Centrum voor Wiskunde en Informatica

REPORT *RAPPORT*

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Report BS-R9426 July 1994

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SMC is sponsored by the Netherlands Organization for Scientific Research (NWO). CWI is a member of ERCIM, the European Research Consortium for Informatics and Mathematics.

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Cross-covariance Functions Characterise Bounded Regular Closed Sets

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Abstract

It is shown that any closed centrally symmetric Borel subset of $\mathbb{R}^d$ is completely known once its covariance function is given. This result does not need any convexity or connectivity assumptions. The theorem is proved by rather simple Fourier techniques. Furthermore we give a geometric proof of the uniform continuity of the covariance function of any bounded Borel set. These results imply a solution to a problem of Pyke concerning the determination of a set from its interpoint distance distribution. Inspired by the result for the covariance function of symmetric sets, we introduce a new function, the *cross-covariance function*, that is shown to determine any bounded regular closed subset of $\mathbb{R}^d$.

AMS Subject Classification (1991): 53C65, 60D05
Keywords and Phrases: covariance function, interpoint distance distribution.

Introduction.

The (set) covariance function was introduced by Matérn in 1960 (see Matérn (1985)) and Matheron (1965). It maps a Borel set A and a vector x onto the volume of the intersection of A with the translation of A by x. The covariance

Report BS-R9426
ISSN 0924-0659
CWI
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function is widely used in spatial statistics to investigate second order properties of the model under study. It can be used for instance to estimate the parameters of a Boolean model (e.g. Hall (1988), Chapter 5).

Theoretically however the covariance function seems to have received less attention. Two interesting questions immediately arise:

(1) how much *geometric* information about a set is contained in its covariance function?

(2) what are the *analytic* properties of covariance functions?

Concerning the first problem, Matheron (1986) conjectured that the set covariance function of a *convex* set determines the set uniquely (up to a translation and a reflection). For *planar symmetric convex* sets, he shows the conjecture, using that for *all* sets the support of the covariance function of a set A is equal to the set $A \oplus (-A)$ and for *convex* symmetric sets: $A \oplus (-A) = 2A$. The reason Matheron restricted attention to convex sets seemed to be due to the analytic approach he used to describe the boundaries of the sets. Later Lešanovský and Rataj (1990) gave an example of two distinct nonconvex sets with the same covariance functions. For convex polygons in the plane the conjecture has recently been proved by Nagel (1993). (There the covariance function is called the covariogram of a set.) Schmitt (1993) gives a reconstruction procedure that also works for a restricted class of nonconvex polygons. Consequently the role of convexity is not yet completely clarified. In this paper we show that *any* regular closed *symmetric* Borel subset of $\mathbb{R}^d$, $d \geq 1$, is determined by its covariance function. Here symmetry refers to the fact that $A = -A$. The symmetry forces the statement to be complete in the sense that we obtain the set itself and not a set up to a translation. Observe that no convexity or connectivity assumptions are made. The question whether convexity is enough to ensure uniqueness of a set given its covariance function (up to translation and reflection) remains unanswered.

In the same paper in which Matheron published his conjecture, Lipschitz continuity of the covariance function of a convex set was proved. We give a geometric proof of continuity of the covariance function of a bounded Borel set (which in itself is a direct consequence of a standard result in Fourier theory, see Zaanen (1989)). Continuity of the function considered as a function acting on sets was proved for certain classes of regular closed sets by Cabo and Baddeley (1993).

This chapter is organised in the following way. In section 1 we give the result underlying the uniqueness theorems about the covariance functions. The main idea is to apply Fourier methods to the convolution of two functions to obtain equality almost everywhere of the functions. In section 2 this is applied to the covariance function. Section 3 is devoted to the continuity property. In section 4 we discuss two related problems of Pyke (1989) and Adler and Pyke (1991). In the latter paper, they asked whether a convex set is uniquely determined (up to translation and reflection) by the distribution of the difference vector of two independent uniformly distributed points in its interior. The well known relation between the covariance function and this distribution immediately shows that the problem of characterising a set by this distribution is equivalent to the problem of characterising a set by its covariance function. The results of section 2 and 3 solve this problem for all regular closed symmetric subsets of $\mathbb{R}^d$. Nagel (1993) solves the latter problem for convex polygons, but again for other sets this problem is still open. As a special case of the abovementioned relation, we have a relation between the covariance function and the interpoint distance distribution (i.e. the distribution of the length of the difference vector). This relation is exploited to prove that a rotation invariant (not–necessarily–convex) regular closed set is uniquely determined by its interpoint distance distribution. This answers a question of Pyke (1989).

In the last section we introduce a new function, the *cross-covariance* that is shown to characterise *any* regular closed Borel set, thus providing an alternative to the so-called three point covariance (see Nagel (1991,1993)).

1. A uniqueness result for convolutions.

1.1 General finite Borel measures.

Let $\mathcal{M}$ denote the space of all finite Borel measures on $\mathbb{R}$, with non-zero total measure; i.e. $\forall \mu \in \mathcal{M} : 0 < \mu(\mathbb{R}) < \infty$. The *moments* μ_n of $\mu \in \mathcal{M}$ are given by

$$\mu_n = \int_{\mathbb{R}} x^n \, \mu(dx), \quad n = 0, 1, 2, \ldots$$

Let $\mathcal{B}(\mathbb{R})$ denote the Borel σ-algebra on $\mathbb{R}$.

Recall that the convolution $\mu * \nu$ of two measures $\mu, \nu \in \mathcal{M}$ is defined by

$$\mu * \nu(B) = \int_{\mathbb{R}} \mu(B - t)\nu(dt), \quad B \in \mathcal{B}(\mathbb{R})$$

where $B - t = \{y - t : y \in B\}$; see e.g. Dudley (1989).

1.1 THEOREM. *Suppose all moments of μ and ν exist:*

$$\mu_n < \infty \quad \textit{and} \quad \nu_n < \infty \quad \textit{for all } n.$$

Then

$$\mu * \mu = \nu * \nu \tag{1.1}$$

implies

$$\mu_n = \nu_n \textit{ for all } n.$$

PROOF. Let ϕ and ψ be the characteristic functions of μ and ν respectively, that is

$$\phi(t) = \int_{\mathbb{R}} e^{itx} \mu(dx)$$

$$\psi(t) = \int_{\mathbb{R}} e^{itx} \nu(dx), \quad t \in \mathbb{R}.$$

It is well-known that the characteristic function of the convolution of two measures equals the product of their characteristic functions. Consequently the characteristic functions of $\mu * \mu$ and $\nu * \nu$ are given by ϕ^2 and ψ^2 respectively. Thus assumption (1.1) is equivalent to

$$\phi(t)^2 = \psi(t)^2 \quad \text{for all } t.$$

Since all moments exist, the k-th derivative $\phi^{(k)}$ of ϕ exists and is a continuous function that is equal to

$$\phi^{(k)}(t) = i^k \int_{\mathbb{R}} x^k e^{itx} \mu(dx),$$

for all k (Feller (1968), XV.4).

Hence for $t = 0$ we obtain

$$\phi^{(k)}(0) = i^k \mu_k \quad \text{for all } k.$$

Moreover (1.1) yields

$$\frac{\partial^n}{(\partial t)^n}\phi(t)^2 = \frac{\partial^n}{(\partial t)^n}\psi(t)^2 \quad \text{for all } t \text{ and } n, \tag{1.2}$$

and thus in particular this is true for $t = 0$. By Leibniz's rule we have

$$\begin{aligned} \frac{\partial^n}{(\partial t)^n}\phi(t)^2|_{t=0} &= \sum_{k=0}^{n} \binom{n}{k} \phi^{(k)}(0)\phi^{(n-k)}(0) \\ &= i^n \sum_{k=0}^{n} \binom{n}{k} \mu_k \mu_{n-k}, \quad \text{for all } n \end{aligned} \tag{1.3}$$

and a similar expression for ψ. We prove the theorem by induction.

For $n = 0$, (1.2) yields

$$\phi(0)^2 = \psi(0)^2,$$

hence

$$\mu_0 = \phi(0) = \psi(0) = \nu_0,$$

since by definition $\phi(0) = \mu(\mathbb{R}) > 0$ and $\psi(0) = \nu(\mathbb{R}) > 0$. Now suppose

$$\mu_k = \nu_k \quad \text{for all } k < n. \tag{1.4}$$

Then (1.2) and (1.3) yield

$$i^n \sum_{k=0}^{n} \binom{n}{k} \mu_k \mu_{n-k} = i^n \sum_{k=0}^{n} \binom{n}{k} \nu_k \nu_{n-k}$$

which is equivalent to

$$2\mu_n\mu_0 + \sum_{k=1}^{n-1} \binom{n}{k} \mu_k\, \mu_{n-k} = 2\nu_n\nu_0 + \sum_{k=1}^{n-1} \binom{n}{k} \nu_k\, \nu_{n-k}.$$

By (1.4) the sums cancel and by the first induction step $\mu_0 = \nu_0 > 0$, so we can divide both sides by μ_0 to obtain

$$\mu_n = \nu_n.$$

Hence by mathematical induction we have

$$\mu_n = \nu_n \qquad \text{for all } n. \qquad \square$$

It is clear that Theorem 1.1 does not depend on the number of dimensions. The analogues for $\mathbb{R}^d$ will be used in the sequel without further comment.

1.2 Remark. Suppose the additional condition is satisfied that the measures are determined by their moments, that is

$$\text{(1.5)} \qquad \mu_n = \nu_n \qquad \text{for all } n \text{ implies } \mu = \nu.$$

A necessary and sufficient condition for this was given by Carleman (see Shohat and Tamarkin (1943)): a finite measure is uniquely determined by its moments *iff*

$$\text{(1.6)} \qquad \sum \mu_{2n}^{-\frac{1}{2n}} = \infty.$$

(See also Feller (1968), VII.3 for an example of a distribution that is not determined by its moments.) For measures concentrated on a compact set K this condition is trivially satisfied. Indeed we can bound the function x^{2n} on K by its supremum M_0^{2n} say, consequently μ_{2n} is bounded above by $M^{2n} = M_0^{2n}\mu(\mathbb{R})$. Hence

$$\sum \mu_{2n}^{-\frac{1}{2n}} \geq \sum \frac{1}{M_0\mu(\mathbb{R})^{\frac{1}{2n}}} = \frac{1}{M_0} \sum \frac{1}{\mu(\mathbb{R})^{\frac{1}{2n}}}.$$

Since the sequence $\{\mu(\mathbb{R})^{\frac{1}{2n}}\}$ converges to 1 the series $\sum \frac{1}{\mu(\mathbb{R})^{\frac{1}{2n}}}$ diverges.

Theorem 1.1 can be rephrased in several ways. For a probabilistic interpretation suppose X and Y are independent random variables with the same distribution P and let Z and W be independent random variables distributed according to Q. Suppose moreover that P and Q satisfy condition (1.5) of remark 1.2. Then Theorem 1.1 states that if the distribution of $X+Y$ coincides with the distribution of $Z+W$, then $P = Q$.

1.2 Absolutely continuous measures.

If the measure μ has a Radon-Nikodym derivative with respect to Lebesgue measure, $f \in L^1$ say, the characteristic function ϕ is the Fourier transform $\hat{f}$ of f. Moreover the convolution of two such functions is defined as

$$f * g(y) = \int_{\mathbb{R}} f(y-x)g(x)\,dx,$$

and it is well-known that $(f*g)\hat{} = \hat{f}\hat{g}$.

1.3 COROLLARY. *Suppose μ and ν are absolutely continuous with respect to Lebesgue measure λ. Let f and g denote their Radon-Nikodym derivatives and suppose f and g have compact supports. Then $(f*f)\hat{}(t) = (g*g)\hat{}(t)$ for all t implies*

$$f = g \text{ for } \lambda\text{-almost all } x.$$

PROOF. By Theorem 1.1, $(f*f)\hat{}(t) = (g*g)\hat{}(t)$ for all t implies equality of all moments, i.e.

$$\int_{\mathbb{R}} f(x)x^n\,dx = \int_{\mathbb{R}} g(x)x^n\,dx, \text{ for all } n = 0,1,2\ldots.$$

Since f and g have compact supports this immediately implies

$$f = g \text{ for } \lambda\text{–almost all } x.$$

□

1.4 Example. Let A and B be compact Borel sets in $\mathbb{R}$ and let $f = 1_A$, $g = 1_B$. Suppose $1_A * 1_A = 1_B * 1_B$. Then $(1_A * 1_A)\hat{}(t) = (1_B * 1_B)\hat{}(t)$ for all t, thus the corollary yields

$$1_A = 1_B \quad \text{almost everywhere.}$$

In the next section we shall give a geometric interpretation of this example.

In the case of two finite absolutely continuous Borel measures on $[0,\infty)$ the conclusion of Theorem 1.1 can be changed into the direct statement that $\mu * \mu = \nu * \nu$ implies $\mu = \nu$, thus avoiding the somewhat unnatural detour via moments. In fact this is a consequence of the following theorem by Titchmarsh (see Mikusiński (1983), Dieudonné (1960)).

1.5 THEOREM (Titchmarsh). *Let f and g be integrable over $[0,T]$. If the convolution of f and g vanishes almost everywhere in $[0,T]$ then there exist two numbers $T_1 > 0$, $T_2 > 0$, such that $T_1 + T_2 > T$ and such that f vanishes a.e. on $[0,T_1]$ and g vanishes a.e. on $[0,T_2]$.*

1.6 THEOREM. *Let μ and ν be two finite Borel measures on $[\alpha, \infty)$ that are absolutely continuous with respect to Lebesgue measure λ on $[\alpha, \infty)$, for some fixed $\alpha \in \mathbb{R}$.*

Then

$$\mu * \mu = \nu * \nu$$

implies

$$\mu = \nu.$$

PROOF. It suffices to give the proof for $\alpha = 0$. Let f and g be the densities of μ and ν respectively. Thus f and g are nonnegative almost everywhere on $[0, \infty)$. From $\mu * \mu = \nu * \nu$, we immediately obtain

$$f * f = g * g \quad \text{a.e. on } [0, \infty).$$

Equivalently

$$(f - g) * (f + g) = 0 \quad \text{a.e. on } [0, \infty).$$

Since f and g are integrable a.e. on $[0, \infty)$, for all $T > 0$ they are integrable a.e. on the interval $[0, T]$. Then Titchmarsh provides us with $T_1 > 0$, $T_2 > 0$ such that $T_1 + T_2 > T$ and such that

$$(i) \qquad f - g = 0 \quad \text{a.e. on } [0, T_1]$$

$$(ii) \qquad f + g = 0 \quad \text{a.e. on } [0, T_2].$$

This implies that $f = g$ a.e. on $[0, \max(T_1, T_2)]$. Indeed, if $T_1 = \max(T_1, T_2)$, (i) implies $f = g$ a.e. on $[0, T_1] = [0, \max(T_1, T_2)]$. If $T_2 = \max(T_1, T_2)$, (ii) implies that $f = g = 0$ a.e. on $[0, T_2] = [0, \max(T_1, T_2)]$, because f, $g \geq 0$ a.e. on $[0, \infty)$. Now note that

$$T < T_1 + T_2 = \min(T_1, T_2) + \max(T_1, T_2) < 2 \max(T_1, T_2),$$

thus $f = g$ a.e. on $[0, \frac{1}{2}T]$, for all $T > 0$. Letting $T \to \infty$, this yields $f = g$ a.e. on $[0, \infty)$. □

In Mikusiński (1987) one can find a higher-dimensional analogue of Theorem 1.5 on the convolution ring of continuous functions with supports in a given cone.

2. Covariance functions determine regular compact symmetric sets.

Denote by $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$ the Borel σ-algebra of $\mathbb{R}^d$. A set $A \in \mathcal{B}$ is *regular closed* if it equals the closure of its interior:

$$\overline{A^\circ} = A.$$

If a regular closed set is bounded, we call it *regular compact.*

2.1 Definition. For a bounded Borel set A the *covariance function* of A is defined as

$$C_A(y) = \lambda(A \cap (A - y)), \quad y \in \mathbb{R}^d. \tag{2.1}$$

Here λ is d-dimensional Lebesgue measure. When we wish to stress the dimension, we use a subscript as in λ_d.

The following properties are easy to check:
(1) $C_A(0) = \lambda(A)$;
(2) C_A is symmetric: $C_A(y) = C_A(-y), \quad \forall y \in \mathbb{R}^d$;
(3) C_A has compact support.

An equivalent definition is obtained by writing (2.1) in terms of convolutions.

$$C_A(y) = (1_A * 1_{-A})(y), \tag{2.2}$$

where $-A = \{-x : x \in A\}$.

As a consequence of Corollary 1.3, we immediately obtain the following uniqueness theorem.

2.2 Theorem. *Let $A, B \in \mathcal{B}(\mathbb{R}^d)$ be regular compact sets such that $A = -A$ and $B = -B$. Then $C_A = C_B$ implies $A = B$.*

Observe that the sets are *completely* determined and not just up to some translation.

Proof. By (2.2) for all $y \in \mathbb{R}^d$

$$C_A(y) = (1_A * 1_{-A})(y) = (1_A * 1_A)(y) \text{ since } A = -A.$$

Hence the assumption is equivalent to $(1_A * 1_A)(y) = (1_B * 1_B)(y)$ for all $y \in \mathbb{R}^d$. Exactly as in Example 1.4 it follows from Corollary 1.3 that the Fourier transforms of the convolutions are equal and thus $1_A = 1_B$ a.e.. Regularity of the sets now implies $A = B$. □

3. Continuity of the covariance function.

In section 4 we need continuity of the covariance function. This is a direct consequence of the continuity of the convolution of an L^1 function and an essentially bounded function (Zaanen (1989)). Here we give a geometric proof for the special case of the covariance function.

3.1 THEOREM. *Let B be a Borel set with $\lambda_d(B) < \infty$. Then the mapping $x \mapsto C_B(x)$ is continuous on $I\!R^d$.*

PROOF. From the definition of the covariance function we easily derive

$$|C_B(x) - C_B(y)| \le 2(C_B(0) - C_B(x-y))$$

(see also Matheron (1986)). Thus it is enough to prove continuity in 0. Let $\epsilon > 0$ be given. By Theorem 11.4 (Billingsley (1979)), there exists a *finite* sequence $\{U_i\}_{i=1}^n$ of hypercubes such that

$$\lambda(B \triangle \cup_i U_i) < \epsilon/2. \tag{3.1}$$

Set $U = \cup_{i=1}^n U_i$. It suffices to prove that

$$\lambda(B) - \lambda(B \cap T_x B) \le \lambda(U) - \lambda(U \cap T_x U) + \lambda(B \triangle U), \tag{3.2}$$

for all $x \in I\!R^d$. Indeed, once we have (3.2)

$$\begin{aligned} \lambda(B) - \lambda(B \cap T_x B) &\le \lambda(U) - \lambda(U \cap T_x U) + \lambda(B \triangle U) \\ &= \lambda(U \cup T_x U) - \lambda(U) + \lambda(B \triangle U) \\ &\le \lambda(U^{\|x\|} \setminus U) + \lambda(B \triangle U) \\ &< \lambda(U^{\|x\|} \setminus U) + \epsilon/2, \quad \text{by (3.1)}, \end{aligned}$$

where $U^{\|x\|}$ consists of all points in $I\!R^d$ at a distance atmost $\|x\|$ from U. Now observe that $\lambda(U^{\|x\|} \setminus U) \le \sum_i \lambda(U_i^{\|x\|} \setminus U_i)$. Since the U_i are convex, we can use the Steiner formula (see Schneider (1993)) to write each $\lambda(U_i^{\|x\|} \setminus U_i)$ as a polynomial in $\|x\|$ with zero constant term and finite positive coefficients determined by U_i.
Now choose $\delta > 0$, such that $\lambda(U^{\|x\|} \setminus U) < \epsilon/2$ for all $\|x\| < \delta$.
Then for $\|x\| < \delta$

$$C_B(0) - C_B(x) = \lambda(B) - \lambda(B \cap T_x B) < \epsilon.$$

So let us proceed with the proof of (3.2). First observe that

$$B = (B \setminus U) \cup (B \cap U). \tag{3.3}$$

By (3.3)

$$\begin{aligned} \lambda(B) - \lambda(B \cap T_x B) &= \lambda(B \setminus U) + \lambda(B \cap U) - \lambda((B \cap U) \cap T_x(B \cap U)) \\ &\quad - \lambda((B \setminus U) \cap T_x(B \setminus U)) \\ &\quad - \lambda((B \setminus U) \cap T_x(B \cap U)) - \lambda((B \cap U) \cap T_x(B \setminus U) \end{aligned} \tag{3.4}$$

Since the last three terms at the right-hand side of (3.4) are nonnegative, we trivially get

$$(3.5)\quad \lambda(B)-\lambda(B\cap T_xB)\leq \lambda(B\setminus U)+\lambda(B\cap U)-\lambda((B\cap U)\cap T_x(B\cap U)).$$

Now rewrite $\lambda((B\cap U)\cap T_x(B\cap U))$ as $2\lambda(B\cap U)-\lambda((B\cap U)\cup T_x(B\cap U))$, using translation invariance of Lebesgue measure. Then the right-hand side of (3.5) is

$$(3.6)\qquad \lambda(B\setminus U)+\lambda(B\cap U)-2\lambda(B\cap U)+\lambda((B\cap U)\cup T_x(B\cap U)).$$

Moreover

$$\begin{aligned}\lambda((B\cap U)\cup T_x(B\cap U)) &\leq \lambda(U\cup T_x(B\cap U))\\ &\leq \lambda(U\cup T_xU),\end{aligned}$$

and

$$\lambda(B\cap U)=\lambda(U)-\lambda(U\setminus B).$$

So (3.6) is not greater than

$$\begin{aligned}\lambda(U\cup T_xU)-\lambda(U)+\lambda(B\setminus U)+\lambda(U\setminus B) &=\\ =\lambda(U)-\lambda(U\cap T_xU)+\lambda(B\Delta U).\end{aligned}$$

□

4. Interpoint distance distributions.

In a sequence of subsequent volumes of the IMS Bulletin (see Pyke (1989)) Pyke asked the following question:

> given two independent points uniformly distributed over the interior of a compact set C, does the distribution of their distance determine C?

Rost (1989) produced an example showing that in general this need not be true and later Lešanovsky and Rataj (1990) published a paper containing among others a result about the structure of these examples. The examples were based on non convex sets and therefore Pyke's question was restricted to *convex* C (see also Adler and Pyke (1991)). However, the following theorem shows that this restriction is not needed as long as symmetric sets are involved. This is a direct consequence of Theorem 2.2 and the continuity of the covariance function.

Recall the relation between interpoint distance distributions and covariance functions derived from Borel's overlap Lemma. (Borel (1925); Sheng (1985). See also Cabo and Baddeley (1993).)

4.1 LEMMA. *For two independent points X, Y uniformly distributed over the interior of a regular compact set $A \in \mathcal{B}(\mathbb{R}^d)$*

$$\mathbb{P}(\|X - Y\| \le \rho) = \frac{1}{\lambda(A)^2} \int_{B(0,\rho)} C_A(w)\, dw, \quad \rho \ge 0. \tag{4.1}$$

First observe that this implies the equivalence of the problem of characterising a set by the distribution of the *difference vector* of two independent uniformly distributed points X, Y in its interior and the problem of characterising a set by its covariance function. Indeed, from (2.2) it is seen that $\lambda(A)^{-2}\, C_A(\cdot)$ is the density of the distribution of the difference vector, since $X - Y = X + (-Y)$. Nagel's result (1993) thus solves the problem for convex polygons in the plane. Our Theorem 2.2 together with the continuity of the covariance function solves the problem for regular closed symmetric subsets of $\mathbb{R}^d$. In the case of the distribution of the *length* of the difference vector, it is clear that the restriction on the sets to be characterised will have to be more severe, since this distribution throws away the information about the directions. Theorem 2.2 yields the following result.

4.2 PROPOSITION. *Let A and B be regular compact Borel sets that are rotation invariant (and hence centrally symmetric). Let X, Y be independent uniform points in the interior of A and let Z, W be two such points in B. Then*

$$\mathbb{P}(\|X - Y\| \le \rho) = \mathbb{P}(\|Z - W\| \le \rho) \quad \textit{for all } \rho \ge 0 \tag{4.2}$$

implies equality of the sets A and B.

PROOF. Rewrite (4.1) as

$$\mathbb{P}(\|X - Y\| \le \rho) = \frac{1}{\lambda(A)^2} \int_0^\rho \int_{S^{d-1}} r\, C_A(ru)\, du dr. \tag{4.3}$$

Since A is rotation invariant, its covariance function only depends on the distance, hence (4.3) is

$$\frac{d\kappa_d}{\lambda(A)^2} \int_0^\rho r\, C_A(ru_0)\, dr$$

where $d\kappa_d$ is the surface area of the unit sphere S^{d-1} in $\mathbb{R}^d$ and $u_0 \in S^{d-1}$ is arbitrary but fixed. By (4.2) we have

$$\frac{1}{\lambda(A)^2} \int_0^\rho r\, C_A(ru_0)\, dr = \frac{1}{\lambda(B)^2} \int_0^\rho r\, C_B(ru_0)\, dr, \quad \text{for all } \rho.$$

Since by continuity of the covariance function we may take the derivative with respect to ρ on both sides, this is

$$\frac{\rho}{\lambda(A)^2} C_A(\rho u_0) = \frac{\rho}{\lambda(B)^2} C_B(\rho u_0) \quad \text{for } \textit{all } \rho \ge 0.$$

Thus

$$\frac{1}{\lambda(A)^2} C_A(\rho u_0) = \frac{1}{\lambda(B)^2} C_B(\rho u_0) \quad \text{for all } \rho > 0, \tag{4.4}$$

which by continuity of the covariance function must then also hold for $\rho = 0$. Setting $\rho = 0$ in (4.4) yields $\lambda(A) = \lambda(B)$ thus

$$C_A(\rho u_0) = C_B(\rho u_0) \quad \text{for } all\ \rho \geq 0.$$

Since u_0 was arbitrary, Theorem 2.2 yields $A = B$. □

5. Cross-covariance functions determine regular compact sets.

In this section we introduce a generalisation of the covariance function, that is shown to characterise every regular closed subset of $\mathbb{R}^d$.

5.1 Definition. Let $A, B \in \mathcal{B}(\mathbb{R}^d)$ be bounded. The *cross-covariance function of A with respect to B* is defined as

$$C_{A,B}(y) = \lambda(A \cap (B + y)), \quad y \in \mathbb{R}^d. \tag{5.1}$$

The following properties are immediate consequences of the definition:

(1) $C_{A,B}(0) = \lambda(A \cap B)$ ($= 0$ if $A \cap B = \emptyset$);
(2) $C_{A,B}$ is anti-symmetric in the following sense:

$$C_{A,B}(y) = C_{B,A}(-y), \quad \forall y \in \mathbb{R}^d;$$

(3) $C_{A,B}$ has compact support;
(4) $C_{A,A}(y) = C_A(-y) = C_A(y)$, $y \in \mathbb{R}^d$, in other words Matérn's covariance function can be considered as an 'autocovariance' in this setup.

We call $C_{A,-A}$ *the cross-covariance of A.*

As for covariance functions, an equivalent formulation in terms of convolutions exists:

$$C_{A,B}(y) = (1_A * 1_{-B})(y). \tag{5.2}$$

Using Theorem 1.1, we now immediately derive the uniqueness result.

5.2 THEOREM. *Let $A, B \in \mathcal{B}(\mathbb{R}^d)$ be regular compact sets. Then $C_{A,-A} = C_{B,-B}$ implies $A = B$.*

Again, observe that the sets are *completely* determined and not just up to some translation.

PROOF. Suppose $C_{A,-A} = C_{B,-B}$, or by (5.2)

$$(1_A * 1_A)(y) = (1_B * 1_B)(y), \quad \forall y \in \mathbb{R}^d.$$

Exactly as in the proof of Theorem 2.2 this implies $1_A = 1_B$ a.e., whence $A = B$ by regularity. □

Remark. This result can be extended slightly to the following situation. Suppose we are given the *normed* cross-covariance functions (for example normed by their maximum value); i.e.

$$\alpha^2 C_{A,-A}(x) = \beta^2 C_{B,-B}(x), \quad \forall x \in \mathbb{R}^d, \tag{5.3}$$

where α, $\beta \in \mathbb{R}_+$. Then $A = B$.

Indeed, writing (5.3) in terms of convolutions yields

$$\alpha 1_A * \alpha 1_A(x) = \beta 1_B * \beta 1_B(x), \quad \text{for all } x.$$

Hence from Theorem 1.6 we get

$$\alpha 1_A(x) = \beta 1_B(x) \quad \text{for almost all } x. \tag{5.4}$$

Let $\mathcal{N}$ denote the null set for which (5.4) does not hold. We now first prove that this implies $\alpha = \beta$.

Suppose $x \in A \setminus \mathcal{N}$. Then (5.4) gives $\alpha = \beta 1_B(x)$ thus $x \in B \setminus \mathcal{N}$, implying $\alpha = \beta$. But then $1_A = 1_B$ a.e. which implies $A = B$ by regularity. □

Next we prove continuity of the cross-covariance function using the result of section 3.

5.3 THEOREM. *Let A and B be Borel sets with finite Lebesgue measure. Then the mapping $x \mapsto C_{A,B}(x)$ is continuous on $\mathbb{R}^d$.*

PROOF. By definition

$$\begin{aligned}|C_{A,B}(x) - C_{A,B}(y)| &= \Big|\int 1_A(t)1_{B+x}(t) - 1_A(t)1_{B+y}(t)\,dt\Big| \\ &\le \int 1_A(t)|1_B(t-x) - 1_B(t-y)|\,dt.\end{aligned}$$

Since obviously $1_A \leq 1$, this is smaller than

$$\int |1_B(t-x) - 1_B(t-y)|\, dt = \int (1_B(t-x) - 1_B(t-y))^2\, dt.$$

By a change of variables and because of the translation invariance of Lebesgue measure this is

$$2(\lambda(B) - C_{B,B}(x-y)) = 2(C_B(0) - C_B(x-y)).$$

Hence the assertion follows from the continuity of the covariance function of B, which was proved in Theorem 3.1. □

Observe that Theorem 5.3 implies *uniform continuity* of the cross-covariance, since $C_{A,B}$ has compact support if A and B are compact.

There does exist another generalisation of the covariance function that also characterises all regular closed sets. It is called the 'three point covariance function' and is defined as follows (see Nagel (1993)):

$$C_A(x,y) = \lambda(A \cap (A-x) \cap (A-y)), \quad x, y \in \mathbb{R}^d.$$

It was proved by Nagel (1991) that it is possible to determine the corresponding regular set from the three point covariance function considered as a function of the two variables x and y *up to translation.* To conclude this paper we would like to remark that these two characterising functions can be considered as special cases of the following unifying entity.

5.5 Definition. For $n = 1, 2 \ldots$ and $A, B \in \mathcal{B}(\mathbb{R}^d)$ bounded let the n-th order *motion covariance function of* A *w.r.t.* B, be defined as

$$\Gamma^{(n)}_{A,B}(\alpha_1, \alpha_2, \ldots \alpha_n; y_1, y_2, \ldots y_n) = \lambda\big(A \cap (\alpha_1 B - y_1) \cap \cdots \cap (\alpha_n B - y_n)\big),$$

where $\alpha_1, \alpha_2, \ldots \alpha_n \in \mathbf{SO}(d)$ denote rotations of $\mathbb{R}^d$ and $y_1, y_2, \ldots y_n \in \mathbb{R}^d$.

Writing the cross-covariance and three point covariance functions in terms of motion covariances, we get

$$C_{A,-A}(x) = \Gamma^{(1)}_{A,A}(\alpha_\pi; -x) = \Gamma^{(2)}_{A,A}(\alpha_\pi, \mathbf{1}; -x, 0);$$

$$C_A(x,y) = \Gamma^{(2)}_{A,A}(\mathbf{1}, \mathbf{1}; x, y),$$

where α_π denotes point reflection w.r.t. the origin and $\mathbf{1}$ is the identity of $\mathbf{SO}(d)$. Hence the results obtained until now may be summarised by saying that for regular closed sets it either suffices to know the motion covariance with two translations, or the one with one 'reflection' and one translation. In these terms Matheron's conjecture is that for convex sets, motion covariance with only one translation – i.e. $\Gamma^{(1)}_{A,A}(\mathbf{1}; x)$ – is sufficient.

Acknowledgements.

We wish to thank A. J. Baddeley, K. Dzhaparidze, P. Groeneboom, W. Nagel and C. Scheffer for helpful comments.

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