Centrum voor Wiskunde en Informatica
REPORTRAPPORT

On the analysis of the symmetrical shortest queue

J.W. Cohen

Department of Operations Reasearch, Statistics, and System Theory

BS-R9420 1994

Report BS-R9420
ISSN 0924-0659

CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

On the Analysis of the Symmetrical Shortest Queue

J.W. Cohen
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

May 1994

Abstract

The symmetrical shortest queueing model has been studied in several papers. It resulted that the two unknown functions needed for the complete analytic description of the solution are meromorphic functions. In the present paper it is shown that this meromorphy can be simple established from the functional equation. Also the zeros and poles of these meromorphic functions are simple to locate. The information so obtained determines these meromorphic functions and leads to simple expressions for the operational characteristics of the symmetrical shortest queue, which are easily calculated with any desired accuracy. helhe

AMS Subject Classification (1991): 60J15, 60K25
Keywords & Phrases: Symmetrical shortest queue, analytic solution, meromorphic functions, queue length distribution, waiting time distribution.
Note: This work was supported in part by the European Grant BRA-QMIPS of CEC DG XIII.

1. Introduction

The "two-servers shortest queueing" model, also called the "two-queues in parallel" model has obtained quite some attention in the literature of Queueing Theory. This attention is not due to the fact that it is an appropriate model for queueing situations which occur frequently in practice, but stems from the analytical problems encountered in the theoretical analysis of the involved stochastic process. The process is a Markov chain on the lattice points with integer-valued coordinates in the first quarter-plane, and actually a rather simple one in this class of Markov chains. As such, in particular the symmetric model, has been used as a starting point to obtain information concerning fruitful techniques for the investigation of the stationary distribution of this type of Markov chain. Presently, several fruitful techniques are available. They fall into two classes, viz. the analytical one and the numerical-iterative one.

In the analytical approach the functional equation for the bivariate generating function of the two-dimensional distribution of the state-variable is the starting point of the analysis. In the numerical-iterative approach the stationary state-probabilities are expressed as series expansions of which the coefficiency are iteratively determined from the Kolmogorov equations.

Concerning the analytic approach we mention here the studies of Kingman [4], Flatto and McKean [5], Fayolle and Iasnogorodski [6], Cohen and Boxma [1]; for the numerical-iterative approach, see Hooghiemstra, e.a. [7], Blanc [8], Adan, e.a. [9].

In [4] and [5] the functional equations for the bivariate generating function is analysed by using the uniformisation of a polynomial of two variables; in [1] and [6] the functional equation is transformed into a Riemann-Hilbert boundary value problem. The outcome of these studies shows that the unknown generating functions in the functional equation are meromorphic functions for the model of the symmetrical shortest queue. In [2] the class of Markov chains with this property has been characterised; this property can be straightforwardly derived from the functional equation. A meromorphic function is an analytic function apart from a finite number of poles in every finite domain. If the number of poles is infinite and if they increase in absolute value sufficiently fast to ∞ then the meromorphic function is apart from a scaling factor completely determined if its zeros and poles are known.

This actually is the case for the random walks studied in [2], and in particular for the symmetrical shortest queue. So the question occurs whether these zeros and poles cannot be determined directly from the functional equation. For the symmetrical shortest queue model this is indeed the case, as it will be shown in the present study. This approach leads to a very simple analysis of the symmetrical shortest queue and provides formulas which can be evaluated numerically quite easily.

In section 2 the functional equation is formulated, see (2.3), for a detailed derivation of it the reader is referred to [2] chapter III.1. In section 3 it is shown that the unknown functions in the functional equation are meromorphic and their zeros and poles are determined. In section 4 the solution of the functional equation is constructed and expressions for the various generating functions and first moments are derived. Appendix A contains some algebra needed for the determination of the zeros and poles of the meromorphic functions. In appendix B a recursive scheme is given for the calculation of these zeros and poles. Further some relations are derived for the estimation of the errors made by approximating infinite products and sums by finite ones. Here are also some numerical results presented for the case with $a = 1$.

The results of the present study have all been derived already in the existing literature. The merit of the present approach is the simplicity of the derivations and the resulting expressions for the generating functions.

2. The functional equation

In the symmetrical shortest queue model the arrival process of the customers is a Poisson process with arrival rate λ. The queueing system consists of two servers, the server times of the customers are independent, negative exponentially distributed stochastic variables with first moment β. An arriving customer joins the shorter queue, if both queues are equal he chooses a queue with probability 1/2. The queue length process has a unique stationary distribution if and only if (cf. [10]),

$$a := \lambda\beta < 2. \tag{2.1}$$

Denote by $(\mathbf{x}_1, \mathbf{x}_2)$ a pair of stochastic variables with joint distribution the stationary distribution just mentioned.

We then have, cf. [1]. p. 242: for $|r_1| \leq 1, |r_2| \leq 1$,

$$\begin{aligned} &\mathrm{E}\{r_1^{\mathbf{x}_1} r_2^{\mathbf{x}_2}(\mathbf{x}_1 > \mathbf{x}_2)\}\{r_2 + \frac{1}{a}(\frac{1}{r_1} + \frac{1}{r_2}) - \frac{2}{a} - 1\} + \\ &\mathrm{E}\{r_1^{\mathbf{x}_1} r_2^{\mathbf{x}_2}(\mathbf{x}_2 > \mathbf{x}_1\}\{r_1 + \frac{1}{a}(\frac{1}{r_1} + \frac{1}{r_2}) - \frac{2}{a} - 1\} + \\ &\mathrm{E}\{r_1^{\mathbf{x}_1} r_2^{\mathbf{x}_2}(\mathbf{x}_1 = \mathbf{x}_2)\}\{\frac{1}{2}(r_1 + r_2) + \frac{1}{a}(\frac{1}{r_1} + \frac{1}{r_2}) - \frac{2}{a} - 1\} + \\ &\mathrm{E}\{r_2^{\mathbf{x}_2}(\mathbf{x}_1 = 0)\}\frac{1}{a}(1 - \frac{1}{r_1}) + \mathrm{E}\{r_1^{\mathbf{x}_1}(\mathbf{x}_2 = 0)\}\frac{1}{a}(1 - \frac{1}{r_2}) = 0. \end{aligned} \tag{2.2}$$

In [1] it has been shown that the relation (2.2) is equivalent with: for $|r_1| \leq 1, |r_2| \leq 1$,

$$\begin{aligned} &\mathrm{E}\{r_1^{\mathbf{x}_1} r_2^{\mathbf{x}_2}(\mathbf{x}_2 > \mathbf{x}_1)\}\{r_1 + \frac{1}{a}(\frac{1}{r_1} + \frac{1}{r_2}) - \frac{2}{a} - 1\} + \\ &\frac{1}{a}(1 - \frac{1}{r_1})\Omega(r_2) + [\frac{1}{2}r_2 + \frac{1}{a}(\frac{1}{r_1} - 1) - \frac{1}{2}]\Phi(r_1 r_2) = 0, \end{aligned} \tag{2.3}$$

with

$$\Omega(r_2) := \mathrm{E}\{r_2^{\mathbf{x}_2}(\mathbf{x}_1 = 0)\}, \quad \Phi(r_1 r_2) := \mathrm{E}\{r_1^{\mathbf{x}_1} r_2^{\mathbf{x}_2}(\mathbf{x}_1 = \mathbf{x}_2)\}. \tag{2.4}$$

Denote by $(\hat{r}_1, \hat{r}_2)$ a zero tuple of the “kernel”

$$r_1 + \frac{1}{a}(\frac{1}{r_1} + \frac{1}{r_2}) - \frac{2}{a} - 1, \quad |r_1| \le 1, \ |r_2| \le 1, \tag{2.5}$$

then it follows because $|\mathrm{E}\{r_1^{\mathbf{x}_1} r_2^{\mathbf{x}_2}(\mathbf{x}_2 > \mathbf{x}_1)\}|$ is finite that: for $|r_1| \le 1$, $|r_2| \le 1$,

$$\frac{1}{a}(1 - \hat{r}_1^{-1})\Omega(\hat{r}_2) + [\frac{1}{2}\hat{r}_2 + \frac{1}{a}(\hat{r}_1^{-1} - 1) - \frac{1}{2}]\Phi(\hat{r}_1 \hat{r}_2) = 0. \tag{2.6}$$

Put

$$t := r_1 r_2, \quad r := r_2, \tag{2.7}$$

then (2.6) may be rewritten as:

$$\Omega(\hat{r}) + k_2(\hat{r}, \hat{t}) \ \Phi(\hat{t}) = 0, \tag{2.8}$$

with

$$k_1(r,t) := at^2 + [1 - (2+a)r]t + r^2, \tag{2.9}$$

$$k_2(r,t) := -1 + \frac{1}{2}a^2 t - \frac{1}{2}ar,$$

and $(\hat{r}, \hat{t})$ a zero tuple of, cf. (2.5) and (2.7),

$$k_1(r,t) = 0, \quad |r| \le 1, \ |t| \le 1.$$

In the derivation of (2.8) from (2.6) the relation

$$\frac{1 - \hat{r}_2}{1 - \hat{r}_1} = \frac{\hat{r}_2}{\hat{r}_1}(a\hat{r}_1 - 1)$$

has been used, this relation follows simply from (2.5).

The relation (2.8) represents the functional equation for $\Omega(r)$ and $\Phi(t)$. Obviously $\Omega(r)$ and $\Phi(t)$ should further satisfy the conditions:

i. *$\Phi(t)$ is regular for $|t| < 1$, continuous for $|t| \le 1$ and its series expansion in powers of t, $|t| \le 1$, has nonnegative coefficients of which at least one is positive;* (2.10)

ii. *$\Omega(r)$ is regular for $|r| < 1$, continuous for $|r| \le 1$, and its series expansion in powers of r, $|r| \le 1$, has nonnegative coefficients of which at least one is positive.*

REMARK 2.1. Because there is a unique stationary distribution it follows readily that $\Omega(r) > 0$ for $0 \le r \le 1$ and $\Phi(t) > 0$ for $0 \le t \le 1$. Note further that

$$\Omega(0) = \Phi(0) > 0, \tag{2.11}$$

since the queueing process is positive recurrent. □

3. ANALYSIS OF THE FUNCTIONAL EQUATION

In appendix A, cf. (a.4), it has been shown that $k_1(r,t)$ has for $|r| \ge 1$ $r \ne 1$, one zero $t_1(r)$ in $|t| \le 1$, and one zero $t_2(r)$ in $|t| \ge 1$, here

$$t_{1,2}(r) = \frac{1}{2a}[(2+a)r - 1 \pm (2+a)\sqrt{(r-\rho_1)(r-\rho_2)}], \quad 0 < \rho_1 < \rho_2 \le 1, \tag{3.1}$$

with the branch points $\rho_{1,2}$ given by (a.8). Hence, $k_1(r,t)$ possesses zero tuples with $|\hat{r}| = 1$, $|\hat{t}| \le 1$. Put

$$r \equiv \hat{r}, \quad t = \hat{t} = t_1(r), \tag{3.2}$$

$$|t_1(r)| < |r| = 1, \quad r \neq 1,$$

$$\mathcal{G} := \{r : \rho_1 \leq r \leq \rho_2\},$$

then we have from (2.8) : for $|r| = 1$, $r \neq 1$,

$$\Omega(r) + k_2(r, t_1(r))\ \Phi(t_1(r)) = 0. \tag{3.3}$$

By noting the $\Omega(r)$ is regular in $|r| < 1$, continuous in $|r| \leq 1$, cf. (2.10)ii, and that $k_2(r, t_1(r)) \neq 0$ for $|r| \leq 1$, cf. (a.11), and regular in $\{r : |r| \leq 1\}\backslash\mathcal{G}$ it follows from (3.3) that $\Phi(t_1(r))$ can be continued analytically out from $|r| = 1$ into $\{r : |r| \leq 1\}\backslash\mathcal{G}$. Because $|\Omega(r)|$ and $|k_2(r, t_1(r))|$ are both finite for $r \in \mathcal{G}$ it follows that the analytic continuation of $\Phi(t_1(r))$ has for r approaching a point of $\mathcal{G}$ a limiting value, this value being dependent on the way r approaches $\mathcal{G}$, viz. from above or below,. At an interior point of $\mathcal{G}$ these values are complex conjugate. Hence (3.3) also holds by continuity for $r \in \mathcal{G}$, and so does its complex conjugate, i.e.

$$\Omega(r) + \overline{k_2(r, t_1(r))}\ \overline{\Phi(t_1(r))} = 0, \quad r \in \mathcal{G}. \tag{3.4}$$

It is readily seen that (3.4) is equivalent with

$$\Omega(r) + k_2(r, t_2(r))\ \Phi(t_2(r)) = 0, \quad r \in \mathcal{G}, \tag{3.5}$$

because

$$\overline{t_1(r)} = t_2(r), \quad r \in \mathcal{G}.$$

The relation (3.5) may be continued analytically out from $\mathcal{G}$ into $\{r : |r| \leq 1\}\backslash\mathcal{G}$ taking proper care of the two-valuedness of $t_2(r)$ for $r \in \mathcal{G}$, note that $k_2(r, t_2(r)) \neq 0$, $|r| \leq 1$. Hence we have: for $|r| \leq 1$,

$$\text{i.} \quad \Omega(r) + k_2(r, t_1(r))\ \Phi(t_1(r)) = 0, \tag{3.6}$$

$$\text{ii.} \quad \Omega(r) + k_2(r, t_2(r))\ \Phi(t_2(r)) = 0.$$

Next note that $k_2(r, t_1(r))$ is regular for all $r \notin \mathcal{G}$, and $|t_1(r)| < 1$ for $|r| = 1$, $r \neq 1$. Because $|t_1(r)| < 1$ for $|r| = 1$, $r \neq 1$ and $t_1(r)$ is regular for all $r \notin \mathcal{G}$ we can continue $t_1(r)$ in $|r| > 1$ and for this continuation $\Phi(t_1(r))$ exists as long as $t_1(r)$ remains bounded by one. Consequently (3.6)i implies that $\Omega(r)$ has an analytic continuation for those $|r| \geq 1$ for which $|t_1(r)| \leq 1$. With the $\Omega(r)$ continued analytically into this region we can by starting from (3.6)ii continue $\Phi(t_2(r))$ analytically, excluding those values of r for which $k_2(r, t_2(r))$ becomes zero, because $\Phi(t_2(r))$ may have poles at those r. It is evident that by repeating the arguments just mentioned $\Omega(r)$ and $\Phi(t)$ can be continued analytically into $|r| > 1$ and $|t| > 1$, respectively, and since $k_2(r, t)$ is a first degree linear form in r and in t the only singularities, which these analytical continuations can have, are poles. Whenever these poles do not have a finite accumulation point then $\Omega(r)$ and $\Phi(t)$ are meromorphic functions. This is indeed the case and will be shown below, cf. also [2].

In appendix A it has been shown that the hyperbola $k_1(r, t) = 0$ and the line $k_2(r, t) = 0$ intersect at two points, viz. (r_0^-, t_0^-) and (r_0^+, t_0^+), see below (a.10), (r_0^-, t_0^-) being located on the left branch of the hyperbola, (r_0^+, t_0^+) on the right branch, see fig. 1 in the appendix.

Starting from the point (r_0^+, t_0^+) the following sequence is constructed

$$\ldots t_{-2}^+ \leftarrow r_{-1}^+ \leftarrow t_{-1}^+ \leftarrow (r_0^+, t_0^+) \rightarrow r_1^+ \rightarrow t_1^+ \ldots r_n^+ \rightarrow t_n^+ \rightarrow r_{n+1}^+ \rightarrow t_{n+1}^+ \ldots . \tag{3.7}$$

It is defined as follows:

i. $\quad r_n^+$ and t_n^+ satisfy (3.8)

$$k_1(r^+_{n+1}, t^+_n) = 0, \qquad r^+_{n+1} > t^+_n, \qquad n = 0, 1, 2, \ldots,$$

$$k_1(r^+_n,\ t^+_n) = 0, \qquad t^+_n > r^+_n, \qquad n = 1, 2, \ldots,$$

ii. $$k_1(r^+_{-n-1}, t^+_{-n}) = 0, \qquad r^+_{-n-1} > t^+_{-n}, \qquad n = 0, 1, 2, \ldots, N^+,$$

$$k_1(r^+_{-n},\ t^+_{-n}) = 0, \qquad t^+_{-n} > r^+_{-n}, \qquad n = 1, 2, \ldots, N^+,$$

where N^+ is the smallest value of n for which r^+_{-n} or t^+_{-n} is less than or equal to one, see fig. 2. Because at least one of the coordinates of the top of the right branch of the hyperbola is less than one it follows that N^+ is finite. Note that, cf. (a.11),

$$t^+_n > 1, \qquad n = 0, 1, 2, \ldots, \tag{3.9}$$

$$r^+_n > 1, \qquad n = 1, 2, \ldots\ ,$$

fig. 2
right branch of the hyperbola

fig. 3
left branch of the hyperbola

A similar sequence is constructed by starting from the point (r_0^-, t_0^-), viz.

$$\dots r_{-2}^- \leftarrow t_{-1}^- \leftarrow r_{-1}^- \leftarrow (r_0^-, t_0^-) \rightarrow t_1^- \rightarrow r_1^- \rightarrow t_2^-, \dots r_{n-1}^- \rightarrow t_n^- \rightarrow r_n^- \rightarrow t_{n+1}^-, \dots,$$

which is defined by

$$\text{i.} \quad k_1(r_n^-, t_n^-) = 0, \qquad r_n^- < t_n^-, \qquad n = 1, 2, \dots, \tag{3.10}$$

$$k_1(r_n^-, t_{n+1}^-) = 0, \qquad t_{n+1}^- < r_n^-, \qquad n = 0, 1, 2, \dots;$$

$$\text{ii.} \quad k_1(r_{-n}^-, t_{-n}^-) = 0, \qquad r_{-n}^- < t_{-n}^-, \qquad n = 0, 1, 2, \dots, N^-,$$

$$k_1(r_{-n-1}^-,\ t_{-n}^-) = 0, \qquad r_{-n-1}^- > t_{-n}^-, \qquad n = 1, 2, \dots, N^-,$$

where N^- is the smallest value of n for which $0 \le r_{-n}^- < 1$. Because $(0,0)$ and $(0, -a^{-1})$ are points of the hyperbola and the top of the left branch of the hyperbola has a positive r-coordinate which is less than $\rho_1 > 0$, see fig. a1, it follows that N^- is finite. Note that, cf. (a.11),

$$r_n^- < -1, \qquad n = 0, 1, 2, \dots,$$

$$t_n^- < 0, \qquad n = 1, 2, \dots .$$

For obvious reasons the sequence in (3.7) will be indicated as the ladder generated by (r_0^+, t_0^+), it consists of an "up"-ladder, formed by the elements $r_1^+, t_1^+, \dots$, and a "down" ladder formed by $t_{-1}^+, r_{-1}^+, \dots$. Similarly (r_0^-, t_0^-) generates a ladder. Actually any point of the hyperbola generates a ladder by using the procedure (3.8) or (3.10) depending on the location of the starting point on the right or left branch of the hyperbola.

For the further analysis of the functions $\Omega(r)$ and $\Phi(t)$ we consider the relations (3.6) at the points of the ladder generated by (r_0^+, t_0^+).

Because $k_2(r_0^+, t_0^+) = 0$, cf. the definition of (r_0^+, t_0^+) in appendix A, it follows from (3.6)i that

$$|\Phi(t_0^+)| < \infty \;\Rightarrow\; \Omega(r_0^+) = 0. \tag{3.11}$$

Since (r_0^+, t_0^+) is the only point at the right branch of the hyperbola for which $k_2(r,t) = 0$ it follows from (3.6)ii that

$$\Omega(r_0^+) = 0 \;\Rightarrow\; \Phi(t_{-1}^+) = 0, \tag{3.12}$$

and similarly

$$\Phi(t_{-n}^+) = 0 \;\Rightarrow\; \Omega(r_{-n}^+) = 0, \tag{3.13}$$

$$\Omega(r_{-n}^+) = 0 \;\Rightarrow\; \Phi(t_{-n-1}^+) = 0.$$

Because N^+ is finite and $0 < r_{-N^+}^+ \le 1$ or $0 < t_{-N^+}^+ \le 1$ it is seen that by assuming $|\Phi(t_0^+)| < \infty$ a contradiction results from (2.10), (3.11) (3.12), (3.13), cf. also remark 2.1. Consequently

$$|\Phi(t_0^+)| = \infty. \tag{3.14}$$

The conclusion (3.14) implies that $|\Omega(r_0^+)| < \infty$. Because if $|\Omega(r_0^+)| = \infty$ then again by using (3.6) and descending along the "down" ladder generated by (r_0^+, t_0^+) it is seen that $\Omega(r)$ or $\Phi(t)$ is not finite at an interior point of $0 < r \le 1$ or $0 < t \le 1$, note $k_2(r,t) \ne 0$ at all points of the ladder generated by (r_0^+, t_0^+), with the point (r_0^+, t_0^+) excepted. Hence $|\Omega(r_0^+)| = \infty$ leads by using (2.10) to a contradiction, so since $r_0^+ > 1$,

$$0 < |\Omega(r_0^+)| < \infty. \tag{3.15}$$

Because $k_2(r,t)=0$ is a linear relation in r and t it follows from (3.14) and (3.15) that

$$\Phi(t) \textit{ has at } t_0 \textit{ a single pole.} \tag{3.16}$$

Again by noting that $k_2(r,t)\neq 0$ at the points of the "up" ladder $r_1^+, t_1^+, r_2^+, t_2^+, \ldots$, it follows from (3.6) that

$$\begin{aligned} &\Omega(r) \quad \textit{has} \quad \textit{at} \quad r_n^+,\ n=1,2,\ldots, \quad \textit{a single pole},\\ &\Phi(t) \quad ,, \quad ,, \quad t_n^+,\ n=0,1,2,\ldots, \quad \textit{a single pole}. \end{aligned} \tag{3.17}$$

Next we consider the relations (3.6) at the points of the ladder generated by (r_0^-, t_0^-).

Starting at (r_0^-, t_0^-), ascending along the "up" ladder, the assumption that $|\Phi(t_0^-)|=\infty$ leads again to the conclusion that $|\Omega(r)|$ or $|\Phi(t)|$ is infinite for a value of r or of t with $|r|\le 1$ or $|t|\le 1$, respectively. This contradicts (2.10). The assumption $\Phi(t_0)=0$ yields $\Omega(r_{-1}^-)=0$, so that by ascending the (r_0^-, t_0^-)-ladder it follows from (3.6) that $\Omega(r)=0$ at r_{-N}^-, and because $0<r_{-N}^-<1$, (2.10) leads again to a contradiction, so

$$0<|\Phi(t_0)|<\infty. \tag{3.18}$$

By descending the ladder generated by (r_0^-, t_0^-) it follows from similar arguments as above that

$$\begin{aligned} &\Omega(r) \quad \textit{has} \quad \textit{at} \quad r_n^-, \quad n=0,1,2,\ldots \quad \textit{a simple zero},\\ &\Phi(t) \quad ,, \quad ,, \quad t_n^-, \quad n=1,2,\ldots \quad \textit{a simple zero}. \end{aligned} \tag{3.19}$$

Next we show that (3.17) and (3.8) list all the zeros and poles of $\Omega(r)$ and $\Phi(t)$.

Suppose that (ρ,τ) is a point not belonging to the set of the points of ladders generated by (r_0^-, t_0^-) or (r_0^+, t_0^+). Consider the ladder generated by (ρ,τ), note that $k_2(r,t)\neq 0$ at all points of this ladder. Consider the case that (ρ,τ) is a point of the right branch of the hyperbola. If $|\Omega(r)|=\infty$ at a point of the (ρ,τ)-ladder then by descending this ladder we reach again a contradiction by using (3.6) and (2.10); similarly if it is supposed that $|\Phi(t)|=\infty$ at a point of this ladder. Further $\Omega(r)\neq 0$ and $\Phi(t)\neq 0$ because at this ladder r and t are both positive. Hence $\Phi(t)$ and $\Omega(r)$ have no zeros nor poles on this (ρ,τ)-ladder. It is readily seen, by using the same arguments as above that $\Phi(t)$ and $\Omega(r)$ have no zeros nor poles at any (ρ,τ)-ladder with (ρ,τ) a point of $k_1(r,t)=0$. Consequently, it follows that

$$\Omega(r) \textit{ and } \Phi(t) \textit{ have no other zeros and poles than those listed in (3.17) and (3.19).} \tag{3.20}$$

4. The solution of the functional equation

With the results obtained in the preceding section $\Omega(r)$ and $\Phi(t)$ can be determined.

It is readily seen that: for $n\to\infty$,

$$r_n^+\to\infty,\ t_n^+\to\infty,\quad r_n^-\to-\infty,\ t_n^-\to-\infty. \tag{4.1}$$

From (3.7), (3.8), (3.10), (3.11) and (a.7) it is seen that

$$\lim_{n\to\infty}\frac{t_n^+}{r_n^+}=\delta,\quad \lim_{n\to\infty}\frac{r_{n+1}^+}{t_n^+}=\delta, \tag{4.2}$$

$$\lim_{n\to\infty}\frac{r_{n+1}^-}{t_n^-}=\delta,\quad \lim\frac{t_n^-}{r_n^-}=\delta,$$

with, cf. (2.1),

$$\delta=\frac{1}{2a}[2+a+\sqrt{a^2+4}]>1. \tag{4.3}$$

Hence

$$\delta^2 = \lim_{n\to\infty} \frac{t^+_{n+1}}{t^+_n} = \lim_{n\to\infty} \frac{r^+_{n+1}}{r^+_n} = \lim_{n\to\infty} \frac{t^-_{n+1}}{t^-_n} = \lim_{n\to\infty} \frac{r^-_{n+1}}{r^-_n}. \tag{4.4}$$

From (3.17), (3.19), (3.20) it is seen that each of the four sequences describing the zeros and poles of $\Omega(r)$ and $\Phi(t)$ have no finite accumulation point. So that by the construction described below (3.6) $\Omega(r)$ can be continued analytically into the domain $|r| \geq 1$, punctured at the points $r^+_n, n = 1, 2, \ldots$. Hence

i. $\Omega(r)$ *is a meromorphic function,* (4.5)

ii. $\Phi(t)$,, ,, ,, ,, ;

the proof of (4.5)ii is analogous to that of (4.5)i.

From (4.2) and (4.3) it follows that each of the series below converge:

$$\sum_{n=1}^{\infty}(r^+_n)^{-1}, \; \sum_{n=0}^{\infty}(t^+_n)^{-1}, \; \sum_{n=0}^{\infty}|r^-_n|^{-1}, \; \sum_{n=1}^{\infty}|t^-_n|^{-1}. \tag{4.6}$$

This together with (4.5) determines $\Omega(r)$ and $\Phi(t)$ uniquely apart from a constant, cf. [3]. p. 296, and we have for $r \neq r^{\pm}_n$, $t \neq t^{\pm}_n$,

$$\Omega(r) = \Omega(1) \frac{\prod_{n=0}^{\infty}(1 - \frac{r}{r^-_n})}{\prod_{n=0}^{\infty}(1 - \frac{1}{r^-_n})} \frac{\prod_{n=1}^{\infty}(1 - \frac{1}{r^+_n})}{\prod_{n=1}^{\infty}(1 - \frac{r}{r^+_n})}, \tag{4.7}$$

$$\Phi(t) = \Phi(1) \frac{\prod_{n=1}^{\infty}(1 - \frac{t}{t^-_n})}{\prod_{n=1}^{\infty}(1 - \frac{1}{t^-_n})} \frac{\prod_{n=0}^{\infty}(1 - \frac{1}{t^+_n})}{\prod_{n=0}^{\infty}(1 - \frac{t}{t^+_n})}.$$

Note that a meromorphic function is completely determined by its poles and zeros if (4.6) holds and that (4.6) guarantees the convergence of the infinite products in (4.7).

It remains to determine $\Omega(1)$ and $\Phi(1)$.

By taking $r_1 = r_2 = 1$ in (2.2) it follows that

$$\Omega(1) = \frac{2-a}{2}. \tag{4.8}$$

From this and from (2.8) with $\hat{t} = \hat{r} = 1$ and with $\hat{r} = 1$, $\hat{t} = a^{-1}$ if $1 \leq a < 2$ it is seen that

i. $\Phi(1) = \dfrac{1}{1+a}$ *for* $a \leq 1$, (4.9)

ii. $\Phi(\frac{1}{a}) = \frac{1}{2}(2-a)$,, $1 \leq a < 2$;

below, cf. (4.15), it will be shown that (4.9)i also holds, for $1 \leq a < 2$, and so the constant factors in (4.7) are known.

Because, cf. (2.4),

$$\Omega(0) = \Phi(0), \tag{4.10}$$

we obtain from (4.7), (4.8) and (4.9) the identity

$$\Phi(0) = \frac{2-a}{a} \frac{\prod_{n=1}^{\infty}(1 - \frac{1}{r^+_n})}{\prod_{n=0}(1 - \frac{1}{r^-_n})} = \Phi(1) \frac{\prod_{n=0}^{\infty}(1 - \frac{1}{t^+_n})}{\prod_{n=1}^{\infty}(1 - \frac{1}{t^-_n})}. \tag{4.11}$$

From (2.3) we obtain

$$\mathrm{E}\{r_1^{\mathbf{x}_1} r_2^{\mathbf{x}_2}(\mathbf{x}_2 > \mathbf{x}_1)\} = r_2 \frac{(1-r_1)\Omega(r_2) - \frac{1}{2}[ar_1r_2 - (2+a)r_1 + 2]\Phi(r_1r_2)}{ar_1^2r_2 + r_2 + r_1 - (2-a)r_1r_2}, \tag{4.12}$$

for all those r_1, r_2 with $|r_1| \le 1$, $|r_2| \le 1$ for which the denominator in (4.12) differs from zero. If it is zero the righthand side of (4.12) has to be replaced by its appriopriate limit, i.e. the limit obtained for $(r_1, r_2) \to (\hat{r}_1, \hat{r}_2)$ with $(\hat{r}_1, \hat{r}_2)$ a zero of the denominator in (4.12).

From (2.7) and (4.12) it is seen that

$$\mathrm{E}\{t^{\mathbf{x}_1} r^{\mathbf{x}_2 - \mathbf{x}_1}(\mathbf{x}_2 > \mathbf{x}_1)\} = \frac{(r-t)\Omega(r) + \frac{1}{2}[at(1-r) + 2(t-r)]\Phi(t)}{at^2 + [1-(2+a)r]t + r^2}, \tag{4.13}$$

for all those r, t which are not poles of $\Omega(r)$ and $\Phi(t)$, respectively, and which are not zeros of the denominator in (4.13), if they are the appropriate limit replaces the righthand side of (4.13). From the analysis above it follows that this limit exists. obviously (4.13) is a meromorphic function in r for fixed t, and also in t for fixed r.

By taking $r_1 = r_2 = r$ in (2.2) and in (4.13) once with $t = 1$ and once with $r = 1$ it follows: for $|r| \le 1$, $|t| \le 1$,

$$\begin{aligned}
&\text{i.} && \mathrm{E}\{r^{\mathbf{x}_1 + \mathbf{x}_2}\} = \frac{2}{2 - ar}\,\Omega(r), \\
&\text{ii.} && \mathrm{E}\{t^{\mathbf{x}_1}(\mathbf{x}_2 > \mathbf{x}_1)\} = \frac{1}{1 - at}[\tfrac{1}{2}(2-a) - \Phi(t)], \\
&\text{iii.} && \mathrm{E}\{t^{\mathbf{x}_2}(\mathbf{x}_2 > \mathbf{x}_1\} = \frac{1}{2}at\Phi(t), \\
&\text{iv.} && \mathrm{E}\{t^{\mathbf{x}_1}(\mathbf{x}_1 = \mathbf{x}_2)\} = \Phi(t), \\
&\text{v.} && \mathrm{E}\{r^{\mathbf{x}_1}(\mathbf{x}_2 = 0)\} = \Omega(r);
\end{aligned} \tag{4.14}$$

for the last two relations in (4.14), see the definitions in (2.4). By taking in (4.14)ii and iii, $t = 1$, a linear relation for $\Phi(1)$ follows from the symmetry of our model, it results that

$$\Phi(1) = \frac{1}{1+a} \; for \; a < 2. \tag{4.15}$$

From (4.8), (4.14) and (4.15) it readily follows that

$$\begin{aligned}
&\text{i.} && \Pr\{\mathbf{x}_1 = \mathbf{x}_2\} = \frac{1}{1+a}, \quad \Pr\{\mathbf{x}_1 > \mathbf{x}_2\} = \Pr\{\mathbf{x}_2 > \mathbf{x}_1\} = \frac{1}{2}\,\frac{a}{1+a}, \\
&\text{ii.} && \mathrm{E}\{\mathbf{x}_2(\mathbf{x}_2 > \mathbf{x}_1)\} = \frac{a}{2(1+a)}[1 + \frac{\Phi^{(')}(1)}{\Phi(1)}], \\
&\text{iii.} && \mathrm{E}\{\mathbf{x}_1(\mathbf{x}_2 > \mathbf{x}_1)\} = \frac{1}{1-a^2}[\frac{1}{2}a^2 - \frac{\Phi^{(')}(1)}{\Phi(1)}] \text{ for } a \neq 1, \\
&\text{iv.} && \mathrm{E}\{\mathbf{x}_1(\mathbf{x}_1 = \mathbf{x}_2)\} = \frac{1}{1+a}\,\frac{\Phi^{(')}(1)}{\Phi(1)}; \\
&\text{v.} && \mathrm{E}\{\mathbf{x}_1\} = \frac{a}{2(2-a)} + \frac{1}{2}\frac{\Omega^{(')}(1)}{\Omega(1)};
\end{aligned} \tag{4.16}$$

and for $a = 1$, from (4.14)ii and (4.16)i,

$$\tag{4.17}$$

i. $$E\{(\mathbf{x}_2 > \mathbf{x}_1)\} = \lim_{t\to 1} \frac{1}{1-t}\{\frac{1}{2} - \Phi(t)\} = \Phi^{(')}(1) = \frac{1}{4},$$

$$= \frac{1}{2}\frac{\Phi^{(')}}{\Phi(1)},$$

ii. $$E\{\mathbf{x}_1(\mathbf{x}_2 > \mathbf{x}_1)\} = [\frac{d}{dt}[\frac{1}{1-t}\{\frac{1}{2} - \Phi(t)\}]]_{t=1} = \frac{1}{4}\frac{\Phi^{('')}(1)}{\Phi(1)}.$$

For the calculation of the derivatives in the formulas above, note that (4.7) implies:

$$\frac{\Phi^{(')}(t)}{\Phi(t)} = \frac{1}{t_0^+ - t} + \sum_{n=1}^{\infty}\{\frac{1}{t_n^+ - t} - \frac{1}{t_n^- - t}\}, \quad |t| \le 1, \tag{4.18}$$

$$\frac{\Omega^{(')}(r)}{\Omega(r)} = \frac{-1}{r_0^- - r} + \sum_{n=1}^{\infty}\{\frac{1}{r_n^+ - r} - \frac{1}{r_n^- - r}\}, \quad |r| \le 1,$$

$$\frac{\Phi^{('')}(t)}{\Phi(t)} = [\frac{\Phi^{(')}(t)}{\Phi(t)}]^2 + \frac{1}{(t_0^+ - t)^2} + \sum_{n=1}^{\infty}[\frac{1}{(t_n^+ - t)^2} - \frac{1}{(t_n^- - t)^2}]^2, \quad |t| \le 1.$$

It is seen from (4.3) and (4.4) that these series converge uniformly, for $|t| \le 1$, and $|r| \le 1$, respectively.
Finally we consider the size of the shorter and of the longer queue, i.e.

$$\mathbf{x}_s := \min(\mathbf{x}_1, \mathbf{x}_2), \quad x_l := \max(\mathbf{x}_1, \mathbf{x}_2). \tag{4.19}$$

It follows from (1.14)i by using the symmetry that for $|t| \le 1$,

$$E\{t^{\mathbf{x}_s}\} = E\{t^{\mathbf{x}_1}(\mathbf{x}_1 < \mathbf{x}_2)\} + E\{t^{\mathbf{x}_2}(\mathbf{x}_2 < \mathbf{x}_1)\} + E\{t^{\mathbf{x}_1}(\mathbf{x}_1 = \mathbf{x}_2)\} \tag{4.20}$$

$$= \frac{2 - a - (1+at)\Phi(t)}{1 - at}, \quad t \ne \frac{1}{a},$$

$$E\{t^{\mathbf{x}_l}\} = (1+at)\Phi(t).$$

From which it follows

$$E\{\mathbf{x}_s\} = \frac{1}{1-a^2}[a^2 - (1+a)\frac{\Phi^{(')}(1)}{\Phi(1)}], \quad a \ne 1, \tag{4.21}$$

$$= \frac{1}{4} + \frac{1}{2}\frac{\Phi^{('')}(1)}{\Phi(1)}], \quad a = 1,$$

$$E\{\mathbf{x}_l\} = \frac{a}{1+a}[1 + a\frac{\Phi^{('')}(1)}{\Phi(1)}].$$

Denote by $\mathbf{w}$ the waiting time of a customer. Since the service times are negative exponentially distributed with moment β it follows readily from (4.20) that: for $\operatorname{Re}\rho \ge 0$,

$$E\{e^{-\rho\mathbf{w}}\} = E\{[\beta(\rho)]^{\mathbf{x}_s}\} = \frac{2 - a + (1 + a\beta(\rho))\Phi(\beta(\rho))}{1 - a\beta(\rho)}, \tag{4.22}$$

with

$$\beta(\rho) = \frac{1}{1+\rho\beta}.$$

It is of interest to note that a customer who can judge the queue lengths at his arrival epoch is confronted with an average queue length $\mathrm{E}\{\mathbf{x}_1(\mathbf{x}_2 \geq \mathbf{x}_1)\}$ and then experiences an average waiting time equal to $\beta\ \mathrm{E}\{\mathbf{x}_1(\mathbf{x}_2 \geq \mathbf{x}_1)\}$.

ACKNOWLEDGEMENT
The author is indebted to Dr. J.P.C. Blanc for a useful remark.

APPENDIX A.
In this appendix we derive some relations for the zero tuples of $k_1(r,t)$. We introduce the stochastic variables $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ with distribution given by

$$\begin{aligned} \Pr\{\boldsymbol{\xi}=2,\boldsymbol{\eta}=0\} &= \frac{a}{2+a}, \\ \Pr\{\boldsymbol{\xi}=1,\boldsymbol{\eta}=0\} &= \frac{1}{2+a}, \\ \Pr\{\boldsymbol{\xi}=0,\boldsymbol{\eta}=2\} &= \frac{1}{2+a}, \end{aligned} \tag{a.1}$$

Then

$$k_1(r,t)=0 \quad \Longleftrightarrow \quad rt=\mathrm{E}\{t^{\boldsymbol{\xi}} r^{\boldsymbol{\eta}}\} \quad \Longleftrightarrow \quad rt=\frac{a}{2+a}t^2+\frac{1}{2+a}t+\frac{1}{2+a}r^2. \tag{a.2}$$

With $t=rp$ it follows

$$k_1(r,rp)=0 \quad \Longleftrightarrow \quad p=\mathrm{E}\{p^{\boldsymbol{\xi}} r^{\boldsymbol{\xi}+\boldsymbol{\eta}-2}\}. \tag{a.3}$$

From (a.1) it is seen that

$$\Pr\{\boldsymbol{\xi}+\boldsymbol{\eta}-2 \leq 0\}=1,$$

and so it follows by using Rouché's theorem that: $k_1(r,rp)$ has for fixed r with $|r| \geq 1$, $r \neq 1$, one and only one zero in $|p| \leq 1$. Because $k_1(r,t)$ has for fixed r two zeros, t_1, t_2, say, it follows that: for $|r| \geq 1$, $r \neq 1$,

$$|t_2(r)| > |r| > |t_1(r)|. \tag{a.4}$$

It is readily seen that $k_1(r,t)=0$ with real r and t represents a hyperbola with center (r_m, t_m) given by

$$r_m=\frac{2+a}{4+a^2}, \qquad t_m=\frac{2}{4+a^2}, \tag{a.5}$$

and asymptotes given by

$$t-t_m=\frac{1}{2a}\,[2+a \pm \sqrt{a^2+4}](r-r_m). \tag{a.6}$$

The zeros of $k_1(r,t)$ are

i. $$t_{1,2}(r)=\frac{1}{2a}[(2+a)r-1 \pm \sqrt{[(2+a)r-1]^2-4ar^2}], \tag{a.7}$$

ii. $$r_{1,2}(t)=\frac{1}{2}\,[(2+a)t \pm \sqrt{(4+a^2)t^2-4t}].$$

The branch points of $t_{1,2}(r)$, i.e. the zeros of the square root in (a.7)i, are given by

$$\rho_{1,2} = [1 + (1 \pm \sqrt{a})^2]^{-1}, \qquad 0 < \rho_1 < \rho_2 \leq 1, \tag{a.8}$$

those of $r_{1,2}(t)$ by

$$\tau_1 = 0, \qquad \tau_2 = \frac{4}{a^2+4}. \tag{a.9}$$

Some special zeros of $k_1(r,t) = 0$ are

$$\begin{aligned}
t_1(0) &= -\frac{1}{a}, & t_2(0) &= 0, & & \\
r_1(0) &= 0, & r_2(0) &= 0, & & \\
t_1(1) &= 1, & t_2(1) &= \frac{1}{a}, & & a \leq 1, \\
&= \frac{1}{a}, & &= 1, & & a \geq 1, \\
r_1(1) &= 1, & r_2(1) &= 1 + a. & &
\end{aligned} \tag{a.10}$$

Consider next the equation $k_2(r,t) = 0$, cf. (2.9), which for real r and t represents a straight line of which the slope lies in between the slopes of the two asymptotes of the hyperbola, cf. (a.6). Hence $k_1(r,t) = 0$, $k_2(r,t) = 0$ intersect at two points (r_0^-, t_0^-) and (r_0^+, t_0^+), say, with (r_0^-, t_0^-) on the left branch and (r_0^+, t_0^+) on the right branch of the hyperbola. It is readily seen cf. fig. a1.,

$$r_0^- < -\frac{2}{a} < -1, \qquad t_0^- < 0, \tag{a.11}$$

$$t_0^+ > \frac{2}{a^2}(1 + \frac{1}{2}a\rho_2) > 1, \qquad r_0^+ > 1.$$

APPENDIX B.

For the numerical evaluation of the results derived in this study the values r_n, t_n of the ladders generated by (r_0^+, t_0^+) and (r_0^-, t_0^-) are needed. For their calculation we described here a simple recursive algorithm.

The points (r_0^-, t_0^-), (r_0^+, t_0^+) are determined, cf. (2.9) and end of appendix A, as the roots of

$$\text{i.} \quad at^2 + [1-(2+a)r]t + r^2 = 0, \tag{b.1}$$

$$\text{ii.} \quad \frac{1}{2}a^2 t - \frac{1}{2}ar - 1 \qquad = 0,$$

here (r_0^-, t_0^-) and (r_0^+, t_0^+) are defined so that

$$r_0^- < -1,\ t_0^- < 1, \qquad r_0^+ > 1,\ t_0^+ > 0. \tag{b.2}$$

From (b.1) it follows that r_0^-, r_0^+ and t_0^-, t_0^+ given by,

$$(r_0^+, t_0^+) = (\tfrac{2}{a}, \tfrac{4}{a^2}), \tag{b.3}$$

$$(r_0^-, t_0^-) = (-1 - \tfrac{2}{a}, -\tfrac{1}{a}).$$

For the points of the (r_0^+, t_0^+)-ladder it follows readily from (3.8), see also fig. 2, by using the properties of the zeros of the quadratic polynomial (b.1)i that for $n = 0, 1, 2, \ldots$,

$$r_n^+ r_{n+1}^+ = a(t_n^+)^2 + t_n^+, \tag{b.4}$$

$$t_n^+ t_{n+1}^+ = \frac{1}{a}(r_{n+1}^+)^2,$$

and

$$r_n^+ + r_{n+1}^+ = (2+a)t_n^+,$$

$$t_n^+ + t_{n+1}^+ = \frac{1}{a}[(2+a)r_{n+1}^+ - 1].$$

Similarly, for the (r_0^-, t_0^-)-ladder: for $n = 0, 1, 2, \ldots$,

$$t_n^- t_{n+1}^- = \frac{1}{a}(r_n^-)^2, \tag{b.5}$$

$$r_n^- r_{n+1}^- = a(t_{n+1}^-)^2 + t_{n+1}^-,$$

and

$$t_n^- + t_{n+1}^- = \frac{1}{a}[(2+a)r_n^- - 1], \tag{b.6}$$

$$r_n^- + r_{n+1}^- = (2+a)t_{n+1}^-.$$

By using (b.3) or (b.4) the values of $r_1^+, t_1^+, r_2^+, t_2^+, \ldots$, are easily computed once r_0^+ or t_0^+ is known. It is noted that for n sufficiently large the asymptotic relations, cf. (4.4),

$$t_{n+1}^{\pm} \sim \delta^2 t_n^{\pm}, \qquad r_{n+1}^{\pm} \sim \delta^2 r_n^{\pm}, \tag{b.7}$$

may be used, here $\delta > 1$.

In order to calculated, cf. (4.17),

$$\Omega(0) = \frac{2-a}{a} \frac{\Pi_{n=1}^{\infty}(1-\frac{1}{r_n^+})}{\Pi_{n=0}^{\infty}(1-\frac{1}{r_n^-})}, \tag{b.8}$$

the infinite products have to be replaced by finite products, and it is of interest to estimate the involved error. First note that

$$\prod_{n=1}^{\infty}(1-\frac{1}{r_n^+}) < \prod_{n=1}^{N}(1-\frac{1}{r_n^+}), \qquad N \geq 1, \tag{b.9}$$

since all $r_n^+ > 1$. Further for $0 < x << 1$ and any $\epsilon > 1$ it follows from

$$e^{-\varepsilon x} < 1 - x < e^{-x}$$

and (b.7) that for N sufficiently large

$$e^{-\varepsilon[r_N^+(\delta^2-1)]^{-1}} < \prod_{n=0}^{\infty}(1-\frac{1}{r_{N+n}^+}) < e^{-[r_N^+(\delta^2-1)]^{-1}},$$

so that for $N \to \infty$.

i. $$\prod_{n=0}^{\infty}(1-\frac{1}{r_{N+n}^+}) \sim 1 - \frac{1}{r_N^+}\frac{1}{\delta^2-1}, \tag{b.10}$$

ii. $$\prod_{n=0}^{\infty}(1+\frac{1}{-r_{N+n}^-}) \sim 1 + \frac{1}{r_{N+n}^+}\frac{1}{\delta^2-1};$$

the derivation of the second relation in (b.10) is analogous to the first one, note that $r_n^- < -1$. Consequently, for N sufficiently large

$$\prod_{n=0}^{\infty}(1-\frac{1}{r_n^+}) \sim \{\prod_{n=1}^{N}(1-\frac{1}{r_n^+})\}(1-\frac{1}{r_N^+}\frac{1}{\delta^2-1}), \tag{b.11}$$

$$\prod_{n=0}^{\infty}(1-\frac{1}{r_n^-}) \sim \{\prod_{n=1}^{N}(1-\frac{1}{r_n^-})\}(1-\frac{1}{r_N^-}\frac{1}{\delta^2-1}).$$

Similarly we obtain for N sufficiently large

$$\sum_{n=1}^{\infty}\frac{1}{r_n^+-1} - \sum_{n=1}^{N}\frac{1}{r_n^+-1} \sim \frac{1}{r_N^+}\frac{1}{\delta^2-1}. \tag{b.12}$$

Finally we present a few numerical results for the case $a = 1$, $\beta = 1$ and compare them with those for the M/M/2 model with $a = 1$. From (b.3) we obtain for $a = 1$,

$$r_0^- = -3,\ t_0^- = -1, \qquad r_0^+ = 2,\ t_0^+ = 4. \tag{b.13}$$

The values of $r_n^{\pm}$ and $t_n^{\pm}$ are listed in the table below, they are obtained from (b.14) and (b.13).

	r_n^+	t_n^+	$-r_n^-$	$-t_n^-$
$n = 0$	2	4	3	1
1	10	25	24	9
2	65	169	168	64
3	442	1156	1155	441
4	3026	7921	7920	3025
5	20737	54289	54288	20736
6	142130	372100	372099	142129

$\delta^2 = 6.854102$

The calculations yield after error correction, cf. (b.11), (b.12),

shortest queue	$M/M/2$
$\dfrac{\Omega^{(')}(1)}{\Omega(1)} = 0.42632 = \mathrm{E}\{\mathbf{x}_s\}$	0.33333
$\dfrac{\Phi^{('')}(1)}{\Phi(1)} = 0.35264$	
$\Phi(0) = 0.31597 = \mathrm{E}\{(\mathbf{x}_1 = \mathbf{x}_2 = 0)\}$	0.28571
$\mathrm{E}\{\mathbf{x}_1(\mathbf{x}_2 \geq \mathbf{x}_1) = 0.33816$	0.33333

and confirm the numerical results obtained in [8], [9].

REFERENCES

1. COHEN, J.W. and BOXMA, O.J., Boundary Value Problems in Queueing System Analysis, North-Holland Publ. Co., Amsterdam, 1983.

2. COHEN, J.W., On a class of two-dimensional nearest neighbour random walks, to appear in J. Appl. Prob.

3. SAKS, S. and ZYGMUND, A., Analytic Fucntions, Nakladem Polskiegu, Warshaw, 1952.

4. KINGMAN, J.F.C., Two similar queues in parallel, Ann. Math. Stat. **32** (1961) 1314-1323.

5. FLATTO, L. and MCKEAN, H.P., Two queues in parallel, Comm. Pure, Appl. Math. **30** (1977) 255-263.

6. FAYOLLE, G. and IASNOGORODSKI, R., Two coupled processors: the reduction to a Riemann-Hilbert problem, Z. Wahrsch. Verw. Gebiete **47** (1979) 325-351.

7. HOOGHIEMSTRA, G., KEANE, M., VAN DE REE, S., Power-series for stationary distributions of coupled processor models, SIAM, J. Appl. Math. **48** (1988) 1159-1166.

8. BLANC, J.P.C., The power-series algorithm applied to the shortest queueing problem, Oper. Res. 40 (1992) 157-167.

9. Adan, I.J.B.F., Wessels, J. and Zijm, W.H.M., Analysis of the symmetric shortest queueing problem, Stochastic Models, **6** (1990) 691-713.

10. Cohen, J.W., Analysis of Random Walks, IOS Press, Amsterdam, 1992.