mathbf{B}\mathbf{u}\\ \left(E+p+\frac{\mathbf{B}\cdot\mathbf{B}}{2}\right)\mathbf{u}-\mathbf{B}\left(\mathbf{u}\cdot\mathbf{B}\right)\end{pmatrix} = -\begin{pmatrix}0\\ \mathbf{B}\\ \mathbf{u}\\ \mathbf{u}\cdot\mathbf{B}\end{pmatrix}\nabla\cdot\mathbf{B}\,. \tag{14}$$

This is a noteworthy result: *the source term that must be added to Equation 2 is proportional to* $\nabla\cdot\mathbf{B}$. At the partial differential equation level, only terms that are equal to zero have been added to the conservative form of the governing equations. So, while technically the equations are no longer in conservative form, the deviations from conservation will be very small. It is only by writing the equations in this slightly non-conservative form that the singularity related to $\nabla \cdot \mathbf{B}$ can be removed. It has been previously noted that solving the momentum equation in non-conservative form can remove instabilities related to non-zero $\nabla \cdot \mathbf{B}$ [1]; the current work hopefully reinforces this earlier result, and sheds further light on the mechanism for stabilizing the equations, as well as applying the idea in a novel way to develop a Riemann solver for multi-dimensional MHD.

It is interesting to note another justification of this particular choice of source term. Rewriting Equation 2 slightly by expanding some of the terms, the following form of the equations

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0 \\
\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) + \nabla \left(p + \frac{\mathbf{B}\cdot\mathbf{B}}{2} \right) - \mathbf{B}\cdot\nabla\mathbf{B} - \underline{\mathbf{B}\nabla\cdot\mathbf{B}} &= 0 \\
\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{B} + \mathbf{B}\nabla\cdot\mathbf{u} - \mathbf{B}\cdot\nabla\mathbf{u} - \underline{\mathbf{u}\nabla\cdot\mathbf{B}} &= 0 \\
\frac{\partial E}{\partial t} + \mathbf{u}\cdot\nabla\left(E + p + \frac{\mathbf{B}\cdot\mathbf{B}}{2}\right) + \left(E + p + \frac{\mathbf{B}\cdot\mathbf{B}}{2}\right)\nabla\cdot\mathbf{u} - & \\
\mathbf{B}\cdot\nabla(\mathbf{u}\cdot\mathbf{B}) - \underline{(\mathbf{u}\cdot\mathbf{B})\nabla\cdot\mathbf{B}} &= 0
\end{aligned} \tag{15}$$

is obtained. The terms that are proportional to $\nabla \cdot \mathbf{B}$ have been underlined; they are exactly the same as the source term defined above. Thus it can be seen that the addition of the source term in Equation 14 simply acts to remove the terms proportional to $\nabla \cdot \mathbf{B}$ that appear in Equation 2.

Another interesting note is what the evolution equation for $\nabla \cdot \mathbf{B}$ is for the two forms of the governing equations. This may be seen by taking the divergence of the evolution equation for the magnetic field in Equations 2 and 14. For the original form of the equations, the evolution equation is

$$\begin{aligned}
\nabla \cdot \left(\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{B} + \mathbf{B}\nabla\cdot\mathbf{u} - \mathbf{B}\cdot\nabla\mathbf{u} - \mathbf{u}\nabla\cdot\mathbf{B} \right) &= 0 \\
\frac{\partial}{\partial t}(\nabla\cdot\mathbf{B}) &= 0\,.
\end{aligned} \tag{16}$$

From the partial differential equation point of view, this might well seem the correct result; $\nabla \cdot \mathbf{B} = 0$ is an initial condition, and this equation guarantees that $\nabla \cdot \mathbf{B} = 0$ is maintained throughout the evolution. For the modified form of the equations, the evolution equation for the magnetic field is

$$\begin{aligned}
\nabla \cdot \left(\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{B} + \mathbf{B}\nabla\cdot\mathbf{u} - \mathbf{B}\cdot\nabla\mathbf{u} \right) &= 0 \\
\frac{\partial}{\partial t}(\nabla\cdot\mathbf{B}) + \nabla\cdot(\mathbf{u}\nabla\cdot\mathbf{B}) &= 0\,.
\end{aligned} \tag{17}$$

Thus the addition of the source term has modified the evolution equation for $\nabla \cdot \mathbf{B}$ so that the quantity $\nabla \cdot \mathbf{B}/\rho$ is treated as a passive scalar. This is clearly the more numerically stable of the two evolution equations; any local $\nabla \cdot \mathbf{B}$ that is created is convected away.

The above derivation gives all the pieces for building an ideal MHD solver that works for two-dimensional problems, without having to resort to non-collocated variables or a projection algorithm. Specifically, a Roe-type approximate Riemann solver has been implemented, where the wave strengths and speeds are derived from the above left eigenvectors and eigenvalues. The eigenvectors are properly normalized to avoid difficulties associated with coinciding wave speeds [5]. The average state needed at cell interfaces is computed by a simple average of left and right states (although a Roe average does exist for the ideal MHD equations [4]). The source term, though small, is calculated in each cell, and added to the residual. The resulting code is first order in space and time.

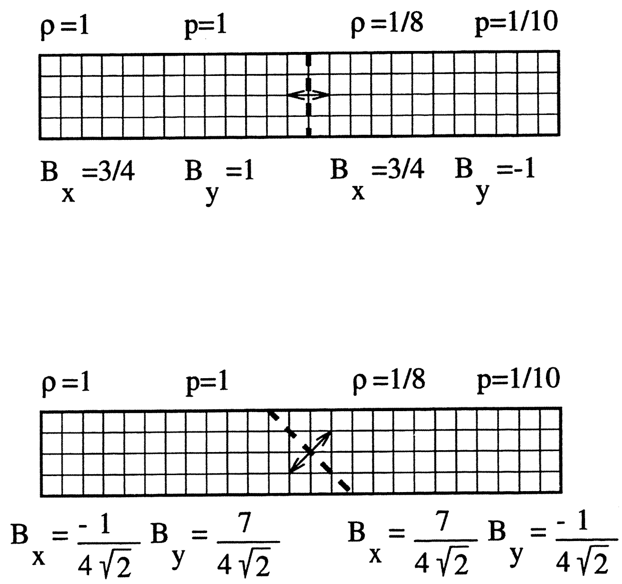

Figure 2: A Test Problem for Two-Dimensional MHD

3 A Test of the Eight-Wave Riemann Solver

Brio and Wu [2] developed a test problem for one-dimensional MHD solvers based on the shock-tube problem of Sod [6]. Two stationary plasmas are separated by a membrane which is removed at $t = 0$, allowing the plasmas to interact. The test problem used here for the two-dimensional MHD solver is a rotated version of the Brio-Wu problem. The left and right input states, and the orientation of propagation of disturbances to the grid, is shown in Figure 2. In the Brio-Wu problem (the top figure), the boundary conditions are that the problem is periodic along a line $y =$ constant; in the current test problem (the bottom figure), the boundary conditions are that the problem is periodic along a line $x + y =$ constant.

Both the rotated and non-rotated problems were run on coarse (600 cells in x) and fine (1200 cells in x) grids. The time step was taken as $\Delta t/\Delta x = 0.2$, which corresponds to a CFL number of approximately 0.8 on the non-rotated problem. The ratio of specific heats, γ, was 2.0. The number of time steps taken on the coarse and fine grids were 100 and 200, respectively. The $x-$axis in the plotted results from the rotated problem was scaled by a factor of $\sqrt{2}$, to account for the fact that the CFL number is lower for the rotated problem than for the non-rotated problem.

Figures 3–7 show comparisons of the results on the fine grid of the non-rotated shock-tube problem with the (scaled) results of the rotated shock-tube problem for

3. density (ρ);

4. pressure (p);

5. velocity component normal to the original discontinuity (u_n);

6. velocity component tangential to the original discontinuity (u_t);

7. magnetic-field component tangential to the original discontinuity (B_t).

As can be seen, the agreement is quite good, with the results of the two cases nearly indistinguishable for all but the normal component of velocity. The errors in u_n are balanced by errors in the magnetic-field component normal to the original discontinuity (B_n). Figure 8 shows B_n for the rotated shock-tube

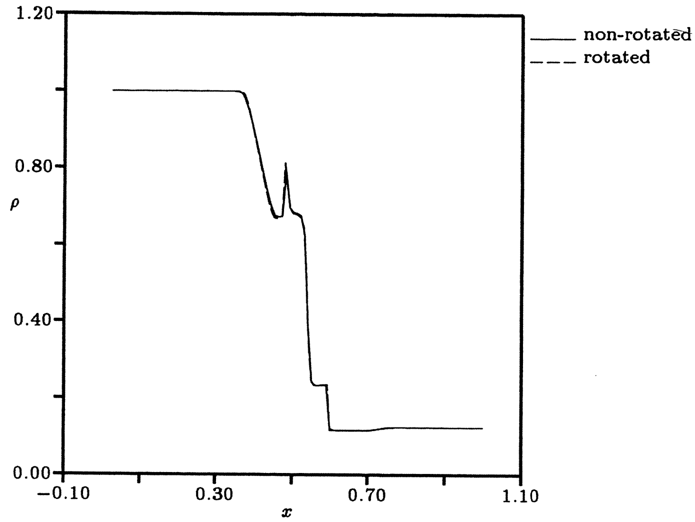

Figure 3: Density in the Rotated and Non-Rotated Shock Tubes

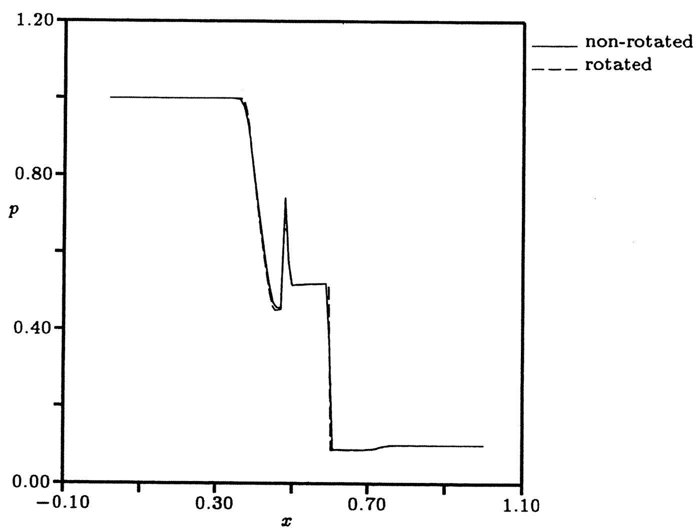

Figure 4: Pressure in the Rotated and Non-Rotated Shock Tubes

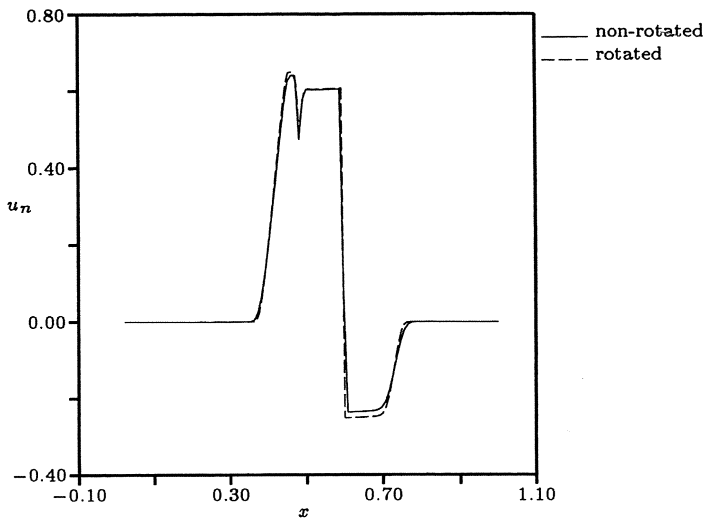

Figure 5: Normal Velocity in the Rotated and Non-Rotated Shock Tubes

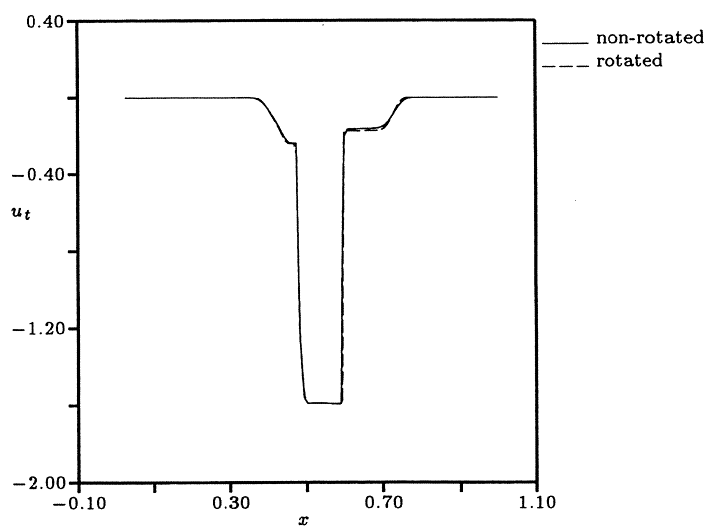

Figure 6: Tangential Velocity in the Rotated and Non-Rotated Shock Tubes

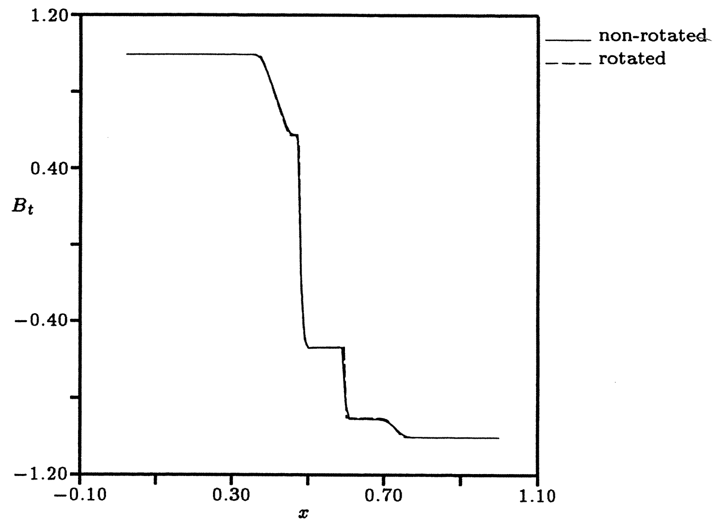

Figure 7: Tangential Magnetic Field in the Rotated and Non-Rotated Shock Tubes

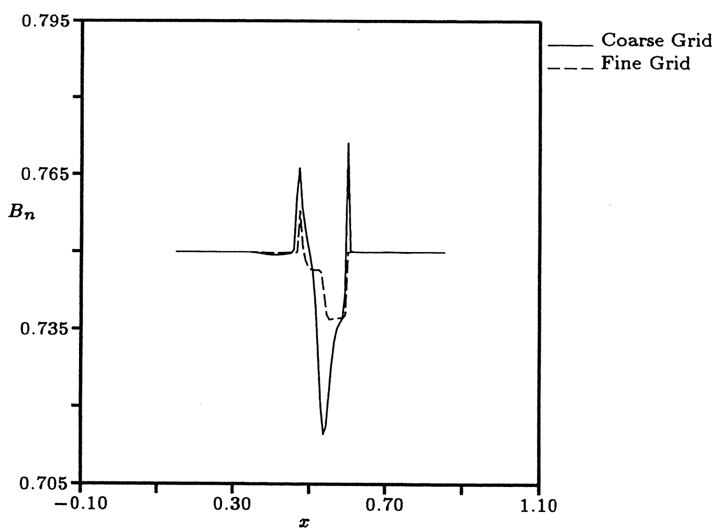

Figure 8: Normal Magnetic Field in the Rotated Shock Tube (Coarse and Fine)

problem on the coarse and fine grids. In the non-rotated problem, $B_n = 0.75$ throughout the tube. As can be seen, there are errors on the order of a few percent in B_n on the coarse grid, but the errors are reduced as the grid is refined.

4 Concluding Remarks

In some respects, this paper presents the development of only one-eighth of a Riemann solver. Seven of the eight waves of the Riemann solver are the same as those used in previous work on upwind methods for MHD. The deceptively simple eighth wave that arises from the analysis, however, is of a different character than the other seven — it arises only in multi-dimensional problems, and it is crucial for understanding and solving those problems. It plays the very important role of stabilizing the numerical method with respect to the small amounts of $\nabla \cdot \mathbf{B}$ generated in solving the discrete MHD equations.

Given the meteoric rise of Riemann solvers in the computation of compressible gas dynamics, it is not very risky to predict that schemes based on Riemann solvers will play an increasingly important role in the computation of compressible conducting flows. The ability of Riemann solvers to capture strong discontinuities robustly and with minimal dissipation, the framework that Riemann solvers provide for implementing stable boundary procedures, and the aesthetically appealing physical basis of Riemann solvers are all strong arguments for their use. The aim of this paper is to remove what is hopefully one of the last major obstacles to the use of Riemann solvers in large-scale codes for computing multi-dimensional conducting flows.

Acknowledgments

The work presented here was carried out at the Institute for Computer Applications in Science and Engineering (ICASE), and the Centrum voor Wiskunde en Informatica (CWI). The author thanks both ICASE and CWI for supporting his visits there, and particularly Dr. Barry Koren, who coordinated the author's visit to CWI. At ICASE, the author's work was sponsored through NASA Contract Number NAS1-19480. The appropriate addresses are: CWI, Kruislaan 413, 1098 SJ Amsterdam, NETHERLANDS; ICASE, NASA Langley Research Center, Hampton VA, 23681. In addition, the author thanks Phil Roe, who was, as always, an inspiration in this work.

References

[1] J. U. Brackbill and D. C. Barnes. The effect of nonzero $\nabla \cdot \mathbf{B}$ on the numerical solution of the magnetohydrodynamic equations. *Journal of Computational Physics*, 35:426–430, 1980.

[2] M. Brio and C. C. Wu. An upwind differencing scheme for the equations of ideal magnetohydrodynamics. *Journal of Computational Physics*, 75:400–422, 1988.

[3] A. Jeffrey and A. Taniuti. *Nonlinear Wave Propagation*. Academic Press, 1964.

[4] P. L. Roe, 1994. Private Communication.

[5] P. L. Roe and D. S. Balsara. Notes on the eigensystem of magnetohydrodynamics. *Journal of Computational Physics*, 1994. To appear.

[6] G. A. Sod. A survey of finite-difference methods for systems of nonlinear conservation laws. *Journal of Computational Physics*, 27(1):1–31, 1978.

[7] P. R. Woodward and W. Dai. A Riemann solver for ideal magnetohydrodynamics. *Journal of Computational Physics*, 1994. To appear.

[8] S. Zachary and P. Colella. A higher-order Godunov method for the equations of ideal magnetohydrodynamics. *Journal of Computational Physics*, 99:341–348, 1992.