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M. de Rijke

Computer Science/Department of Software Technology

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P.O. Box 94079, 1090 GB Amsterdam (NL)
Kruislaan 413, 1098 SJ Amsterdam (NL)
Telephone +31 20 592 9333
Telefax +31 20 592 4199

A Lindström Theorem for Modal Logic

Maarten de Rijke
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

Email: `mdr@cwi.nl`

Abstract

A modal analogue of Lindström's characterization of first-order logic is proved. Basic modal logics are characterized as the only modal logics that have a notion of finite rank, or, equivalently, as the strongest modal logic whose formulas are preserved under ultraproducts over ω.

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1. Introduction

In the semantics of concurrent programs modal logics are used to give logical descriptions of bisimulations and other notions of process equivalence (see [8]). Hence, from a computational point of view it is important to gain a thorough understanding of the relation between modal logic and bisimulations. Independently, the connection between equivalence relations on classes of models and notions of logical equivalence is an important topic in model theory (see [1], especially Chapter XIX).

Van Benthem [2] characterizes modal formulas as the fragment of first-order formulas whose truth is preserved by bisimulations between models. And De Rijke [12] shows that two models are modally equivalent iff they have bisimilar ultrapowers. The present paper adds a further characterization result to this list: it uses bisimulations to prove a modal analogue of Lindström's [10] well-known characterization of first-order logic.

Lindström's result states that, given a suitable explication of a 'classical logic', first-order logic is the strongest logic to possess the Compactness and Löwenheim-Skolem properties. To prove an analogous characterization result for modal logic we need to agree on a number of things. We need to determine the logic we want to characterize; to this end we define *basic modal logic* in §2 below. Next, we have to agree on a notion of abstract modal logic; to this end we introduce bisimulations in §3. Then, in §4, we isolate the property distinguishing basic modal logic: the property of *having a notion of finite rank*. In §5 a notion of abstract modal logic is defined; in this definition bisimulations play an essential role. We then prove that basic modal logic is the only modal logic that has a notion of finite rank in §6; we show that this property

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CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

is equivalent to preservation under ultraproducts over ω. We conclude with some comments and questions.

2. Basic modal logic

When interpreted on models (as opposed to frames) modal formulas live inside a fragment of first-order logic. So to specify a modal language we need some notation from first-order logic. We use τ, τ_1, ... to denote (relational) vocabularies of classical languages; and for τ a classical vocabulary, a τ-structure is a tuple of the form $\mathfrak{A} = (A, R, \ldots)$, where A is a non-empty domain, and the Rs interpret all the relation symbols in τ; $\mathrm{Str}[\tau]$ denotes the class of τ-structures. We write $R^{\mathfrak{A}}$ to denote the interpretation of R in the model $\mathfrak{A}$.

Definition 2.1 (Languages) For τ a classical vocabulary with unary predicate symbols, the *finitary basic modal language over* τ is the modal language denoted $\mathcal{BML}(\tau)$ having proposition letters p_0, p_1, ... corresponding to the unary predicate symbols in τ, and also having n-ary modal operators $\#$ with patterns specifying their truth-conditions:

$$\delta_{\#} = \lambda x.\, \exists x_1 \ldots \exists x_n \left(Rxx_1 \ldots x_n \wedge P_1(x_1) \wedge \ldots \wedge P_n(x_n)\right),$$

for every $(n+1)$-ary relation symbol R in τ. In addition $\mathcal{BML}(\tau)$ has the usual Boolean connectives, and constants $\bot$ and $\top$.

The *standard modal language* is $\mathcal{BML}(\tau)$ where τ only contains a single binary predicate R (in addition to a collection of unary predicates); in the standard modal language we write $\Diamond$ ('diamond') rather than $\#$ for the modal operator.

Definition 2.2 (Models) We interpret basic modal languages on τ-structures of the form $(W, R_1, R_2, \ldots, P_1, P_2, \ldots)$, where P_1, P_2, ... interpret the proposition letters of the modal language. As usual we will let *valuations* V take care of proposition letters; thus we will write $(W, R_1, R_2, \ldots, V)$, where $V(p_i) = P_i$. Then, the relation $\mathfrak{A}, a \models \phi$ is defined as follows:

$$\begin{array}{rcl}
\mathfrak{A}, a \models p & \text{iff} & a \in V(p) \\
\mathfrak{A}, a \models \bot & \text{iff} & a \neq a \\
\mathfrak{A}, a \models \top & \text{iff} & a = a \\
\mathfrak{A}, a \models \neg\phi & \text{iff} & \mathfrak{A}, a \not\models \phi \\
\mathfrak{A}, a \models \phi \wedge \psi & \text{iff} & \mathfrak{A}, a \models \phi \text{ and } \mathfrak{A}, a \models \psi \\
\mathfrak{A}, a \models \#(\phi_1, \ldots, \phi_n) & \text{iff} & \exists b_1 \ldots b_n \,(R_{\#}ab_1 \ldots b_n \wedge \bigwedge_i(\mathfrak{A}, b_i \models \phi_i)).
\end{array}$$

As an aside, using the above truth definition, a translation ST can be defined that takes modal formulas to formulas in the classical language in which those patterns live. The translation ST maps proposition letters onto unary predicates, it commutes with the Booleans, and to translate modal operators it uses their patterns. The result is that for all modal formulas ϕ: $(W, R_1, R_2, \ldots, V), a \models \phi$ iff $(W, R_1, R_2, \ldots, V) \models ST(\phi)[a]$ (see [2] for more details).

Convention 2.3 (Pointed models) Throughout this paper models for modal languages are always *pointed models* of the form $(\mathfrak{A}, a)$, where $\mathfrak{A}$ is a relational structure and a is an element of $\mathfrak{A}$ (its *distinguished point*) at which evaluation takes place.

Our main reasons for adopting this convention are the following. First, the basic semantic unit in modal logic simply *is* a structure together with a distinguished node at which evaluation takes place. Second, some of the results below admit smoother formulations when we adopt the *local* perspective of pointed models. Of course, this local perspective dates back (at least) to Kripke's original publication [9]. The usual *global* perspective ('$\mathfrak{A} \models \phi$ iff for all a in $\mathfrak{A}$: $\mathfrak{A}, a \models \phi$') is obviously definable using the local point of view.

3. Bisimulations

In this section we define bisimulations. One of the defining properties of an abstract classical logic is the Isomorphism property which states that it is impossible to distinguish isomorphic structures by means of formulas from the abstract logic. In abstract modal logic this property is replaced by a Bisimilarity property which states that bisimilar structures are be indistinguishable by modal means. In addition to this, bisimulations will play an important role below as a technical tool.

Definition 3.1 (Bisimulations) For τ a classical vocabulary and $\mathfrak{A}$, $\mathfrak{B} \in \mathrm{Str}[\tau]$, we say that $(\mathfrak{A}, a)$ and $(\mathfrak{B}, b)$ are *τ-bisimilar*, $(\mathfrak{A}, a) \leftrightarroweq_\tau (\mathfrak{B}, b)$, if there exists a non-empty relation Z between the elements of $\mathfrak{A}$ and $\mathfrak{B}$ (called a *τ-bisimulation*, and written $Z : (\mathfrak{A}, a) \leftrightarroweq_\tau (\mathfrak{B}, b)$) such that the following hold.

1. Z links the distinguished points of $(\mathfrak{A}, a)$ and $(\mathfrak{B}, b)$: Zab.
2. For all unary predicate symbols P in τ and a_0 in $\mathfrak{A}$ and b_0 in $\mathfrak{B}$, Za_0b_0 implies $a_0 \in P^{\mathfrak{A}}$ iff $b_0 \in P^{\mathfrak{B}}$.
3. If Za_0b_0, $a_1, \ldots, a_n \in \mathfrak{A}$ and $(a_0, a_1, \ldots, a_n) \in R^{\mathfrak{A}}$, then there are $b_1, \ldots, b_n \in \mathfrak{B}$ such that $(b_0, b_1, \ldots, b_n) \in R^{\mathfrak{B}}$ and Za_ib_i, where $1 \leq i \leq n$ and R is an $(n+1)$-ary relation symbol in τ (*forth condition*).
4. If Za_0b_0, $b_1, \ldots, b_n \in \mathfrak{B}$ and $(b_0, b_1, \ldots, b_n) \in R^{\mathfrak{B}}$, then there are $a_1, \ldots, a_n \in \mathfrak{A}$ such that $(a_0, a_1, \ldots, a_n) \in R^{\mathfrak{A}}$ and Za_ib_i, where $1 \leq i \leq n$ and R is an $(n+1)$-ary relation symbol in τ (*back condition*).

Many familiar constructions on relational structures arise as special examples of bisimulations: isomorphisms, disjoint unions, p-morphism, and generated submodels. For the first three the reader is referred to Goldblatt [7] for definitions; as we will need generated submodels in the sequel, we will define that construction here. $(\mathfrak{A}, a)$ is a *generated submodel* of $(\mathfrak{B}, b)$ whenever (i) $a = b$, (ii) the domain of $\mathfrak{A}$ is a subset of the domain of $\mathfrak{B}$, (iii) $R^{\mathfrak{A}}$ is simply the restriction of $R^{\mathfrak{B}}$ to $\mathfrak{A}$, and (iv) if $a_0 \in \mathfrak{A}$ and $R^{\mathfrak{B}}a_0b_1 \ldots b_n$, then $b_1, \ldots, b_n$ are in $\mathfrak{A}$. If X is a subset of the domain of $\mathfrak{A}$, then the *submodel generated by* X is the smallest generated submodel of $\mathfrak{A}$ whose domain includes X; if X is a singleton $\{a\}$ we simply refer to the *submodel generated by* a rather than $\{a\}$. If $(\mathfrak{A}, a)$ is a generated submodel of $(\mathfrak{B}, b)$, there is a τ-bisimulation $Z : (\mathfrak{A}, a) \leftrightarroweq_\tau (\mathfrak{B}, b)$ defined by Zxy iff $x = y$.

Definition 3.2 (In-degree) For $\mathfrak{A}$ a model, c in $\mathfrak{A}$, the *in-degree* of c is

$$\Big|\{\vec{a} \in \mathfrak{A}^{<\omega} \mid \text{ for some } R \in \tau \text{ and } i > 1,\ c = a_i \text{ and } R^{\mathfrak{A}} a_1 \dots a_i \dots a_n \ \}\Big|.$$

Thus, the in-degree of c is the number of times it occurs as an argument in a relation: $Rx \dots c \dots$.

Definition 3.3 (Depth) The second notion we need measures the distance from a given element in a model to its distinguished point. Let $(\mathfrak{A}, a)$ be a τ-structure; the *τ-hulls H^n_τ* around a are defined as follows

- $H^0_\tau(\mathfrak{A}, a) = \{a\}$,
- $H^{n+1}_\tau(\mathfrak{A}, a) = H^n_\tau(\mathfrak{A}, a) \cup \{b \text{ in } \mathfrak{A} \mid \text{ for some } R \in \tau,\ u \in H^n_\tau(\mathfrak{A}, a) \text{ and } v_1, \dots, v_n$ in $\mathfrak{A}$: b is one of the v_i and $R^{\mathfrak{A}} u v_1 \dots v_n\}$.

So, the τ-hull H^n_τ around a contains all elements in $\mathfrak{A}$ that can be reached from a in at most n relational steps.

For c in $(\mathfrak{A}, a)$, the *depth of c in* $(\mathfrak{A}, a)$ is the smallest n such that $c \in H^n_\tau(\mathfrak{A}, a)$.

For $n \in \omega$, the model $(\mathfrak{A} \restriction n, a)$ is the *restriction of* $(\mathfrak{A}, a)$ *to points of depth* n; it is defined as the submodel of $(\mathfrak{A}, a)$ whose domain is $H^n(\mathfrak{A}, a)$.

Below we will want to get models that have nice properties, such as a low in-degree or finite depth for each of its elements. To obtain such models the notion of forcing comes in handy. Fix a vocabulary τ. A property P of models is *$\leftrightarroweq_\tau$-enforceable*, or *enforceable*, iff for every $(\mathfrak{A}, a) \in \mathrm{Str}[\tau]$, there is a $(\mathfrak{B}, b) \in \mathrm{Str}[\tau]$ with $(\mathfrak{A}, a) \leftrightarroweq_\tau (\mathfrak{B}, b)$ and $(\mathfrak{B}, b)$ has P.

Proposition 3.4 *The property "every element has finite depth" is enforceable.*

Proof. Let $\mathfrak{A} \in \mathrm{Str}[\tau]$. Let $(\mathfrak{B}, a)$ be the submodel of $\mathfrak{A}$ whose domain is $\bigcup_n H^n_\tau(\mathfrak{A}, a)$. In $(\mathfrak{B}, a)$ every element has finite depth. Moreover, $(\mathfrak{A}, a) \leftrightarroweq_\tau (\mathfrak{B}, a)$. $\dashv$

Proposition 3.5 below generalizes the *unraveling construction* from standard modal logic with a single diamond $\Diamond$ (Sahlqvist [13]) to arbitrary vocabularies.

Proposition 3.5 *The property "every element has in-degree at most 1" is enforceable.*

Proof. We may assume that $(\mathfrak{A}, a)$ is generated by a. Expand τ to a vocabulary τ^+ that has constants $\bar{c}$ for all elements c in $\mathfrak{A}$. Define a *path conjunction* to be a first-order formula that is a conjunction of closed atomic formulas (over τ^+) taken from the smallest set X such that

(i) $\overline{a} = \overline{a}$ is in X;

(ii) $R\overline{a}\overline{c}_1 \ldots \overline{c}_n$ is in X for any R and $c_1, \ldots, c_n$ such that $R^{\mathfrak{A}}ac_1 \ldots c_n$; and

(iii) if $\alpha \wedge R\overline{c}\overline{c}_1 \ldots \overline{c}_n$ is in X and for some S and i, $S^{\mathfrak{A}}c_i d_1 \ldots d_m$, then the conjunction $\alpha \wedge R\overline{c}\overline{c}_1 \ldots \overline{c}_n \wedge S\overline{c}_i\overline{d}_1 \ldots \overline{d}_m$ is in X.

A path conjunction $\alpha \equiv \alpha' \wedge S\overline{d}\overline{d}_1 \ldots \overline{d}_m$ is admissible for a constant c in $\tau^+ \setminus \tau$ if c is one of the d_i occurring in the last conjunct of α.

Define a model $\mathfrak{B}$ whose domain contains, for every constant $\overline{c}$ in $\tau^+ \setminus \tau$, a copy c_α, for every α that is admissible for $\overline{c}$. Define

$$R^{\mathfrak{B}}\overline{c}_\alpha \overline{c}^1_{\alpha_1} \ldots \overline{c}^n_{\alpha_n} \text{ iff } \alpha_1 \equiv \ldots \equiv \alpha_n \equiv \alpha \wedge R\overline{c}\overline{c}^1 \ldots \overline{c}^n.$$

And define a valuation V' on $\mathfrak{B}$ by putting $c_\alpha \in V'(p)$ iff $c \in V(p)$. Finally, define a relation Z between $\mathfrak{A}$ and $\mathfrak{B}$ by putting Zxy iff $y = x_\alpha$ for some path conjunction α. Then $Z : (\mathfrak{A}, a) \leftrightarrows_\tau (\mathfrak{B}, a_{\overline{a}=\overline{a}})$. $\dashv$

A short historical note to conclude this section: in modal logic bisimulations were introduced by Van Benthem [2] as p-relations. In the computational tradition bisimulations date back to Park [11]. In essence bisimulations are trimmed down versions of the Ehrenfeucht-Fraïssé games found in classical logic (see Barwise and Feferman [1]). Further references, on modal and computational aspects of bisimulations, can be found in Van Benthem and Bergstra [3].

4. Properties of basic modal logic

We will characterize basic modal logic by showing that it is the only modal logic satisfying a modal counterpart of the original Lindström conditions: having a notion of finite rank. First, we need to show that modal formulas are invariant under bisimulations.

For $\mathcal{BML}(\tau)$ a basic modal language over τ, let $(\mathfrak{A}, a) \equiv_{\mathcal{BML}(\tau)} (\mathfrak{B}, b)$ denote that $(\mathfrak{A}, a)$ and $(\mathfrak{B}, b)$ satisfy the same $\mathcal{BML}(\tau)$-formulas.

Proposition 4.1 *Let τ be a classical vocabulary, and let $\mathcal{BML}(\tau)$ be a basic modal language over τ. Then $\leftrightarrows_\tau \subseteq \equiv_{\mathcal{BML}(\tau)}$.*

One of the distinguishing features of basic modal logic is that it has a notion of *finite rank* which gives a fixed upperbound on the depth of the elements that need to be considered to verify a formula.

Definition 4.2 Define the *rank* of a basic modal formula, rank(ϕ), as follows:

$$\begin{aligned}
\mathrm{rank}(p) &= 0\\
\mathrm{rank}(\neg\phi) &= \mathrm{rank}(\phi)\\
\mathrm{rank}(\phi \wedge \psi) &= \max(\{\mathrm{rank}(\phi), \mathrm{rank}(\psi)\})\\
\mathrm{rank}(\#(\phi_1, \ldots, \phi_n)) &= 1 + \max\{\mathrm{rank}(\phi_i) \mid 1 \leq i \leq n\}.
\end{aligned}$$

Proposition 4.3 *Let ϕ be a basic modal formula with* $\mathrm{rank}(\phi) \leq n$*. Then* $(\mathfrak{A}, a) \models \phi$ *iff* $(\mathfrak{A} \restriction n, a) \models \phi$.

We write $(\mathfrak{A}, a) \equiv^n_{\mathcal{BML}(\tau)} (\mathfrak{B}, b)$ for $(\mathfrak{A}, a)$ and $(\mathfrak{B}, b)$ verify the same $\mathcal{BML}(\tau)$-formulas of rank at most n.

Lemma 4.4 *Let τ be a finite vocabulary. Then, modulo logical equivalence, there are only finitely many basic modal formulas with a fixed finite rank.*

Proof. The proof is by a induction on rank. For $n = 0$, there are only finitely many proposition letters. For the induction step, choose a set Σ of modal formulas of rank $\leq n$ such that every every such formula has an equivalent in Σ. Now consider disjunctive normal forms over 'atoms' $\#_1(\phi_{1_1}, \ldots, \phi_{m_1})$, …, $\#_k(\phi_{k_1}, \ldots, \phi_{m_k})$, where all ϕ_j are in Σ and the $\#_1$, …, $\#_k$ are all the modal operators in the finite language. $\dashv$

Proposition 4.5 *Let τ be a finite vocabulary. Let $(\mathfrak{A}, a)$, $(\mathfrak{B}, b)$ be two models such that every element has in-degree at most 1 and depth at most n. If $(\mathfrak{A}, a) \equiv^n_{\mathcal{BML}(\tau)} (\mathfrak{B}, b)$, then $(\mathfrak{A}, a) \leftrightarrows_\tau (\mathfrak{B}, b)$.*

Proof. Define $Z \subseteq A \times B$ by

$$xZy \quad \text{iff} \quad \mathrm{depth}(x) = \mathrm{depth}(y) = m \quad \text{and} \quad (\mathfrak{A}, x) \equiv^{n-m}_{\mathcal{BML}(\tau)} (\mathfrak{B}, y).$$

We claim that $Z : (\mathfrak{A}, a) \leftrightarrows_\tau (\mathfrak{B}, b)$. To prove this, we only show the forth condition. Assume xZy and $R^{\mathfrak{A}}xx_1 \ldots x_k$, where $\mathrm{depth}(x) = \mathrm{depth}(y) = m$. Then $n - m \geq 1$. Let $\#$ be the modal operator whose semantics is based on R.

As τ is finite there are only finitely many non-equivalent formulas of rank at most $n - m - 1$. Let ψ_i be the conjunction of all non-equivalent basic modal formulas of rank at most $n - m - 1$ that are true at x_i $(1 \leq i \leq k)$. Then $(\mathfrak{A}, x) \models \#(\psi_1, \ldots, \psi_k)$ and $\#(\psi_1, \ldots, \psi_k)$ has rank $n - m$. Hence, as xZy, $(\mathfrak{B}, , y) \models \#(\psi_1, \ldots, \psi_k)$. So there exist y_1, …, y_k in $\mathfrak{B}$ such that $R^{\mathfrak{B}}yy_1 \ldots y_k$ and $(\mathfrak{B}, y_i) \models \psi_i$ $(1 \leq i \leq k)$.

Now, as all states have in-degree at most 1, $\mathrm{depth}(x_i) = \mathrm{depth}(y_i) = m + 1$ and $(\mathfrak{A}, x_i) \equiv^{n-(m+1)}_{\mathcal{BML}(\tau)} (\mathfrak{B}, y_i)$ $(1 \leq i \leq k)$. Hence $(\mathfrak{A}, x_i) \leftrightarrows_\tau (\mathfrak{B}, , y_i)$. This proves the forth condition. $\dashv$

5. Abstract modal logic

Lindström's Theorem starts from a definition of an abstract classical logic as a pair $(\mathcal{L}, \models_{\mathcal{L}})$ consisting of a set of formulas $\mathcal{L}$ and a satisfaction relation $\models_{\mathcal{L}}$ between $\mathcal{L}$-structures and $\mathcal{L}$-formulas that satisfies three 'book keeping' conditions, an Isomorphism property, and a Relativization property which allows one to consider definable submodels (cf. Chang and Keisler [5, Definition 2.5.1]). Then, an abstract logic extending first-order logic coincides with first-order logic iff it satisfies the Compactness and Löwenheim-Skolem properties. We will now set up our modal analogue of Lindström's Theorem along similar lines.

Somewhat analogous to an abstract classical logic an abstract modal logic is characterized by three properties: two book keeping properties, and a Bisimilarity property to replace the Isomorphism property.

Definition 5.1 (Abstract modal logic) An *abstract modal logic* is a pair $(\mathcal{L}, \models_{\mathcal{L}})$ with the following properties; $\mathcal{L}$ is its set of formulas, and $\models_{\mathcal{L}}$ is its satisfaction relation, that is, a relation between (pointed) models and $\mathcal{L}$-formulas.

(i) Occurrence property. For each ϕ in $\mathcal{L}$ there is an associated finite language $\mathcal{L}(\tau_\phi)$. The relation $(\mathfrak{A}, a) \models_{\mathcal{L}} \phi$ is a relation between $\mathcal{L}$-formulas ϕ and structures $(\mathfrak{A}, a)$ for languages $\mathcal{L}$ containing $\mathcal{L}(\tau_\phi)$. That is, if ϕ is in $\mathcal{L}$, and $\mathfrak{A}$ is an $\mathcal{L}$-model, then the statement $(\mathfrak{A}, a) \models_{\mathcal{L}} \phi$ is either true or false if $\mathcal{L}$ contains $\mathcal{L}(\tau_\phi)$, and undefined otherwise.

(ii) Expansion property. The relation $(\mathfrak{A}, a) \models_{\mathcal{L}} \phi$ depends only on the reduct of $\mathfrak{A}$ to $\mathcal{L}(\tau_\phi)$. That is, if $(\mathfrak{A}, a) \models_{\mathcal{L}} \phi$ and $(\mathfrak{B}, a)$ is an expansion of $(\mathfrak{A}, a)$ to a larger language, then $(\mathfrak{B}, b) \models_{\mathcal{L}} \phi$.

(iii) Bisimilarity property. The relation $(\mathfrak{A}, a) \models_{\mathcal{L}} \phi$ is preserved under basic bisimulations: if $(\mathfrak{A}, a) \leftrightarroweq_\tau (\mathfrak{B}, b)$ and $(\mathfrak{A}, a) \models_{\mathcal{L}} \phi$, then $(\mathfrak{B}, b) \models_{\mathcal{L}} \phi$.

A few remarks are in order. First, to define an abstract modal logic we only need two bookkeeping properties (the Occurrence and Expansion properties), whereas three are needed to define an abstract classical logic; in particular we don't need modal counterparts of the Renaming and Relativization properties (Chang and Keisler [5, page 128]).

Comparing the above definition to the list of properties defining an abstract classical logic, we see that it's the Bisimilarity property that determines the *modal* character of an abstract modal logic.

Remark 5.2 The language of standard temporal logic has operators F, with $x \models F\phi$ iff for some y, both Rxy and $y \models \phi$, and P, with $x \models P\phi$ iff for some y, both Ryx and $y \models \phi$. The pattern for F is just a basic modal pattern in the sense of Definition 2.1, but the one for P isn't. As this language 'looks back and forth' along the relation R it violates the Bisimilarity property, hence it is not a abstract modal logic.

Next, we need to say what we mean by '$(\mathcal{L}, \models_{\mathcal{L}})$ extends basic modal logic' and by closure under negation.

Definition 5.3 We say that $(\mathcal{L}, \models_{\mathcal{L}})$ *extends basic modal logic* if for every basic modal formula there exists an equivalent $\mathcal{L}$-formula, that is, if for each basic modal formula ϕ there exists an $\mathcal{L}$-formula ψ such that for any model $(\mathfrak{A}, a)$, $(\mathfrak{A}, a) \models \phi$ iff $(\mathfrak{A}, a) \models_{\mathcal{L}} \psi$.

We say that $(\mathcal{L}, \models_{\mathcal{L}})$ is *closed under negation* if for all $\mathcal{L}$-formulas ϕ there exists an $\mathcal{L}$-formula $\neg\phi$ such that for all models $(\mathfrak{A}, a)$, $(\mathfrak{A}, a) \models \phi$ iff $(\mathfrak{A}, a) \not\models \neg\phi$.

Logics in the sense of Definition 5.1 deal with the same class of pointed models as basic modal logic, and only the formulas and satisfaction relation may be different. This implies, for example, that intuitionistic logic or the nominal modal logic of Blackburn [4], whose repertoire contains special proposition symbols, is not an abstract modal logic: their models need to satisfy special constraints. The original Lindström characterization of first-order logic suffers from similar limitations (by not allowing ω-logic as a logic, for example).

We will use the property of having a finite rank to single out the (finitary) basic modal language $\mathcal{BML}$ among its extensions.

Definition 5.4 An abstract modal logic has a *notion of finite rank* if there is a function $\text{rank}_{\mathcal{L}} : \mathcal{L} \to \mathbb{N}$ such that for all $(\mathfrak{A}, a)$, all ϕ in $\mathcal{L}$,

$$(\mathfrak{A}, a) \models_{\mathcal{L}} \phi \quad \text{iff} \quad \Big((\mathfrak{A}, a) \restriction \{x \in \mathfrak{A} \mid \text{depth}(x) \leq \text{rank}(\phi)\}\Big), a \models_{\mathcal{L}} \phi.$$

Observe that by Proposition 4.3 $\mathcal{BML}$ has a notion of finite rank.

If $\mathcal{L}$ extends basic modal logic, we assume that $\text{rank}_{\mathcal{L}}$ behaves regularly with respect to standard modal operators and proposition letters. That is, for $\#$ a modal operator as defined in §2, $\text{rank}_{\mathcal{L}}(p) = 0$ and $\text{rank}_{\mathcal{L}}(\#(\phi_1, \ldots, \phi_n)) = 1 + \max\{\text{rank}_{\mathcal{L}}(\phi_i) \mid 1 \leq i \leq n\}$.

Two models $(\mathfrak{A}, a)$ and $(\mathfrak{B}, b)$ for the same language are *$\mathcal{L}$-equivalent* if for every ϕ in $\mathcal{L}$, $(\mathfrak{A}, a) \models \phi$ iff $(\mathfrak{B}, b) \models \phi$.

Remark 5.5 Having a finite rank is a very restrictive property, which is not implied by the finite model property (FMP). To see this recall that **PDL** (Propositional Dynamic Logic) has the FMP: it has the property that every satisfiable formula ϕ is satisfiable on a model of size at most $|\phi|^3$, where ϕ is the length of ϕ, cf. [7]. However, it does not have a notion of finite rank. To see this, consider the model (ω, R_a, V), where R_a is the successor relation and V is an arbitrary valuation, and let $\phi = [a^*]\langle a\rangle\top$; clearly $(\omega, R_a, V), 0 \models \phi$. But for no $n \in \omega$ does the restriction $(\omega, R_a, V) \restriction n$ satisfy ϕ at 0. It follows that **PDL** does not have a notion of finite rank.

6. Characterizing modal logic

We are almost ready now to prove our characterization result. The following lemma is instrumental.

Lemma 6.1 *Let $(\mathcal{L}, \models_{\mathcal{L}})$ be an abstract modal logic which which is closed under negation. Assume $\mathcal{L}$ has a notion of finite rank* $\text{rank}_{\mathcal{L}}$. *Let ϕ be an $\mathcal{L}$-formula with* $\text{rank}_{\mathcal{L}}(\phi) = n$. *Then, for any two models $(\mathfrak{A}, a)$, $(\mathfrak{B}, b)$ such that $(\mathfrak{A}, a) \equiv^n_{\mathcal{BML}} (\mathfrak{B}, b)$, $(\mathfrak{A}, a) \models \phi$ implies $(\mathfrak{B}, b) \models \phi$.*

Proof. Assume that the conclusion of the Lemma does not hold. Let $(\mathfrak{A}, a)$, $(\mathfrak{B}, b)$ be such that $(\mathfrak{A}, a) \equiv^n_{\mathcal{BML}} (\mathfrak{B}, b)$ but $(\mathfrak{A}, a) \models \phi$ and $(\mathfrak{B}, b) \models \neg\phi$.

By the Occurrence and Expansion properties we may assume that $\mathcal{L} = \mathcal{L}(\tau_\phi)$, where $\mathcal{L}(\tau_\phi)$ is the finite language in which ϕ lives.

By Proposition 3.5 we can assume that $(\mathfrak{A}, a)$ and $(\mathfrak{B}, b)$ have in-degree at most 1. Then $(\mathfrak{A} \upharpoonright n, a) \equiv^n_{\mathcal{BML}} (\mathfrak{B} \upharpoonright n, b)$, and $(\mathfrak{A} \upharpoonright n, a) \models \phi$ but $(\mathfrak{B} \upharpoonright n, b) \models \neg\phi$. In addition $(\mathfrak{A} \upharpoonright n, a)$ and $(\mathfrak{B} \upharpoonright n, b)$ both have in-degree 1. By Proposition 4.5 it follows that $(\mathfrak{A} \upharpoonright n, a) \leftrightarrows_{\tau_\phi} (\mathfrak{B} \upharpoonright n, b)$ — but now we have a contradiction as $(\mathfrak{A} \upharpoonright n, a)$ and $(\mathfrak{B} \upharpoonright n, b)$ are bisimilar but don't agree on ϕ. $\dashv$

Theorem 6.2 *Let $(\mathcal{L}, \models_\mathcal{L})$ extend basic modal logic. If $(\mathcal{L}, \models_\mathcal{L})$ has a notion of finite rank, then it is equivalent to the basic modal logic $\mathcal{BML}$.*

Proof. We must show that every $\mathcal{L}$-formula ϕ is $\mathcal{L}$-equivalent to a basic modal formula ψ, that is, for all $(\mathfrak{A}, a)$, $(\mathfrak{A}, a) \models_\mathcal{L} \phi$ iff $(\mathfrak{A}, a) \models_\mathcal{L} \psi$. As before, by the Occurrence and Expansion properties we may restrict ourselves to a finite language. Moreover, ϕ has a basic modal equivalent iff it has such an equivalent with the same rank; so we have to locate the equivalent we are after among the basic modal formulas whose rank equals the $\mathcal{L}$-rank of ϕ.

Assume $n = \mathrm{rank}_\mathcal{L}(\phi)$. By Lemma 4.4 there are only finitely many (non-equivalent) basic modal formulas whose rank equals n; assume that they are all contained in Γ_n. It suffices to show the following

$$\text{if } (\mathfrak{A}, a),\ (\mathfrak{B}, b) \text{ agree on all formulas in } \Gamma_n, \text{ then they agree on } \phi. \tag{6.1}$$

For then, ϕ will be equivalent to a Boolean combination of formulas in Γ_n. To see this reason as follows. The relation 'satisfies the same formulas in Γ_n' is an equivalence relation on the class of all models; as Γ_n is finite, there can only be finitely many equivalence classes. Choose representatives $(\mathfrak{A}_1, a_1), \ldots, (\mathfrak{A}_m, a_m)$, and for each i, with $1 \leq i \leq m$, let ψ_i be the conjunction of all formulas in Γ_n that are satisfied by $(\mathfrak{A}_i, a_i)$. Then ϕ is equivalent to $\bigvee\{\psi_i \mid (\mathfrak{A}_i, a_i) \models \phi\}$.

Now to conclude the proof of the theorem we need only observe that (6.1) is exactly the content of Lemma 6.1. $\dashv$

Our next aim is to present a more algebraic version of Theorem 6.2. This alternative formulation is based on an observation due to Ian Hodkinson that having a notion of finite rank is equivalent to being preserved under suitable ultraproducts.

We need two lemmas.

Lemma 6.3 *Let τ be a countable vocabulary. For each $n \in \omega$, let $(\mathfrak{A}_n, a_n)$ and $(\mathfrak{B}_n, b_n)$ be τ-models with $(\mathfrak{A}_n, a_n) \equiv^n_{\mathcal{BML}} (\mathfrak{B}_n, b_n)$. Let U be a non-principal ultrafilter. Then the ultraproducts $(\prod \mathfrak{A}_n/U, \prod a_n/U)$ and $(\prod \mathfrak{B}_n/U, \prod b_n/U)$ are τ-bisimilar.*

Proof. By Łoś' Theorem $(\prod \mathfrak{A}_n/U, \prod a_n/U)$ and $(\prod \mathfrak{B}_n/U, \prod b_n/U)$ agree on all modal formulas. By Chang and Keisler [5, Chapter 6] the two ultraproducts are ω_1-saturated. From these two facts it follows that $(\prod \mathfrak{A}_n/U, \prod a_n/U) \leftrightarrows_\tau (\prod \mathfrak{B}_n/U, \prod b_n/U)$. (See De Rijke [12] for details.) $\dashv$

Lemma 6.4 *Let $(\mathcal{L}, \models_{\mathcal{L}})$ be an abstract modal logic that is closed under negation. The following are equivalent.*

1. *$(\mathcal{L}, \models_{\mathcal{L}})$ has a notion of finite rank.*

2. *$\mathcal{L}$-formulas are preserved under ultraproducts over ω. That is: if ϕ is an $\mathcal{L}$-formula, and for each $i \in \omega$, $(\mathfrak{A}_i, a_i)$ is a model with $(\mathfrak{A}_i, a_i) \models \phi$, then, for any ultrafilter U over ω, $(\prod \mathfrak{A}_i/U, \prod a_i/U) \models \phi$.*

Proof. To prove the implication $2 \Rightarrow 1$, assume that $\mathcal{L}$-formulas are preserved under ultraproducts over ω, but that $\mathcal{L}$ has no notion of finite rank; we will derive a contradiction. As $\mathcal{L}$ does not have a notion of finite rank, there is an $\mathcal{L}$-formula ϕ 'without rank', that is: there exists ϕ such that for all $i \in \omega$ there is a τ_ϕ-model $(\mathfrak{A}_i, a_i)$ with

$$(\mathfrak{A}_i, a_i) \models \phi \text{ iff } (\mathfrak{A}_i \restriction i, a_i) \models \neg\phi.$$

(The restriction to τ_ϕ uses the Expansion and Occurrence properties.)

Now, let U be a non-principal ultrafilter over ω, and consider the ultraproducts $\mathfrak{B} = (\prod \mathfrak{A}_i/U, \prod a_i/U)$ and $\mathfrak{B}' = (\prod(\mathfrak{A}_i \restriction i)/U, \prod a_i/U)$. We want to show that

$$\mathfrak{B} \models \phi \text{ iff } \mathfrak{B}' \models \neg\phi. \tag{6.2}$$

To see this, take an $S \in U$ such that for all $i \in S$ it holds that $(\mathfrak{A}_i, a_i) \models \phi$ but $(\mathfrak{A}_i \restriction i, a_i) \models \neg\phi$. Let $U' = \{X \cap S \mid X \in U\}$. Then

$$\mathfrak{B} \cong (\prod_{i \in S} \mathfrak{A}_i/U', \prod_{i \in S} a_i/U') \models \phi \quad \text{and} \quad \mathfrak{B}' \cong (\prod_{i \in S}(\mathfrak{A}_i \restriction i)/U', \prod_{i \in S} a_i/U') \models \neg\phi,$$

and this establishes (6.2). As $(\mathfrak{A}_i, a_i) \equiv^i_{\mathcal{BML}} (\mathfrak{A}_i \restriction i, a_i)$ for all $i \in \omega$, it follows from Lemma 6.3 that $\mathfrak{B} \leftrightarroweq_{\tau_\phi} \mathfrak{B}'$. But because $\mathfrak{B}$ and $\mathfrak{B}'$ don't agree on ϕ by (6.2), this is a contradiction.

For the implication $1 \Rightarrow 2$, assume $\mathcal{L}$ has a notion of finite rank. Let ϕ be an $\mathcal{L}$-formula. Suppose that for all $i \in \omega$, $(\mathfrak{A}_i, a_i) \models \phi$, and let U be an ultrafilter over ω. We have to show that $(\prod \mathfrak{A}_i/U, \prod a_i/U) \models \phi$. By the Expansion and Occurrence properties we can work in a finite vocabulary τ_ϕ. Assume that $\text{rank}_{\mathcal{L}}(\phi) = n$. Then $(\mathfrak{A}_i \restriction n, a_i) \models \phi$, for all $i \in \omega$. As τ_ϕ is finite, there are finitely many first-order sentences $\alpha_1, \ldots, \alpha_k$ (involving constants to denote the distinguished point of each model) such that any model of depth n satisfies one of $\alpha_1, \ldots, \alpha_k$, and such that for any j $(1 \leq j \leq k)$ any two models of α_j are τ_ϕ-bisimilar. It follows that for some j, with $1 \leq j \leq k$, the set $S = \{i \in \omega \mid (\mathfrak{A}_i \restriction n, a_i) \models \alpha_j\}$ is in U. Therefore $(\prod(\mathfrak{A}_i \restriction n)/U, \prod a_i/U) \models \alpha_j$. And this in turn implies that for any $s \in S$

$$(\prod(\mathfrak{A}_i \restriction n)/U, \prod a_i/U) \leftrightarroweq_{\tau_\phi} (\mathfrak{A}_s \restriction n, s). \tag{6.3}$$

As $(\mathfrak{A}_s \restriction n, s) \models \phi$, (6.3) implies that $(\prod(\mathfrak{A}_i \restriction n)/U, \prod a_i/U) \models \phi$. Finally, as

$$(\prod(\mathfrak{A}_i \restriction n)/U, \prod a_i/U) = ((\prod \mathfrak{A}_i/U) \restriction n, \prod a_i/U),$$

we find that $((\prod \mathfrak{A}_i/U) \upharpoonright n, \prod a_i/U) \models \phi$, and so $(\prod \mathfrak{A}_i/U, \prod a_i/U) \models \phi$. $\dashv$

Theorem 6.5 *Let $(\mathcal{L}, \models_{\mathcal{L}})$ extend basic modal logic. If $\mathcal{L}$-formulas are preserved under ultraproducts over ω, then $(\mathcal{L}, \models_{\mathcal{L}})$ is equivalent to the basic modal logic $\mathcal{BML}$.*

Proof. By Theorem 6.2 and Lemma 6.4. $\dashv$

To conclude this section we briefly discuss examples and extensions of the above results.

Specific vocabularies. First of all, in the proof of the Lindström Theorem the basic modal formula ψ that is found as the equivalent of the abstract modal formula ϕ is in the same vocabulary as ϕ. This means, for example, that the only abstract modal logic over a binary relation that has a notion of finite rank is the standard modal logic with a single modal operator $\Diamond$.

As a second example, by choosing $\{\lambda xy.\, x = x\}$ or $\{\lambda xy.\, x \neq y\}$ as one's classical vocabulary, the Lindström theorem characterizes the modal logic with the existential modality E (where $x \models E\phi$ iff there exists y with $y \models \phi$) and the modal logic with a difference operator D (where $x \models D\phi$ iff for some $y \neq x$, $y \models \phi$) as the (finitary) basic modal languages over those vocabularies, respectively.

Beyond the basic pattern. So far we have only covered the basic modal pattern consisting of a finite prefix of existential quantifiers followed by a conjunction of atomic formulas; in some cases extensions beyond this pattern can easily be obtained.

As a first example, consider the standard temporal language with operators F and P, where $x \models Fp$ ($x \models Pp$) iff for some y, Rxy and $y \models \phi$ (Ryx and $y \models \phi$). Consider *temporal bisimulations* in which one not only looks forward along the binary relation, but also backward, and adopt the notion of depth accordingly. Given the obvious definition of an *abstract temporal logic*, standard temporal logic is the only temporal logic over a single binary relation that has a notion of finite rank.

By tinkering with the notion of model and, more specifically, by allowing models with constraints on the values assigned to certain atomic symbols, one can obtain a Lindström style characterization of, for example, *nominal modal logic* in which special atomic symbols called *nominals* are constrained to denote (at most) a singleton.

7. Discussion

In this paper we gave a Lindström style characterization of basic modal logic starting from the assumption that bisimulations are a fundamental tool in the model theory of modal logic. Extensions of our result to languages beyond the basic modal format were briefly discussed, but a lot remains to be done. Here are some open issues.

In his important 1969 paper, Lindström proves that whenever $\mathcal{L}$ is a classical logic that has the Löwenheim-Skolem property and is recursively enumerable for validity, then $\mathcal{L}$ is effectively included in first-order logic. What about a modal analogue of this result?

Another question is to give Lindström style characterization results for modal languages differing from the basic modal language, such as **PDL**.

In an unpublished manuscript Albert Visser considers bisimulations and notions of rank fine-tuned for dealing with models that have special properties, such as reflexivity or transitivity. It remains to be seen to which extent the results of this paper can be extended to that setting.

As basic modal logic has the finite model property, it follows from our main result that whenever an abstract modal logic extends basic modal logic and has a notion of finite rank, then it has the finite model property — but what is the general relation between the two properties? And, which modal logic is the strongest modal logic with the FMP?

Further, in a recent manuscript Johan van Benthem characterizes the (first-order) formulas defining operations on relations that preserve bisimilarity. What is the connection between this 'safety result' and the characterization results obtained here?

And finally, throughout this paper we have concentrated on *pointed* models with a distinguished element for evaluation. This suggests that the classical languages in which our modal languages live be equipped with a constant to denote the distinguished point. And this, in turn, suggests that one adds an operator like Hans Kamp's NOW to our modal languages, where $x \models \mathrm{NOW}\phi$ iff for a the distinguished point of the model one has $a \models \phi$. In a recent manuscript Johan van Benthem shows that the standard modal results and techniques go through in this extended format; what about our characterization result?

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