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On the Determination of the Stationary Distribution of a Symmetric Clocked Buffered Switch

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Abstract

The analysis of a symmetric clocked buffered switch leads to the study of a nearest neighbour random walk on the lattice in the first quadrant. The bivariate generating function of the stationary distribution of this random walk has to be obtained as a solution of a functional equation of a type which frequently occurs in the analysis of performance models. In this study it is illustrated that the functional equation for the present model can be solved very simply by using elementary properties of meromorphic functions.

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1. Introduction

In [1] Jaffe studies a stochastic model for the symmetric clocked buffered switch. The stochastic model used in [1] is actually a special case of the symmetric random walk $\mathbf{z}_n = (\mathbf{x}_n, \mathbf{y}_n)$, $n = 1, 2, \ldots$, on the set of lattice points with integral-valued, nonnegative coordinates and with structure described by: for $n = 0, 1, 2, \ldots$,

$$\begin{aligned} \mathbf{x}_{n+1} &= [\mathbf{x}_n - 1]^+ + \boldsymbol{\xi}_n, \\ \mathbf{y}_{n+1} &= [\mathbf{y}_n - 1]^+ + \boldsymbol{\eta}_n; \end{aligned} \tag{1.1}$$

here $(\boldsymbol{\xi}_n, \boldsymbol{\eta}_n)$, $n = 0, 1, 2, \ldots$, is a sequence of i.i.d. stochastic vectors with state space $\{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\}$, and with $\boldsymbol{\xi}_n, \boldsymbol{\eta}_n$ exchangeable variables for every n.

Put

$$\phi(p, q) := \mathrm{E}\{p^{\boldsymbol{\xi}_n} q^{\boldsymbol{\eta}_n}\}, \quad |p| \le 1,\ |q| \le 1, \tag{1.2}$$

so that the symmetry implies

$$\phi(p, q) = \phi(q, p), \quad |p| \le 1,\ |q| \le 1, \tag{1.3}$$

then for the model discussed in [1] $\phi(p, q)$ is given by

$$\phi(p, q) = [1 - a + \frac{1}{2}a(p + q)]^2, \quad 0 < a < 1, \tag{1.4}$$

so that

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$$\Pr\{\xi_n = 2, \eta_n = 2\} = 0,$$
$$\Pr\{\xi_n = 2, \eta_n = 1\} = 0 \tag{1.5}$$
$$\Pr\{\xi_n = 1, \eta_n = 2\} = 0.$$

Hence the random walk $\mathbf{z}$ is a nearest neighbour random walk for which the one-step transition probabilities to the North, to the North-East and the East are all zero.

The symmetric random walk $\mathbf{z}_n$, cf. (1.1), has been analysed extensively without specifying the bivariate generating functions $\phi(p_1, p_2)$ in [2] and, consequently, it requires a rather complicated analysis. In [3] it has been shown that a simpler analysis is available for the case that the $\mathbf{z}_n$-process is a nearest neighbour random walk with the property (1.5). However, the random walk $\mathbf{z}_n$ defined above is a special case of those studied in [3], and it will be shown in the present study that the analysis can be again simplified. We shall illustrate this for the random walk $\mathbf{z}_n$ described above, so that it may be easily compared with that in [1], see also [4] where an analogous approach has been applied.

It is mentioned here that our present analysis can be easily extended to asymmetric random walks of the type (1.1) which are nearest neighbour random walks with the property (1.5); in particular this applies for the model of the asymmetric clocked buffered switch as studied in [7].

In section 2 we formulate the functional equation for the bivariate generating function $\Phi(p,q)$ of the stationary distribution of the inherent random walk $\mathbf{z}_n$, $n = 0, 1, 2, \ldots$, which is assumed to be positive recurrent, cf. (2.3). From this functional equation we derive the equation for the functions $\Phi(p,0)$ and $\Phi(0,q)$, cf. (2.12). Theorem 2.1 formulates the explicit expression for these functions and that for $\Phi(p,q)$ can be readily obtained from (2.3). Section 3 provides the essential part of the proof of theorem 2.1.

In the present study we shall refrain from a further analysis of the bivariate generating function $\Phi(p,q)$. Its analysis can proceed along the same lines as in [1], [4] and [6], because our main purpose here is the development of a simple analysis of the functional equation (2.3) which is of a type that so frequently occurs in the study of nearest neighbour random walks.

REMARK 1.1. Jaffe analysis the functional equation (2.3) for $\Phi(p,q)$ in [1] by using an automorphic technique, and in [6] by applying the approach developed in [2]. In this analysis he implicitly assumes that the regularity of $\phi(p,0)\Phi(p,0)$ in $|p| < 1$ implies that of $\Phi(p,0)$ in $|p| < 1$. In general this is not true, and so the correctness of his results may be questioned. However, it may be shown directly from the functional equation that indeed $|\Phi(p,0)| < \infty$ for p a zero of $\phi(p,0)$, cf. (3.10). With the addition of this argument it is readily seen that the derivation of the solution of the functional equation as given in [1] and [6] is correct. □

2. THE FUNCTIONAL EQUATION

The random walk $\mathbf{z}_n$ defined in the preceding section possesses a unique stationary distribution because, cf. (1.4),

$$0 < a < 1, \tag{2.1}$$

guarantees that the $\mathbf{z}_n$-process is positive recurrent.

This statement is readily proved by using the general ergodicity conditions as described in [5], section II.2.6.

Let $(\mathbf{x}, \mathbf{y})$ be a stochastic vector with distribution the stationary distribution of the random walk

$\mathbf{z}_n$, and put

$$\Phi(p,q) := \mathrm{E}\{p^{\mathbf{x}}q^{\mathbf{y}}\}, \quad |p| \leq 1, \ |q| \leq 1. \tag{2.2}$$

It then readily follows, cf. [2], section II.2.16, that for: $|p| \leq 1$, $|q| \leq 1$,

$$[pq - \phi(p,q)]\Phi(p,q) = (1-p)(1-q)[\phi(0,0)\Phi(0,0) - \frac{\phi(p,0)}{1-p}\Phi(p,0) - \frac{\phi(0,q)}{1-q}\Phi(0,q)]. \tag{2.3}$$

For the analysis of the functional equation (2.3) we need some properties of the zero tuples $(\hat{p},\hat{q})$ of the kernel

$$K(p,q) := pq - \phi(p,q), \tag{2.4}$$

which is obviously well defined for all p and q, cf.(1.4).

It is readily shown by using Rouché's theorem, cf. for a proof [3], Lemma A.2, that the kernel $K(p,q)$, which is a biquadratic form in p and q has for fixed p two zeros q_1 and q_2 with the property that

$$|q_1| < |p| < |q_2| \quad \text{for} \quad |p| \geq 1, \ p \neq 1, \tag{2.5}$$

here

$$q_1 \equiv q_1(p), \qquad q_2 \equiv q_2(p);$$

analogously $p_1(q)$ and $p_2(q)$ are defined. It is also readily verified that the only branch points of $q_1(p)$, and also those of $q_2(p)$, are given by

$$p = 0 \quad \text{and} \quad p = \frac{2a}{1+a} < 1. \tag{2.6}$$

Further

$$q_1(1) = 1, \qquad q_2(1) = \left(\frac{2-a}{a}\right)^2 > 1, \tag{2.7}$$

and $\phi(p,0)$ has only one zero $p = p_0$, this zero has multiplicity two,

$$p_0 = 2 - 2a^{-1}, \quad q_1(p_0) = 0, \quad q_2(p_0) = 8a^{-3}(a-1) < 0, \tag{2.8}$$

$$\begin{aligned} p_0 &\in (-\infty, -1) \quad \text{for} \quad 0 < a < \frac{2}{3}, \\ &\in [-1, 0) \quad \text{for} \quad \frac{2}{3} \leq a < 1. \end{aligned}$$

Obviously, $K(p,q)=0$ with p and q real is a hyperbola with center (p_c, q_c) and asymptotic directions given by

$$p_c = q_c = \frac{a}{1+a}, \qquad \frac{q}{p} = \frac{1}{a^2}[2-a^2 \pm 2\sqrt{1-a^2}] = \left[\frac{1}{a}\{1 \pm \sqrt{1-a^2}\}\right]^2, \tag{2.9}$$

note that

$$\delta := \frac{1}{a^2}[2-a^2+2\sqrt{1-a^2}] > 1, \qquad 0 < \frac{1}{a^2}[2-a^2-2\sqrt{1-a^2}] < 1. \tag{2.10}$$

Note that the axis $q=0$ is a tangent of the hyperbola at $p=p_0$, similarly $p=0$ at $q=p_0=2-2a^{-1}$, so that the left branch of the hyperbola lies in the third quadrant.

From the definition of the bivariate generating function $\Phi(p,q)$ of the stationary distribution of the positive recurrent $\mathbf{z}_n$-process, cf. (2.1), it follows that

i. $\Phi(p,q)$ is for fixed q with $|q|\le 1$, regular in $|p|<1$, continuous in $|p|\le 1$, analogously with p and q interchanged, and its double series expansion in powers of p and q has nonnegative coefficients, (2.11)

ii. $$|\Phi(p,q)| \le 1 \text{ for } |p|\le 1,\ |q|\le 1,$$

$$\Phi(1,1)=1.$$

Let $(\hat{p},\hat{q})$ be a zero tuple of $K(p,q)$, $|p|\le 1$, $|q|\le 1$, that is $K(\hat{p},\hat{q})=0$, then it follows from (2.3), and (2.12) that

i. $$\frac{\phi(\hat{p},0)}{1-\hat{p}}\Phi(\hat{p},0)+\frac{\phi(0,\hat{q})}{1-\hat{q}}\Phi(0,\hat{q})=\phi(0,0)\Phi(0,0),\ \hat{p}\ne 1,\ \hat{q}\ne 1, \tag{2.11}$$

ii. $\Phi(p,0)$ is regular in $|p|<1$, continuous in $|p|\le 1$, similarly for $\Phi(0,q)$,
iii. the coefficients in the series expansion of $\Phi(p,0)$ in powers of p, $|p|\le 1$, are nonnegative, similarly for $\Phi(0,q)$. Note that the symmetry, cf. (1.3) implies

$$\Phi(p,0)=\Phi(0,p), \quad |p|\le 1. \tag{2.13}$$

Further we have

i. $$\Phi(1,0)=\frac{1-a}{(1-\frac{1}{2}a)^2}, \tag{2.14}$$

ii. $$0<\Phi(0,0)<1.$$

To prove (2.14) take $p=1$ in (2.3), divide the resulting relation by $1-q$ and let $q\to 1$, then (2.14)i follows by using $\Phi(1,1)=1$. Since the $\mathbf{z}_n$-process is positive recurrent (2.14)ii should hold.

The relations (2.12), (2.13) and (2.14) describe the conditions to be satisfied by $\Phi(p,0)$. In the next section we shall describe the construction of the solution of the functional equation (2.12)i satisfying the conditions (2.11), (2.12)ii, iii, (2.13) and (2.14). Here we state the solution, to do so we introduce the following sequences, see (2.15) and (2.18).

The sequence

$$p_n^+,\ n = 0, 1, 2, \ldots \tag{2.15}$$

is defined by, cf. (2.5),

$$p_0^+ := q_0^+ = 1,\ q_1^+ = q_2(p_0^+) = q_2(1) = (2-a)^2 a^{-2}, \tag{2.16}$$

$$q_n^+ = q_2(p_{n-1}^+) = p_n^+ = p_2(q_{n-1}^+), \qquad n = 1, 2, \ldots,$$

note that for $n = 1, 2, \ldots$,

$$\begin{aligned} &K(p_n^+, q_{n-1}^+) = 0, \quad p_n^+ > q_{n-1}^+, \\ &K(p_{n-1}^+, q_n^+) = 0, \quad q_n^+ > p_{n-1}^+. \end{aligned} \tag{2.17}$$

The sequence

$$p_n^-,\ n = 0, 1, 2, \ldots, \tag{2.18}$$

is defined by, cf. (2.8),

$$p_0^- := 2 - 2a^{-1}, \quad q_1^- = q_2(p_0^-) = 8(a-1)a^{-3}, \tag{2.19}$$

$$p_n^- = p_2(q_{n-1}^-) = q_n^- = q_2(p_{n-1}^-), \quad n = 1, 2, 3, \ldots;$$

note that for $n = 1, 2, 3, \ldots$,

$$\begin{aligned} &K(p_n^-, q_{n-1}^-) = 0, \quad |p_n^-| > |q_{n-1}^-|, \\ &K(p_{n-1}^-, q_n^-) = 0, \quad |q_n^-| > |p_{n-1}^-|. \end{aligned} \tag{2.20}$$

It is readily seen from (2.10), (2.16) and (2.19) that

i. $\quad 1 = p_0^+ < p_1^+ < p_2^+ < \ldots,$ (2.21)

$\quad 0 > p_0^- > p_1^- > p_2^- > \ldots,$

ii. $\quad \lim_{n\to\infty} \dfrac{p_{n+1}^+}{p_n^+} = \lim_{n\to\infty} \dfrac{p_{n+1}^-}{p_n^-} = \delta > 1.$

THEOREM 2.1. For $0 < a < 1$ the function $\Phi(p, 0)$ is a meromorphic function of p and

$$\Phi(p,0)=\Phi(0,p)=\frac{1-a}{(1-\frac{1}{2}a)^2}\,\frac{\prod\limits_{n=1}^{\infty}(1-\frac{1}{p_n^+})\ \prod\limits_{n=1}^{\infty}(1-\frac{p}{p_{2n}^+})^2}{\prod\limits_{n=1}^{\infty}(1-\frac{p}{p_n^+})\ \prod\limits_{n=1}^{\infty}(1-\frac{1}{p_{2n}^+})^2}. \tag{2.22}$$

The essential part of the proof of this theorem is given in the next section. Here we show that the righthand side of (2.22) is indeed a meromorphic function which satisfies (2.12)ii, iii, and (2.14).

From (2.21)ii it is readily seen that the infinite products in (2.22) all converge absolutely and that the poles and zeros of the righthand side of (2.22) have no finite accumulation point. Consequently, this righthand side is a meromorphic function, which is regular for $|p| < 1$ and continuous for $|p| \leq 1$, because all its poles p_n^+ satisfy $|p_n^+| > 1$, so (2.12)ii is satisfied. Further (2.12)iii is also satisfied since the p_n^- are negative and the p_n^+ are positive. Further (2.14) is seen to hold (take $p = 1$ in the righthand side of (2.22)).

3. PROOF OF THEOREM 2.1.

From the last paragraph of the previous section it is seen that for the proof of theorem 2.1 it remains to show that $\Phi(p,0)$ as given by (2.22) satisfies (2.12)i. To do so take in (2.12), cf. (2.5),

$$\hat{q} = q_1(p), \quad \hat{p} = p \text{ with } |p| = 1, \quad p \neq 1, \tag{3.1}$$

so that

$$\frac{\phi(p,0)}{1-p}\Phi(p,0) + \frac{\phi(0,q_1(p))}{1-q_1(p)}\Phi(0,q_1(p)) = \phi(0,0)\Phi(0,0). \tag{3.2}$$

The points $p = 0$ and $p = 2a(1+a)^{-1}$, cf. (2.6), are the only branch points of $q_1(p)$, and so it is readily seen that $\phi(0,q_1(p))$ and $1-q_1(p)$ are regular in $\{p : |p| \leq 1\}\backslash\mathcal{G}$ with

$$\mathcal{G} := \{p :\ 0 \leq |p| \leq 2a(1+a)^{-1}\}. \tag{3.3}$$

Because $\phi(p)$, $1-p$ and $\Phi(p,0)$ are regular in $|p| < 1$, continuous in $|p| \leq 1$, it follows that the second term in (3.2) can be continued analytically into $\{p : |p| \leq 1,\ p \neq 1\}\backslash\mathcal{G}$ and that the principle of permanence implies that for this analytic continuation (3.2) remains valid. Because of the continuity of the first term in (3.2) it follows that (3.2) also holds for a point $\tilde{p} \in int\mathcal{G}$, if p approaches $\tilde{p}$ from above, i.e. Im$p > 0$, or from below, i.e. Im$p < 0$; but then the second term in (3.2) takes in $p = \tilde{p}$ the conjugate value. Hence, since

$$q_1(\tilde{p}) = \overline{q_2(\tilde{p})},$$

and $\tilde{p}$ is real it follows readily that: for $p = \tilde{p}$,

$$\frac{\phi(p,0)}{1-p}\Phi(p,0) + \frac{\phi(0,q_2(p))}{1-q_2(p)}\Phi(0,q_2(p)) = \phi(0,0)\Phi(0,0). \tag{3.4}$$

The relation (3.3) can again be continued analytically out from point $\tilde{p} \in int\mathcal{G}$ into $|p| \leq 1$, $p \neq 1$, and this leds to: for $|p| \leq 1$, $p \neq 1$,

$$\frac{\phi(p,0)}{1-p}\Phi(p,0)+\frac{\phi(0,q_2(p))}{1-q_2(p)}\Phi(0,q_2(p))=\phi(0,0)\Phi(0,0). \tag{3.5}$$

Analogously we have: for $|q|=1$, $q\neq 1$,

$$\frac{\phi(0,q)}{1-q}\Phi(0,q)+\frac{\phi(p_2(q),0)}{1-p_2(q)}\Phi(p_2(q),0)=\phi(0,0)\Phi(0,0), \tag{3.6}$$

$$\frac{\phi(0,q)}{1-q}\Phi(0,q)+\frac{\phi(p_1(q),0)}{1-p_1(q)}\Phi(p_1(q),0)=\phi(0,0)\Phi(0,0). \tag{3.7}$$

Because $|q_2(p)|>1$ for $|p|=1$, it follows from the analytic continuation above, which has lead to (3.4), that $\Phi(0,q)$ is regular for, cf. (2.5),

$$|q|<\delta_2:=\sup_{|p|=1}|q_2(p)|, \tag{3.8}$$

note that $\delta_2>1$.

From this, from the fact that $p_2(q)$ has no branch points in $|q|\geq 1$, and from $|q_2(p)|>1$ for $|p|=1$, $p\neq 1$, it follows by using (3.6) that $\Phi(p_2(q),0)$ can be continued analytically for all those q satisfying (3.8) *except* for those q for which $p_2(q)=1$; here $\Phi(p_2(q),0)$ then has a pole. Since $p_2(q)=1$, has a simple zero, at q_1^+, cf. (2.16), it is seen that $p_2^+=p_2(q_1^+)$ is a simple pole of $\Phi(p,0)$. Hence, since $|p_2(q)|>|q|$ for $1<|q|<\delta_2$ it follows that $\Phi(p,0)$ can be continued analytically into $|p|>\delta_2$, $|p|\neq p_2^+$. For this continuation it is then seen that $\Phi(0,q)$ should have a simple pole at that q for which $q=q_2(p_2^+)$, i.e. $q=q_3^+$ and so it follows from (3.6) and (3.7) that $\Phi(p_2(q),0)$ has a simple pole at $p_4^+=p_2(q_3^+)$, cf. (2.16). By repeating the arguments used above and by using the symmetry, cf. (2.13), we have:

$\Phi(p,0)$ has a unique analytic continuation in $|p|\geq 1$, except for $p=p_n^+$, $n=1,2,\ldots$, where it has simple poles. (3.9)

Next we consider the zero $p=p_0$ of $\phi(p,0)$, cf. (2.8), and we show that

$$0\neq|\Phi(p_0,0)|<\infty. \tag{3.10}$$

Since $p_0^-=p_0$ is a zero of multiplicity two of $K(p,q)=0$, cf. (2.8), we have

$$\left.\frac{\mathrm{d}q_i(\mathrm{p})}{\mathrm{dp}}\right|_{p=p_0}=0,\quad q_1(p_0)=0. \tag{3.11}$$

It follows that: for $p\sim p_0$,

$$q_1(p)=\frac{1}{2}(p-p_0)^2\left\{\frac{\mathrm{d}^2}{\mathrm{dp}^2}q_1(p)\right\}_{p=p_0}+o((p-p_0)^2), \tag{3.12}$$

$$\phi(p,0)=\frac{1}{2}(p-p_0)^2\left\{\frac{\mathrm{d}^2}{\mathrm{dp}^2}\phi(p,0)\right\}_{p=p_0}=-\frac{1}{4}(p-p_0)^2a^2.$$

Further, cf. (3.2): for $q \sim 0$,

$$\phi(0,q)\Phi(0,q) - (1-q)\phi(0,0)\Phi(0,0) = \tag{3.13}$$

$$q[\Phi(0,0)\frac{\mathrm{d}}{\mathrm{dq}}\phi(0,q)\Big|_{q=0} + \phi(0,0)\frac{\mathrm{d}}{\mathrm{dq}}\Phi(0,q)\Big|_{q=0} + \phi(0,0)\Phi(0,0)] + \dot{O}(q^2).$$

Hence from (3.2), (3.11), (3.12) and (3.13): for $p_1 \to p_0$,

$$\frac{\Phi(p_0,0)}{1-p_0^-}\frac{\frac{\mathrm{d}^2}{\mathrm{dp}^2}\phi(p,0)}{\frac{\mathrm{d}^2}{\mathrm{dp}^2}q_1(p)}\Bigg|_{p=p_0} + \phi(0,0)\frac{\mathrm{d}}{\mathrm{dq}}\Phi(0,q)\Bigg|_{q=0} + \phi(0,0)\Phi(0,0) = 0, \tag{3.14}$$

so that, cf. (2.12)iii, (2.14)ii and (3.14),

$$\Phi(0,0) > 0,\ \frac{\mathrm{d}}{\mathrm{dq}}\Phi(0,q)\Big|_{q=0} \geq 0 \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{dp}^2}q_1(p)\Big|_{p=p_0} \neq 0,$$

implies (3.10).

Since $p_0^- = p_0$ and $\phi(p_0,0) = 0$, it follows from (2.19) and (3.4) that $q = q_1^-$ is a double zero of

$$\frac{\phi(0,q)}{1-q}\Phi(0,q) - \phi(0,0)\Phi(0,0). \tag{3.15}$$

Take in (3.7) $q = q_1^-$, i.e. $p_2(q) = p_2^-$, cf. (2.19), then (3.15) implies, since $\phi_2(p_2^-,0) \neq 0$, that:

$$\Phi(p,0)\ \textit{has a zero of multiplicity two at } p = p_2^-. \tag{3.16}$$

Hence from (2.19), (3.7) and (3.16), q_3^- is a zero of multiplicity two of

$$\frac{\phi(0,q)}{1-q}\Phi(0,q) - \phi(0,0)\Phi(0,0),$$

from which it follows that p_4 is a double zero of

$$\Phi(p,0)$$

and generally, by noting that $\Phi(p,q)$ is regular for $p \neq p_n^+$ and that $p_n^- \neq p_m^+$ for all n and m, that

$$\Phi(p,0) \quad \textit{has zeros of multiplicity two at } p = p_{2n}^-, \quad n = 1,2,\ldots, \tag{3.17}$$

$$\frac{\phi(p,0)}{1-p}\Phi(p,0) - \phi(0,0)\Phi(0,0) \tag{3.18}$$

has double zeros at $p = p_{2n-1}^-$, $n = 1, 2, \ldots$.

From (3.9) it follows that $\Phi(p, 0)$ is a meromorphic function, since its poles p_n^+, $n = 1, 2, \ldots$, have no finite accumulation point. Also its zeros p_{2n}^- have no such point. Consequently the function

$$w(p) := \frac{\prod_{n=1}^{\infty}(1 - \frac{1}{p_n^+}) \prod_{n=1}^{\infty}(1 - \frac{p}{p_{2n}^-})^2}{\prod_{n=1}^{\infty}(1 - \frac{p}{p_n^+}) \prod_{n=1}^{\infty}(1 - \frac{1}{p_{2n}^-})^2} C, \tag{3.19}$$

with C a nonzero constant satisfies the relations (3.2) and (3.5), since a meromorphic function, for which the sum of the inverses of its poles p_n^+ and also that of the inverses of its zeros p_{2n}^- both converge absolutely, is uniquely determined apart from a constant factor. Hence it is given by (3.19); note that the zero p_{2n}^- has multiplicity two. Because $p_n^+ > 1$ it is seen that $w(p)$ is regular in $|p| < 1$, continuous in $|p| \leq 1$. Because the $\mathbf{z}_n$-process is assumed to be positive recurrent there is only one function $\Phi(p, q)$ which satisfies (2.3) and (2.11). Hence there is also one function $\Phi(p, 0)$, note (2.13), which satisfies (2.12). Hence, there is only one function $\Phi(p, 0) = \Phi(0, p)$ which satisfies (3.2) and (3.5), or equivalently (3.6) and (3.7). Note that the principle of permanence implies that (2.12) is equivalent to (3.2) and (3.5). Consequently, the proof of theorem 2.1 is complete.

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