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A Limit Theorem for Scaled Vacancies of the Boolean Model

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Abstract

The vacancy of the high-intensity Boolean model (or mosaic process) is considered. The main result gives a limit distribution of a scaled connected vacant component. It is shown that the limiting set is the polyhedron generated by (in general anisotropic) Poisson network of hyperplanes driven by the expected surface measure of the grain. The associated zonoid of the network is equal to the projection body of the so-called Blaschke expectation of the typical grain. In difference to earlier results obtained by P.Hall, few geometrical and no isotropy assumptions are imposed on the grain and the proofs are based on the translative integral geometric formula instead of direct analytical computations.

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1. Introduction

The Boolean model is the most famous random set model in stochastic geometry. Its simplest stationary Euclidean variant is defined as the union of independent identically distributed random sets $\Xi_1, \Xi_2, \ldots$ (grains) in the Euclidean space $\mathbf{R}^d$ shifted by points $\{x_1, x_2, \ldots\}$ (germs) of the stationary Poisson point process Ψ_λ in $\mathbf{R}^d$:

$$\Xi = \bigcup_{i:x_i \in \Psi_\lambda} (x_i + \Xi_i)\,, \tag{1.1}$$

see [4, 6, 23, 39, 44]. The set Ξ is called sometimes the mosaic process [11, 14]. Equivalently, it is possible to define Ξ to be the union set of the Poisson point process

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in the space $\mathcal{K}$ of compact sets in $\mathbf{R}^d$ [23, 50]. The Boolean model is, thereupon, a particular case of the so-called germ-grain model [15, 16].

The intensity λ of the Poisson point process of germs is said to be the *intensity* of the Boolean model. The random set Ξ_0 which has the same distribution as the grains $\Xi_1, \Xi_2, \ldots$ is called the *typical grain*. To get a non-trivial set Ξ in (1.1), the typical grain must be not "very large" [16, 44]. For instance, it is sufficient to assume that the d-th moment of the circumscribed diameter of the typical grain is finite.

The Boolean model is used in the random sets theory and spatial statistics to model stationary sets, see [6, 39, 44] and references therein. The corresponding problems there are related to the estimation techniques and hypotheses testing.

At the same time, the Boolean model serves as the basic model in continuum percolation theory, see [12, 14, 32, 24]. The main problem is the existence of unbounded connected covered (or uncovered) sets.

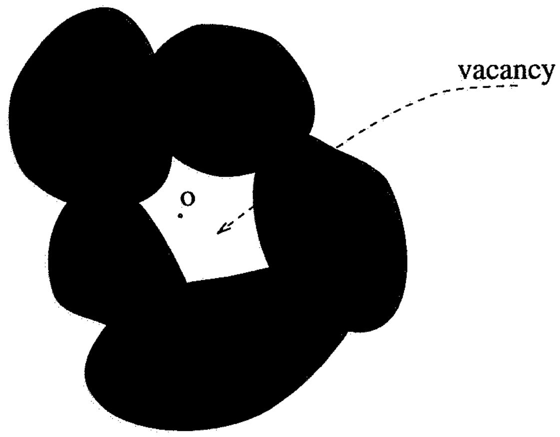
Another application of the Boolean model is related to coverage problems, see [9, 11, 13, 14, 20, 30, 43]. The principal aim is to find the probability of coverage of a fixed set K by Ξ. Due to some biological applications one considers often the coverage problem for the Boolean model on the circle or sphere [19, 21, 25, 40, 41]. Another family of coverage problems is related to the study of the time of total coverage for time-dependent Boolean models and Johnson-Mehl tessellations [3, 5, 45]. The complement to the Boolean model is of the most interest in the theory of coverage [1, 14]. Under some conditions and the isotropy assumption Hall [11, 14] proved that the typical uncovered region is distributed like the polyhedron bounded by the stationary and isotropic Poisson net of hyperplanes. He also found the asymptotic probability of total coverage for the Boolean model with high intensity and small grains. Further steps in this direction were done by Chiu [3].

All possible limiting results for the Boolean model and its complement (vacant region) are of two types. *Global* results deal with the limiting behaviour of the set in the "whole", e.g., with the number of connected vacant regions inside a fixed set for the Boolean model with a high intensity. A typical example is Theorem 2 of [11], which states that the number of connected uncovered regions is approximately Poisson, or results of Zähle [52], who proved that, under some conditions, the complement of unions of Boolean models (with open grains) can have a fractal limit.

On the other hand, *local* limit theorems deal with the structure of the vacant component in "small", although such vacancies can be quite rare in the space. A typical example is Theorem 1 of [11], which states that the scaled typical uncovered region converges in distribution to the typical cell generated by a stationary isotropic Poisson net of hyperplanes.

This paper deals with local limit theorems for the *vacant region* of the Boolean model. The results presented here include as particular cases earlier local limit theorems for scaled vacancies. In contrast to previous papers where proofs relied essentially on analytical methods, here the integral geometric approach is used. This makes it possible

Figure 1.1: A clump and the corresponding elementary vacancy.



to prove limit theorems without exploiting shape conditions on the typical grain, and also in the anisotropic case.

The paper is organized as follows. Section 2 recalls the notion of the weak convergence of random closed sets. In Section 3 some properties of the Boolean model are mentioned. This section brings also necessary information on the translative integral geometric formula. Section 4 contains the central result of the paper, which gives the limit distribution of the typical uncovered region. Then some corollaries and earlier results are discussed.

Some notations will be used throughout the paper without comments. We denote by $\mathbf{R}^d$ the d-dimensional Euclidean space with the corresponding metric $\rho(\cdot,\cdot)$, the norm $\|\cdot\|$ and the scalar product $\langle u, x\rangle$. Furthermore, $\mathbf{S}^{d-1}$ is the unit sphere in $\mathbf{R}^d$, $B_r(x)$ is the ball of radius r centered at x. For the sake of brevity we write $B_r = B_r(o)$ and $B = B_1(o)$, where o is the origin. The Minkowski (or element-wise) sum of two sets is denoted by

$$F_1 \oplus F_2 = \{x + y \colon\ x \in F_1,\ y \in F_2\}\ .$$

For any set K we write

$$K^r = K \oplus B_r = \{x \colon\ \rho(x, K) \leq r\}$$

for the r-neighborhood (or r-parallel set) of K.

The intrinsic volumes [35, 37, 38, 46] of a convex set K in $\mathbf{R}^d$ are denoted by $V_i(K)$, $1 \leq i \leq d$. Note that $V_d(K)$ is exactly the d-dimensional Lebesgue measure, μ_d, of K, $V_{d-1}(K)$ is the surface area of K, $V_1(K)$ is equal up to a constant to the mean width of K [35, p.210], and $V_0(K) = 1$. We use the same notations for the additive extensions of the intrinsic volumes onto the convex ring (family of finite unions of convex sets), see [35, 46]. Extended onto the convex ring the functional V_0 becomes equal to the

Euler-Poincaré characteristic χ of the corresponding set. The extended functionals V_{d-1} and V_d are still the surface area and the volume of the corresponding set from the convex ring.

We always write dx, where the integration with respect to the Lebesgue measure μ_d is meant. Furthermore, for any set $M \subset \mathbf{R}^d$ we denote by $\bar{M}$, $\operatorname{Int} M$, ∂M, $\mathrm{c}(M)$ and $\operatorname{conv}(M)$ respectively the closure, the interior, the boundary, the complement and the convex hull of M.

Some letter conventions are worth mentioning. As a rule we use the letter X (with or without indices) to denote general random closed sets and reserve Greek Ξ for Boolean models and their grains. The letter Y is used for random open sets. Furthermore, F and K stand for some closed and compact subsets of $\mathbf{R}^d$.

2. Weak Convergence of Random Sets

A random closed set X is a random element in the space $\mathcal{F}$ of all closed subsets of $\mathbf{R}^d$. The measurability is ensured by the condition that $\{X \cap K \neq \emptyset\}$ is a random event for any compact set K, see [23]. The distribution of the random closed set X is determined by the hitting probabilities

$$T_X(K) = \mathbf{P}\{X \cap K \neq \emptyset\} \tag{2.1}$$

for K running through the family $\mathcal{K}$ of all compact sets. The functional $T_X : \mathcal{K} \mapsto [0, 1]$ from (2.1) is said to be the *capacity functional* of X, see [6, 23, 44].

Random open sets were also introduced in [23]. Namely, a random element Y is a random open set if it takes values in the family $\mathcal{G}$ of all open sets and $\{K \subset Y\}$ is measurable for each $K \in \mathcal{K}$. The distribution of Y is determined by the *containment functional*

$$I_Y(K) = \mathbf{P}\{K \subset Y\}, \quad K \in \mathcal{K}.$$

The *weak convergence* (or the convergence in distribution) of random closed (open) sets is a particular case of the weak convergence of general random elements. It can be characterized through the pointwise convergence of the corresponding capacity functionals. Namely, X_n converges weakly to X if

$$T_{X_n}(K) \to T_X(K) \quad \text{as} \quad n \to \infty \tag{2.2}$$

for all K from the family

$$\mathcal{T}_X = \{K \in \mathcal{K}\colon\ T_X(K) = T_X(\operatorname{Int} K)\}$$

of the "continuity points" of the limiting capacity functional T_X, see [29, 31, 34]. An important problem is to reduce the family of compact sets such that the pointwise convergence in (2.2) on this reduced family ensures the weak convergence of the corresponding random sets [7, 29, 31, 34]. It is usual to take the class $\mathcal{M}$ of finite unions of balls of positive radii as such a reduction.

On the other hand, it is sometimes natural to consider other functionals rather than the capacity functional (2.1). For example, the probability $\mathbf{P}\{K \subset X\}$ (the *containment functional* of the random closed set X) is of interest. Unfortunately, it is not always possible to determine the distribution of X by such probabilities. For example, if X is a singleton with an absolutely continuous distribution, then these containment probabilities are identically equal to zero. To exclude such trivial cases we consider an a.s. *regular closed* set X, which, by definition, almost surely coincides with the closure of its interior, i.e., $X = \overline{\mathrm{Int}\, X}$ a.s. The following result simply follows from the regular closedness property.

Theorem 2.1. *If X is a.s. regular closed, then its distribution is determined uniquely by the containment functional*

$$I_X(K) = \mathbf{P}\{K \subset \mathrm{Int}\, X\}\,, \quad K \in \mathcal{K}. \tag{2.3}$$

Also it is possible to replace $\mathcal{K}$ by the class $\mathcal{M}$.

As in (2.2), the weak convergence of regular closed random sets can be characterized by the pointwise convergence of their containment functionals. It is easy to give a proof following the classical line in [2, Theorem 2.2].

Theorem 2.2. *A sequence X_n, $n \geq 1$, of regular closed sets converges weakly to the regular closed set X if*

$$I_{X_n}(K) \to I_X(K) \quad \text{as} \quad n \to \infty \tag{2.4}$$

for all K from the class

$$\mathcal{I}_X = \{K \in \mathcal{K}\colon\ I_X(K) = I_X(\mathrm{Int}\, K)\}\,.$$

A similar result is valid for a sequence of random open sets Y_n, $n \geq 1$, and the corresponding containment functionals.

We are interested sometimes in *conditional distributions* of random sets. We need this notion in its simplest form, see also [17, 18, 23]. For any event $\mathcal{A}$ of positive probability the functional $\mathbf{P}\{K \cap X \neq \emptyset | \mathcal{A}\}$ is again the capacity functional of a random closed set. The weak convergence of such conditional sets is characterized through the pointwise convergence of the corresponding *conditional* capacity or containment functionals in the same way as in (2.2) and (2.4). Conditional random open sets are defined similarly.

3. The Boolean Model and Its Distribution

The Boolean model is defined in (1.1). We suppose that its grains are random closed sets. Then the complement of the Boolean model is a random open set. Sometimes the grains are supposed to be open. Then the Boolean model is also open, and it is possible to handle the complement as a random closed set, see [11, 52] and [3]. For us this is not so important, as we work mostly with containment functionals and regular closed sets.

Theorem 3.1. *Let Ξ_0 be a random set from the convex ring. Then the Boolean model Ξ' with the grain* $\mathrm{Int}\,\Xi_0$ *has the a.s. regular closed complement.*

PROOF. If the complement is not a.s. regular closed, then, with a positive probability, there exists a point $x \notin \Xi'$ such that $B_r(x) \cap \mathrm{Int}\,(\mathrm{c}(\Xi')) = \emptyset$ for all sufficiently small $r > 0$. Furthermore, $\mathrm{Int}\,(\mathrm{c}(\Xi')) \supset \mathrm{c}(\Xi)$ yields $B_r(x) \subset \Xi$. Thus, with a positive probability, x lies on the boundary of a grain, which is impossible due to assumptions. □

For the Boolean model Ξ given by (1.1) the capacity functional has a simple form:

$$T_\Xi(K) = 1 - \exp\{-\lambda \mathbf{E}\mu_d(\Xi_0 \oplus \check{K})\}\,. \tag{3.1}$$

Here $\oplus$ is the Minkowski (element-wise) addition of sets, and $\check{K} = \{-x\colon\ x \in K\}$ is the mirrored variant of K, i.e.,

$$\Xi_0 \oplus \check{K} = \{x - y\colon\ x \in \Xi_0\,,\ y \in K\}\ .$$

Consider a sequence of Boolean models $\Xi^{(n)}$, $n \geq 1$, with the corresponding intensities λ_n and typical grains $\Xi_0^{(n)}$, that is

$$\Xi^{(n)} = \bigcup_{i:x_i \in \Psi_{\lambda_n}} (x_i + \Xi_i^{(n)})\,. \tag{3.2}$$

We assume that the typical grain $\Xi_0^{(n)}$ takes values from the convex ring, i.e., $\Xi_0^{(n)}$ is an a.s. finite union of convex sets. Without loss of generality it is possible to assume that $\Xi_0^{(n)}$ contains the origin for all n. Otherwise the appropriately shifted grain must be taken. It is necessary to impose a condition which ensures that the union set in (3.2) is not trivial, see [16]. The latter is true if and only if

$$\mathbf{E}\mu_d(\Xi_0^{(n)} \oplus B_r) < \infty$$

for any $r > 0$. For this, it is sufficient to suppose that $\mathbf{E}\|\Xi_0^{(n)}\|^d < \infty$, where

$$\|F\| = \sup\{\|x\|\colon\ x \in F\}$$

is the norm of the set F. On the other hand, the existence of the Boolean model yields $\mathbf{E}\|\Xi_0^{(n)}\| < \infty$.

The dilatation of $\Xi^{(n)}$ brings little new, since $c\Xi^{(n)} = \left\{cx\colon\ x \in \Xi^{(n)}\right\}$ is the Boolean model with the typical grain $c\Xi_0^{(n)}$ and the intensity $c^{1/d}\lambda_n$. However, sometimes the use of such a normalization is justified by the better form of results.

It follows from (3.1) that the Boolean model $\Xi^{(n)}$ has the capacity functional

$$T_{\Xi^{(n)}}(K) = 1 - \exp\{-\lambda_n \mathbf{E}\mu_d(\Xi_0^{(n)} \oplus \check{K})\}\,. \tag{3.3}$$

It is easy to see that the weak limit of the set $\Xi^{(n)}$ is always trivial for the case of growing intensity. It follows from the fact that in the "worst" case when the grains are singletons, the corresponding Poisson point process Ψ_{λ_n} of germs converges weakly to the whole plane. Namely,

$$T_{\Xi^{(n)}}(K) \geq \mathbf{P}\{K \cap \Psi_{\lambda_n} \neq \emptyset\} = 1 - \exp\{-\lambda_n \mu_d(K)\} \to 1 \quad \text{as} \quad n \to \infty$$

for any K from the class $\mathcal{M}$ (since $\mu_d(K) > 0$). Therefore, the weak limit of $\Xi^{(n)}$ in the space of closed sets is the whole plane.

The subsequent presentation will be based on the translative integral geometric formula. Let us recall briefly a basic and simplified results of this kind, see [36, 48] for detailed discussions.

Let K and K' be convex regular closed compact sets. The translative integral geometric formula yields

$$\int_{\mathbf{R}^d} \chi(K \cap (K' + x))dx = V_0(K)V_d(K') + \sum_{k=1}^{d-1} V_k(K, K') + V_d(K)V_0(K')\,. \tag{3.4}$$

Here the functionals $V_k(\cdot, K')$ and $V_k(K, \cdot)$ are additive; the first is homogeneous of degree k, while the second is homogeneous of degree $(d-k)$. For K and K' being polytopes it is possible to calculate the corresponding functionals explicitly [36].

The functional of the order $(d-1)$ is the most important for us. It can be represented as

$$V_{d-1}(K, K') = \int_{\mathbf{S}^{d-1}} h(K', u) S_{d-1}(K; du)\,, \tag{3.5}$$

where

$$h(K', u) = \sup\{\langle u, x\rangle\colon\ x \in K'\}$$

is the support function of K', and $S_{d-1}(K; du)$ is the surface measure of K, see [35, Section 4.2]. In the smooth case $S_{d-1}(K; \Gamma)$ can be defined to be the $(d-1)$-dimensional surface measure of the set of all points on ∂K such that the corresponding unit outer normal vectors lie in the set $\Gamma \subset \mathbf{S}^{d-1}$.

The additivity of the functionals V_k and the surface measure makes it possible to extend the translative integral geometric formula for K and K' belonging to the convex ring. Then the functional $V_0(K)$ becomes the Euler-Poincaré characteristic $\chi(K)$, and $V_d(K) = \mu_d(K)$.

Furthermore, note that for convex K and K' the integral in the left-hand side of (3.4) is equal to the volume of the set $K \oplus \check{K}'$. Indeed,

$$\chi(K \cap (K' + x)) = 1_{\{K \cap K' + x \neq \emptyset\}}$$

for convex K and K', whence

$$\int \chi(K \cap (K' + x))dx = \int 1_{\{K \cap K' + x \neq \emptyset\}} dx = \mu_d(K \oplus \check{K}'). \tag{3.6}$$

Unfortunately, for non-convex summands the integral in the left-hand side is not always equal to the volume of the corresponding Minkowski sum, i.e., (3.6) is no longer valid. However, exactly the right-hand side of (3.6) appears in the formula for the capacity functional of the Boolean model with K replaced by $\Xi_0^{(n)}$ and K' by K. Fortunately, sometimes $\mathbf{E}\mu_d(\Xi_0^{(n)} \oplus \check{K})$ is *asymptotically* equal to the integral (3.4). In the following we will use the condition

$$\lambda_n \mathbf{E}\left[\int_{\mathbf{R}^d} \chi(\Xi_0^{(n)} \cap (\lambda_n^{-1} K + x))dx - \mu_d(\Xi_0^{(n)} \oplus \lambda_n^{-1}\check{K})\right] \to 0 \quad \text{as} \quad n \to \infty \tag{3.7}$$

for all compact sets K from the family which determines the weak convergence, for example, for all finite unions of balls. Assumption (3.7) makes it possible to use the translative integral formula to decompose $\mu_d(\Xi_0^{(n)} \oplus \check{K})$ in (3.3). Thus, (3.7) may be thought as a Boolean model analogue of the "tightness" condition for random processes. Another tightness condition for the Boolean model was used in [16] to study the "global" convergence of Boolean models.

Some special cases where (3.7) is valid will be considered later on. Let us only note that for convex $\Xi_0^{(n)}$ and K it holds automatically. Furthermore, (3.7) is valid if

$$\lambda_n \mathbf{E}\left[\nu_n \mu_d\left(\left\{x\colon\ \chi(\Xi_0^{(n)} \cap (\lambda_n^{-1} K + x)) \neq 1_{\{\Xi_0^{(n)} \cap (\lambda_n^{-1} K + x) \neq \emptyset\}}\right\}\right)\right] \to 0 \quad \text{as} \quad n \to \infty, \tag{3.8}$$

where ν_n is the number of convex components of $\Xi_0^{(n)}$. The condition of type (3.7) (for non-random and fixed $\Xi_0^{(n)}$) was first used by Rataj [33] to estimate the mixed area measure of a planar set from the convex ring.

4. A Limit Theorem for the Normalized Uncovered Region

The uncovered (or vacant) region is the complement $\mathsf{c}(\Xi^{(n)})$ to the Boolean model $\Xi^{(n)}$. The whole uncovered region can be represented as the union of its connected components (vacancies). Considered as translated realizations of a random closed set, these vacancies describe the distribution of the so-called "typical" uncovered region of the Boolean model.

Let us denote by Y_n the conditional random open set which is equal to the uncovered connected region containing the origin provided that the origin is not covered. The normalized set $\lambda_n Y_n$ has the containment functional given by

$$\begin{aligned} I_n(K) &= \mathbf{P}\{K \subset \lambda_n Y_n\} \\ &= \mathbf{P}\left\{\lambda_n^{-1} K \cap \Xi^{(n)} = \emptyset |\, o \notin \Xi^{(n)}\right\} \\ &= \exp\{-\lambda_n[\mathbf{E}\mu_d(\Xi_0^{(n)} \oplus \lambda_n^{-1}\check{K}) - \mathbf{E}\mu_d(\Xi_0^{(n)})]\} \end{aligned}$$

for K containing the origin. We are interested in the limiting behaviour of the set $\lambda_n Y_n$ (or, equivalently, of the containment functional $I_n(K)$) as $n \to \infty$.

The following general theorem includes some technical conditions. It will be shown below that these technical conditions can be replaced in most cases by simpler conditions of geometric nature.

Theorem 4.1. *Suppose that $\lambda_n \to \infty$ and, for each finite-point set K,*

$$\Delta_n(K) = \lambda_n \left(\mathbf{E}\mu_d(\Xi_0^{(n)} \oplus \lambda_n^{-1}\mathrm{conv}(K)) - \mathbf{E}\mu_d(\Xi_0^{(n)} \oplus \lambda_n^{-1}K)\right) \to 0 \qquad (4.1)$$

as $n \to \infty$. Furthermore, let (3.7) be valid for all convex compact sets K. If the expected surface measure $\mathbf{E}S_{d-1}(\Xi_0^{(n)};\cdot)$ converges weakly to the measure $\nu(\cdot)$ on the unit sphere, then $\lambda_n Y_n$ (normalized conditional vacancy) converges weakly to the random open convex set $\tilde{Y}$ with the containment functional

$$\tilde{I}(K) = \mathbf{P}\left\{K \subset \tilde{Y}\right\} = \exp\left\{-\int_{\mathbf{S}^{d-1}} |h(K,u)|\nu(du)\right\}. \qquad (4.2)$$

Proof. Clearly, (4.1) yields

$$I_n(K) - I_n(\mathrm{conv}(K)) \to 0 \quad \text{as} \quad n \to \infty \qquad (4.3)$$

for any finite-point set K. Therefore, (4.3) is valid for all K from the class of finite unions of polyhedrons. Since this class determines the weak convergence of random open sets, the set $\lambda_n Y_n$ is asymptotically convex, and (4.3) is valid for each compact set K. Indeed, the limiting containment functionals of K and $\mathrm{conv}(K)$ coincides, i.e.,

$$\mathbf{P}\left\{K \subset \tilde{Y}\right\} = \mathbf{P}\left\{\mathrm{conv}(K) \subset \tilde{Y}\right\} = \mathbf{P}\left\{\mathrm{conv}(K) \subset \mathrm{conv}(\tilde{Y})\right\}.$$

Since the limit is asymptotically convex and the sets Y_n contain the origin, it is sufficient to consider only convex compact sets K containing the origin. Then

$|h(K,u)| = h(K,u)$ and (3.7) yields

$$\lim_{n\to\infty} I_n(K) = \lim_{n\to\infty} \exp\left\{-\lambda_n \mathbf{E}\left[\int_{\mathbf{R}^d} \chi(\Xi_0^{(n)} \cap (\lambda_n^{-1}K + x))dx - \mu_d(\Xi_0^{(n)})\right]\right\}$$

$$= \lim_{n\to\infty} \exp\left\{-\lambda_n \mathbf{E}\left[V_{d-1}(\Xi_0^{(n)}, \lambda_n^{-1}K) + \sum_{k=1}^{d-2} V_k(\Xi_0^{(n)}, \lambda_n^{-1}K) + V_0(\Xi_0^{(n)})\mu_d(\lambda_n^{-1}K)\right]\right\}$$

$$= \lim_{n\to\infty} \exp\left\{-\mathbf{E}V_{d-1}(\Xi_0^{(n)}, K)\right\}$$

due to the homogeneity property of the functionals V_k. Now the statement follows from the representation (3.5). □

Remark. It is well-known (see, e.g., [22]) that

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}(\mu_d(L \oplus \varepsilon K) - \mu_d(L)) = \int_{\mathbf{S}^{d-1}} h(K,v) S_{d-1}(L; dv)$$

for convex K and L with $o \in K$. For convex grains the proof of Theorem 4.1 can be derived from this fact (it is no longer valid for general non-convex sets K and L).

The comparison of the formula (4.2) for the containment functional of $\tilde{Y}$ with the representation of the distribution function of the *Poisson polyhedron* (see [23, Section 6.2]) shows that $\tilde{Y}$ is the Poison polyhedron generated by the network of hyperplanes driven by the measure $\nu(\cdot)$. Namely, $\tilde{Y}$ is the open polyhedron bounded by the planes from the network and containing the origin. This network is determined by the Poisson point process on the space $\mathbf{S}^{d-1} \times [0,\infty)$ with the intensity measure $\nu(\cdot) \times \mu_1$. Each point of this point process determines the corresponding hyperplane from the network in such a way that the first coordinate gives the normal vector to the plane, while the second one is the distance between the plane and the origin.

Simple properties of $\tilde{Y}$ and the corresponding network of hyperplanes follow directly from those established in [23] for Poisson polyhedrons and networks of hyperplanes. For example, if the measure $\nu(\cdot)$ is symmetric, then the corresponding network is stationary. If ν is rotation-invariant and, thereupon, is proportional to the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}_{d-1}$ (surface area measure) on $\mathbf{S}^{d-1}$, then $\tilde{Y}$ is an isotropic random polyhedron. Furthermore, the set $\tilde{Y}$ is a.s. bounded if the linear hull of the support of the measure ν coincides with the whole space.

The measure $\nu(\cdot)$ determines the central symmetric convex set Π_ν called the *associated zonoid* (or Steiner compact) of the corresponding network, see [10, 23, 35, 46]. The set Π_ν has the support function given by

$$h(\Pi_\nu, u) = \frac{1}{2} \int_{\mathbf{S}^{d-1}} |\langle u, v\rangle| \nu(dv),$$

whence

$$\tilde{I}(\{u\}) = \exp\{-2h(\Pi_\nu, u)\}\,.$$

Clearly, Π_ν is a ball if ν is rotation invariant.

The measure ν is obtained as the weak limit of the surface measures of the grains. On the other hand, the mean surface measure of the grain $\Xi_0^{(n)}$ coincides with the surface measure of the so-called Blaschke mean of $\Xi_0^{(n)}$, see [47, 49]. This Blaschke mean, $\mathbf{E}_{\mathrm{B}}\Xi_0^{(n)}$, is the convex set having the surface measure equal to $\mathbf{E}S_{d-1}(\Xi_0^{(n)};\cdot)$. Since the surface measures are continuous with respect to the convergence of convex sets in the Hausdorff metric [35, Theorem 4.2.1], the associated zonoid is the *projection body* [10] of the weak limit of the sequence of Blaschke means of the grains, i.e.,

$$\Pi_\nu = \lim_{n\to\infty} \Pi_{d-1}(\mathbf{E}_{\mathrm{B}}(\Xi_0^{(n)}))\,.$$

The projection body $\Pi_{d-1}(K)$ of a convex set K has the support function $h(\Pi_{d-1}(K), u)$ given by the $(d-1)$-dimensional measure of the projection of K onto the linear subspace orthogonal to u, see [35, p.296]. In particular, if $\Xi_0^{(n)} = \Xi_0$, then $\Pi_\nu = \Pi_{d-1}(\mathbf{E}_{\mathrm{B}}\Xi_0)$.

If the measure ν is proportional to the surface area measure, then the corresponding Poisson polyhedron is isotropic and the limiting functional in Theorem 4.1 is

$$\tilde{I}(K) = \exp\left\{-\frac{\lambda}{\omega_{d-1}} \int_{\mathbf{S}^{d-1}} |h(K,u)|\mathcal{H}_{d-1}(du)\right\},$$

where

$$\lambda = \lim_{n\to\infty} \mathbf{E}\mathcal{H}_{d-1}(\Xi_0^{(n)})\,, \tag{4.4}$$

is the intensity of the corresponding isotropic network of hyperplanes and ω_{d-1} is the surface area of $\mathbf{S}^{d-1}$. Note that

$$|h(K,u)| = h(K_0,u)\,,$$

where $K_0 = \mathrm{conv}(K \cup \{o\})$ is the convex hull of K and the origin. Since the mean value of the support function $h(K_0,u)$ over the unit sphere is the half of the mean width $\bar{b}(K_0)$ of K_0, we get

$$\tilde{I}(K) = \exp\{-\lambda\bar{b}(K_0)/2\} = \exp\{-\lambda V_1(K_0)\}\,.$$

Some mean values of the set $\tilde{Y}$ are well-known in the isotropic case [23, 26, 27, 42]. For the general case, it was shown in [51] that the expected volume of the set $\tilde{Y}$ (anisotropic Poisson polyhedron) is given by

$$\mathbf{E}\mu_d(\tilde{Y}) = d!2^{-d}\mu_d(\Pi_\nu^*)\,,$$

where

$$\Pi_\nu^* = \left\{ x \in \mathbf{R}^d \colon \langle x, y \rangle \le 1 \text{ for all } y \in \Pi_\nu \right\}$$

is the polar reciprocal set to Π_ν. Further results and inequalities can be found in [51].

The formula (4.4) for the intensity of the Poisson network was obtained in [11, 14] under the assumption $\Xi_0^{(n)} = \Xi_0$ for special isotropic random compact sets Ξ_0 (balls, polygons, parallelepipeds). Now it is clear that this formula has quite general nature and can be derived without referring to the geometrical shape of the grain (cf. [11, 14]). Clearly, the isotropy of the grain yields $\nu(\cdot) = \lambda \mathcal{H}_{d-1}(\cdot)/\omega_{d-1}$ (the rotation-invariance of ν), while the reverse statement is not true. Two disjoint deterministic balls provide a simple example of a set which have rotation invariant surface measure, but is, evidently, not isotropic itself.

The limit of $\lambda_n Y_n$ is the polyhedron containing the origin and bounded by the corresponding Poisson network of hyperplanes. This corresponds to the so-called *volume* law of the Poisson polyhedron (see the discussion in [23, p.168]). It is also possible to consider all polyhedrons formed by the network to be realizations of a certain random polyhedron, which obeys the *number* law. This situation is actually a multidimensional variant of the length-biased sampling. Hall [11] considered also an arbitrary vacancy formed by the mosaic processes. In the limit it yields the number law of the Poisson polyhedron, while the weak limit of the vacancy containing the origin gives the volume law. The same holds also in the framework of Theorem 4.1.

Note that $\lambda_n Y_n$ is the conditional vacancy of the Boolean model with the grain $\lambda_n \Xi_0^{(n)}$ and the intensity $\lambda_n^{1-1/d}$. Thus, Theorem 4.1 can be reformulated for the non-normalized conditional vacancy (i.e., Y_n instead of $\lambda_n Y_n$) if $\lambda_n \to \infty$ and the mean surface measure of $\lambda_n^{-d/(d-1)} \Xi_0^{(n)}$ has the weak limit ν.

Theorem 4.1 suggests using limit approximations to make inferences about mean bodies related to anisotropic high intensity Boolean models. To perform the estimation procedure the grid of points and the Boolean model Ξ must be superimposed. Let $P_1, \ldots, P_n$ be the connected components of $\mathsf{c}(\Xi)$ containing the points $x_1, \ldots, x_n$ of the grid. The measure ν can be estimated up to a constant from the integral equation

$$\hat{I}(\{u\}) = \exp\left\{ - \int_{\mathbf{S}^{d-1}} |\langle u, v \rangle| \nu(du) \right\}, \tag{4.5}$$

relating the empirical containment functional

$$\hat{I}(\{u\}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{x_i + u \in P_i}$$

to the mean surface measure of the grain. Note that the estimator $\hat{I}(\{u\})$ of the functional $I(\{u\})$ is uniformly strong consistent on the class of singletons, see [28].

This method can be used if other methods fail (for example if the intensity is very high and the volume fraction is large). However, it is not easy to solve the integral equation (4.5) directly, see [10, Section 8].

5. Corollaries and Earlier Results

In this section we will check the conditions of Theorem 4.1 for some special grains and compare our results with the earlier ones. Let us suppose that $\Xi_0^{(n)} = \Xi_0$ for all $n \geq 1$. The following result follows from Theorem 4.1.

Theorem 5.1. *If the grain Ξ_0 is convex, then the random open set $\lambda_n Y_n$ (the normalized conditional vacancy) converges weakly to the random open set $\tilde{Y}$ with the containment functional given by*

$$\tilde{I}(K) = \exp\left\{ - \int_{\mathbf{S}^{d-1}} h(K,u)\mathbf{E}S_{d-1}(\Xi_0; du) \right\}, \quad o \in K. \tag{5.1}$$

PROOF. The condition (3.7) is, evidently, valid. It is sufficient to check (4.1) for $K = \{x_1, \ldots, x_m\}$. First, note that, for the convex set F and sufficiently small ε,

$$\begin{aligned} &\mu_d(F \oplus \varepsilon \mathrm{conv}(K)) - \mu_d(F \oplus K) \leq \\ &\qquad \leq \sum_i \left[\mu_d(F \oplus \varepsilon \mathrm{conv}(K_i)) - \mu_d(F \oplus K_i)\right], \end{aligned}$$

where the sum stretches over all $(d-1)$-dimensional faces K_i of $\mathrm{conv}(K)$.

Thus, it is possible to assume (4.1) for sets K lying in $(d-1)$-dimensional subspaces of $\mathbf{R}^d$. If $d = 2$, then it is sufficient to check this for all two-point sets $K = \{o, x\}$, so that

$$\begin{aligned} &\Delta_n(\{o,x\}) = \\ &\quad = \mathbf{E}\left[\lambda_n \left(\mu_d(\Xi_0^{(n)} \oplus [o, \lambda_n^{-1}x]) - \mu_d(\Xi_0^{(n)} \oplus \{o, \lambda_n^{-1}x\})\right)\right] \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{5.2}$$

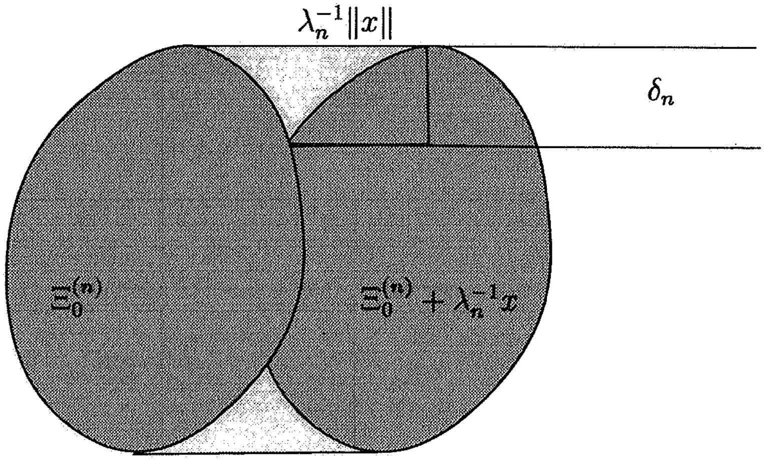
This is valid if the set $\Xi_0^{(n)} = \Xi_0$ is convex and does not depend on n. Indeed,

$$\mu_d(\Xi_0^{(n)} \oplus [o, \lambda_n^{-1}x]) - \mu_d(\Xi_0^{(n)} \oplus \{o, \lambda_n^{-1}x\}) \leq 2\lambda_n^{-1}\|x\|\delta_n \tag{5.3}$$

with $\mathbf{E}\delta_n \to 0$ as $n \to \infty$. The latter follows from the monotone convergence theorem, since $\mathbf{E}\|\Xi_0\| < \infty$. The same result holds also in higher dimensions provided the grain $\Xi_0^{(n)} = \Xi_0$ is convex. □

Earlier results for the grain being ball or parallelepiped were considered in [11, 14]. If Ξ_0 is a ball of radius ξ, then all conditions of Theorem 5.1 are valid so that the

Figure 5.1: Geometric meaning of the inequality (5.3).



limiting random set $\tilde{Y}$ is the Poisson polyhedron generated by the isotropic Poisson network of hyperplanes of intensity $\mathbf{E}\mathcal{H}_{d-1}(\Xi_0) = \omega_{d-1}\mathbf{E}\xi^{d-1}$. This corresponds to Theorem 1 of [11].

If the grain is a random parallelepiped with the facets parallel to coordinate axes, then Theorem 5.1 is applicable with the measure $\mathbf{E}S_{d-1}(\Xi_0; du)$ concentrated at the points $\{\pm e_1, \dots, \pm e_d\}$ (the basis in $\mathbf{R}^d$ containing unit vectors orthogonal to the facets of the grain). Furthermore, $\mathbf{E}S_{d-1}(\Xi_0; \pm e_i)$ is equal to the surface area of the facet orthogonal to $\pm e_i$. The corresponding Poisson network contains only hyperplanes parallel to coordinate planes. The intensity of each network containing such parallel hyperplanes is equal to $\mathbf{E}S_{d-1}(\Xi_0; e_i)$. The Blaschke expectation of the grain is again the parallelepiped with the same orientations of the facets with the surface areas equal to the mean surface areas of the facets of Ξ_0.

We have already seen that for convex random sets the conditions of Theorem 4.1 are evidently satisfied. Let us check these conditions for one important and quite general case of non-convex grains. Suppose that $\Xi_0^{(n)} = \Xi_0$ does not depend on n, and

$$\Xi_0 = A_1 \cup \cdots \cup A_N \tag{5.4}$$

for random convex sets $A_1, \dots, A_N$ and a positive integer random variable N. These sets and N can be dependent.

Theorem 5.2. *Suppose that Ξ_0 has the form (5.4) with N fixed. Then the conditions (3.7) and (4.1) are valid.*

PROOF. First, check the condition (4.1). It is easy to see that, for all fixed N,

$$\Delta_n(K) \le \sum_{i=1}^{N} \lambda_n \mathbf{E}\left[\mu_d(A_i \oplus \lambda_n^{-1}\mathrm{conv}(K)) - \mu_d(A_i \oplus \lambda_n^{-1}K)\right] .$$

Each of the terms in the sum tends to zero as $n \to \infty$ by the same arguments used in the proof of Theorem 5.1.

Let us check the condition (3.7). The following relies essentially on a result of Rataj [33] who proved that (3.7) is valid for each fixed set from the convex ring. Thus, (3.7) is valid for almost all realizations of Ξ_0. To show the convergence of expectations we bound the left-hand side of (3.7). Without loss of generality suppose that $K \subset B$. Then

$$\lambda_n^{-1}\left|\int_{\mathbf{R}^d} \chi(\Xi_0 \cap (\lambda_n^{-1}K + x))dx - \mu_d(\Xi_0 \oplus \lambda_n^{-1}\check{K})\right| \le$$

$$\le \lambda_n^{-1}\left|\int_{\mathbf{R}^d} \left(\chi(\Xi_0 \cap (\lambda_n^{-1}K + x))dx - \mathbf{1}_{\Xi_0 \cap (\lambda_n^{-1}K+x)}\right) dx\right|$$

$$\le N\lambda_n^{-1}\mu_d((\partial\Xi_0)^t)$$

$$\le N\lambda_n^{-1}2\left[\sum_{i=1}^{N} \mu_d(A_i^t \setminus A_i)\right] .$$

Since all means of the intrinsic volumes of A_i are finite, (3.7) follows from the the bounded convergence theorem. □

If N is fixed or bounded a.s. from above, then the statement of Theorem 4.1 is valid. In this case the grain Ξ_0 can be represented as a union of no more than finitely many convex components. The result is especially simple if the grain is fixed. Namely, each fixed grain from the convex ring satisfies the conditions of Theorem 4.1. Then $\mathbf{E}_B\Xi_0$ is the so-called convexification of the set Ξ_0. Thus, the associated zonoid of the limiting Poisson network is the projection body of the convexification of Ξ_0.

Hall [11] noticed that the same result is valid for the unions of a random number of isotropic random polygons. We shall see that the same argumentation is appropriate for the union of a random number of general convex set (5.4).

Theorem 5.3. *Suppose that Ξ_0 has the form (5.4) and*

$$\sup_{0<t<1} t^{-1}\mathbf{E}\left[\mu_d(\Xi_0^t \setminus \Xi_0)\mathbf{1}_{N>m}\right] \to 0 \quad as \quad m \to \infty\,. \tag{5.5}$$

Then the normalized conditional vacancy converges weakly to the random open set with the containment functional given by (4.2) for the measure ν equal to the mean surface measure of Ξ_0.

Remark. The condition (5.5) follows from

$$\sup_{0<t<1} t^{-1}\mathbf{E}\left[\sum_{i=m}^{N} \mu_d(A_i^t \setminus A_i)\right] \to 0 \quad \text{as} \quad m \to \infty\,.$$

Furthermore, it is possible to replace 1 in (5.5) by an arbitrary $\varepsilon > 0$.

Remark. A similar condition to (5.5) was imposed in [11] for the case of random polygons defined as follows. Let $P_1, P_2, \dots$ be a sequence of non-random sets obtained as the unions of finite numbers of polyhedrons. Then the random set $X = P_N$ for a random number N is said to be a *random polygon.* Hall [11] considered its isotropized variant (after applying a random rotation) to be the grain of the Boolean model. Now it is easy to see that the condition

$$\sup_{0<t<1} t^{-1}\mathbf{E}\left[\mu_d(P_N^t \setminus P_N)\mathbf{1}_{N>m}\right] \to 0 \quad \text{as} \quad m \to \infty \tag{5.6}$$

imposed in [11] is equivalent to (5.5) for random polyhedral grains.

PROOF. The argumentation in [11] remains the same, since *Step (i)* of the proof of Theorem 1 in [11] does not use the polygonal assumption. We give below a similar proof using above introduced notations.

Let us write $q(\Xi_0) = m$ if $\Xi_0 = A_1 \cup \dots \cup A_m$ for some $m \geq 1$. Furthermore, $\Xi = \Xi'(m) \cup \Xi''(m)$, where

$$\begin{aligned}
\Xi'(m) &= \bigcup_{x_i \in \Psi_\lambda, q(\Xi_i) \leq m} (x_i + \Xi_i)\,, \\
\Xi''(m) &= \bigcup_{x_i \in \Psi_\lambda, q(\Xi_i) > m} (x_i + \Xi_i)
\end{aligned}$$

are two independent sets. Then

$$\begin{aligned}
p(\lambda, m) &= \mathbf{P}\left\{\Xi''(m) \cap B_{r/\lambda} = \emptyset | o \notin \Xi\right\} \\
&= \mathbf{P}\left\{o \notin \Xi'(m)\right\} \mathbf{P}\left\{\Xi''(m) \cap B_{r/\lambda} = \emptyset\right\} / \mathbf{P}\left\{o \notin \Xi\right\} \\
&= \frac{\exp\left\{-\lambda \sum_{i=1}^m \mathbf{E}\left[\mu_d(\Xi_0)\mathbf{1}_{N=i}\right]\right\} \exp\left\{-\lambda \sum_{i=m+1}^\infty \mathbf{E}\left[\mu_d(\Xi_0^{r/\lambda})\mathbf{1}_{N=i}\right]\right\}}{\exp\left\{-\lambda \sum_{i=1}^\infty \mathbf{E}\left[\mu_d(\Xi_0)\mathbf{1}_{N=i}\right]\right\}} \\
&= \exp\left\{-\lambda \sum_{i=1}^\infty \mathbf{E}\left[\mathbf{E}\mu_d(\Xi_0^{r/\lambda} \setminus \Xi_0)\mathbf{1}_{N=i}\right]\right\} \\
&= \exp\left\{-\lambda \mathbf{E}\left[\mu_d(\Xi_0^{r/\lambda} \setminus \Xi_0)\mathbf{1}_{N>m}\right]\right\}.
\end{aligned}$$

In view of (5.5), given $\varepsilon > 0$ we may choose m_0 so large that

$$\mathbf{E}\left[\mu_d(\Xi_0^t \setminus \Xi_0)\mathbf{1}_{N>m_0}\right] < \varepsilon r^{-1} t$$

for all $0 < t < 1$. Therefore,

$$p(\lambda, m) \geq e^{-\varepsilon} > 1 - \varepsilon$$

for sufficiently large λ. This result means that, in fact, the elementary vacancy is bounded by grains with $N \leq m$ with probability going to 1 as $m \to \infty$. Thus, the set

$\Xi''(m)$ (for N greater than m) adds nothing in the limit as $m \to \infty$, while the result for $N \leq m$ follows from Theorem 5.2. □

If the grain depends on n, then the conditions above must be uniform with respect to n. In particular, all results remain the same for the case $\Xi_0^{(n)} = r_n \Xi_0$ for $r_n \to r_0$ as $n \to \infty$ and Ξ_0 satisfying the above conditions.

6. Concluding Remarks

Let us enlist several open problem related to the contents of this paper.

1. It has been shown that the associated zonoid of the network is related to the $(d-1)$-projection body of the Blaschke mean of the grain. It would be interesting to find a normalization such that the Steiner compact of the limiting set is the jth projection body of Blaschke mean with $j < d - 1$.

2. The concept of the Boolean model can be easily reformulated for the maxima of random functions, see [39]. In this case the properties of cones determined by these functions are of interest. The corresponding technique should involve the variations of the translative integral formula which must give a possibility to compute quantities like

$$\mu_d((K_1 \oplus L_1) \cap (K_2 \oplus L_2)) .$$

3. The translative integral geometric formula is very naturally applied to study the vacancy of the Boolean model. It is interesting to find interpretations in the language of vacancies of the iterated translative integral formula developed in [48].

4. It is intuitively clear (see the comments in [13]) that the limiting results will look similar for the coverage processes on spheres. However, on more general spaces of non-constant curvatures other interesting effects can appear.

5. The condition (3.7) appear as well in the study of *global limit* theorems for the Boolean model. They are of interest if the intensity of the model tends to zero. In this case it is possible to get Poisson networks as limits of Boolean models with low intensities and not very large grains (e.g., for grains being very long and thin rectangles). Although this topic lays beyond the framework of this paper, the tightness condition (3.7) is still very natural to impose. Note again that the condition (3.7) is different from the tightness condition in [16], since the main topic in [16] was to consider the convergence of germ-grain models to the limit which is also a germ-grain model, while here the limits are no longer germ-grain models.

6. If the germ process is not Poisson, then the capacity functional of the Boolean model can be expressed through the probability generating functional $G_\Psi[\cdot]$ of the germ process [8],

$$T_\Xi(K) = 1 - G_\Psi[1 - T_{\Xi_0}(K - (\cdot))] ,$$

see [16]. Then the capacity functional of the normalized conditional vacancy $\lambda_n Y_n$ is given by

$$I_n(K) = G_n[1 - T_{\Xi_0^{(n)}}(\lambda_n^{-1}K - (\cdot))]/G_n[1 - T_{\Xi_0^{(n)}}(\{o\} - (\cdot))]\,,$$

where $G_n[\cdot]$ is the probability generating functional of the point process Ψ_{λ_n}. Thus, the limit of the conditional vacancy can be obtained if some differentiability properties on the probability generating functional are imposed.

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