Centrum voor Wiskunde en Informatica

REPORT*RAPPORT*

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BS-R9431 1994

Report BS-R9431
ISSN 0924-0659

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Part I: State-Space and Polynomial Representations

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Abstract

A 'switched' or 'multimode' system is one that can switch between various modes of operation. We consider here switched systems in which the modes of operation are characterized as linear finite-dimensional systems, not necessarily all of the same McMillan degree. When a switch occurs from one of the modes to another of lower McMillan degree, the state space collapses and an impulse may result, followed by a smooth evolution under the new regime. This paper is concerned with the description of such impulsive-smooth behavior on a typical interval. We propose an algebraic framework, modeled on the class of impulsive-smooth distributions as defined by Hautus. Both state-space and polynomial representations are considered, and we discuss transformations between the two forms.

AMS Subject Classification (1991): 93B20, 93B12, 46F05.
Keywords & Phrases: impulsive-smooth distributions, multimode systems, hybrid systems, behaviors, realization theory.
Note: The research of Dr. Geerts was supported in part by the Dutch organisation for scientific research (NWO), CWI, and Tilburg University.

1 Introduction

In this paper we will be concerned with some aspects of 'switched' or 'multimode' systems. In general, a multimode system may be defined as one that may switch, either by external or by internal causes, between a finite number of possible modes of operation. Such systems occur very frequently. Examples include:

- electrical circuits incorporating switches (note that an ideal diode may be seen as a current-controlled switch)
- mechanical linkages; the different modes may relate for instance to contact forces being active or not (see [1] for a simple example)
- hydraulic systems; here valves take the place of switches.

Switching is an important part of many practical control systems. Stagewise gain scheduling can be considered as an example; one may also think of the gear shifting in motor vehicles. There are applications in which control is exerted exclusively through switching, such as in power electronics [15]. Note also that sliding mode control [14, 17] is based on switching. Multimode systems can be viewed as a class of *hybrid systems* (cf. for instance [2]); indeed they combine logic with dynamics, and the switching can be viewed as a timed discrete-event process that influences a continuous-time system. It is easy to think of situations in which the switching is influenced by the dynamics; diodes in electrical networks provide an example of this.

Multimode systems give rise to a number of interesting modeling problems. In this paper, which is the first part of two, we shall be concerned with situations in which switches take place between modes of operation that can be described as finite-dimensional linear time-invariant systems, resulting in what might be called a 'piecewise linear system' (cf. [1]). In particular we shall be concerned with the description of the dynamics in the case that not all of the constituent systems are of the same McMillan degree. When a switch takes place from one of the modes to another of a lower degree, there is an instantaneous collapse of the state space and an impulse may occur. Clearly, a description in terms of smooth functions would not be satisfactory in such situations. Here we shall work with a space of generalized functions that is large enough to cover impulses of arbitrary order at isolated instants, yet small enough to allow a fairly algebraic treatment. This space is based on the class of impulsive-smooth distributions introduced by M. L. J. Hautus [9]. Following the lead of J. C. Willems [16], our aim will be to describe the 'behavior' of a multimode system by specifying the set of trajectories of external variables. We shall consider both state space and polynomial representations.

The present Part I is organized as follows. In the next section, we consider a simple example to motivate the development, and we introduce the mathematical framework that we shall use. A proposal for a formal specification of piecewise linear systems is made in section 3. Then we concentrate on the description of the behavior on a typical interval between switches. First-order representations are discussed in section 4, and polynomial representations follow in section 5. The conclusions are summarized in section 6. In Part II, we focus on minimality of representations and obtain a state space isomorphism theorem for impulsive-smooth behaviors.

In this paper, the following terms will be used interchangeably for rational matrices $M(s)$: $M(s)$ has full generic column rank / has full column rank as a rational matrix / is left invertible (as a rational matrix). Also, the following terms will be used interchangeably for polynomial matrices: $M(s)$ has full column rank for all $s \in \mathbb{C}$ / is left unimodular. The following facts are well-known: a polynomial matrix $M(s)$ has a polynomial left inverse if and only if $M(s)$ is left unimodular; a proper rational matrix $M(s)$ has a proper rational left inverse if and only if the constant matrix $M(\infty)$ has full column rank. Similar remarks hold with 'column' replaced by 'row' and 'left' by 'right'.

2 A BEHAVIORAL FRAMEWORK FOR PIECEWISE LINEAR SYSTEMS

A simple example of a multimode system in which switching takes place between modes of different McMillan degrees is the electrical network in Fig. 1 below. As external variables, we might for instance take the current i and the voltage V at the terminals. The 'behavior' of the system, in the terminology of J. C. Willems, is the set of compatible trajectories of these variables. Of course, we want to describe the behavior by means of equations. On the open intervals between switches, the evolution may clearly be given by means of differential equations as usual. Specifically, for intervals on which the switch is open the equations relating $V(t)$ and $i(t)$ can be written in 'pencil' form (cf. [11]) as

$$\begin{aligned} \dot{z}_1(t) &= \tfrac{1}{C_1} z_3(t) \\ \dot{z}_2(t) &= \tfrac{-1}{RC_2} z_2(t) \\ V(t) &= z_1(t) \\ i(t) &= z_3(t) \end{aligned} \tag{2.1}$$

where the z_i's are 'internal' or 'auxiliary' variables. For intervals on which the switch is closed the equations are

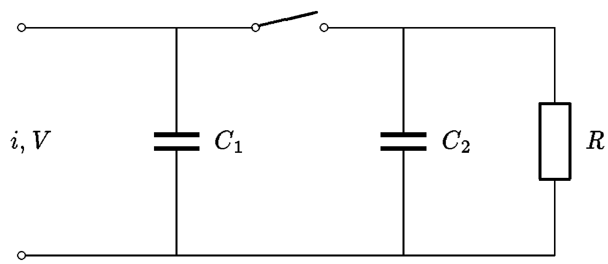

FIGURE 1. A multimode system

$$\begin{aligned} \dot{z}'_1(t) &= \tfrac{-1}{R(C_1+C_2)} z'_1(t) + \tfrac{1}{C_1+C_2} z'_2(t) \\ V(t) &= z'_1(t) \\ i(t) &= z'_2(t). \end{aligned} \tag{2.2}$$

where the z'_i's are auxiliary variables. These are the two modes of the overall system; clearly, their McMillan degrees do not agree. At the closing of the switch, there is a collapse from a two-dimensional to a one-dimensional state space. It is possible to embed the smaller state space in the larger one, and certainly in this case there is a natural way to do so; still, to describe the exact behavior, one needs 'jump relations' [1] which tell how the state will jump from some position in the larger space to a corresponding position in the subspace. Just as the differential equations describing the system, the jump relations have to be obtained either from identification or from physical principles. In the case of the circuit of Fig. 1, the law of conservation of charge leads to the following transition relation for a closing of the switch at time t_0 [4, p. 98]:

$$z'_1(t_0^+) = \frac{C_1}{C_1+C_2} z_1(t_0^-) + \frac{RC_2}{C_1+C_2} z_2(t_0^-) \tag{2.3}$$

where the variable at the left hand side refers to (2.2) and the variables at the right hand side refer to (2.1).

The behavioral framework as described for instance in [16] requires first of all the construction of a *universum* of which the to-be-described behaviors are subsets. If one wants to hold on to a simple description in terms of ideal elements for examples such as the above, it is clear that one cannot make do with a universum consisting of continuous functions. The example shows that jumps have to be allowed, and in general one may even have delta functions or higher-order impulses at switching instants. This calls for a *distributional* framework. On the other hand, it would be worthwile to limit the set of considered distributions so as to make a fairly algebraic treatment possible. Below we shall describe a framework that allows impulses of arbitrary order at isolated (switching) points but that assumes smooth behavior between those points. Moreover, we shall describe a calculus that allows one to write down equations that are valid on a *semi*-open interval $[t_0, t_1)$. By connecting these intervals, one obtains a complete description of the evolution of the real-valued variables in the system. The choice of intervals that are closed on the left hand side and open on the right hand side is arbitrary from a mathematical point of view; an analogous theory could be developed using intervals of the form $(t_0, t_1]$. Physically speaking, however, one may argue that the impulses that occur in the mathematical description are idealizations of (very) fast behavior that occurs *after* a switch has been closed, which makes it more natural to work with left-closed and right-open intervals.

Since our calculus will be based on the theory of impulsive-smooth distributions (cf. [9, 10]), let us first quickly recall the main points from this theory. Let $\mathcal{D}_-$ denote the space of test functions with upper-bounded support, and let $\mathcal{D}'_+$ denote the dual space of distributions on $\mathcal{D}_-$. With the convolution $*$ as multiplication, $\mathcal{D}'_+$ is a commutative algebra over $\mathbb{R}$ with unit element δ, defined by $\langle \delta, \phi \rangle = \phi(0)$ ($\phi \in \mathcal{D}_-$). It is convenient to apply the notational conventions associated with multiplication to the operation of convolution on $\mathcal{D}'_+$; in particular, $a\delta$ ($a \in \mathbb{R}$) is then denoted by a and fg stands for $f * g$.

The space of locally integrable functions with lower-bounded support can be embedded as a subspace in $\mathcal{D}'_+$ by the standard identification $\langle u, \phi \rangle = \int_{-\infty}^{\infty} u(t)\phi(t)\,dt$. A *smooth* distribution is defined as one that arises in this way from a function u that is zero on $(-\infty, 0)$ and smooth on $[0, \infty)$, meaning that $u(t)$ is arbitrarily often differentiable on $(0, \infty)$ and is such that $\lim_{t \downarrow 0} u^{(k)}(t)$ exists for all $k \geq 0$. The space of smooth distributions will be denoted by $\mathcal{C}_{\text{sm}}(0, \infty)$. For $u \in \mathcal{C}_{\text{sm}}(0, \infty)$, the function $\dot{u}$ is defined by $\dot{u}(t) = 0$ ($t < 0$), $\dot{u}(t) = \frac{du}{dt}(t)$ ($t > 0$). Obviously the mapping $u \mapsto \dot{u}$ is a linear mapping which takes $\mathcal{C}_{\text{sm}}(0, \infty)$ into itself. On $\mathcal{C}_{\text{sm}}(0, \infty)$ we also introduce the linear functional $u \mapsto u(0^+)$, defined by $u(0^+) = \lim_{t \downarrow 0} u(t)$.

A second subspace of $\mathcal{D}'_+$ is the space $\mathcal{C}_{\text{p-imp}}(0)$ of *purely impulsive* distributions, which is defined as the linear space generated by δ and its derivatives. If the first derivative of δ is denoted by p, the k-th derivative is simply p^k if we write convolution as multiplication, so that the general form of an element of $\mathcal{C}_{\text{p-imp}}(0)$ is $\sum_{k=0}^{n} c_k p^k$, $c_k \in \mathbb{R}$.

The direct sum of $\mathcal{C}_{\text{sm}}(0, \infty)$ and $\mathcal{C}_{\text{p-imp}}(0)$ is denoted by $\mathcal{C}_{\text{imp}}(0, \infty)$ and is called the space of *impulsive-smooth distributions*. This space is a subalgebra of $\mathcal{D}'_+$. For a distribution $u \in \mathcal{C}_{\text{imp}}(0, \infty)$ with decomposition $u = u_{\text{p-imp}} + u_{\text{sm}}$ in impulsive and smooth parts, we have the fundamental formula

$$pu = pu_{\text{p-imp}} + u_{\text{sm}}(0^+) + \dot{u}_{\text{sm}}. \tag{2.4}$$

The convolution equation $pu = au + u_0$ ($a \in \mathbb{R}$) generalizes the pair consisting of the differential equation $\dot{u} = au$ and the initial condition $u(0) = u_0$, in the sense that the solution u of the convolution equation corresponds via the standard identification to the solution of the differential equation that satisfies $u(0^+) = u_0$. A similar correspondence exists also for systems of linear differential and algebraic equations, as we shall see below (cf. also [8]).

The class of distributions as introduced by Schwartz is a wide one and the set of impulsive-smooth distributions is only a small subclass of it. This limited class might in fact be introduced in a more algebraic way. For this purpose, let $\mathcal{C}(0, \infty)$ denote the set of all real-valued C^∞-functions on $(0, \infty)$ all of whose derivatives have left-hand limits at 0, and let (as usual) $\mathbb{R}[p]$ denote the ring of polynomials in p with real coefficients. The set of vectors with entries in the product $\mathbb{R}[p] \times \mathcal{C}(0, \infty)$ obviously has the structure of a linear space over $\mathbb{R}$, and we can make it into an $\mathbb{R}[p]$-module by *defining* multiplication by p by the formula (2.4). This framework is sufficient for models based on linear vector differential equations with constant coefficients. Note that $(\mathbb{R}[p] \times \mathcal{C}(0, \infty))^k$ with multiplication by p defined by (2.4) is isomorphic as an $\mathbb{R}[p]$-module to $\mathcal{C}^k_{\text{imp}}(0, \infty) := (\mathcal{C}_{\text{imp}}(0, \infty))^k$.

Motivated by this development, let us now set up an analogous framework on a half-open interval. Consider $t_0, t_1 \in \mathbb{R} \cup \{-\infty, +\infty\}$ with $t_0 < t_1$, and denote by $\mathcal{C}(t_0, t_1)$ the set of restrictions of $C^\infty(\mathbb{R})$-functions to (t_0, t_1). If $t_0 > -\infty$, the linear space $(\mathbb{R}[p] \times \mathcal{C}(t_0, t_1))^k$ can be equipped with an $\mathbb{R}[p]$-module structure by using (2.4) to *define* multiplication by p, and we shall denote the space $(\mathbb{R}[p] \times \mathcal{C}(t_0, t_1))^k$ by $\mathcal{C}^k_{\text{imp}}(t_0, t_1)$. To cover the case $t_0 = -\infty$, we set $\mathcal{C}^k_{\text{imp}}(-\infty, t_1) = \mathcal{C}^k(-\infty, t_1)$, and understand mupltiplication by p as ordinary differentiation. For impulsive-smooth distributions on an interval (t_0, t_1) with values in a vector space Z we shall also use the notation $\mathcal{C}_{\text{imp}}(t_0, t_1; Z)$.

The space $\mathcal{C}^k_{\text{imp}}(t_0, t_1)$ will be used below to construct a 'universum' in the sense of [16] for the purpose of describing behaviors of piecewise linear multimode systems. First however we need to introduce some notation and terminology related to the discrete aspects of multimode systems. Let us introduce a set $\mathcal{S}$ of functions that indicate switching times.

Definition 2.1 A function τ from $\mathbb{Z}$ to the extended real line $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$ is called a *timing* if it is strictly increasing in the sense that

(i) for all k, if $-\infty < \tau(k) < +\infty$, then $\tau(k-1) < \tau(k) < \tau(k+1)$;

(ii) $\lim_{k\to\infty}\tau(k) = +\infty$ and $\lim_{k\to-\infty}\tau(k) = -\infty$.

The set of all timings will be denoted by $\mathcal{S}$. Two timings τ_1 and τ_2 will be said to be *equivalent* if there exists an $\ell \in \mathbb{Z}$ such that $\tau_1(k) = \tau_2(k+\ell)$ for all k. The set of *switching instants* associated with a timing τ is

$$T(\tau) = \{t \in \mathbb{R} \mid \exists k \in \mathbb{Z} \text{ s. t. } t = \tau(k)\} \tag{2.5}$$

and the collection of *intervals bounded by switches* is

$$I(\tau) = \{(t_1,t_2) \mid t_1 \in \mathbb{R}\cup\{-\infty\}, t_2 \in \mathbb{R}\cup\{\infty\}, \exists k \in \mathbb{Z} \text{ s.t. } t_1 = \tau(k), t_2 = \tau(k+1)\}. \tag{2.6}$$

Note that the number of switching instants can be either finite (even zero) or infinite. The definition is such that the set of switching instants can have no limit points, although no *a priori* lower bound is imposed on the distance between two switching instants and in fact a nonzero lower bound not necessarily exists for a given timing. It is easily seen that two timings define the same set of switching instants if and only if they are equivalent. By gluing together spaces of the form $\mathcal{C}_{\text{imp}}(t_1,t_2)$ we obtain a space of vector-valued 'switched functions' on $\mathbb{R}$:

$$\mathcal{C}^k_{\text{imp}}(\mathbb{R};\tau) = \prod_{(t_1,t_2)\in I(\tau)} \mathcal{C}^k_{\text{imp}}(t_1,t_2). \tag{2.7}$$

For an element w of this space, its component on the interval (t_1,t_2) will be denoted by $w|_{t_1,t_2}$. The union of the spaces $\mathcal{C}^k_{\text{imp}}(\mathbb{R};\tau)$ for all timings τ, given by

$$\mathcal{U} \;=\; \bigcup_{\tau\in\mathcal{S}} \mathcal{C}^k_{\text{imp}}(\mathbb{R};\tau), \tag{2.8}$$

provides a convenient *universum* in which we can now describe specific behaviors of piecewise linear systems.

3 Formal specification

In this section we propose a specification of the class of (finite-dimensional, time-invariant) piecewise linear systems in terms of a particular class of representations. First of all, we need a vector space W in which the external variables take their values, as in [16]. The vector w of external variables contains both inputs and outputs; since causality relations may vary from one constituent system to another, it seems preferable not to introduce any labeling of the external variables in order to distinguish inputs and outputs. To model the switching from one constituent system to another, we use a graph Γ consisting of a finite set of vertices V and a set of directed edges $E \subset V \times V$. Attached to each vertex v there is a continuous system in first-order representation, given by a tuple of linear spaces and linear mappings $\Sigma_v = (Z_v, X_v; F_v, G_v, H_v)$ with $F_v : Z_v \to X_v$, $G_v : Z_v \to X_v$, $H_v : Z_v \to W$. For each given interval (t_1,t_2), the representation specifies a set of *behavioral equations* (in 'pencil' form, cf. [11]):

$$\begin{aligned} p_{t_1} G_v z &= F_v z + x_{\text{in}} \\ w &= H_v z \end{aligned} \tag{3.1}$$

where p_{t_1} denotes the operator p defined by (2.4) with the time 0 replaced by the time t_1. If $t_1 = -\infty$, p_{t_1} denotes ordinary differentiation and x_{in} vanishes. The *full behavior* on an interval (t_1,t_2) associated with the representation Σ_v is the set

$$\begin{aligned} \mathcal{B}_{\text{f}}(t_1,t_2;\Sigma_v) = \{(z,w,x_{\text{in}},x_{\text{out}}) \in \mathcal{C}_{\text{imp}}(t_1,t_2;Z) \times \mathcal{C}_{\text{imp}}(t_1,t_2;W) \times X_v \times X_v \mid \\ \text{eqn. (3.1) holds, and } x_{\text{out}} = G_v z(t_2)\}. \end{aligned} \tag{3.2}$$

To each edge $e = (v_1, v_2)$ there is a linear mapping $J_e : X_{v_1} \to X_{v_2}$ that determines the transition relations.

Formally then, a piecewise linear system (in first-order representation) is a four-tuple (Γ, Σ, J, W) where W is a finite-dimensional vector space, $\Gamma = (V, E)$ is a directed graph, Σ is a mapping assigning to each $v \in V$ a tuple of linear spaces and linear mappings $\Sigma_v = (Z_v, X_v; F_v, G_v, H_v)$ with $F_v : Z_v \to X_v$, $G_v : Z_v \to X_v$, $H_v : Z_v \to W$, and J is a mapping assigning to each $e = (v_1, v_2)$ a linear mapping $J_e : X_{v_1} \to X_{v_2}$. The *behavior* associated to (Γ, Σ, J, W) is specified as follows. Consider a signal w in the universum $\mathcal{U}$, and let τ denote its associated timing (which is unique up to equivalence). The signal w belongs to $\mathcal{B}(\Gamma, \Sigma, J, W)$ if to each switching point $t \in T(\tau)$ there is an edge $e(t)$ together with 'matching vectors' $x_{\text{in}}(t)$ and $x_{\text{out}}(t)$, and to each interval $(t_1, t_2) \in I(\tau)$ there is a vertex $v(t_1, t_2)$, such that the following conditions hold:

(i) for all t_1, t_2, t_3 such that both (t_1, t_2) and (t_2, t_3) belong to $I(\tau)$, we have $e(t_2) = (v(t_1, t_2), v(t_2, t_3))$

(ii) for each $(t_1, t_2) \in I(\tau)$, there is a $z \in \mathcal{C}_{\text{imp}}(t_1, t_2; Z)$ such that (3.1) holds with $x_{\text{in}} = x_{\text{in}}(t_1)$ if $t_1 > -\infty$ and $x_{\text{in}} = 0$ otherwise, and $G_v z(t_2) = x_{\text{out}}(t_2)$

(iii) for all $t \in T(\tau)$, we have $x_{\text{in}}(t) = J_{e(t)} x_{\text{out}}(t)$.

The first condition specifies that a transition from vertex v_1 to vertex v_2 can only take place if there is a directed edge in Γ connecting v_1 and v_2. The second condition describes the behavior on the intervals between switches, and the third one gives the transition relations.

For example, the behavior of the circuit in Fig. 1 can be represented as follows. The graph Γ consists of two vertices 0 and 1 and two edges $(0, 1)$ and $(1, 0)$. The systems associated with the vertices are given by

$$G_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad F_0 = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} \\ 0 & \frac{-1}{RC_2} & 0 \end{bmatrix}, \quad H_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.3}$$

and

$$G_1 = [1 \quad 0], \qquad F_1 = [\frac{-1}{R(C_1 + C_2)} \quad \frac{1}{C_1 + C_2}], \qquad H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.4}$$

respectively. The transition relations are given by

$$J_{(0,1)} = [\frac{C_1}{C_1 + C_2} \quad \frac{RC_2}{C_1 + C_2}], \qquad J_{(1,0)} = \begin{bmatrix} 1 \\ R^{-1} \end{bmatrix}. \tag{3.5}$$

In this example we took the real-valued variables current and voltage as external variables. In other applications, it may be of interest to consider also or only discrete-valued variables (positions of the switches) as external; this would bring us close to the point of view from which hybrid systems are studied in computer science (cf. for instance [12, 13]). In the models studied in computer science, however, one does not usually consider continuous inputs.

Our main object of study in this paper will be the specification of the impulsive-smooth dynamics on a typical interval. We aim in particular at finding necessary and sufficient conditions for minimality of first-order representations of impulsive-smooth behaviors on a semi-open interval, and at determining the relation between equivalent representations. We will consider polynomial representations as well; these will be useful as a technical tool, but are also interesting by themselves.

Remark 3.1 It should be noted that matching conditions may have an impact on the formulation of the notion of minimality. This is already seen in the example given above. The equations that we wrote down for intervals on which the switch is open are non-minimal from the point of view of describing the current/voltage relation at the external port, as the variable z_2 in (2.1), which is the voltage V_2 across the second capacitor, evolves autonomously; nevertheless the value of V_2 is important when a switch occurs, since the transition relations depend on it (see (2.3)). Therefore one might argue that it is not allowed to remove the equation for V_2 from the system's description. On the other hand, the value of V_2 at a time t_2 when the switch is closed is determined completely by the initial data at the preceding time t_1 when the switch was opened and by the length of the interval (t_1, t_2), so that one might still remove the autonomous part if one would allow the transition relation to depend not only on $z(t_2)$ but also on $x_{\text{in}}(t_1)$ and the time difference $t_2 - t_1$. We will not address these modeling issues here, but simply define minimality in terms of the external variables. A safe but possibly conservative way to ensure that this also suffices to describe the matching conditions is to include among the external variables all variables that play a role in the transition relations.

4 First-order representations

In this paper we are mainly concerned with the description of impulsive-smooth behavior on a typical switching interval $[t_{\text{in}}, t_{\text{out}})$. We shall assume from now on that such an interval has been fixed, and specific reference to the points t_{in} and t_{out} will be avoided as much as possible to ease the notation. On this interval we are concerned with one system of the form (3.1), and so we shall also drop the index v. Therefore, we shall consider impulsive-smooth behaviors that can be described as follows.

Definition 4.1 For a matrix triple (F, G, H) ($F,\ G \in \mathbb{R}^{n\times(n+m)}$, $H \in \mathbb{R}^{q\times(n+m)}$), we define

$$\mathcal{B}(F, G, H) = \{w \in \mathcal{C}^q_{\text{imp}} \mid \exists z \in \mathcal{C}^{n+m}_{\text{imp}},\ x_0 \in \mathbb{R}^n \text{ s.t. } pGz = Fz + x_0,\ w = Hz\}. \tag{4.1}$$

Since we allow arbitrary redundancy at this stage, the integer m may be negative, but it will be shown later that in minimal representations m must be nonnegative. The following statement is immediately seen to be true:

Lemma 4.2 *If S and T are invertible nonsingular matrices, then $\mathcal{B}(SFT, SGT, HT) = \mathcal{B}(F, G, H)$.*

The representation (4.1) is derived from the 'pencil' form that was proposed for smooth linear systems in [11]. It deviates from the more standard (cf. for instance [5, 3, 7]) descriptor representation

$$\begin{aligned} pEx &= Ax + Bu + Ex_0 \\ y &= Cx + Du \end{aligned} \tag{4.2}$$

in two respects. Firstly, the descriptor representation puts inputs and outputs on an unequal footing, whereas we have chosen to treat all external variables alike, at least *a priori*. It may be noted that in the most general case ($sE - A$ not necessarily invertible and possibly nonsquare) there is from the equations no incentive to treat inputs and outputs asymmetrically. Secondly, re-writing the above equations in the 'external variable' form via the usual transformations

$$G = [E \quad 0], \quad F = [A \quad B], \quad H = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \tag{4.3}$$

leads in first instance to a representation of the following type, which we shall refer to as the 'conventional' representation.

Definition 4.3 For a matrix triple (F, G, H) ($F,\ G \in \mathbb{R}^{n\times(n+m)}$, $H \in \mathbb{R}^{q\times(n+m)}$), we define

$$\mathcal{B}_c(F, G, H) = \{w \in \mathcal{C}^q_{\text{imp}} \mid \exists z \in \mathcal{C}^{n+m}_{\text{imp}},\ z_0 \in \mathbb{R}^{n+m} \text{ s.t. } pGz = Fz + Gz_0,\ w = Hz\}. \tag{4.4}$$

The expressive power of both representations is the same, as the following proposition shows.

Proposition 4.4 *For every triple (F, G, H) as above, there exists a triple $(\tilde{F}, \tilde{G}, \tilde{H})$ such that $\mathcal{B}(\tilde{F}, \tilde{G}, \tilde{H}) = \mathcal{B}_c(F, G, H)$; and vice versa.*

Proof Let $F, G : Z \to X$, $H : Z \to W$ be given. Denote $\tilde{X} = \operatorname{im} G$, $\tilde{Z} = F^{-1}[\operatorname{im} G]$; then both F and G map $\tilde{Z}$ into $\tilde{X}$. For $\tilde{F}$ and $\tilde{G}$, take the induced mappings from $\tilde{Z}$ into $\tilde{X}$; and for $\tilde{H}$, take the restriction of H to $\tilde{Z}$. We now have to show that $\mathcal{B}(\tilde{F}, \tilde{G}, \tilde{H}) = \mathcal{B}_c(F, G, H)$.

First, take $w \in \mathcal{B}(\tilde{F}, \tilde{G}, \tilde{H})$. Then there exist an impulsive-smooth $z \in \mathcal{C}_{\text{imp}}(F^{-1}[\operatorname{im} G])$ and a vector $x_0 \in \operatorname{im} G$ such that $p\tilde{G}z = \tilde{F}z + x_0$ and $w = \tilde{H}z$. If we now write $x_0 = Gz_0$, then $pGz = Fz + Gz_0$ and $w = Hz$ so that $w \in \mathcal{B}_c(F, G, H)$. For the reverse inclusion, take $w \in \mathcal{B}_c(F, G, H)$. Then there exist z and z_0 such that $pGz = Fz + Gz_0$, and $w = Hz$. But it follows from the first equation that z actually takes values in $F^{-1}[\operatorname{im} G]$, so that we have $w \in \mathcal{B}(\tilde{F}, \tilde{G}, \tilde{H})$.

Conversely, suppose that we start with a pencil representation $\mathcal{B} = \mathcal{B}(F_{11}, G_{11}, H_1)$. A conventional representation is then obtained by taking

$$F = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}, \quad G = \begin{bmatrix} G_{11} & G_{12} \\ 0 & 0 \end{bmatrix}, \quad H = [H_1 \;\; H_2] \tag{4.5}$$

where G_{12} is any matrix such that $[G_{11} \;\; G_{12}]$ has full row rank, F_{22} is any matrix of full column rank, and F_{12} and H_2 are any matrices of the appropriate dimensions. To see this, note that the equations for the conventional representation as just defined are of the form

$$pG_{11}z_1 + pG_{12}z_2 = F_{11}z_1 + F_{12}z_2 + G_{11}z_{10} + G_{12}z_{20} \tag{4.6}$$
$$0 = F_{22}z_2 \tag{4.7}$$
$$w = H_1z_1 + H_2z_2. \tag{4.8}$$

Because F_{22} was taken to be of full column rank, equation (4.7) is equivalent to $z_2 = 0$ and so the equations (4.6–4.8) are equivalent to

$$pG_{11}z_1 = F_{11}z_1 + G_{11}z_{10} + G_{12}z_{20} \tag{4.9}$$
$$w = H_1z_1. \tag{4.10}$$

Finally note that since $[G_{11} \;\; G_{12}]$ has full row rank, it is possible for any given x_0 to find z_{10} and z_{20} such that $G_{11}z_{10} + G_{12}z_{20} = x_0$. □

To obtain a representation in conventional form in the way described in the proof, one may for instance take $G_{12} = I$ and $F_{22} = I$, but in this way one may introduce more variables and equations than is strictly necessary.

To compare the distributional framework with the more standard framework of differential equations, let us consider the 'smooth' behaviors that might be associated to a triple (F, G, H) in one of the following two ways.

Definition 4.5 For a matrix triple (F, G, H) (F, $G \in \mathbb{R}^{n \times (n+m)}$, $H \in \mathbb{R}^{q \times (n+m)}$), we define

$$\mathcal{B}^{\text{d/s}}(F, G, H) = \{w \in \mathcal{C}^q_{\text{sm}} \mid \exists z \in \mathcal{C}^{n+m}_{\text{imp}},\; x_0 \in \mathbb{R}^n \text{ s.t. } pGz = Fz + x_0,\; w = Hz\}. \tag{4.11}$$

Definition 4.6 For a matrix triple (F, G, H) (F, $G \in \mathbb{R}^{n \times (n+m)}$, $H \in \mathbb{R}^{q \times (n+m)}$), we define

$$\mathcal{B}^{\text{s/s}}(F, G, H) = \{w \in \mathcal{C}^q_{\text{sm}} \mid \exists z \in \mathcal{C}^{n+m}_{\text{sm}},\; x_0 \in \mathbb{R}^n \text{ s.t. } pGz = Fz + x_0,\; w = Hz\}. \tag{4.12}$$

The relation between the smooth and the impulsive-smooth behavior is given as follows.

Proposition 4.7 *For any triple (F, G, H), we have*

$$\mathcal{B}^{\text{d/s}}(F, G, H) = \mathcal{B}^{\text{s/s}}(F, G, H) = \mathcal{B}(F, G, H) \cap \mathcal{C}^q_{\text{sm}}. \tag{4.13}$$

PROOF (Compare [7, Lemma 2.5].) It is clear that $\mathcal{B}^{s/s}(F,G,H) \subset \mathcal{B}^{d/s}(F,G,H) \subset \mathcal{B}(F,G,H) \cap \mathcal{C}^q_{\mathrm{sm}}$, so that it only remains to prove the inclusion $\mathcal{B}(F,G,H) \cap \mathcal{C}^q_{\mathrm{sm}} \subset \mathcal{B}^{s/s}(F,G,H)$. Take $w \in \mathcal{B}(F,G,H) \cap \mathcal{C}^q_{\mathrm{sm}}$, and let $z \in \mathcal{C}^{n+m}_{\mathrm{imp}}$ and $x_0 \in \mathbb{R}^n$ be such that $pGz = Fz + x_0$, $w = Hz$. Write z as the sum of an impulsive part $z_{\text{p-imp}} = \sum_{k=0}^N z_k p^k$ and a smooth part z_{sm}. The equation $pGz = Fz + x_0$ now reads

$$\sum_{k=0}^{N} G z_k p^{k+1} + G z_{\mathrm{sm}}(t_{\mathrm{in}}^+) \; = \; \sum_{k=0}^{N} F z_k p^k + x_0 \tag{4.14}$$

$$G \dot{z}_{\mathrm{sm}} \; = \; F z_{\mathrm{sm}}. \tag{4.15}$$

We already know that $Hz_{\text{p-imp}} = 0$ because w must be smooth, and so it follows that the pair $(z_{\mathrm{sm}}, Fz_0 + x_0)$ produces the same w as the pair (z, x_0) does. Consequently, w is in $\mathcal{B}^{s/s}(F,G,H)$. □

5 POLYNOMIAL REPRESENTATIONS

Until now we have only considered first-order expressions in p. Since p is a linear operator mapping the space of impulsive-smooth distributions $\mathcal{C}_{\mathrm{imp}}$ into itself, one can also consider polynomials in p. To describe the action of a polynomial in p on an element of $\mathcal{C}_{\mathrm{imp}}$, we need to introduce a certain shift operator on polynomials. For $r(s) = r_k s^k + \cdots + r_1 s + r_0$, we define

$$(\sigma r)(s) \; = \; \frac{r(s) - r(0)}{s} \; = \; r_k s^{k-1} + \cdots + r_2 s + r_1. \tag{5.1}$$

We can then give an explicit expression for $r(p)w$ if r is a polynomial and w belongs to $\mathcal{C}_{\mathrm{imp}}$.

LEMMA 5.1 *For $r(s) \in \mathbb{R}[s]$ and $w = w_{\text{p-imp}} + w_{\mathrm{sm}}$, we have*

$$r(p)w \; = \; r(p) w_{\text{p-imp}}(p) + \sum_{k=1}^{\infty} (\sigma^k r)(p) w_{\mathrm{sm}}^{(k-1)}(t_{\mathrm{in}}^+) + r(\frac{d}{dt}) w_{\mathrm{sm}}. \tag{5.2}$$

Note that the summation is actually finite, since $\sigma^k r = 0$ for all sufficiently large k. The proof of the lemma is a straightforward induction with respect to the degree of r and will be omitted. It is also straightforward although somewhat tedious to verify the following (cf. [9, Thm. 3.7]).

LEMMA 5.2 *The mapping $r(s) \mapsto r(p)$ from $\mathbb{R}[s]$ to the ring of linear operators mapping $\mathcal{C}_{\mathrm{imp}}$ into itself is a ring homomorphism.*

A number of useful properties of the space $\mathcal{C}_{\mathrm{imp}}$ are discussed (in a slightly different formal context) in [9] and [7]. We shall in particular need the fact that every rational function $f(s)$ determines uniquely a linear operator $f(p)$ mapping $\mathcal{C}_{\mathrm{imp}}$ into itself, and that the set of operators obtained in this way is isomorphic as a field to the field of rational functions $\mathbb{R}(s)$. To give an example (cf. [9, Thm. 3.11]), for $a \in \mathbb{R}$ and $v = v_{\text{p-imp}} + v_{\mathrm{sm}} \in \mathcal{C}_{\mathrm{imp}}$, with $v_{\text{p-imp}}(p) = r_v(p)$ ($r_v(s)$ a polynomial), the purely impulsive part of $w = (p-a)^{-1} v \in \mathcal{C}_{\mathrm{imp}}$ is $w_{\text{p-imp}} = r_w(p)$ where the polynomial r_w is determined by

$$r_w(s) = \frac{r_v(s) - r_v(a)}{s - a}, \tag{5.3}$$

whereas the smooth part coincides on $(t_{\mathrm{in}}, t_{\mathrm{out}})$ with the solution x of the initial value problem

$$\dot{x} = ax + v_{\mathrm{sm}}, \quad x(0) = r_v(a). \tag{5.4}$$

In an approach which makes more use of distribution theory than we do here and which therefore allows to put a ring structure on $\mathcal{C}_{\mathrm{imp}}$ rather than only a module structure, one may also look at the operator $f(p)$ as the operation of convolution by a 'fractional impulse' [7].

Of course it is possible to extend what has been said above to the vector/matrix case in the obvious way. In particular, every rational matrix $R(s)$ of size $p \times q$ determines a linear mapping $R(p)$ from

$\mathcal{C}^q_{\mathrm{imp}}$ to $\mathcal{C}^p_{\mathrm{imp}}$. A number of elementary properties of mappings of this type follow as in [7, Cor. 2.4]); in particular, the mapping $R(p)$ from $\mathcal{C}^q_{\mathrm{imp}}$ to $\mathcal{C}^p_{\mathrm{imp}}$ is surjective/injective/invertible if and only if the rational matrix $R(s)$ has full row rank / has full column rank / is nonsingular.

For a matrix $R(s) \in \mathbb{R}^{p\times q}(s)$, the kernel and image of the associated mapping $R(p)$ from $\mathcal{C}^q_{\mathrm{imp}}$ to $\mathcal{C}^p_{\mathrm{imp}}$ will be denoted by $\ker R(p)$ and $\mathrm{im}\, R(p)$ respectively; so $\ker R(p)$ (the 'solution space') is a subspace of $\mathcal{C}^q_{\mathrm{imp}}$ and $\mathrm{im}\, R(p)$ is a subspace of $\mathcal{C}^p_{\mathrm{imp}}$. We shall need the following results concerning such subspaces; the converses of these results are also true but will not be used below.

Lemma 5.3 *If $R_1(s) \in \mathbb{R}^{p_1\times q}(s)$ and $R_2(s) \in \mathbb{R}^{p_2\times q}(s)$ satisfy $\ker R_1(s) = \ker R_2(s)$, then $\ker R_1(p) = \ker R_2(p)$.*

Proof If $\ker R_1(s) = \ker R_2(s)$ then there exist rational matrices $X_1(s)$ and $X_2(s)$ such that $R_1(s) = X_1(s)R_2(s)$ and $R_2(s) = X_2(s)R_1(s)$. It follows from this that $\ker R_1(p) = \ker R_2(p)$. □

Lemma 5.4 *If $R_1(s) \in \mathbb{R}^{p\times q_1}(s)$ and $R_2(s) \in \mathbb{R}^{p\times q_2}(s)$ satisfy $\mathrm{im}\, R_1(s) = \mathrm{im}\, R_2(s)$, then $\mathrm{im}\, R_1(p) = \mathrm{im}\, R_2(p)$.*

Proof The proof is similar to the one above. □

Lemma 5.5 *If $R(s) \in \mathbb{R}^{p\times q}(s)$ and $T(s) \in \mathbb{R}^{q\times r}(s)$ satisfy $\ker R(s) = \mathrm{im}\, T(s)$, then $\ker R(p) = \mathrm{im}\, T(p)$.*

Proof If $w = T(p)v$ for some v, then clearly $R(p)w = 0$ since $R(s)T(s) = 0$, so $\mathrm{im}\, T(p) \subset \ker R(p)$. Conversely, suppose that $R(p)w = 0$. By the previous lemmas, we may assume without loss of generality that $R(s)$ has full row rank and that $T(s)$ has full column rank. We can then choose matrices $\hat{R}(s)$ and $\hat{T}(s)$ such that

$$\begin{bmatrix} \hat{R}(s) \\ R(s) \end{bmatrix} [T(s) \quad \hat{T}(s)] = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{5.5}$$

Define

$$\begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{R}(p) \\ R(p) \end{bmatrix} w, \tag{5.6}$$

then

$$w = [T(p) \quad \hat{T}(p)] \begin{bmatrix} v \\ 0 \end{bmatrix} = T(p)v, \tag{5.7}$$

so that we also have $\ker R(p) \subset \mathrm{im}\, T(p)$. □

We now introduce *polynomial representations* for impulsive-smooth behaviors in the following way.

Definition 5.6 Let $R(s) \in \mathbb{R}^{p\times q}[s]$ and $V(s) \in \mathbb{R}^{n\times q}[s]$. We define

$$\mathcal{B}(R, V) = \{w \mid R(p)w \in \mathrm{span}_{\mathbb{R}}\, V(p)\}. \tag{5.8}$$

The relation to first-order representations is as follows.

Lemma 5.7 *If one has*

$$\mathrm{im} \begin{bmatrix} sG - F \\ H \end{bmatrix} = \ker [-V(s) \quad R(s)] \tag{5.9}$$

as an equality between rational vector spaces, then $\mathcal{B}(F, G, H) = \mathcal{B}(R, V)$.

Proof Suppose $w \in \mathcal{B}(F,G,H)$, so there exists an impulsive-smooth z and a constant vector x_0 such that $pGz = Fz + x_0$ and $w = Hz$. By Lemma 5.5, we then have

$$\begin{bmatrix} x_0 \\ w \end{bmatrix} = \begin{bmatrix} pG - F \\ H \end{bmatrix} z \in \operatorname{im} \begin{bmatrix} pG - F \\ H \end{bmatrix} = \ker \begin{bmatrix} -V(p) & R(p) \end{bmatrix} \tag{5.10}$$

so that $R(p)w = V(p)x_0$, and consequently $w \in \mathcal{B}(R,V)$. The reverse inclusion is obtained by reversing this reasoning. □

Not every pair of polynomial matrices $(R(s), V(s))$ is such that $\ker [-V(s) \;\; R(s)]$ may be written in the special form appearing at the left hand side of (5.9). To describe a class of pairs which do have this property, we need the following definitions. The first of these is from [11] and builds on the work of Fuhrmann [6].

Definition 5.8 For a polynomial matrix $R(s) \in \mathbb{R}^{p\times q}[s]$, define

$$X_R = \{f(s) \in \mathbb{R}^p[s] \mid f(s) = R(s)g(s) \text{ for some strictly proper } g(s)\}. \tag{5.11}$$

This space of polynomials is associated with the 'smooth' part of the behavior (as will be shown more explicitly in Part II of this paper). We now introduce spaces of polynomials that are related to the 'impulsive' part.

Definition 5.9 An $\mathbb{R}$-linear subspace of $\mathbb{R}^k[s]$ will be called *shift invariant* if it is closed under the operation $\sigma: f(s) \mapsto (f(s) - f(0))/s$. A polynomial matrix $L(s)$ will be called a *minimal impulse generator* if its columns are independent over $\mathbb{R}$ and $\operatorname{span}_{\mathbb{R}} L(s)$ is shift invariant.

Example 5.10 The space $\operatorname{span}_{\mathbb{R}} \{s^2, s, 1\}$ is shift invariant, the space $\operatorname{span}_{\mathbb{R}} \{s^3, s, 1\}$ is not.

Definition 5.11 A pair of polynomial matrices $(R(s), V(s)) \in \mathbb{R}^{p\times q}[s] \times \mathbb{R}^{n\times q}[s]$ is called *eligible* if the following conditions hold:

(i) $[-V(s) \;\; R(s)]$ has full row rank for all $s \in \mathbb{C}$;

(ii) $R(s)$ has full row rank as a rational matrix;

(iii) the columns of $V(s)$ are linearly independent over $\mathbb{R}$;

(iv) $\operatorname{span}_{\mathbb{R}} V(s) = X_R + R(s) \operatorname{span}_{\mathbb{R}} T(s)$ where $T(s)$ is a minimal impulse generator.

Lemma 5.12 *Let $(R(s), V(s))$ be an eligible pair. If $U(s)$ is unimodular and S is a nonsingular constant matrix, then the pair $(U(s)R(s), U(s)V(s)S)$ is also eligible, and $\mathcal{B}(UR, UVS) = \mathcal{B}(R,V)$.*

Proof This is immediate from the definitions. Note in particular that $X_{UR} = UX_R$ if U is unimodular. □

Remark 5.13 Other transformations that will not affect the set of solutions to the equations $R(p)w = V(p)x_0$ are the following: left multiplication of $V(s)$ and $R(s)$ by a nonsingular rational matrix, addition of zero rows to both $V(s)$ and $R(s)$, and right multiplication of $V(s)$ by a constant matrix of full row rank. It follows from Thm. 5.15 below that every pair of polynomial matrices (R,V) that can be obtained from an eligible pair by a sequence of operations of the types just mentioned is such that $\mathcal{B}(R,V) = \mathcal{B}(F,G,H)$ for some triple of constant matrices (F,G,H). It seems likely that also the converse statement is true, but we do not prove this here.

We next show that minimal impulse generators have a first-order representation.

Lemma 5.14 *A polynomial matrix $L(s) \in \mathbb{R}^{p\times n}[s]$ is a minimal impulse generator if and only if there exist matrices $H \in \mathbb{R}^{p\times n}$ and $G \in \mathbb{R}^{n\times n}$ such that G is nilpotent, $\ker\left[\begin{smallmatrix} G \\ H \end{smallmatrix}\right] = \{0\}$, and $L(s) = H(sG-I)^{-1}$. Moreover, this representation is unique.*

Proof Let $L(s)$ be a minimal impulse generator; then there exists a matrix $G \in \mathbb{R}^{g\times g}$ such that $(\sigma L)(s) = (L(s) - L(0))/s = L(s)G$. The matrix G is uniquely determined because the columns of $L(s)$ are linearly independent. Since $L(s)G^k = (\sigma^k L)(s) = 0$ for sufficiently large k, the matrix G is nilpotent. We have $L(s) = L(0) + sL(s)G$ and hence $L(s) = -L(0)(sG-I)^{-1} = H(sG-I)^{-1}$ with $H = -L(0)$. Finally, if x is such that both $Gx = 0$ and $Hx = 0$, then $L(s)x = (sL(s)G + H)x = 0$ so $x = 0$.

Conversely, suppose that G and H are matrices as in the statement of the lemma. Because G is nilpotent, the matrix $(sG-I)^{-1}$ is polynomial so $L(s) = H(sG-I)^{-1}$ is indeed a polynomial matrix. Moreover, this matrix satisfies $(L(s) - L(0))/s = (H(sG-I)^{-1} + H)/s = H(sG-I)^{-1}G$. To show that the columns of $L(s)$ are independent over $\mathbb{R}$, suppose there would be a nonzero constant vector x such that $H(sG-I)^{-1}x = 0$. Then there would exist a polynomial vector $y(s) = y_k s^k + \cdots + y_0$ with $y_k \neq 0$ such that $(sG-I)y(s) = x$. This equation implies $Gy_k = 0$, whereas we also have $Hy_k = 0$ because $Hy(s) = H(sG-I)^{-1}x = 0$. Because of the assumption $\ker G \cap \ker H = \{0\}$ we have a contradiction. □

We now show that eligible pairs are such that the relation (5.9) holds for some triple (F, G, H) of constant matrices. The next result may be seen as a realization theorem.

Theorem 5.15 *For every eligible pair $(R(s), V(s))$, there exists a matrix triple (F, G, H) such that $\mathcal{B}(R, V) = \mathcal{B}(F, G, H)$.*

Proof Let $R(s)$ have size $p \times q$. The construction in the proof of [11, Thm. 4.2] produces a triple (F_1, G_1, H_1) of constant matrices and a polynomial matrix $V_1(s)$ such that

$$[-V_1(s) \quad R(s)] \begin{bmatrix} sG_1 - F_1 \\ H_1 \end{bmatrix} = 0 \tag{5.12}$$

and the following properties hold:

(i) F_1 and G_1 have size $n_{\mathrm{sm}} \times (n_{\mathrm{sm}} + m)$, where $n_{\mathrm{sm}} := \dim X_R$ and $m := q - p$, and H_1 has size $q \times (n_{\mathrm{sm}} + m)$;

(ii) $\mathrm{span}_{\mathbb{R}}\, V_1(s) = X_R$;

(iii) G_1 has full row rank, $\ker \begin{bmatrix} G_1 \\ H_1 \end{bmatrix} = \{0\}$, and $\begin{bmatrix} sG_1 - F_1 \\ H_1 \end{bmatrix}$ is left unimodular.

By replacing the given matrix $V(s)$ with one of the same linear span if necessary, we may assume that $V(s) = [V_1(s) \quad V_2(s)]$ where $V_2(s) = R(s)T(s)$ and $T(s)$ is a minimal impulse generator. Let $n_{\text{p-imp}}$ denote the number of columns of $V_2(s)$. By the previous lemma, we may write $T(s) = H_2(sG_2 - I)^{-1}$ where G_2 is nilpotent and $\ker G_2 \cap \ker H_2 = \{0\}$; note that H_2 has size $q \times n_{\text{p-imp}}$. By construction, we have

$$[-V_1(s) \quad -V_2(s) \quad R(s)] \begin{bmatrix} sG_1 - F_1 & 0 \\ 0 & sG_2 - I \\ H_1 & H_2 \end{bmatrix} = 0. \tag{5.13}$$

So if we define

$$F = \begin{bmatrix} F_1 & 0 \\ 0 & I \end{bmatrix}, \quad G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}, \quad H = [H_1 \quad H_2], \tag{5.14}$$

we have

$$\operatorname{im}\begin{bmatrix} sG-F \\ H \end{bmatrix} \subset \ker\,[-V(s) \quad R(s)]. \tag{5.15}$$

To prove equality, it suffices to show that the dimensions of the two subspaces are equal. The matrix appearing on the left has full column rank by property (iii) above and by the fact that $sG_2 - I$ is nonsingular; its number of columns is $n_{\text{sm}} + n_{\text{p-imp}} + m$. The matrix $[-V(s) \quad R(s)]$ has p rows and $n_{\text{sm}} + n_{\text{p-imp}} + q$ columns, and it has full row rank because $R(s)$ has full row rank. Since $m = q - p$, the equality of the dimensions is established and the proof is complete by Lemma 5.7. □

It will be shown in Part II of this paper that, conversely, to every matrix triple (F, G, H) there exists an eligible pair $(R(s), V(s))$ such that $\mathcal{B}(F, G, H) = \mathcal{B}(R, V)$.

Remark 5.16 The realization that has been constructed in the proof above enjoys a number of special properties. First of all, by the fact that G_1 has full row rank it follows that the matrix

$$sG - F \;=\; \begin{bmatrix} sG_1 - F_1 & 0 \\ 0 & sG_2 - I \end{bmatrix}$$

has full row rank as a rational matrix. Secondly, suppose that

$$\begin{bmatrix} sG_1 - F_1 & 0 \\ 0 & sG_2 - I \\ H_1 & H_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;=\; 0$$

for some constant vector $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ and some s. Because $sG_2 - I$ is invertible for all s, the component x_2 must be zero, and the left unimodularity of $\left[\begin{smallmatrix} sG_1 - F_1 \\ H_1 \end{smallmatrix}\right]$ then implies that also x_1 vanishes. This shows that $\left[\begin{smallmatrix} sG-F \\ H \end{smallmatrix}\right]$ is left unimodular. Thirdly, we can also show that $\ker\left[\begin{smallmatrix} G \\ H \end{smallmatrix}\right] = \{0\}$. Indeed, suppose that $\left[\begin{smallmatrix} G \\ H \end{smallmatrix}\right]x = 0$ for some constant vector x. From the equality $(R(s)H - V(s)(sG - F))x = 0$ it then follows that $V(s)Fx = 0$, which implies that $Fx = 0$ since the columns of $V(s)$ are linearly independent. But then $\left[\begin{smallmatrix} sG-F \\ H \end{smallmatrix}\right]x = 0$ and so $x = 0$.

The three properties that we have established will be shown in Part II to characterize minimality of representations of the form (4.1). So the proof actually gives the construction of a *minimal* first-order representation corresponding to an eligible pair $(R(s), V(s))$.

Remark 5.17 Of course it is also possible to get a conventional representation (i. e. one of the form (4.4)) for an eligible pair $(R(s), V(s))$, by applying the corresponding construction in the proof of Prop. 4.4.

6 Conclusions

In this paper we have presented an approach to the modelling of linear multimode systems, based on the behavioral framework of J.C. Willems and the functional setting of the class of impulsive distributions as introduced by M.L.J. Hautus. The approach will in particular be appropriate in cases where transitions between different modes are not always smooth; such behavior is to be expected when the dynamics corresponding to various modes do not all have the same McMillan degree. We have proposed a specification in first-order form for finite-dimensional time-invariant piecewise linear behaviors. Such a specification consists of a description of the system's behavior in a particular mode on an interval between switches, together with jump conditions that describe the transitions from one mode to another.

A detailed analysis has been made of possible descriptions for the behavior between switches, taking into account the possibility of impulsive behavior at the switching instant. Two types of first-order representations were studied; one motivated by the standard approach to singular systems in which the initial condition is always located in the same subspace as the state derivative, and one in which this requirement need not hold, so that in fact it might be preferable to speak about 'initial data' rather than about an initial condition in the sense of differential equations. The first type we called the conventional form, the second type we named pencil form after a similar representation used in [11]. We showed that the two representations types have the same descriptive power (that is, they describe the same class of behaviors), by explicitly transforming conventional representations into pencil representations and vice versa. While there is thus no distinction between the two representations from the point of view of expressive power, it might be said that pencil representations are in general more economical than conventional representations in the sense that the number of variables and equations is generally less.

Next to first-order representations, we have also considered polynomial representations of impulsive-smooth behavior. Such representations are convenient in a mathematical analysis of minimality conditions, as will be shown more extensively in Part II of this paper; but they also hold an interest of their own, since often system properties can be expressed most concisely in terms of a polynomial representation. We have defined polynomial representations of impulsive-smooth behaviors by using a pair of polynomial matrices, which should be in a very specific relation to each other in order to make sure that the corresponding behavior can also be represented in first-order form. This relation is specified in the notion of eligibility (Def. 5.11).

The emphasis in this paper has been on the representation of behavior on intervals between switches, including a possible impulse at the switching instant. Jump conditions have only been discussed insofar as they are needed in the specification of the full piecewise linear behavior. Certainly there is more to be said about the relation between jump conditions and mode dynamics; this issue will be addressed in future work. The analysis of representations of impulsive-smooth behaviors is not complete in the sense that we have not discussed yet under what conditions representations are minimal, and how minimal representations of the same behavior are related to each other. That task will be taken up in Part II of this paper, in which in particular we obtain a state space isomorphism theorem for pencil representations.

References

[1] R.W. Brockett. Smooth multimode control systems. In L. Hunt and C. Martin, editors, *Proc. 1983 Berkeley-Ames Conference on Nonlinear Problems in Control and Fluid Mechanics*, pages 103–110. Math. Sci. Press, 1984.

[2] R.W. Brockett. Hybrid models for motion control systems. In H.L. Trentelman and J.C. Willems, editors, *Essays on Control. Perspectives in the Theory and its Applications* (lectures and the minicourses of the European control conference (ECC'93), Groningen, the Netherlands, June/July 1993), pages 29–53. Birkhäuser, 1993.

[3] D. Cobb. On the solution of linear differential equations with singular coefficients. *J. Diff. Eq.*, 46:310–323, 1982.

[4] C.A. Desoer and E.S. Kuh. *Basic Circuit Theory.* McGraw-Hill, New York, 1969.

[5] G. Doetsch. *Introduction to the Theory and Application of the Laplace Transformation.* Springer, Berlin, 1974.

[6] P.A. Fuhrmann. *Linear Systems and Operators in Hilbert Space.* McGraw-Hill, New York, 1981.

[7] T. Geerts. Invariant subspaces and invertibility properties for singular systems: the general case. *Lin. Alg. Appl.*, 183:61–88, 1993.

[8] T. Geerts. Solvability conditions, consistency and weak consistency for linear differential-algebraic equations and time-invariant linear systems: the general case. *Lin. Alg. Appl.*, 181:111–130, 1993.

[9] M.L.J. Hautus. The formal Laplace transform for smooth linear systems. In G. Marchesini and S.K. Mitter, editors, *Mathematical Systems Theory*, Lect. Notes Econ. Math. Syst. 131, pages 29–47. Springer, New York, 1976.

[10] M.L.J. Hautus and L.M. Silverman. System structure and singular control. *Lin. Alg. Appl.*, 50:369–402, 1983.

[11] M. Kuijper and J.M. Schumacher. Realization of autoregressive equations in pencil and descriptor form. *SIAM J. Contr. Optimiz.*, 28:1162–1189, 1990.

[12] O. Maler, Z. Manna, and A. Pnueli. From timed to hybrid systems. In J.W. de Bakker *et al.*, editor, *Real-time: theory in practice* (Proc. REX workshop, Mook, the Netherlands, June 1991), Lect. Notes Comp. Sci. 600, pages 447–484. Springer, Berlin, 1991.

[13] X. Nicollin, J. Sifakis, and S. Yovine. From ATP to timed graphs and hybrid systems. In J.W. de Bakker *et al.*, editor, *Real-time: theory in practice* (Proc. REX workshop, Mook, the Netherlands, June 1991), Lect. Notes Comp. Sci. 600, pages 549–572. Springer, Berlin, 1991.

[14] V.I. Utkin. Variable structure systems with sliding modes. *IEEE Trans. Automat. Contr.*, AC-22:212–222, 1977.

[15] G.C. Verghese, M. Ilić-Spong, and J.H. Lang. Modeling and control challenges in power electronics. In *Proc. 25th IEEE Conf. Dec. Contr.* (Athens, Greece, Dec. 1986), pages 39–45, 1986.

[16] J.C. Willems. Paradigms and puzzles in the theory of dynamical systems. *IEEE Trans. Automat. Contr.*, AC-36:259–294, 1991.

[17] A.S.I. Zinober, editor. *Variable Structure and Lyapunov Control*. Lect. Notes Contr. Inform. Sci. 193. Springer, Berlin, 1994.