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# Equilibrium and Transient System Effectiveness of a Production Unit with an Output Buffer

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**Abstract**

We present equilibrium and transient analysis of a system that consists of a failure prone production unit with a buffered output. We give an exact analysis of the equilibrium system effectiveness by using the equivalence of the buffer contents in the production system to the work in progress in a GI/G/1 queue with bounded virtual waiting time. Two approximation methods are introduced for the computation of equilibrium and transient performance measures.
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*Keywords & Phrases*: System effectiveness, preventive maintenance, transient behaviour.

## 1 Introduction

For production systems where a number of machines are employed to process (intermediate) materials, it has widely been recognized that buffers are necessary to overcome temporary disturbances that are caused by failures of the machines. A buffer may prevent a downstream machine from starvation when an upstream machine breaks down, and vice versa, may prevent an upstream machine from blocking when a downstream machine breaks down. In this way buffers can improve the system effectiveness of the production plant.

Since buffers can reduce the effects of failures, i.e. unscheduled downtimes, on system performance, it might also be possible to use them to introduce scheduled downtimes, i.e. preventive maintenance, without disturbing the output of the production process. In this paper we present a quantitative analysis of production lines with buffers.

We consider production lines where the flow of raw and intermediate materials can be represented by a fluid flow. We assume that there is no shortage of input to the production machine. There is also a deterministic and constant demand for the output of the machine. The production rate of the machine, when operational, is larger than the demand rate, and the surplus output can be stored in a finite buffer. When the machine has failed and is under repair, then the demand can temporarily be satisfied from this buffer, until it runs empty. Backlogging is not allowed and the demand occurring during a period when the production unit is down and the buffer is empty, is lost.

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The main performance measure that we are interested in, is *system effectiveness*: the fraction of demand that can be satisfied by the output process. In our model the production unit cannot work at a speed slower than the demand rate, and the system effectiveness is thus the fraction of time that the plant is capable of meeting the demand, or equivalently, the fraction of time that the buffer is not empty. *Long term* or *equilibrium system effectiveness* is important in the *design phase* of the installation, where one must determine, for instance, the optimal sizes for the buffers, the best layout of the plant, or calculate the effect of improved redundancy by introducing parallel machines. *Interval system effectiveness* is a performance measure of the *transient* behaviour of the system: given the state at a starting point, determine the distribution function of the total lost demand over a finite interval. This measure is important at the operational planning level.

This paper is organised as follows. In Section 2 we present an overview on the literature of this subject. In Section 3 the model is introduced. The equilibrium and the transient analysis are presented in Section 4 and 5, respectively. Preventive maintenance strategies are discussed in Section 6. We conclude the paper with numerical results in Section 7.

## 2 Literature Overview

In the literature one finds quite a few references to production lines with buffers and unscheduled downtimes, but to the best of our knowledge there are few references that deal with the interaction between buffers and maintenance.

We have found a number of references to models of inventory systems with continuous buffers. In De Kok et al. [14] a single machine production/inventory system is considered, where the production unit produces at a constant rate, while demands are generated by a compound Poisson process. The authors present a two-moments approximation for the control rule for starting up and shutting down the production. In De Kok et al. [15] a similar model is discussed, but here the production can be switched between two rates, buffer storage capacity is infinite and backlog is allowed. In [14] and [15] one can also find more references to papers on controlled production/inventory models where production rate is the control variable. In Vanneste and Van der Duyn Schouten [27] a model is discussed for scheduling preventive maintenance of the production unit. Vanneste derives some structural properties for the optimal policy, and an analysis of nearly optimal policies.

In computer communications literature the interest in fluid models has increased considerably during the last few years, mainly because of applications in high speed networking technology. In this field an important model was introduced by Kosten [16]. In the model an infinite number of independent sources are alternating between an up and a down state. When a source is up, then it generates a constant input flow into an infinite buffer that is replenished by a server at a constant rate. The up and down times of the sources are mutually independent and exponentially distributed. This model is extended in Kosten [17] and Kosten [18] to include Erlang and hyperexponentially distributed up times, respectively. Kosten provides asymptotic results for large buffer contents in [19].

Anick et al. [3] consider a generalization of the model of Kosten with a finite number of sources, that have independent exponentially distributed up and down times. After the publication of [3] this model has been extended by several authors. Tucker considers the case of finite buffers in [26]. The analysis can be carried out in the same manner as in the original model with the exception that the finiteness of the buffer introduces extra boundary conditions. A more substantial generalization is presented by Mitra in [22]. He considers a model with $m$ machines that provide an input flow into a finite buffer and $n$ machines that deplete the buffer. Both the input and output machines are subject to failures that are generated from exponential distributions.

The model of Mitra [22] is extended by Stern and Elwalid in [23] to include more general distributions for the up and down times of the machines. They assume that the input and output flows are driven by independent Markov Modulated Rate Processes. In particular this means that the state of each source is modelled as an independent Markov process. They give special numerical algorithms for the case when these underlying processes are reversible and they provide approximations. An extension to a model with a number of independent subsystems for input and output can be dealt with via a decomposition approach.

A related non-fluid model is presented in Elwalid et al. [9]. In this model the buffer contents are discrete and the input sources are given as a number of independent Markov Modulated Poisson sources. Related models can be also be found in De Koster [8].

The general trend in these studies is to put the emphasis on the probability of buffer overflow, since that is an appropriate performance measure in the telecommunication applications for which these models were originally developed. An exception is the model for a production line that is presented in Mitra [22]. In that paper one of the numerical examples deals with the computation of system effectiveness. It seems unlikely that the fluid models in these papers allow a straightforward introduction of repair strategies.

Another class of models was introduced in Akella and Kumar [2]. They address the problem of controlling the production rate of a machine feeding a buffer that is depleted at a constant rate. The up and down times of the machine are exponentially distributed, the cost function is a convex function of the buffer contents and the objective is to minimize the discounted inventory costs. They show that the optimal policy is a so-called hedging point policy: if the buffer contents are less than the hedging point, then production should be maximal (if possible) and if the buffer contents are equal to the hedging point, then the production rate should be equal to the demand rate. As a result the buffer may be considered finite under such a policy.

In Chen and Yao [4] a model is considered where the inflow into an infinite buffer is constant, and the processing machine that feeds from the buffer is subject to failures. In its most general form the up and down times $\{d_i, u_i\}$ form a sequence of i.i.d. *pairs* of random variables — so the up and down time in one cycle need not be independent — that may have a general distribution. An approach with regenerative processes is used to derive stability conditions and a functional equation for the equilibrium buffer contents distribution. They also consider the special cases where either the down times or both the up and down times are exponentially distributed.

Chen and Yao also prove that the fluid model under consideration can be viewed as the limit of a sequence of GI/G/1 queuing models with disruptions. Similar results are presented in Kella and Whitt [11] and Hu and Xiang [10]. The latter reference deals with a model where the machine's processing rate is operated under a hedging point policy, so in effect they are dealing with a model that has a finite buffer. They show that the production surplus (or rather the difference between the production surplus and the maximum buffer contents) embedded at the moments when the machines comes up (goes down) is stochastically equivalent to the customer waiting (sojourn) time in a GI/G/1 queue. By using up and down crossings arguments they show how the equilibrium buffer contents can be related to the equilibrium virtual waiting time process of a GI/G/1 queue. Their model allows a demand backlog, i.e.

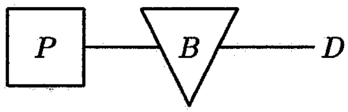


Figure 1: Model of a production unit and a buffer

negative buffer contents, but it's likely that results from single server queues with a limited waiting room (cf. for instance De Kok and Tijms [13]) can be used to overcome this limitation.

As a final remark we refer to recent research on the application of the theory of large deviations on fluid models (cf. Weiss [29], Kesidis et al. [12], Weber [28] and Courcoubetis and Weber [6]). Again these models typically deal primarily with telecommunication applications, where the rare event of a buffer overflow is the main concern. It's unclear whether these results can be applied for the evaluation of system effectiveness.

The analysis in all the papers that were mentioned deal with the equilibrium performance of the considered models. An exception is Malhamé and Boukas [20] who introduce a generalization of the model of Akella and Kumar [2]. They describe how the transient evolution of a single production manufacturing system under a hedging point policy can be characterized by a system of coupled partial differential equations.

Although we do not consider networks of production systems, we refer to Dallery and Gershwin [7] for more information on this subject.

## 3 A model with one machine and one buffer

The model that we want to consider consists of one machine $P$ and one buffer $B$ (see Figure 1). If the machine is operational, then it produces fluid at a constant rate and this is put into a finite buffer. If the machine is operational and the buffer is full, then the excess production is lost. The buffer is depleted with a constant demand, represented by $D$.

As long as the buffer is not empty, the demand can be fulfilled. The production rate, when the machine is up, is larger than or equal to the demand rate, so the only time that demand cannot be satisfied is when the buffer is empty (which can of course only happen when the machine is down). Backlog of demand is not allowed, so the buffer contents cannot become negative. In particular this means that system effectiveness is completely determined by the fraction of time that the buffer is empty.

Before we proceed with the analysis we introduce the following notation:

$F_u$ : Distribution function of the life or up time of the machine (mean $E_u$).

$F_c$ : Distribution function of the corrective maintenance or repair time (mean $E_c$).

$K$ : Maximum buffer contents.

$d$ : Demand rate.

$r$ : Production rate when the machine is operational ($r > d$).

We shall discuss the equilibrium analysis of this model in Section 4 and the transient analysis in Section 5. In both sections we restrict our attention to systems without preventive maintenance. The analysis of preventive maintenance strategies is reported in Section 6.

# 4 Equilibrium System Effectiveness

In this section we describe how the system effectiveness of the buffer-model can be computed. System effectiveness is defined as follows. We let $\{X(t) \mid t \geq 0\}$ denote the process of the buffer contents. *Interval system effectiveness* $\{S(t) \mid t \geq 0\}$ is the stochastic process that is derived from $X$ as

$$S(t) = \frac{1}{t} \int_0^t 1(X(u) > 0)\, du, \tag{1}$$

i.e. the fraction of time during the interval $[0, t]$ that the buffer is not empty. In our model this is equal to the fraction of demand that can be satisfied, which is the usual definition found in literature. In production/inventory models this quantity is also referred to as *service level*, i.e. the fraction of demand that can be met from stock on hand (cf. for instance Tijms [25]).

The *equilibrium system effectiveness* $S$ is defined as

$$S = \lim_{t \to \infty} E\left\{ \frac{1}{t} \int_0^t 1(X(u) > 0)\, du \right\}, \tag{2}$$

that is, the expected longrun fraction of time that the buffer is not empty.

Finally we introduce for both the interval and the equilibrium version the convention that *system ineffectiveness* is defined as $1 - S$.

In the remainder of this section we describe one exact and two approximative methods for the computation of the equilibrium system effectiveness. The exact method is based on the equivalence of the buffer contents and the virtual waiting time in a finite capacity queue. It allows the complete analysis of models in which either the up times or the down times are exponentially distributed.

The two approximation methods use an approach with alternating renewal processes. They can be applied to models with general life and repair time distributions.

## 4.1 An Equivalent M/G/1 Model

In this section we show how the stochastic process of the buffer contents is equivalent to the workload process in a GI/G/1 queue with uniformly bounded virtual waiting times. This equivalence has been described already in Cohen [5, Chapter III.5], and has gained new popularity recently in Hu and Xiang [10] and Kella and Whitt [11].

This model is also discussed in Meyer *et al.* [21] for exponentially and deterministic distributions of the life times and repair times. Although they don't recognize the equivalence to the GI/G/1 queueing model, they derive expressions for the equilibrium system effectiveness and for the average buffer inventory level.

The equivalence can be constructed as follows. Consider a realization of the buffer contents $X(t)$ as in Figure 2. In this figure the dark grey areas represent time intervals when the machine is up, and the light grey areas correspond to machine repairs. We construct two new processes $\{X_c(t) \mid t > 0\}$ and $\{X_u(t) \mid t > 0\}$ as follows: $X_c$ has the same behaviour as $X$, but in $X_c$ time is halted when the machine is up. Analogously, $X_u$ is constructed by stopping the time clock when the machine is down (cf. Figure 2). By writing down the recurrence relations for the process $X$ embedded at the time instants when the machine comes up or goes down, we get relations in the form of the well known Lindley equations for the GI/G/1 queue (cf. Hu and Xiang [10]). From these equation it follows that that $X_c$ corresponds to the virtual waiting time process of a $GI/G/1$ queue, with interarrival times distribution function

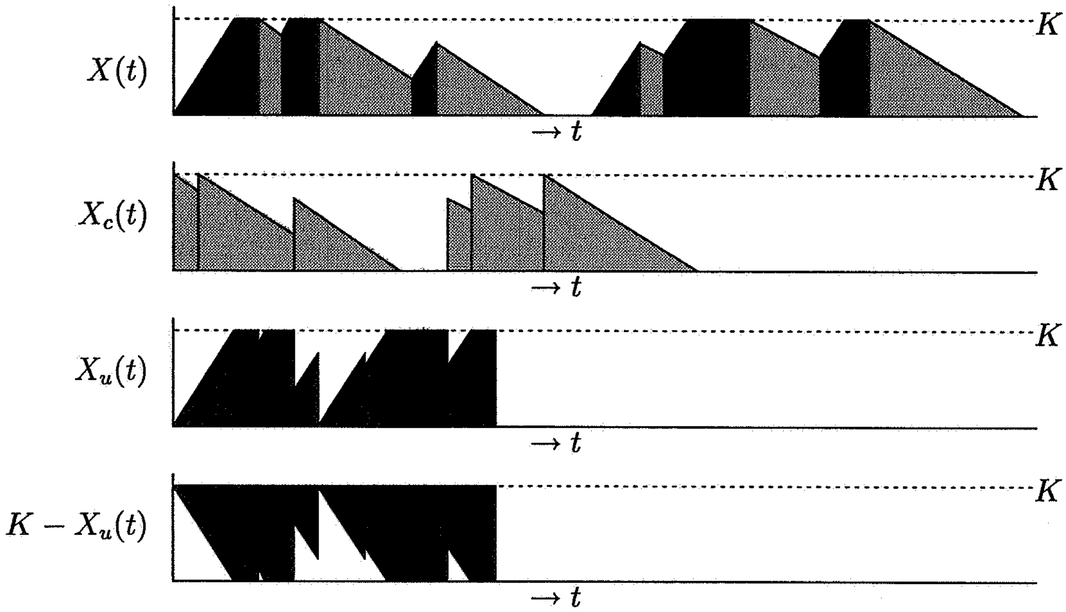


Figure 2: A sample path of the buffer contents and the corresponding dual queueing processes

$F_c$ (scaled with a factor $d$) and service time distribution $F_u$ (scaled with a factor $r-d$), and with a uniform bound on the virtual waiting time. Analogously $Y_u \equiv K - X_u$, is the virtual waiting time process of a $GI/G/1$ queue with uniformly bounded virtual waiting time, and it can be constructed from $X_c$ by interchanging the interarrival and service time distributions. This duality was already observed in Cohen [5, Chapter III.5].

The equilibrium distribution function $F^X$ of $X$ can be reconstructed from the distribution function $F^{X_c}$ of $X_c$ (or equivalently that of $X_u$). By using an up and down crossings counting argument, one can show that:

$$F^X(0) = \frac{E_c}{E_u + E_c} F^{X_c}(0), \tag{3}$$

$$F^X(t) = F^X(0) + \frac{rE_c}{(r-d)(E_u+E_c)}(F^{X_c}(t) - F^{X_c}(0)), \quad \text{if } 0 < t < K, \tag{4}$$

$$F^X(K) = 1. \tag{5}$$

Note that $F^X(0)$ is the equilibrium probability that the buffer is empty, or equivalently, the equilibrium system ineffectiveness.

### Exponential down times

For general distribution functions it is difficult to find a solution for $F^{X_c}$, but the analysis becomes feasible when one of the distributions is exponential. Assume, e.g., that the down times are exponentially distributed, then the $X_c$ process corresponds to an $M/G/1$ queue (with bounded virtual waiting time) and the distribution function $F^{X_c}$ can be constructed from the distribution function for the *unbounded* $M/G/1$ queue, by truncation (cf. Cohen [5, p.536]):

$$F^{X_c}(x) = \frac{W(x)}{W(K)}, \quad 0 \leq x \leq K. \tag{6}$$

Here $W$ is the waiting time distribution function of the $M/G/1$ queue with unbounded virtual waiting times. $W$ can of course be determined by inversion from its Laplace-Stieltjes transform (LST) $\mathcal{W}(s)$:

$$\mathcal{W}(s) = \frac{(1-a)\alpha s}{\beta(s) - 1 + \alpha s} \tag{7}$$

where

$$\begin{aligned} \alpha &= d * E_c && (8) \\ \beta &= (r-d) * E_u && (9) \\ a &= \beta/\alpha && (10) \end{aligned}$$

and the LST $\beta(s)$ of the service time distribution is related to the LST $\mathcal{F}_u$ of $F_u$ via $\beta(s) = \mathcal{F}_u(s(r-d))$. In terms of the queue parameters, $\alpha$ is the mean interarrival time, $\beta$ the mean service time, and $a$ the traffic load.

For the computation of the longterm system ineffectiveness $F^X(0)$, we thus need to compute only $F^{X_c}(0)$ and (cf. Cohen [5, Chapter III.5,§5]) this is equal to

$$F^{X_c}(0) = \left[\frac{1}{2\pi i}\int_{C_\eta} \frac{e^{\eta K}\alpha}{\beta(\eta) + \alpha\eta - 1} d\eta,\right]^{-1} \tag{11}$$

where $C_\eta$ is a suitably chosen contour that includes all zeroes of $\beta(\eta) + \alpha\eta - 1$.

In the general case, when the lifetime distribution has a rational LST $\beta(s) = \beta_1(s)/\beta_2(s)$, with $\beta_1$ and $\beta_2$ polynomials and $n_1 = \text{degree}(\beta_1) < n_2 = \text{degree}(\beta_2)$, then the denominator of the integrand of (11) has exactly $n_2 + 1$ zeroes $\eta_0, \ldots, \eta_{n_2}$. Typically $\eta_0 = 0$ is a zero. In most cases the remaining zeroes are distinct, so $F^{X_c}(0)$ is given by

$$F^{X_c}(0) = \left[\sum_{j=0}^{n_2} \text{Res}_{\eta=\eta_j}\left\{\frac{e^{\eta K}\alpha\beta_2(\eta)}{\beta_1(\eta) + (\alpha\eta - 1) * \beta_2(\eta)}\right\}\right]^{-1}, \tag{12}$$

and thus the complexity of the computation of the system effectiveness is equivalent to that of finding the zeroes of an $n_2$-degree polynomial.

If, for instance, the up times are exponentially distributed, then $\beta(s) = 1/(1+\beta s) = 1/(1+(r-d)E_u s)$ and the long term system ineffectiveness is

$$F^{X_c}(0) = \begin{cases} \dfrac{1-a}{1 - a * \exp(-(1-a)K/\beta)}, & \text{if } a \neq 1, \\ \dfrac{1}{(1+K/\beta)}, & \text{if } a = 1. \end{cases} \tag{13}$$

Note that the zeroes of $\beta(\eta) + \alpha\eta - 1$ are $\eta_0 = 0$ and $\eta_1 = (\beta - \alpha)/\alpha\beta$, and thus $\eta = 0$ is a zero with multiplicity two when $\alpha = \beta$, or equivalently $a = 1$.

For Erlang-2 distributed up times, we get $\beta(s) = 1/(1+\beta s/2)^2$, and

$$F^{X_c}(0) = \frac{9}{8 + 12K/\beta + \exp(-3K/\beta)}, \quad \text{if } a = 1, \tag{14}$$

and

$$F^{X_c}(0) = \frac{\beta^2\eta_1\eta_2(\eta_1-\eta_2)}{4(\eta_1-\eta_2)+4\eta_2(1+\beta\eta_1/2)^2e^{\eta_1 K}-4\eta_1(1+\beta\eta_2/2)^2e^{\eta_2 K}}, \text{ if } a \neq 1, \tag{15}$$

where $\eta_0 = 0$, $\eta_1$ and $\eta_2$ are the zeroes of the polynomial $1-(1-\alpha\eta)(1+\beta\eta/2)^2$. Again in this case $\eta_0 = 0$ has multiplicity two when $\alpha = \beta$.

If the up times have a Hyperexponential distribution of order 2, then $\beta(s) = p/(1+\gamma_1 s) + (1-p)/(1+\gamma_2 s)$, for some $p \in (0,1)$, $\gamma_1 > 0$, $\gamma_2 > 0$, and thus for $a = 1$

$$F^{X_c}(0) = \frac{\gamma_1\gamma_2 C^2}{\exp(C*K)(1+\gamma_1 C)(1+\gamma_2 C)-1-C(K+\gamma_1+\gamma_2)} \tag{16}$$

where $C = 1/\gamma_1 + 1/\gamma_2 - 1/\alpha$, while for $a \neq 1$ we get

$$F^{X_c}(0) = \frac{\gamma_1\gamma_2\eta_1\eta_2(\eta_1-\eta_2)}{(\eta_1-\eta_2)+(1+\gamma_1\eta_1)(1+\gamma_2\eta_1)\eta_2 e^{\eta_1 K}-(1+\gamma_1\eta_2)(1+\gamma_2\eta_2)\eta_1 e^{\eta_2 K}}, \tag{17}$$

where $\eta_1$ and $\eta_2$ are the zeroes of the polynomial

$$p(1+\gamma_2\eta)+(1-p)(1+\gamma_1\eta)+(a\eta-1)(1+\gamma_2\eta)(1+\gamma_1\eta).$$

**Remark 4.1** Note that the system effectiveness can be computed very trivially for the degenerate cases of no buffer ($K = 0$) and infinite buffer ($K = \infty$). When $K = 0$, then demand cannot be fulfilled whenever the machine goes down, and thus the equilibrium system effectiveness is equal to the availability of the production machine. When $K = \infty$ and $a < 1$, then the corresponding M/G/1 queue is a "genuine" M/G/1 queue with unlimited virtual waiting time and thus $F^{X_c}(0) = 1-a$. As a result for $K = \infty$ and $a \geq 1$ the buffer process is unstable.

In Figures 6–8 the distribution function $F^X$ is depicted for four different up time distribution types. The parameters that are used for these plots are $E_u = 10$–$40$, $E_c = 1$, $r = 1.05$, $d = 1$, $K = 1$. The service time distributions of these examples are discussed in more detail in Section 7.

### Exponential up times

When the up times of the machine are exponentially distributed, then the translation to the M/G/1 queue proceeds via the $Y_u(t) \equiv K - X_d(t)$ process of Figure 2. The process $Y_u$ then corresponds to the uniformly bounded virtual waiting time process of an M/G/1 queue with interarrival time distribution function $F_u$ (scaled with a factor $r-d$) and service time distribution $F_c$ (scaled with a factor $d$). Analogously to the construction of $F^{X_c}$ for exponential down times, the distribution function $F^{Y_u}$ of $Y_u$ is derived from the waiting time distribution of the unbounded M/G/1 queue by truncation at $K$.

Using up and down crossing arguments we then reconstruct the equilibrium distribution function $F^Y$ of $Y$ as:

$$F^Y(0) = \frac{E_u}{E_u+E_c}F^{Y_u}(0), \tag{18}$$

$$F^Y(t) = F^Y(0) + \frac{rE_u}{d(E_u+E_c)}(F^{Y_u}(t)-F^{Y_u}(0)), \quad \text{if } 0<t<K, \tag{19}$$

$$F^Y(K) = 1. \tag{20}$$

Note that here the mean interarrival time is $\alpha = (r-d) * E_u$ and the mean service time $\beta = d * E_c$. Since $Y \equiv K - X$, the buffer is empty when $Y(t) = K$, so the equilibrium system effectiveness is given by $\lim_{t\to\infty} P(Y(t) < K)$. For the process $Y_u(t)$ we know that $F^{Y_u}(K-) = 1$, so the system ineffectiveness is

$$1 - F^Y(K-) = 1 - \frac{E_u}{E_u + E_c}\left\{(1 - \frac{r}{d})F^{Y_u}(0) + \frac{r}{d}\right\}. \tag{21}$$

### 4.2 An Alternating Renewal Process Approach

In this section we give an approximation method that models the stochastic process of the buffer contents as an alternating renewal process. It was originally introduced to provide an approximate analysis of the transient behaviour (see Section 5), but it appeared to give fairly accurate approximations for the equilibrium system effectiveness too. As a bonus it allows more general distribution types than the M/G/1 model, since it does not require that either of the two distributions is exponential.

The approximation is based on the following assumption:

**Approximation Assumption:** The parameters of the model are such that every up time is long enough to fill the buffer, even when it is empty at the beginning of the up time.

It is more likely that the assumption will be satisfied when the buffer size is not too large when compared with $E_u * (r - d)$, the expected net buffer growth during one up time. Of course the assumption is also more likely to be met when the up times have a behaviour that is close to deterministic.

#### 4.2.1 Approach A

Under the approximation assumption we can derive expressions for the long term system effectiveness. For this purpose we describe the dynamics of the buffer contents by the alternating renewal process of the intervals when the machine is up and down (see Figure 3). Here the up time intervals are denoted as $u_1, u_2, \ldots$, and the down time intervals as $c_1, c_2, \ldots$. Each pair $(u_i, c_i)$, $i \in I\!N$, represents a cycle of an up and a down time, and at the end of the down time interval there is an interval $e_i$ during which the buffer may become empty. Note that if the down time $c_i$ is too short to empty the buffer, then $e_i$ has length zero (cf. $e_1$, $e_2$, $e_4$ and $e_5$ in Figure 3). We call the $e_i$'s *empty buffer intervals.* If no confusion can arise, we also use the notation $u_i$, $c_i$, $e_i$, to denote the lengths of the corresponding intervals.

Due to the approximation assumption the buffer is always full when the machine breaks down, and thus the probability that the buffer may become empty during a down time is the same for every repair. This probability is equal to $1 - F_c(K/d)$. Since the buffer need not become empty during a down time, the length of an empty buffer interval has a positive probability mass at zero.

Under the approximation assumption we do not know the buffer contents at the start of an up time interval $u$, but we do know that at the end of the up time the buffer is full, so the end of every $u_i$ is a regeneration point for $\{X(t) \mid t > 0\}$. From the theory of regenerative processes we have for the equilibrium system ineffectiveness:

$$1 - S = \frac{E[\text{empty interval length}]}{E[\text{cycle length}]} = \frac{E[e_i]}{E[u_i] + E[c_i]} = \frac{E[e_i]}{E_u + E_c}. \tag{22}$$

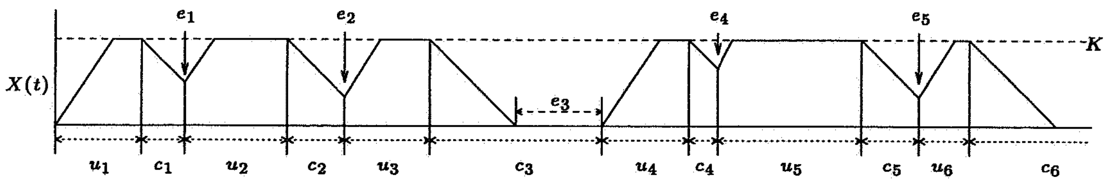


Figure 3: Alternating renewal process approach 1

The expected length of the interval that the buffer is empty, is equal to

$$E[e_i] = \int_0^\infty \max(t - K/d, 0)\ dF_c(t) = \int_{K/d}^\infty (1 - F_c(t))dt. \tag{23}$$

The main computational effort in Approach 1A lies in (23), but this is fairly simple calculus when the down times for instance have a mixed-Erlang distribution type.

### 4.2.2 Approach B

We already mentioned that the approximation assumption is likely to be violated when the buffer becomes large, as the probability that the buffer is not full at the end of an up time will become significant. It's much more likely that the contents of the buffer at the time of a failure of the machine is in the order of $E_u * (r - d)$, the average net buffer growth during an up time. We thus introduce the following approximation for $E[e_i]$:

$$E[e_i] = \int_0^\infty \max(t - T^*, 0)\ dF_c(t) = \int_{T^*}^\infty (1 - F_c(t))dt, \tag{24}$$

where

$$T^* = \min\{K/d, E_u * (r - d)/d\}. \tag{25}$$

This approach appears to give better approximations for system effectiveness (see Section 7), provided that $E_u * (r - d) < E_d * d$, in other words: the mean total surplus produced during an up time is smaller than the mean total demand during a down time. Numerical results of both approximation methods are discussed extensively in Section 7.

## 5 Interval System Effectiveness

The approach with alternating renewal processes, as depicted in the preceding section, is also quite suitable for transient analysis. It is of course only valid under Assumption A and it proceeds as follows. Consider the buffer at time $t = 0$, when the machine comes up and the buffer is empty (cf. Figure 4). Denote the interval until the buffer becomes empty again as $l_1$. In analogy with queueing models we shall call this a *busy period.* Such an interval is comprised of

- a stochastic number $N$ of combined up+down intervals ($(u_1, c_1)$ and $(u_2, c_2)$ in the first busy period in Figure 4),
- one up period $u_{N+1}$ ($u_3$ in Figure 4),
- one interval of deterministic length $K/d$.

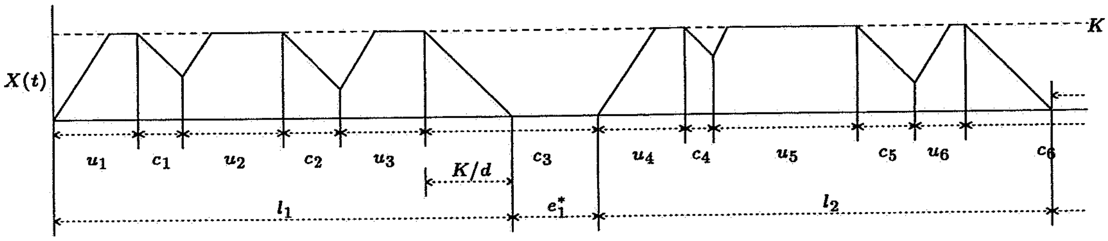


Figure 4: Alternating renewal process approach 2

After interval $l_1$ follows an interval $e_1^*$ during which the buffer is empty. At the end of $e_1^*$ the machine comes up again and the buffer fills up again for an interval $l_2$ and this process is repeated indefinitely.

We know that an arbitrary down time starts with a full buffer, so for any combination pair $(u_i, c_i)$ of an up and a down interval, the probability that the buffer will become empty during $c_i$ is $1 - F_c(K/d)$. If we denote $p = F_c(K/d)$, then the number $L$ of up and down combinations $(u_i, c_i)$ in an interval $l_i$, is geometrically distributed with parameter $p$. This means that the LST $\mathcal{F}_l$ of the distribution function for $l_i$, for each $i$, is

$$\begin{aligned} \mathcal{F}_l(s) &= E\Big[e^{-sl_1}\Big] \\ &= Ee^{-s\left[\sum_{j=1}^{N}(u_j+c_j)+u_{N+1}+K/d\right]} \\ &= \sum_{i=0}^{\infty}(1-p)p^i Ee^{-s\left[\sum_{j=1}^{i}(u_j+\bar{c}_j)+u_{i+1}+K/d\right]} \\ &= \sum_{i=0}^{\infty}(1-p)p^i(\mathcal{F}_u(s)\mathcal{F}_{\bar{c}}(s))^i\mathcal{F}_u(s)e^{-sK/d} \\ &= \frac{(1-p)\mathcal{F}_u(s)e^{-sK/d}}{1-p\mathcal{F}_u(s)\mathcal{F}_{\bar{c}}(s)} \end{aligned} \tag{26}$$

where $\mathcal{F}_{\bar{c}}$ is the Laplace transform of the repair time distribution function that is truncated at $K/d$, i.e.

$$F_{\bar{c}}(x) = \begin{cases} F_c(x)/F_c(K/d), & \text{if } x < K/d, \\ 1, & \text{if } x \geq K/d. \end{cases} \tag{27}$$

We have to use the truncated distribution for the $c_i$'s, since we know that the repair intervals in $l_i$ that are part of a $(u_i, c_i)$ combination, are *not* long enough to empty the buffer. The distribution function $F_l$ of $l_i$ can, in principle, be reconstructed from (26) by numerical inversion.

We also need to know the distribution of the length of the interval $e_i^*$ that the buffer is empty. Note that this is different from the distribution of $e_i$ in Figure 3, since $e_i^*$ in general has *no* probability mass at zero. In fact, the distribution function of $e_i^*$ is

$$P(e_i^* \leq x) = P(c_i \leq x + K/d \mid c_i > K/d) = \frac{F_c(x+K/d) - F_c(K/d)}{1 - F_c(K/d)}. \tag{28}$$

If we want to compute the mean and variance of $l_i$, then this could be done by differentiation of (26), but it's easier to follow a direct approach. The length of the first busy period can be written as

$$l_1 = \sum_{i=1}^{N}(u_i + \bar{c}_i) + u_{N+1} + K/d, \tag{29}$$

where

$$I\!\!P(N = n) = (1-p)p^n, \quad n = 0, 1, \ldots, \tag{30}$$

and since all the up intervals $u_i$ in the busy period have the same distribution, as do the (truncated) down times $\bar{c}_i$, the first two moments of $l_1$ are

$$\begin{aligned}
E(l_1) &= E\Big(\sum_{i=1}^{N}(u_i + \bar{c}_i) + u_{N+1} + K/d\Big) \\
&= \sum_{n=0}^{\infty} I\!\!P(N = n)\Big\{\sum_{i=1}^{n} E(u_i + \bar{c}_i) + E(u_{n+1}) + K/d\Big\} \\
&= \sum_{n=0}^{\infty} I\!\!P(N = n)\,\{n * E(u_1 + \bar{c}_1) + E(u_1) + K/d\} \\
&= E(N)[E(u_1) + E(\bar{c}_1)] + E(u_1) + K/d
\end{aligned} \tag{31}$$

and in a similar way

$$\begin{aligned}
E(l_1^2) &= E\Big(\sum_{i=1}^{N}(u_i + \bar{c}_i) + u_{N+1} + K/d\Big)^2 \\
&= \sum_{n=0}^{\infty} I\!\!P(N = n) E\Big[\Big(\sum_{i=1}^{n}(u_i + \bar{c}_i) + u_{n+1} + K/d\Big)^2\Big] \\
&= \sum_{n=0}^{\infty} I\!\!P(N = n) E\Big[\Big(\sum_{i=1}^{n}(u_i + \bar{c}_i)\Big)^2 + (u_{n+1} + K/d)^2 + 2(u_{n+1} + K/d)\sum_{i=1}^{n}(u_i + \bar{c}_i)\Big] \\
&= \sum_{n=0}^{\infty} I\!\!P(N = n) E\Big[\sum_{i=1}^{n}(u_i + \bar{c}_i)^2 + 2\sum_{i=1}^{n}\sum_{j \neq i}^{n}(u_i + \bar{c}_i)(u_j + \bar{c}_j) + \\
&\qquad + (u_{n+1} + K/d)^2 + 2(U_{n+1} + K/d)\sum_{i=1}^{n}(u_i + \bar{c}_i)\Big] \\
&= \sum_{n=0}^{\infty} I\!\!P(N = n)\Big[nE(u_1 + \bar{c}_1)^2 + n(n-1)E^2(u_1 + \bar{c}_1) + \\
&\qquad + E(u_{N+1} + K/d)^2 + 2nE(u_1 + K/d)E(u_1 + \bar{c}_1)\Big] \\
&= E(N)E(u_1 + \bar{c}_1)^2 + E(N(N-1))E^2(u_1 + \bar{c}_1) + E(u_{N+1} + K/d)^2 +
\end{aligned}$$

$$+2E(N)E(u_1+K/d)E(u_1+\overline{c}_1) \tag{32}$$

and thus

$$\sigma^2(l_1) = E(N)E(u_1+\overline{c}_1)^2 + E(N(N-1))E^2(u_1+\overline{c}_1) + E(u_{N+1}+K/d)^2 +$$

$$+2E(N)E(u_1+K/d)E(u_1+\overline{c}_1) - E^2(N)E^2(u_1+\overline{c}_1)+$$

$$-E^2(u_1+K/d) - 2E(N)E(u_1+K/d)E(u_1+\overline{c}_1)$$

$$= E(N)\sigma^2(u_1+\overline{c}_1) + \sigma^2(N)E^2(u_1+\overline{c}_1) + \sigma^2(u_1+K/d)$$

$$= E(N)\sigma^2(u_1+\overline{c}_1) + \sigma^2(N)E^2(u_1+\overline{c}_1) + \sigma^2(u_1) \tag{33}$$

with

$$E(N) = \frac{p}{1-p}, \tag{34}$$

$$\sigma^2(N) = \frac{p}{(1-p)^2}. \tag{35}$$

Once we have computed the distribution functions of the $l_i$'s and $e_i^*$'s, then we can use an approach of Takács [24] to compute the transient or interval system effectiveness. Denote the distribution functions of $l_i$ and $e_i^*$ as $F_l$ and $F_{e^*}$, and their Laplace transforms as $\mathcal{F}_l$ and $\mathcal{F}_{e^*}$, respectively. Consider a system where at time 0 the buffer is empty and the machine comes up. If we fix $t$, $t > 0$, then the total length of the intervals $e_i^*$ — i.e., $t$ times the system ineffectiveness during $[0, t]$ — has the distribution function

$$\Omega(t,x) = \sum_{n=0}^{\infty} F_{e^*}^{(n)}(t-x)[F_l^{(n)}(x) - F_l^{(n+1)}(x)] \tag{36}$$

where the superscript $(n)$ denotes the $n$-th iterated convolution. In practice the evaluation of (36) is less severe than it looks at first sight, since the sum will not be infinite due to Approximation Assumption A.

We can also use a result from [24] that gives an expression for the moments of interval system effectiveness. If we denote by $M_1(t)$ and $M_2(t)$ the first and second moment of system ineffectiveness over the interval $[0, t]$ then their Laplace transforms are given by:

$$\mathcal{M}_1(s) = \int_0^\infty e^{-st} dM_1(t) = \frac{1}{s}\left[1 - \frac{1-\mathcal{F}_l(s)}{1-\mathcal{F}_l(s)\mathcal{F}_{e^*}(s)}\right] \tag{37}$$

and

$$\mathcal{M}_2(s) = \int_0^\infty e^{-st} dM_2(t) = \frac{2}{s}\left[1 - \frac{1-\mathcal{F}_l(s)}{1-\mathcal{F}_l(s)\mathcal{F}_{e^*}(s)} + \frac{s[1-\mathcal{F}_l(s)]\mathcal{F}_l(s)\mathcal{F}'_{e^*}(s)}{(1-\mathcal{F}_l(s)\mathcal{F}_{e^*}(s))^2}.\right] \tag{38}$$

The first and second moment of the interval system effectiveness can be computed for both approximation methods from (37) and (38) by numerical inversion (see for instance Abate and Whitt [1]).

# 6 Analysis of Preventive Maintenance Strategies

In this section we describe how the alternating renewal process approach, that was introduced for the analysis of the transient behaviour, can be used to study the performance of two preventive maintenance (PM) strategies.

Under a PM strategy the production unit can undergo preventive maintenance, for instance, at fixed time intervals. During such a PM activity the machine is not available for production, and thus the buffer contents will then decrease at a rate $d$. In this sense a PM action can be viewed as a special kind of down time, with a distribution that is different from an ordinary down time. We denote the distribution function for the length of a PM activity as $F_p$ and its mean as $E_p$. We assume that PM activities are mutually independent.

As we mentioned before, the analysis of PM strategies is based on the alternating renewal process model of the buffer contents. We thus implicitly assume that we operate in a region of the model parameters where the approximation assumption is satisfied, i.e. $E_u * (r - d)$ must be in the order of the buffer size $K$.

We first discuss block replacement in Section 6.1, and in Section 6.2 we shall deal with age replacement.

## 6.1 Block replacement

Under a block replacement strategy a preventive maintenance action is carried out at fixed intervals, say of length $T$, $T > 0$. After a PM action is concluded, the machine is as good as new, and it will be operated under the standard corrective maintenance strategy — fix it when it breaks down — for another $T$ time units, after which it will undergo PM again. This means that the behaviour between two successive PM actions is equivalent to the transient behaviour of the model of Section 5, restricted to the time interval $[0, T]$. Specifically this means that the long term system effectiveness can be calculated from the interval system effectiveness over $[0, T]$, since according to the approximation assumption the process $X$ regenerates itself at the beginning of every interval $[nT, (n + 1)T]$ (see Figure 5). Note that each regeneration interval starts with a PM action.

Each cycle thus consists of a preventive maintenance interval followed by an interval in which only corrective maintenance is performed. The expected system effectiveness over the second interval can be computed by numerical inversion of the Lapace transform of (37). For the ineffectiveness over the the PM interval we can choose two alternatives. Firstly we can assume that at the beginning of this PM interval the buffer is full, and thus the expected system ineffectiveness over this interval is

$$\int_0^\infty \max(t - K/d, 0)dF_p(t). \tag{39}$$

The alternative is to assume that during the entire PM interval the demand process cannot be satisfied, and thus the entire PM interval is 'ineffective'.

## 6.2 Age replacement

Under an age replacement strategy the machine is halted for preventive maintenance whenever its up time exceeds a threshold $T$, for some fixed $T > 0$. For convenience we denote the buffer contents process under age replacement as $X^a = \{X^a(t) \mid t \geq 0\}$.

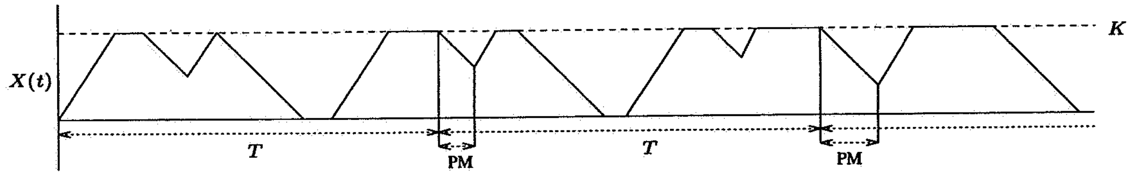


Figure 5: Block replacement preventive maintenance

Both the equilibrium and transient analysis of $X^a$ can proceed along the same lines as the analysis of $X$ in the model without preventive maintenance (Sections 4 and 5). This is done as follows. The machine state alternates between up and down, but there are two major differences with the original model:

- The length of an up interval cannot be longer then the age replacement threshold $T$.
- The length of a down interval is dependent of the length of the preceding up interval. If the preceding up interval is smaller than $T$, then the length of the down interval has distribution function $F_c$, and if the length of the up interval equals $T$, then the down interval has distribution $F_p$.

### 6.2.1 Equilibrium Analysis

The equilibrium system effectiveness can be computed in the same way as was done in Section 4.2. In the model without PM the buffer process $X$ regenerates itself at the end of every up period. In the PM case we cannot use these points as regeneration points, since the up and down periods of disjoint cycles would become dependent. However, for the computation of system effectiveness we do not need to describe the behaviour of $X^a$, but it suffices to describe the evolution of $I^a = \{I(X^a(t) > 0) \mid t \geq 0\}$. $I^a$ is a regenerative process too, and, because of the approximation assumption, we can choose the *start* of an up period as regeneration point. If we denote the up and down intervals under the age replacement strategy as $u_i^a$ and $c_i^a$, respectively, then the mean cycle length is

$$
\begin{aligned}
E[u_i^a + c_i^a] &= E\Big[E[u_i^a + c_i^a \mid u_i^a]\Big] \\
&= \int_0^{T-} (x + E_c) dF_u(x) + (1 - F_u(T-))(T + E_p) \\
&= E_c F_u(T-) + E_p(1 - F_u(T-)) + \int_0^{T-} \int_0^x dy \cdot dF_u(x) + \int_0^{T-} (1 - F_u(T-)) dy \\
&= E_c F_u(T-) + E_p(1 - F_u(T-)) + \int_0^{T-} (1 - F_u(y)) dy, \qquad (40)
\end{aligned}
$$

and the expected length of the interval during which the buffer is empty, in one regeneration cycle, is

$$
E[e_i^a] = F_u(T-) \int_{K/d}^{\infty} (1 - F_c(y)) dy + (1 - F_u(T-)) \int_{K/d}^{\infty} (1 - F_p(y)) dy. \qquad (41)
$$

### 6.2.2 Transient Analysis

The transient analysis of the $X^a$ process proceeds in a way that's analogous to that of Section 5. For this we need to determine the distribution functions (or their LST's) of the busy period length $l_i^a$ and the empty buffer interval $e_i^{a*}$ (see also Figure 4). This is done as follows.

The up intervals $u_i^a$ are independent and identically distributed. Because of the nature of an age replacement strategy, an up time cannot be larger than $T$, so the distribution function $F_u^a$ of $u_i^a$ is a truncated version of the original life time distribution $F_u$:

$$F_u^a(x) = P(u_i^a \leq x) = \begin{cases} F_u(x), & \text{if } x < T, \\ 1, & \text{if } x \geq T, \end{cases} \tag{42}$$

so $\mathbb{P}(u_i^a = T) = 1 - F_u(T)$.

Consider an arbitrary pair $(u_i^a, c_i^a)$ of an up and a down interval. Here $c_i^a$ may be a down time caused by either corrective or preventive maintenance. Denote by $p^*$ the probability that at the end of an arbitrary down interval the buffer content is not empty. Under the age replacement strategy $p^*$ plays the same role as $p$ in Section 5. We can compute $p^*$ by conditioning on $u_i^a$:

$$\begin{aligned} p^* &= P(c_i^a \leq K/d) \\ &= \int_0^\infty P(c_i^a \leq K/d \mid u_i^a = t)\, dF_u^a(t) \\ &= \int_0^{T-} P(c_i^a \leq K/d \mid u_i^a = t)\, dF_u(t) + \int_{T-}^\infty P(c_i^a \leq K/d \mid u_i^a = T)\, dF_u(t) \\ &= F_c(K/d)F_u(T-) + F_p(K/d)[1 - F_u(T-)]. \end{aligned} \tag{43}$$

In the next step we determine the LST $\mathcal{F}_l$ of the busy period length $l_i$, $i \in \mathbb{N}$. The derivation of the LST proceeds analogously to Section 5, with a modification to account for dependence between the up and down intervals. Like in the case when there's only corrective maintenance, a busy period is comprised of (see also Figure 4):

- A stochastic number $N$ of combinations of up and down intervals $(u_i^a, c_i^a)$, where $N$ is geometrically distributed with parameter $p^*$,
- One up interval $u_{N+1}$,
- One interval of deterministic length $K/d$.

Next we determine the distribution of $u_i^a + c_i^a$, $1 \leq i \leq N$. The distribution of $c_i^a$ depends on whether the downtime is corrective or preventive, and thus it depends on the length of the preceding up time $u_i^a$. If $u_i^a$ is strictly smaller than $T$, then it must have been ended by a breakdown, and thus $c_i^a$ will correspond to a corrective repair. Moreover, the down interval $c_i^a$ is smaller than $K/d$, since it is part of a $(u_i^a, c_i^a)$ pair that does not end with an empty buffer ($i \leq N$!), and thus its distribution function $F_c^a(\cdot \mid u_i^a < T)$ is $F_{\bar{c}}(\cdot)$ as in (27). If $u_i^a = T$, then $c_i^a$ is in fact the length of a preventive maintenance action, also truncated to the interval $[0, K/d]$, and its distribution function is

$$F_{\bar{p}}(x) := F_c^a(x \mid u_i^a = T) = \begin{cases} F_p(x)/F_p(K/D), & \text{if } x < K/d, \\ 1, & \text{if } x \geq K/d. \end{cases} \tag{44}$$

Removing the condition on $u_i^a$ we get the following expression for the LST $\mathcal{F}_l$ of $l_i$:

$$\mathcal{F}_l(s) = E[e^{-sl_i^a}] = \frac{(1-p)\mathcal{F}_u^a(s)e^{-sK/d}}{1-p\mathcal{F}_{u+c}^a(s)} \tag{45}$$

with

$$\mathcal{F}_u^a(s) = \int_0^\infty e^{-st} dF_u^a(t) = \int_0^{T-} e^{-st} dF_{\overline{u}}(t) + (1 - F_u(T-))e^{-sT} \tag{46}$$

and

$$\begin{aligned} \mathcal{F}_{u+c}^a(s) &= \int_0^\infty E[e^{-s(u_i^a + c_i^a)} \mid u_i^a = x] \, dF_u^a(x) \\ &= \mathcal{F}_{\overline{c}}(s) \int_0^{T-} e^{-sx} \, dF_u(x) + e^{-sT} \mathcal{F}_{\overline{p}}(s)(1 - F_u(T-)), \end{aligned} \tag{47}$$

where $\mathcal{F}_{\overline{c}}$ and $\mathcal{F}_{\overline{p}}$ are the LST's of $F_{\overline{c}}$ and $F_{\overline{p}}$ (see equations (27) and (44)), respectively. There is no general expression that simplifies the first term in (46) and (47). It can, however, be computed for most distribution functions. For exponentially distributed up times, for instance, the integral would become

$$\int_0^{T-} e^{-sx} dF_u^a(x) = \int_0^T \frac{1}{E_u} e^{-(s+1/E_u)x} dx = \frac{1}{E_u s + 1}[1 - e^{-(s+1/E_u)T}] \tag{48}$$

Numerical results for both preventive maintenance strategies are reported in Section 7.3.

# 7 Numerical Results

In this section we present some numerical results. Specifically we investigate the impact of the distribution type on the system effectiveness. We also report on the accuracy of the two proposed approximation methods.

In the first examples we present the long term system ineffectiveness of a production-buffer model with exponentially distributed repair times. The parameters of the model are

| | |
|---|---|
| Mean life time ($E_u$) | 1 - 60 |
| Mean repair time ($E_c$) | 1 |
| Production rate ($r$) | 1.05 |
| Demand rate ($d$) | 1 |
| Buffer contents ($K$) | $0 - \infty$ |

## 7.1 Equilibrium behaviour

For the up times we use four different distributions: Erlang-2 (denoted as E2), a $K_2$ distribution with gamma normalization (see Tijms [25, Appendix B]) with squared coefficient of variation $c^2$ equal to 0.75, an exponential distribution (M), and a $H_2$ hyper-exponential distribution with balanced means and $c^2 = 1.5$ (cf. Tijms [25]). The equilibrium system ineffectiveness, computed through the M/G/1 equivalent model is depicted in Figures 9–13. Note that the traffic load of the M/G/1 queue is given by $a = E_u * (r-d)/d * E_c$, so $a$ varies from

0.05 for $E_u = 1$ to $a = 3$ when $E_u = 60$. We have not included any examples with $a > 2$, since they all show the same behaviour as $a = 1$, i.e. the system ineffectiveness goes to 0 as the buffer size tends to infinity. The M/G/1 model can be evaluated, even if $a > 1$, provided $K < \infty$. Infinite buffers can be treated as well, but may be of little practical value, since the buffer contents will go to infinity with probability 1 when $a > 1$.

#### 7.1.1 Sensitivity analysis

From Figures 9–13 we conclude that the system ineffectiveness is somewhat sensitive to the up time distribution type, but when the coefficient of variation is around 1, then the differences may be neglected. Furthermore, the sensitivity decreases when the load of the M/G/1 queue becomes larger (compare $a = 0.05$ in Figure 9 versus $a = 3$ in Figure 13). Recall from the definition of the M/G/1 parameters (see equations (8)– (10)) that the laod $a$ corresponds to the ratio of the mean production surplus during an up period over the mean total demand during a down period.

In Figures 14–18 we present some examples for exponential up times and E2, K2, M and H2 repair times. Note that when the mean life time is small, then the system ineffectiveness is almost insensitive to the type of the repair time distribution.

#### 7.1.2 Approximation methods

Next we investigate the accuracy of the two approximation methods from Section 4.2. We consider the same parameter sets as in Figures 9–13. In Tables 1–5 we find the equilibrium system ineffectiveness for the M/G/1 and for the two approximation methods (A and B). For both methods the approximation of the equilibrium system inefectivenss depends only on the mean of the up time (cf. (22 and (23)), so the approximative values are the same for the four distribution types. We see that for $a < 1$ ($E_u < 20$) method A gives an ineffectiveness value of 0 for large $K$, as was to be expected from (23). This estimate is obviously not correct for the real system and method B compensates for this (albeit in a crude way: it just replaces all the estimates of A for large $K$ with that of the model with $K = E_u(r - d)$).

When the load $a \geq 1$, then the situation reverses: in the real system and under A the ineffectiveness goes to 0 when $K$ becomes large, while under B it is a strictly positive constant.

### 7.2 Transient behaviour

The next results concern the transient behaviour of the system. We have run simulations on a system with exponentially distributed down times and four different types of up times, viz. deterministic, Erlang-2, exponential and Hyper-exponential of order 2. For each simulation we collected the data over 50000 realizations of busy periods. The estimates of the first and second moment, plus the comparisons to the two approximation methods, are depicted in Tables 6–25.

In most cases the approximations for the mean length of the busy period are quite accurate, provided that the buffer size is not larger than roughly $E_u(r - d)$. The approximations are also more accurate when the coefficient of variation of the up times is small. This is all, of course, in accordance with the approximation assumption. In fact, it appears from the simulation results that the busy period length is fairly insensitive to the type of the up time distribution. Compare for instance the examples of Erlang-2 and Hyper-exponential-2 distributions, with a squared coefficient of variation equal to 0.5 and 1.5, respectively. In the region where the approximation methods work well — $k < Eu * (r - d)$ — Approximation method B is by definition the same as A, so for the transient analysis we do not gain much from the refinement of B.

The approximations for the second moment of the busy period length are less accurate, unfortunately. First, from the simulations it appears that the second moment *is* sensitive to the distribution type. In all examples the busy periods tend to become more random than the original up time distribution, in the sense that the coefficient of variation for the busy period length is larger than the coefficient for the up time distribution.

From the simulations we observed that for large buffer sizes $K$ the coefficient of variation of the busy period is maximal when $E_u * (r - d) = E_d * d$. Two example are summarized in Tables 26 and 27 for $K = 5$ and $K = 10$. As an explanation for this behaviour one can think of the following. If $E_u * (r - d) < E_c * d$, then the expected net growth during an up time is smaller than the expected total demand during a down time and thus the buffer contents process $X$ will fluctuate above 0. A busy period will then typically consist of one up and down pair $(u_1, c_1)$ and its coefficient of variation will be about that of $u_1 + c_1$. If $E_u * (r - d) > E_c * d$, then $X$ will mainly fluctuate below K. If $E_u * (r - d) = E_c * d$, then $X$ will typically fluctuate over the entire range $[0, K]$, and, since the end of a busy period coincides with $X$ hitting the boundary at 0, the fluctuation of the busy period length will be maximal in this case. Moreover, this effect becomes more pronounced if $K$ gets large.

## 7.3 Preventive maintenance

The numerical results for the block replacement preventive maintenance strategy are depicted in Tables 28–30. The system under consideration has exponentially distributed down times (for both corrective and preventive maintenance). The mean preventive down time $E_p$ is one tenth of the mean corrective down time $E_c$. For the up time distribution we consider only Erlang-2 and K-2 and for each value of the mean up time $E_u$ we consider two possible values of the block replacement threshold $T$. We have compared the approximations with simulation results, which are shown in the same tables. We don't have simulation results for K-2 distributed up times because of the special structure of these distribution functions.

In each table we give two figures for the approximation of the system ineffectiveness. The first figure is derived by using equation (39) as the expected ineffectiveness over the Pm interval, the second figure comes from assuming that the entire PM interval has an empty buffer (cf. also the discussion in Section 6.1).

In Tables 31–33 we present some results on system effectiveness under an age replacement strategy. The systems under consideration are the same as for the block replacement strategy. We conclude that the age replacement strategy gives a comparable or slightly better performance than block replacement, which was to be expected.

In Table 34 we present a final approximation for the system ineffectiveness under age replacement. Here we use the system effectiveness of the M/M/1 model, where we take adjusted values for the mean up time and the mean down time. The mean up time is approximated by the expected up time under age replacement:

$$\int_0^T x \, dF_u(x) + T * (1 - F_u(T-)) \tag{49}$$

and the mean down time is approximated by the mean down time under age replacement:

$$E_c F_u(T-) + E_p(1 - F_u(T-)). \tag{50}$$

The M/M/1 model gives surprisingly good results for small buffer sizes, considering the crudeness of this approximation

### Conclusions

In general both approximation methods give good results for the computation of the longrun system effectiveness. For transient performance measures the approximations are good only when the parameter values of the model are such that the approximation assumption is not violated. Typically this means that $K < C * E_u * (r - d)$, for some constant $C$ that may range from $C = 1$ for deterministic up times to $C = 0.5$ for a Hyper-exponential distribution with $c^2 = 1.5$. The approximation errors for the first moment of the busy period are moderate, but the errors for the second moment may be considerable.

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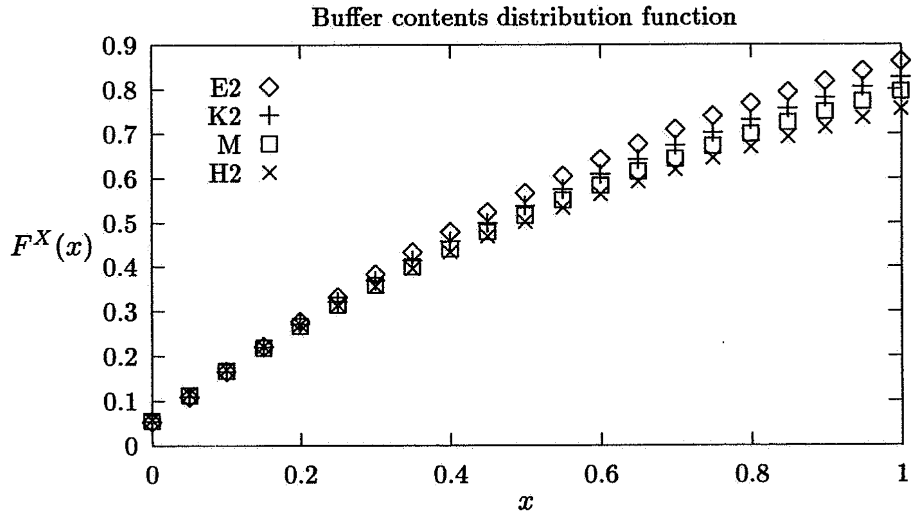


Figure 6: Distribution function of the buffer contents in the M/G/1 model, $E_u = 10$

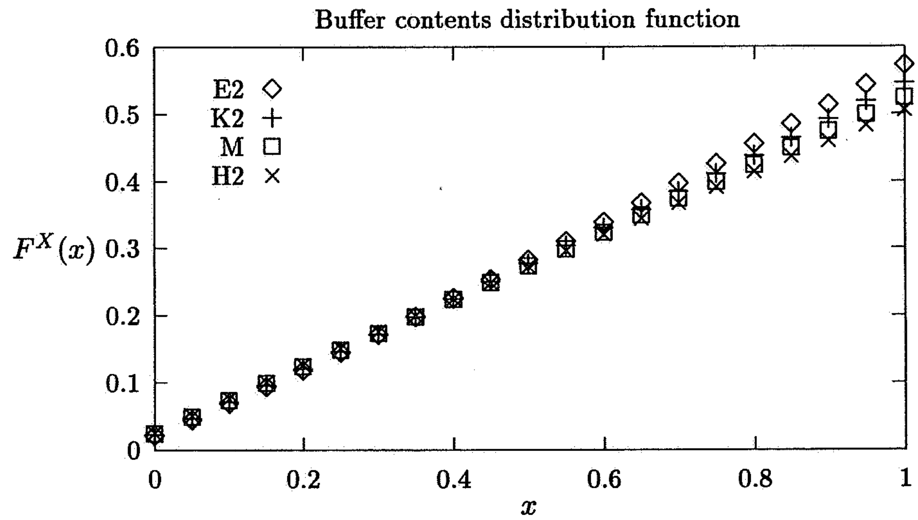


Figure 7: Distribution function of the buffer contents in the M/G/1 model, $E_u = 20$

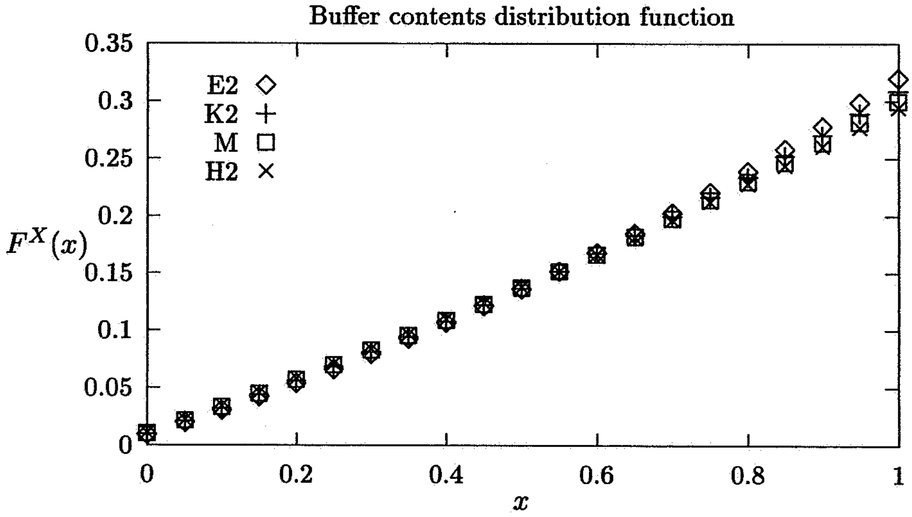


Figure 8: Distribution function of the buffer contents in the M/G/1 model, $E_u = 40$

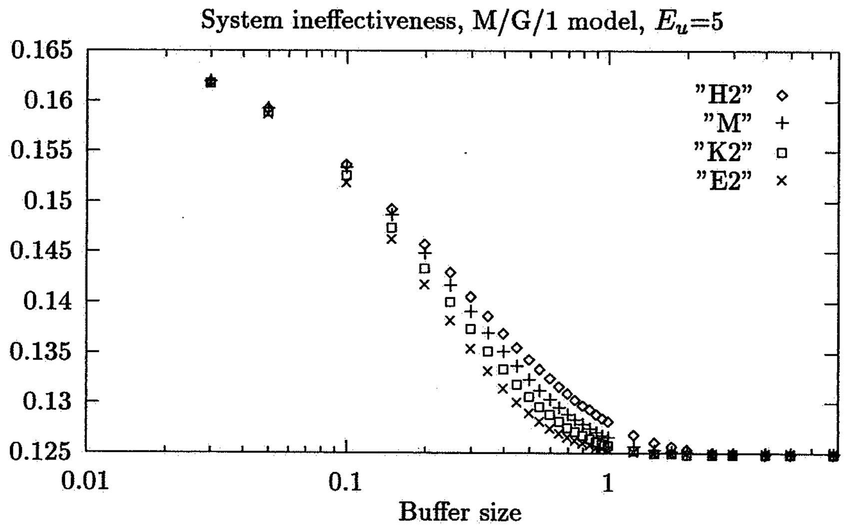


Figure 9: Equilibrium System Ineffectiveness in the M/G/1 model, $E_u = 5$

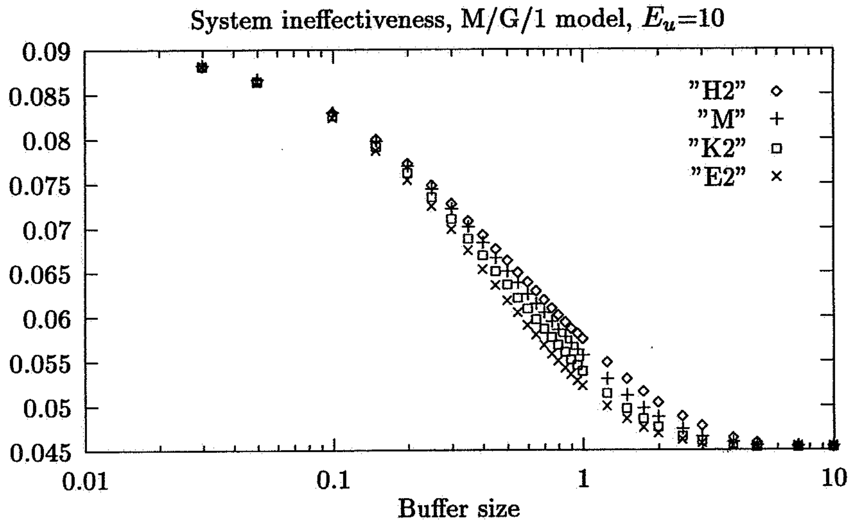


Figure 10: Equilibrium System Ineffectiveness in the M/G/1 model, $E_u = 10$

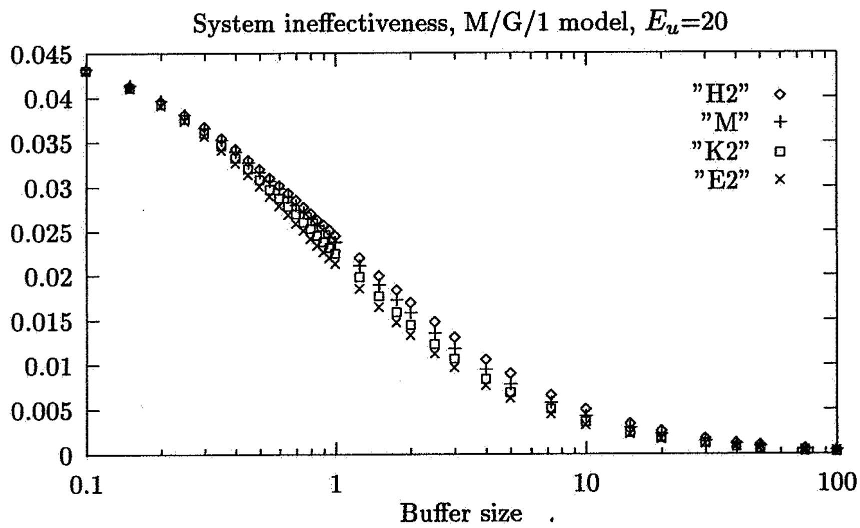


Figure 11: Equilibrium System Ineffectiveness in the M/G/1 model, $E_u = 20$

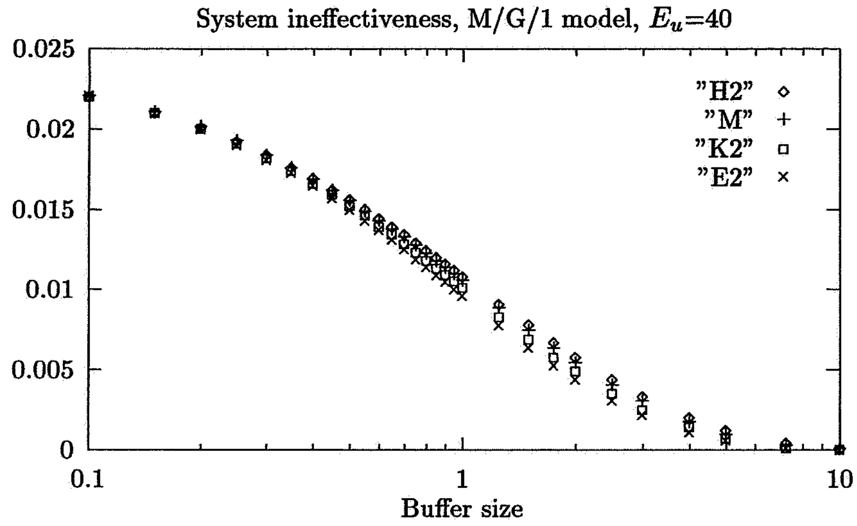


Figure 12: Equilibrium System Ineffectiveness in the M/G/1 model, $E_u = 40$

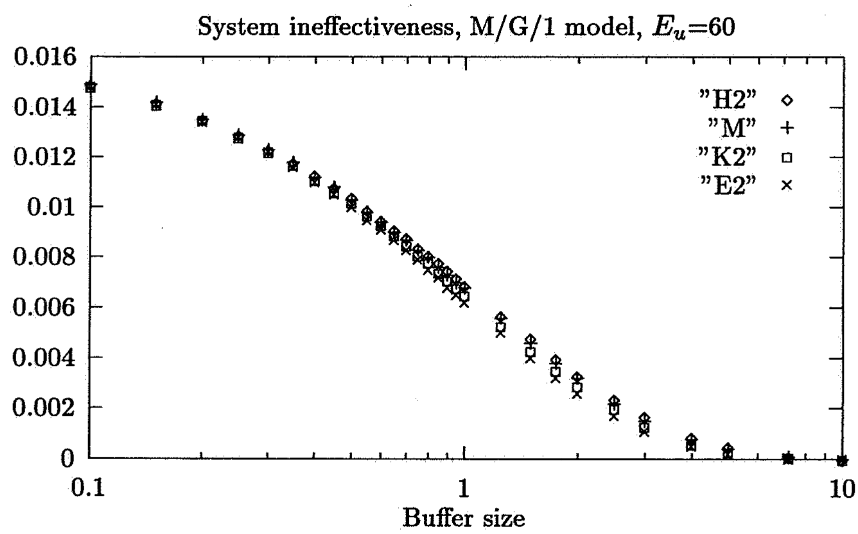


Figure 13: Equilibrium System Ineffectiveness in the M/G/1 model, $E_u = 60$

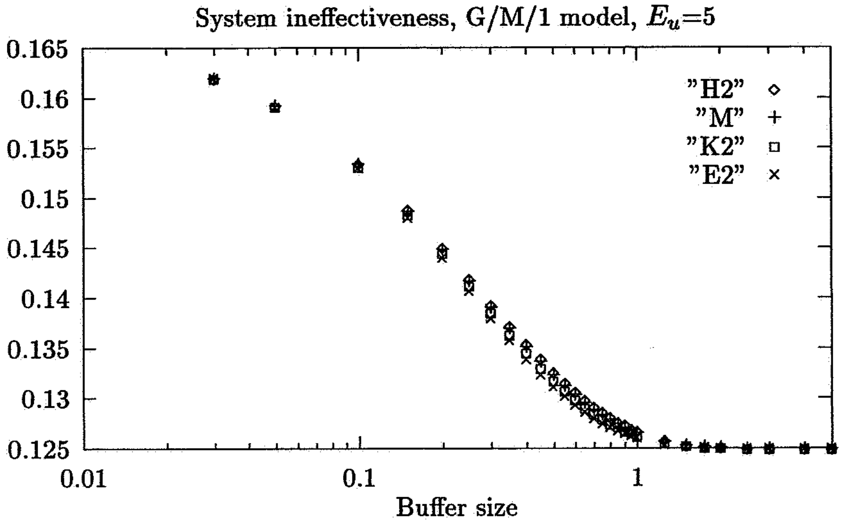


Figure 14: Equilibrium System Ineffectiveness in the G/M/1 model, $E_u = 5$

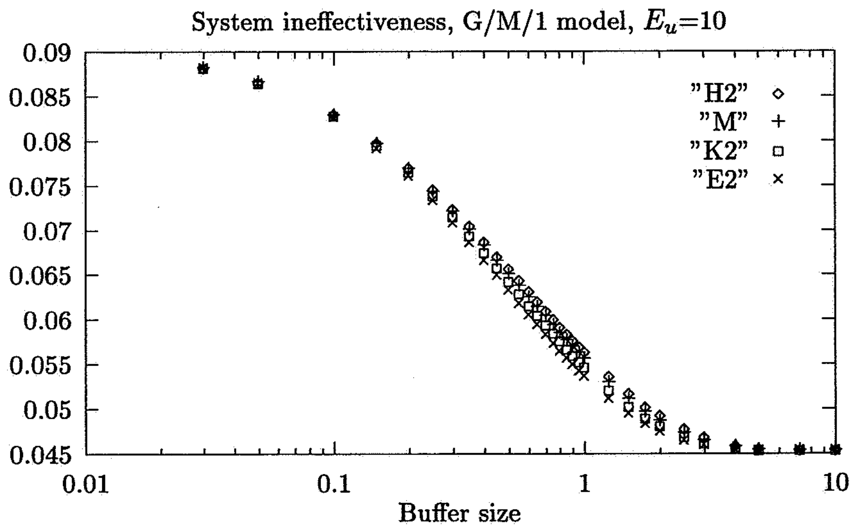


Figure 15: Equilibrium System Ineffectiveness in the G/M/1 model, $E_u = 10$

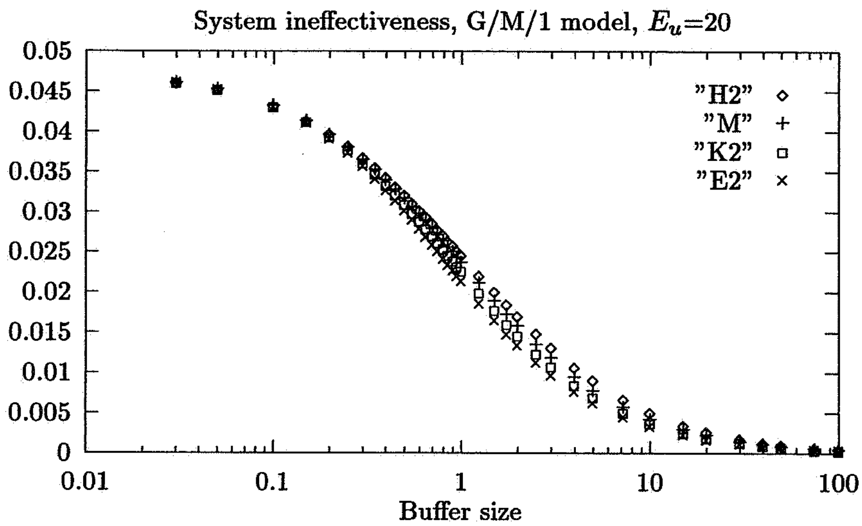


Figure 16: Equilibrium System Ineffectiveness in the G/M/1 model, $E_u = 20$

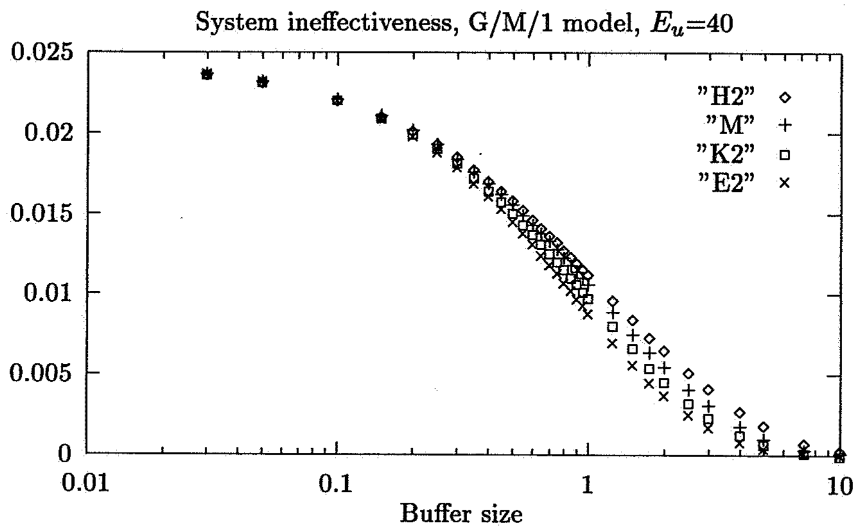


Figure 17: Equilibrium System Ineffectiveness in the G/M/1 model, $E_u = 40$

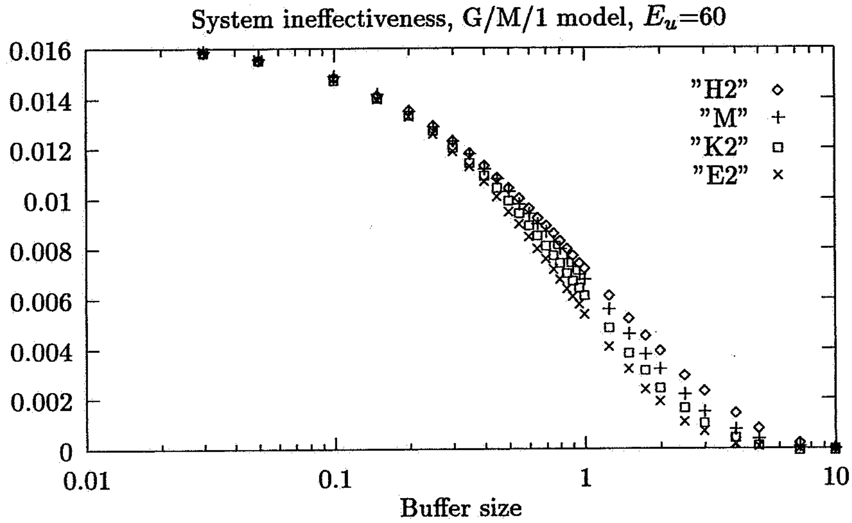


Figure 18: Equilibrium System Ineffectiveness in the G/M/1 model, $E_u = 60$

| System ineffectiveness | | | | | | |
|---|---|---|---|---|---|---|
| | Exact | | | | Approx. | |
| $K$ | E-2 | K-2 | M | H-2 | A | B |
| 0.00 | .166 | .166 | .166 | .166 | .167 | .167 |
| 0.10 | .153 | .153 | .153 | .153 | .151 | .151 |
| 0.20 | .144 | .144 | .144 | .145 | .137 | .137 |
| 0.30 | .138 | .138 | .139 | .139 | .123 | .130 |
| 0.40 | .133 | .134 | .135 | .135 | .112 | .130 |
| 0.50 | .131 | .131 | .132 | .132 | .101 | .130 |
| 0.60 | .129 | .129 | .130 | .130 | .091 | .130 |
| 0.70 | .128 | .128 | .128 | .129 | .083 | .130 |
| 0.80 | .127 | .127 | .127 | .128 | .075 | .130 |
| 0.90 | .126 | .126 | .127 | .127 | .068 | .130 |
| 1.00 | .126 | .126 | .126 | .126 | .061 | .130 |

Table 1: System ineffectiveness, General up time distribution, Exponential down time, $E_u = 5$

| System ineffectiveness | | | | | | |
|---|---|---|---|---|---|---|
| | Exact | | | | Approx. | |
| $K$ | E-2 | K-2 | M | H-2 | A | B |
| 0.00 | .090 | .090 | .090 | .090 | .091 | .091 |
| 0.10 | .082 | .082 | .083 | .083 | .082 | .082 |
| 0.20 | .076 | .076 | .077 | .077 | .074 | .074 |
| 0.30 | .070 | .071 | .072 | .072 | .067 | .067 |
| 0.40 | .066 | .067 | .068 | .068 | .061 | .061 |
| 0.50 | .063 | .064 | .065 | .065 | .055 | .055 |
| 0.60 | .060 | .061 | .062 | .063 | .050 | .055 |
| 0.70 | .058 | .059 | .060 | .061 | .045 | .055 |
| 0.80 | .056 | .057 | .058 | .059 | .041 | .055 |
| 0.90 | .055 | .056 | .057 | .057 | .037 | .055 |
| 1.00 | .053 | .054 | .055 | .056 | .033 | .055 |
| 2.00 | .047 | .048 | .048 | .049 | .012 | .055 |
| 5.00 | .045 | .045 | .045 | .045 | .001 | .055 |

Table 2: System ineffectiveness, General up time distribution, Exponential down time, $E_u = 10$

| System ineffectiveness | | | | | | |
|---|---|---|---|---|---|---|
| | Exact | | | | Approx. | |
| $K$ | E-2 | K-2 | M | H-2 | A | B |
| 0.00 | .047 | .047 | .047 | .047 | .048 | .048 |
| 0.20 | .039 | .039 | .039 | .039 | .039 | .039 |
| 0.40 | .032 | .033 | .034 | .034 | .032 | .032 |
| 0.60 | .027 | .028 | .029 | .030 | .026 | .026 |
| 0.80 | .024 | .025 | .026 | .027 | .021 | .021 |
| 1.00 | .021 | .022 | .023 | .024 | .018 | .018 |
| 2.00 | .013 | .014 | .015 | .017 | .006 | .018 |
| 3.00 | .009 | .010 | .011 | .013 | .002 | .018 |
| 4.00 | .007 | .008 | .009 | .010 | .001 | .018 |
| 5.00 | .006 | .007 | .007 | .009 | .000 | .018 |
| 10.00 | .003 | .003 | .004 | .005 | .000 | .018 |

Table 3: System ineffectiveness, General up time distribution, Exponential down time, $E_u = 20$

| System ineffectiveness | | | | | | |
|---|---|---|---|---|---|---|
| | Exact | | | | Approx. | |
| $K$ | E-2 | K-2 | M | H-2 | 1A | 1B |
| 0.00 | .024 | .024 | .024 | .024 | .024 | .024 |
| 0.20 | .019 | .020 | .020 | .020 | .020 | .020 |
| 0.40 | .016 | .016 | .016 | .017 | .016 | .016 |
| 0.60 | .013 | .013 | .014 | .014 | .013 | .013 |
| 0.80 | .010 | .011 | .012 | .012 | .011 | .011 |
| 1.00 | .008 | .009 | .010 | .011 | .009 | .009 |
| 2.00 | .003 | .004 | .005 | .006 | .003 | .003 |
| 3.00 | .001 | .002 | .003 | .004 | .001 | .003 |
| 4.00 | .000 | .001 | .001 | .002 | .000 | .003 |
| 5.00 | .000 | .000 | .001 | .001 | .000 | .003 |
| 10.00 | .000 | .000 | .000 | .000 | .000 | .003 |

Table 4: System ineffectiveness, General up time distribution, Exponential down time, $E_u = 40$

| System ineffectiveness | | | | | | |
|---|---|---|---|---|---|---|
| | Exact | | | | Approx. | |
| $K$ | E-2 | K-2 | M | H-2 | 1A | 1B |
| 0.00 | .016 | .016 | .016 | .016 | .016 | .016 |
| 0.20 | .013 | .013 | .013 | .013 | .013 | .013 |
| 0.40 | .010 | .011 | .011 | .011 | .011 | .011 |
| 0.60 | .008 | .009 | .009 | .009 | .009 | .009 |
| 0.80 | .006 | .007 | .008 | .008 | .007 | .007 |
| 1.00 | .005 | .006 | .006 | .007 | .006 | .006 |
| 2.00 | .001 | .002 | .003 | .004 | .002 | .002 |
| 3.00 | .000 | .001 | .001 | .002 | .001 | .001 |
| 4.00 | .000 | .000 | .000 | .001 | .000 | .001 |
| 5.00 | .000 | .000 | .000 | .000 | .000 | .001 |
| 10.00 | .000 | .000 | .000 | .000 | .000 | .001 |

Table 5: System ineffectiveness, General up time distribution, Exponential down time, $E_u = 60$

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 5.00 ±0.00 | 0.00 – 0.00 | 5.00 | 0.00 | 5.00 | 0.00 |
| 0.01 | 5.06 ±0.00 | 0.01 – 0.01 | 5.06 | 0.01 | 5.06 | 0.01 |
| 0.03 | 5.18 ±0.01 | 0.03 – 0.03 | 5.18 | 0.03 | 5.18 | 0.03 |
| 0.05 | 5.30 ±0.01 | 0.04 – 0.05 | 5.31 | 0.05 | 5.31 | 0.05 |
| 0.10 | 5.62 ±0.01 | 0.09 – 0.10 | 5.63 | 0.09 | 5.63 | 0.09 |
| 0.15 | 5.95 ±0.02 | 0.13 – 0.14 | 5.97 | 0.14 | 5.97 | 0.14 |
| 0.20 | 6.32 ±0.02 | 0.17 – 0.18 | 6.33 | 0.18 | 6.33 | 0.18 |
| 0.25 | 6.69 ±0.03 | 0.20 – 0.22 | 6.70 | 0.21 | 6.70 | 0.21 |
| 0.30 | 6.78 ±0.03 | 0.23 – 0.25 | 7.10 | 0.25 | 6.70 | 0.21 |
| 0.35 | 6.85 ±0.03 | 0.25 – 0.27 | 7.51 | 0.28 | 6.70 | 0.21 |
| 0.40 | 6.90 ±0.03 | 0.27 – 0.29 | 7.95 | 0.31 | 6.70 | 0.21 |

Table 6: Transient analysis of busy period, Up times: Deterministic, $E_u$=5

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 10.00 ±0.00 | 0.00 – 0.00 | 10.00 | 0.00 | 10.00 | 0.00 |
| 0.05 | 10.55 ±0.02 | 0.05 – 0.05 | 10.56 | 0.05 | 10.56 | 0.05 |
| 0.10 | 11.13 ±0.03 | 0.09 – 0.10 | 11.16 | 0.09 | 11.16 | 0.09 |
| 0.15 | 11.75 ±0.04 | 0.13 – 0.14 | 11.78 | 0.14 | 11.78 | 0.14 |
| 0.20 | 12.41 ±0.05 | 0.17 – 0.19 | 12.44 | 0.18 | 12.44 | 0.18 |
| 0.25 | 13.10 ±0.05 | 0.21 – 0.22 | 13.12 | 0.22 | 13.12 | 0.22 |
| 0.30 | 13.82 ±0.06 | 0.25 – 0.26 | 13.85 | 0.25 | 13.85 | 0.25 |
| 0.35 | 14.57 ±0.07 | 0.28 – 0.30 | 14.61 | 0.29 | 14.61 | 0.29 |
| 0.40 | 15.36 ±0.08 | 0.31 – 0.33 | 15.41 | 0.32 | 15.41 | 0.32 |
| 0.45 | 16.19 ±0.08 | 0.34 – 0.36 | 16.25 | 0.35 | 16.25 | 0.35 |
| 0.50 | 17.08 ±0.09 | 0.37 – 0.39 | 17.14 | 0.38 | 17.14 | 0.38 |
| 0.55 | 17.44 ±0.10 | 0.40 – 0.43 | 18.07 | 0.41 | 17.14 | 0.38 |
| 0.60 | 17.79 ±0.10 | 0.44 – 0.47 | 19.04 | 0.44 | 17.14 | 0.38 |
| 0.65 | 18.11 ±0.11 | 0.47 – 0.50 | 20.07 | 0.46 | 17.14 | 0.38 |

Table 7: Transient analysis of busy period, Up times: Deterministic, $E_u$=10

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 20.00 ±0.00 | 0.00 – 0.00 | 20.00 | 0.00 | 20.00 | 0.00 |
| 0.10 | 22.15 ±0.06 | 0.09 – 0.10 | 22.21 | 0.09 | 22.21 | 0.09 |
| 0.20 | 24.60 ±0.09 | 0.17 – 0.19 | 24.65 | 0.18 | 24.65 | 0.18 |
| 0.30 | 27.29 ±0.12 | 0.25 – 0.27 | 27.35 | 0.26 | 27.35 | 0.26 |
| 0.40 | 30.23 ±0.15 | 0.31 – 0.34 | 30.33 | 0.33 | 30.33 | 0.33 |
| 0.50 | 33.52 ±0.18 | 0.37 – 0.40 | 33.62 | 0.39 | 33.62 | 0.39 |
| 0.60 | 37.24 ±0.22 | 0.43 – 0.46 | 37.26 | 0.44 | 37.26 | 0.44 |
| 0.70 | 41.33 ±0.26 | 0.48 – 0.52 | 41.29 | 0.49 | 41.29 | 0.49 |
| 0.80 | 45.92 ±0.30 | 0.52 – 0.57 | 45.74 | 0.54 | 45.74 | 0.54 |
| 0.90 | 50.78 ±0.34 | 0.56 – 0.60 | 50.65 | 0.58 | 50.65 | 0.58 |
| 1.00 | 56.31 ±0.39 | 0.59 – 0.64 | 56.08 | 0.62 | 56.08 | 0.62 |
| 1.25 | 65.81 ±0.51 | 0.76 – 0.83 | 72.30 | 0.70 | 56.08 | 0.62 |
| 1.50 | 76.17 ±0.65 | 0.92 – 1.00 | 93.12 | 0.76 | 56.08 | 0.62 |

Table 8: Transient analysis of busy period, Up times: Deterministic, $E_u$=20

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 40.00 ±0.00 | 0.00 – 0.00 | 40.00 | 0.00 | 40.00 | 0.00 |
| 0.10 | 44.19 ±0.12 | 0.09 – 0.10 | 44.31 | 0.09 | 44.31 | 0.09 |
| 0.20 | 48.98 ±0.18 | 0.17 – 0.19 | 49.08 | 0.18 | 49.08 | 0.18 |
| 0.30 | 54.24 ±0.24 | 0.25 – 0.27 | 54.34 | 0.26 | 54.34 | 0.26 |
| 0.40 | 59.97 ±0.30 | 0.31 – 0.34 | 60.16 | 0.33 | 60.16 | 0.33 |
| 0.50 | 66.39 ±0.36 | 0.37 – 0.40 | 66.60 | 0.39 | 66.60 | 0.39 |
| 0.75 | 86.09 ±0.55 | 0.51 – 0.55 | 85.80 | 0.52 | 85.80 | 0.52 |
| 1.00 | 110.89 ±0.77 | 0.60 – 0.65 | 110.45 | 0.63 | 110.45 | 0.63 |
| 1.25 | 143.00 ±1.05 | 0.68 – 0.74 | 142.10 | 0.71 | 142.10 | 0.71 |
| 1.50 | 182.91 ±1.41 | 0.74 – 0.81 | 182.75 | 0.77 | 182.75 | 0.77 |
| 1.75 | 235.75 ±1.87 | 0.79 – 0.86 | 234.94 | 0.82 | 234.94 | 0.82 |
| 2.00 | 303.34 ±2.46 | 0.82 – 0.89 | 301.95 | 0.86 | 301.95 | 0.86 |
| 2.50 | 461.98 ±4.06 | 0.96 – 1.05 | 498.48 | 0.91 | 301.95 | 0.86 |
| 3.00 | 710.73 ±6.56 | 1.06 – 1.16 | 822.51 | 0.95 | 301.95 | 0.86 |

Table 9: Transient analysis of busy period, Up times: Deterministic, $E_u$=40

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 60.00 ±0.00 | 0.00 – 0.00 | 60.00 | 0.00 | 60.00 | 0.00 |
| 0.20 | 73.37 ±0.27 | 0.17 – 0.19 | 73.51 | 0.18 | 73.51 | 0.18 |
| 0.40 | 89.71 ±0.45 | 0.31 – 0.34 | 90.00 | 0.33 | 90.00 | 0.33 |
| 0.60 | 110.07 ±0.65 | 0.43 – 0.47 | 110.15 | 0.45 | 110.15 | 0.45 |
| 0.80 | 135.29 ±0.88 | 0.53 – 0.57 | 134.76 | 0.55 | 134.76 | 0.55 |
| 1.00 | 165.48 ±1.15 | 0.60 – 0.65 | 164.82 | 0.63 | 164.82 | 0.63 |
| 1.25 | 213.26 ±1.58 | 0.68 – 0.74 | 211.91 | 0.71 | 211.91 | 0.71 |
| 1.50 | 272.63 ±2.11 | 0.75 – 0.81 | 272.38 | 0.77 | 272.38 | 0.77 |
| 1.75 | 351.25 ±2.80 | 0.79 – 0.86 | 350.03 | 0.82 | 350.03 | 0.82 |
| 2.00 | 451.81 ±3.67 | 0.83 – 0.90 | 449.73 | 0.86 | 449.73 | 0.86 |
| 2.50 | 737.78 ±6.17 | 0.87 – 0.95 | 742.13 | 0.91 | 742.13 | 0.91 |
| 3.00 | 1220.32 ±10.40 | 0.91 – 0.98 | 1224.22 | 0.95 | 1224.22 | 0.95 |
| 4.00 | 3165.45 ±28.24 | 0.99 – 1.08 | 3329.49 | 0.98 | 1224.22 | 0.95 |
| 5.00 | 8172.74 ±74.70 | 1.04 – 1.14 | 9052.20 | 0.99 | 1224.22 | 0.95 |

Table 10: Transient analysis of busy period, Up times: Deterministic, $E_u$=60

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 5.00 ±0.03 | 0.49 – 0.52 | 5.00 | 0.50 | 5.00 | 0.50 |
| 0.01 | 5.06 ±0.03 | 0.49 – 0.52 | 5.06 | 0.50 | 5.06 | 0.50 |
| 0.03 | 5.19 ±0.03 | 0.50 – 0.53 | 5.18 | 0.51 | 5.18 | 0.51 |
| 0.05 | 5.31 ±0.03 | 0.50 – 0.54 | 5.31 | 0.51 | 5.31 | 0.51 |
| 0.10 | 5.58 ±0.04 | 0.53 – 0.56 | 5.63 | 0.53 | 5.63 | 0.53 |
| 0.15 | 5.83 ±0.04 | 0.55 – 0.59 | 5.97 | 0.54 | 5.97 | 0.54 |
| 0.20 | 6.05 ±0.04 | 0.59 – 0.63 | 6.33 | 0.56 | 6.33 | 0.56 |
| 0.25 | 6.23 ±0.04 | 0.63 – 0.68 | 6.70 | 0.57 | 6.70 | 0.57 |
| 0.30 | 6.37 ±0.05 | 0.66 – 0.71 | 7.10 | 0.58 | 6.70 | 0.57 |
| 0.35 | 6.50 ±0.05 | 0.70 – 0.76 | 7.51 | 0.59 | 6.70 | 0.57 |
| 0.40 | 6.59 ±0.05 | 0.73 – 0.80 | 7.95 | 0.61 | 6.70 | 0.57 |
| 0.45 | 6.67 ±0.05 | 0.76 – 0.83 | 8.41 | 0.62 | 6.70 | 0.57 |
| 0.50 | 6.74 ±0.05 | 0.79 – 0.86 | 8.89 | 0.63 | 6.70 | 0.57 |

Table 11: Transient analysis of busy period, Up times: Erlang-2, $E_u$=5

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 10.01 ±0.06 | 0.49 – 0.52 | 10.00 | 0.50 | 10.00 | 0.50 |
| 0.05 | 10.57 ±0.07 | 0.51 – 0.54 | 10.56 | 0.52 | 10.56 | 0.52 |
| 0.10 | 11.12 ±0.07 | 0.53 – 0.57 | 11.16 | 0.54 | 11.16 | 0.54 |
| 0.15 | 11.69 ±0.08 | 0.55 – 0.59 | 11.78 | 0.56 | 11.78 | 0.56 |
| 0.20 | 12.24 ±0.08 | 0.58 – 0.62 | 12.44 | 0.57 | 12.44 | 0.57 |
| 0.25 | 12.78 ±0.09 | 0.61 – 0.65 | 13.12 | 0.59 | 13.12 | 0.59 |
| 0.30 | 13.29 ±0.09 | 0.63 – 0.68 | 13.85 | 0.61 | 13.85 | 0.61 |
| 0.35 | 13.77 ±0.10 | 0.67 – 0.72 | 14.61 | 0.62 | 14.61 | 0.62 |
| 0.40 | 14.25 ±0.11 | 0.70 – 0.76 | 15.41 | 0.63 | 15.41 | 0.63 |
| 0.45 | 14.72 ±0.11 | 0.73 – 0.79 | 16.25 | 0.65 | 16.25 | 0.65 |
| 0.50 | 15.11 ±0.12 | 0.76 – 0.83 | 17.14 | 0.66 | 17.14 | 0.66 |
| 0.55 | 15.51 ±0.12 | 0.79 – 0.86 | 18.07 | 0.67 | 17.14 | 0.66 |

Table 12: Transient analysis of busy period, Up times: Erlang-2, $E_u$=10

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 20.01 ±0.12 | 0.49 – 0.52 | 20.00 | 0.50 | 20.00 | 0.50 |
| 0.10 | 22.17 ±0.14 | 0.53 – 0.57 | 22.21 | 0.54 | 22.21 | 0.54 |
| 0.20 | 24.52 ±0.17 | 0.57 – 0.62 | 24.65 | 0.58 | 24.65 | 0.58 |
| 0.30 | 26.94 ±0.19 | 0.62 – 0.66 | 27.35 | 0.62 | 27.35 | 0.62 |
| 0.40 | 29.55 ±0.22 | 0.67 – 0.72 | 30.33 | 0.65 | 30.33 | 0.65 |
| 0.50 | 32.25 ±0.24 | 0.71 – 0.76 | 33.62 | 0.68 | 33.62 | 0.68 |
| 0.60 | 34.96 ±0.27 | 0.76 – 0.82 | 37.26 | 0.71 | 37.26 | 0.71 |
| 0.70 | 37.63 ±0.30 | 0.80 – 0.87 | 41.29 | 0.73 | 41.29 | 0.73 |
| 0.80 | 40.47 ±0.34 | 0.86 – 0.93 | 45.74 | 0.75 | 45.74 | 0.75 |
| 0.90 | 43.24 ±0.37 | 0.91 – 0.99 | 50.65 | 0.77 | 50.65 | 0.77 |
| 1.00 | 46.05 ±0.40 | 0.96 – 1.05 | 56.08 | 0.79 | 56.08 | 0.79 |

Table 13: Transient analysis of busy period, Up times: Erlang-2, $E_u$=20

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 40.03 ±0.25 | 0.49 – 0.52 | 40.00 | 0.50 | 40.00 | 0.50 |
| 0.10 | 44.26 ±0.29 | 0.53 – 0.57 | 44.31 | 0.55 | 44.31 | 0.55 |
| 0.20 | 48.93 ±0.33 | 0.57 – 0.62 | 49.08 | 0.59 | 49.08 | 0.59 |
| 0.30 | 53.99 ±0.38 | 0.61 – 0.66 | 54.34 | 0.62 | 54.34 | 0.62 |
| 0.40 | 59.58 ±0.43 | 0.65 – 0.71 | 60.16 | 0.66 | 60.16 | 0.66 |
| 0.50 | 65.74 ±0.49 | 0.68 – 0.74 | 66.60 | 0.69 | 66.60 | 0.69 |
| 0.60 | 72.33 ±0.55 | 0.72 – 0.78 | 73.71 | 0.72 | 73.71 | 0.72 |
| 0.70 | 79.35 ±0.61 | 0.75 – 0.81 | 81.56 | 0.74 | 81.56 | 0.74 |
| 0.80 | 86.84 ±0.69 | 0.79 – 0.85 | 90.25 | 0.76 | 90.25 | 0.76 |
| 0.90 | 94.82 ±0.77 | 0.82 – 0.89 | 99.84 | 0.78 | 99.84 | 0.78 |
| 1.00 | 103.37 ±0.85 | 0.85 – 0.92 | 110.45 | 0.80 | 110.45 | 0.80 |
| 1.25 | 127.25 ±1.10 | 0.93 – 1.01 | 142.10 | 0.84 | 142.10 | 0.84 |
| 1.50 | 155.18 ±1.40 | 1.02 – 1.11 | 182.75 | 0.88 | 182.75 | 0.88 |

Table 14: Transient analysis of busy period, Up times: Erlang-2, $E_u$=40

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 60.04 ±0.37 | 0.49 – 0.52 | 60.00 | 0.50 | 60.00 | 0.50 |
| 0.20 | 73.35 ±0.50 | 0.57 – 0.62 | 73.51 | 0.59 | 73.51 | 0.59 |
| 0.40 | 89.53 ±0.65 | 0.65 – 0.71 | 90.00 | 0.66 | 90.00 | 0.66 |
| 0.60 | 109.28 ±0.82 | 0.71 – 0.77 | 110.15 | 0.72 | 110.15 | 0.72 |
| 0.80 | 132.51 ±1.04 | 0.76 – 0.83 | 134.76 | 0.77 | 134.76 | 0.77 |
| 1.00 | 159.73 ±1.29 | 0.81 – 0.88 | 164.82 | 0.81 | 164.82 | 0.81 |
| 1.25 | 200.42 ±1.68 | 0.88 – 0.95 | 211.91 | 0.85 | 211.91 | 0.85 |
| 1.50 | 249.77 ±2.17 | 0.94 – 1.03 | 272.38 | 0.88 | 272.38 | 0.88 |
| 1.75 | 311.09 ±2.78 | 1.00 – 1.08 | 350.03 | 0.91 | 350.03 | 0.91 |
| 2.00 | 385.98 ±3.53 | 1.04 – 1.14 | 449.73 | 0.93 | 449.73 | 0.93 |

Table 15: Transient analysis of busy period, Up times: Erlang-2, $E_u$=60

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 4.97 ±0.04 | 0.96 – 1.04 | 5.00 | 1.00 | 5.00 | 1.00 |
| 0.01 | 5.02 ±0.04 | 0.96 – 1.04 | 5.06 | 1.00 | 5.06 | 1.00 |
| 0.03 | 5.13 ±0.04 | 0.95 – 1.04 | 5.18 | 0.99 | 5.18 | 0.99 |
| 0.05 | 5.23 ±0.05 | 0.95 – 1.04 | 5.31 | 0.98 | 5.31 | 0.98 |
| 0.10 | 5.46 ±0.05 | 0.97 – 1.05 | 5.63 | 0.96 | 5.63 | 0.96 |
| 0.15 | 5.67 ±0.05 | 0.99 – 1.07 | 5.97 | 0.95 | 5.97 | 0.95 |
| 0.20 | 5.85 ±0.05 | 1.02 – 1.11 | 6.33 | 0.94 | 6.33 | 0.94 |
| 0.25 | 6.00 ±0.05 | 1.05 – 1.14 | 6.70 | 0.93 | 6.70 | 0.93 |
| 0.30 | 6.13 ±0.06 | 1.08 – 1.19 | 7.10 | 0.92 | 6.70 | 0.93 |
| 0.35 | 6.24 ±0.06 | 1.12 – 1.23 | 7.51 | 0.91 | 6.70 | 0.93 |
| 0.40 | 6.33 ±0.06 | 1.15 – 1.27 | 7.95 | 0.90 | 6.70 | 0.93 |

Table 16: Transient analysis of busy period, Up times: Exponential, $E_u$=5

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 9.94 ±0.09 | 0.96 – 1.04 | 10.00 | 1.00 | 10.00 | 1.00 |
| 0.05 | 10.45 ±0.09 | 0.96 – 1.04 | 10.56 | 0.99 | 10.56 | 0.99 |
| 0.10 | 10.94 ±0.10 | 0.96 – 1.05 | 11.16 | 0.98 | 11.16 | 0.98 |
| 0.15 | 11.43 ±0.10 | 0.97 – 1.06 | 11.78 | 0.97 | 11.78 | 0.97 |
| 0.20 | 11.91 ±0.11 | 0.99 – 1.08 | 12.44 | 0.97 | 12.44 | 0.97 |
| 0.25 | 12.32 ±0.11 | 1.01 – 1.10 | 13.12 | 0.96 | 13.12 | 0.96 |
| 0.30 | 12.72 ±0.12 | 1.04 – 1.13 | 13.85 | 0.96 | 13.85 | 0.96 |
| 0.35 | 13.11 ±0.12 | 1.07 – 1.17 | 14.61 | 0.95 | 14.61 | 0.95 |
| 0.40 | 13.48 ±0.13 | 1.10 – 1.20 | 15.41 | 0.95 | 15.41 | 0.95 |
| 0.45 | 13.80 ±0.13 | 1.12 – 1.23 | 16.25 | 0.95 | 16.25 | 0.95 |
| 0.50 | 14.14 ±0.14 | 1.16 – 1.27 | 17.14 | 0.94 | 17.14 | 0.94 |

Table 17: Transient analysis of busy period, Up times: Exponential, $E_u$=10

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 19.87 ±0.17 | 0.96 – 1.04 | 20.00 | 1.00 | 20.00 | 1.00 |
| 0.05 | 20.88 ±0.18 | 0.96 – 1.04 | 21.08 | 1.00 | 21.08 | 1.00 |
| 0.10 | 21.89 ±0.19 | 0.96 – 1.05 | 22.21 | 0.99 | 22.21 | 0.99 |
| 0.15 | 22.93 ±0.20 | 0.97 – 1.05 | 23.40 | 0.99 | 23.40 | 0.99 |
| 0.20 | 24.00 ±0.21 | 0.98 – 1.07 | 24.65 | 0.98 | 24.65 | 0.98 |
| 0.25 | 25.01 ±0.22 | 0.99 – 1.08 | 25.96 | 0.98 | 25.96 | 0.98 |
| 0.30 | 26.01 ±0.23 | 1.01 – 1.10 | 27.35 | 0.98 | 27.35 | 0.98 |
| 0.35 | 27.03 ±0.25 | 1.03 – 1.12 | 28.80 | 0.98 | 28.80 | 0.98 |
| 0.40 | 28.03 ±0.26 | 1.04 – 1.14 | 30.33 | 0.97 | 30.33 | 0.97 |
| 0.45 | 29.04 ±0.27 | 1.06 – 1.16 | 31.93 | 0.97 | 31.93 | 0.97 |
| 0.50 | 30.09 ±0.28 | 1.09 – 1.19 | 33.62 | 0.97 | 33.62 | 0.97 |
| 0.55 | 31.15 ±0.29 | 1.11 – 1.21 | 35.40 | 0.97 | 35.40 | 0.97 |
| 0.60 | 32.16 ±0.31 | 1.14 – 1.24 | 37.26 | 0.97 | 37.26 | 0.97 |
| 0.65 | 33.17 ±0.32 | 1.16 – 1.26 | 39.23 | 0.97 | 39.23 | 0.97 |
| 0.70 | 34.18 ±0.33 | 1.18 – 1.29 | 41.29 | 0.97 | 41.29 | 0.97 |
| 0.75 | 35.25 ±0.35 | 1.21 – 1.32 | 43.46 | 0.97 | 43.46 | 0.97 |
| 0.80 | 36.25 ±0.36 | 1.23 – 1.35 | 45.74 | 0.97 | 45.74 | 0.97 |

Table 18: Transient analysis of busy period, Up times: Exponential, $E_u$=20

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 39.74 ±0.35 | 0.96 – 1.04 | 40.00 | 1.00 | 40.00 | 1.00 |
| 0.10 | 43.78 ±0.38 | 0.96 – 1.05 | 44.31 | 1.00 | 44.31 | 1.00 |
| 0.20 | 48.18 ±0.43 | 0.97 – 1.06 | 49.08 | 0.99 | 49.08 | 0.99 |
| 0.30 | 52.60 ±0.47 | 0.99 – 1.08 | 54.34 | 0.99 | 54.34 | 0.99 |
| 0.40 | 57.39 ±0.52 | 1.01 – 1.10 | 60.16 | 0.99 | 60.16 | 0.99 |
| 0.50 | 62.43 ±0.57 | 1.04 – 1.14 | 66.60 | 0.99 | 66.60 | 0.99 |
| 0.60 | 67.76 ±0.63 | 1.06 – 1.16 | 73.71 | 0.98 | 73.71 | 0.98 |
| 0.70 | 73.44 ±0.69 | 1.09 – 1.18 | 81.56 | 0.98 | 81.56 | 0.98 |
| 0.80 | 79.56 ±0.75 | 1.11 – 1.22 | 90.25 | 0.98 | 90.25 | 0.98 |
| 0.90 | 85.52 ±0.82 | 1.15 – 1.25 | 99.84 | 0.98 | 99.84 | 0.98 |
| 1.00 | 92.04 ±0.90 | 1.18 – 1.29 | 110.45 | 0.98 | 110.45 | 0.98 |

Table 19: Transient analysis of busy period, Up times: Exponential, $E_u$=40

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 59.61 ±0.52 | 0.96 – 1.04 | 60.00 | 1.00 | 60.00 | 1.00 |
| 0.10 | 65.67 ±0.58 | 0.96 – 1.05 | 66.42 | 1.00 | 66.42 | 1.00 |
| 0.20 | 72.39 ±0.64 | 0.97 – 1.06 | 73.51 | 0.99 | 73.51 | 0.99 |
| 0.30 | 79.26 ±0.70 | 0.98 – 1.07 | 81.34 | 0.99 | 81.34 | 0.99 |
| 0.40 | 86.83 ±0.77 | 0.99 – 1.08 | 90.00 | 0.99 | 90.00 | 0.99 |
| 0.50 | 94.89 ±0.86 | 1.02 – 1.11 | 99.57 | 0.99 | 99.57 | 0.99 |
| 0.60 | 103.79 ±0.94 | 1.03 – 1.12 | 110.15 | 0.99 | 110.15 | 0.99 |
| 0.70 | 113.17 ±1.04 | 1.05 – 1.14 | 121.84 | 0.99 | 121.84 | 0.99 |
| 0.80 | 123.53 ±1.14 | 1.07 – 1.16 | 134.76 | 0.99 | 134.76 | 0.99 |
| 0.90 | 133.65 ±1.25 | 1.09 – 1.19 | 149.04 | 0.99 | 149.04 | 0.99 |
| 1.00 | 145.18 ±1.37 | 1.11 – 1.21 | 164.82 | 0.99 | 164.82 | 0.99 |

Table 20: Transient analysis of busy period, Up times: Exponential, $E_u$=60

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 4.99 ±0.05 | 1.39 – 1.56 | 5.00 | 1.50 | 5.00 | 1.50 |
| 0.05 | 5.25 ±0.06 | 1.37 – 1.53 | 5.31 | 1.45 | 5.31 | 1.45 |
| 0.10 | 5.48 ±0.06 | 1.36 – 1.52 | 5.63 | 1.40 | 5.63 | 1.40 |
| 0.15 | 5.66 ±0.06 | 1.38 – 1.54 | 5.97 | 1.36 | 5.97 | 1.36 |
| 0.20 | 5.83 ±0.06 | 1.40 – 1.56 | 6.33 | 1.32 | 6.33 | 1.32 |
| 0.25 | 5.96 ±0.06 | 1.43 – 1.60 | 6.70 | 1.28 | 6.70 | 1.28 |
| 0.30 | 6.07 ±0.07 | 1.46 – 1.63 | 7.10 | 1.25 | 6.70 | 1.28 |
| 0.35 | 6.17 ±0.07 | 1.50 – 1.67 | 7.51 | 1.22 | 6.70 | 1.28 |
| 0.40 | 6.26 ±0.07 | 1.52 – 1.70 | 7.95 | 1.20 | 6.70 | 1.28 |

Table 21: Transient analysis of busy period, Up times: Hyper-2, $E_u$=5

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 9.97 ±0.11 | 1.39 – 1.56 | 10.00 | 1.50 | 10.00 | 1.50 |
| 0.05 | 10.49 ±0.11 | 1.37 – 1.54 | 10.56 | 1.46 | 10.56 | 1.46 |
| 0.10 | 10.98 ±0.12 | 1.36 – 1.52 | 11.16 | 1.43 | 11.16 | 1.43 |
| 0.15 | 11.45 ±0.12 | 1.36 – 1.52 | 11.78 | 1.39 | 11.78 | 1.39 |
| 0.20 | 11.86 ±0.12 | 1.37 – 1.52 | 12.44 | 1.36 | 12.44 | 1.36 |
| 0.25 | 12.26 ±0.13 | 1.38 – 1.54 | 13.12 | 1.34 | 13.12 | 1.34 |
| 0.30 | 12.67 ±0.14 | 1.41 – 1.57 | 13.85 | 1.31 | 13.85 | 1.31 |
| 0.35 | 13.04 ±0.14 | 1.43 – 1.59 | 14.61 | 1.29 | 14.61 | 1.29 |
| 0.40 | 13.35 ±0.14 | 1.45 – 1.61 | 15.41 | 1.26 | 15.41 | 1.26 |
| 0.45 | 13.64 ±0.15 | 1.47 – 1.63 | 16.25 | 1.24 | 16.25 | 1.24 |
| 0.50 | 13.93 ±0.15 | 1.50 – 1.67 | 17.14 | 1.22 | 17.14 | 1.22 |

Table 22: Transient analysis of busy period, Up times: Hyper-2, $E_u$=10

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 19.94 ±0.21 | 1.39 – 1.56 | 20.00 | 1.50 | 20.00 | 1.50 |
| 0.05 | 20.97 ±0.22 | 1.38 – 1.54 | 21.08 | 1.47 | 21.08 | 1.47 |
| 0.10 | 21.99 ±0.23 | 1.36 – 1.52 | 22.21 | 1.44 | 22.21 | 1.44 |
| 0.15 | 23.02 ±0.24 | 1.35 – 1.51 | 23.40 | 1.41 | 23.40 | 1.41 |
| 0.20 | 23.95 ±0.25 | 1.35 – 1.50 | 24.65 | 1.39 | 24.65 | 1.39 |
| 0.25 | 25.03 ±0.26 | 1.35 – 1.51 | 25.96 | 1.36 | 25.96 | 1.36 |
| 0.30 | 26.05 ±0.27 | 1.36 – 1.51 | 27.35 | 1.34 | 27.35 | 1.34 |
| 0.35 | 26.98 ±0.28 | 1.36 – 1.51 | 28.80 | 1.32 | 28.80 | 1.32 |
| 0.40 | 27.93 ±0.29 | 1.37 – 1.52 | 30.33 | 1.30 | 30.33 | 1.30 |
| 0.45 | 28.91 ±0.31 | 1.38 – 1.53 | 31.93 | 1.28 | 31.93 | 1.28 |
| 0.50 | 29.88 ±0.32 | 1.40 – 1.55 | 33.62 | 1.26 | 33.62 | 1.26 |

Table 23: Transient analysis of busy period, Up times: Hyper-2, $E_u$=20

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 39.89 ±0.42 | 1.39 – 1.56 | 40.00 | 1.50 | 40.00 | 1.50 |
| 0.10 | 43.99 ±0.46 | 1.36 – 1.52 | 44.31 | 1.45 | 44.31 | 1.45 |
| 0.20 | 48.24 ±0.50 | 1.34 – 1.49 | 49.08 | 1.40 | 49.08 | 1.40 |
| 0.30 | 52.86 ±0.55 | 1.33 – 1.48 | 54.34 | 1.35 | 54.34 | 1.35 |
| 0.40 | 57.49 ±0.59 | 1.32 – 1.47 | 60.16 | 1.32 | 60.16 | 1.32 |
| 0.50 | 62.34 ±0.64 | 1.32 – 1.46 | 66.60 | 1.28 | 66.60 | 1.28 |
| 0.60 | 67.51 ±0.70 | 1.34 – 1.48 | 73.71 | 1.25 | 73.71 | 1.25 |
| 0.70 | 73.04 ±0.76 | 1.35 – 1.49 | 81.56 | 1.23 | 81.56 | 1.23 |
| 0.80 | 78.67 ±0.83 | 1.37 – 1.51 | 90.25 | 1.20 | 90.25 | 1.20 |
| 0.90 | 84.70 ±0.90 | 1.39 – 1.53 | 99.84 | 1.18 | 99.84 | 1.18 |
| 1.00 | 91.14 ±0.97 | 1.41 – 1.55 | 110.45 | 1.16 | 110.45 | 1.16 |

Table 24: Transient analysis of busy period, Up times: Hyper-2, $E_u$=40

| Mean and squared coefficient of variation of busy period | | | | | | |
|---|---|---|---|---|---|---|
| | Simulation | | Approx. 1A | | Approx. 1B | |
| K | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ | $E(\|l\|)$ | $c^2(\|l\|)$ |
| 0.00 | 59.83 ±0.64 | 1.39 – 1.56 | 60.00 | 1.50 | 60.00 | 1.50 |
| 0.10 | 66.00 ±0.69 | 1.36 – 1.52 | 66.42 | 1.45 | 66.42 | 1.45 |
| 0.20 | 72.55 ±0.75 | 1.33 – 1.48 | 73.51 | 1.40 | 73.51 | 1.40 |
| 0.30 | 79.71 ±0.82 | 1.32 – 1.46 | 81.34 | 1.36 | 81.34 | 1.36 |
| 0.40 | 87.18 ±0.90 | 1.31 – 1.45 | 90.00 | 1.32 | 90.00 | 1.32 |
| 0.50 | 95.07 ±0.97 | 1.29 – 1.43 | 99.57 | 1.29 | 99.57 | 1.29 |
| 0.60 | 103.55 ±1.06 | 1.30 – 1.43 | 110.15 | 1.26 | 110.15 | 1.26 |
| 0.70 | 112.97 ±1.16 | 1.30 – 1.43 | 121.84 | 1.23 | 121.84 | 1.23 |
| 0.80 | 122.66 ±1.26 | 1.31 – 1.44 | 134.76 | 1.21 | 134.76 | 1.21 |
| 0.90 | 133.15 ±1.37 | 1.32 – 1.45 | 149.04 | 1.19 | 149.04 | 1.19 |
| 1.00 | 144.25 ±1.49 | 1.33 – 1.46 | 164.82 | 1.17 | 164.82 | 1.17 |

Table 25: Transient analysis of busy period, Up times: Hyper-2, $E_u$=60

| Squared coefficient of variation of busy period length | | | | |
|---|---|---|---|---|
| $E_u$ | D | E2 | M | H2 |
| 5 | 0.3 – 0.4 | 1.0 – 1.1 | 1.6 – 1.8 | 2.2 – 2.5 |
| 10 | 0.9 – 1.1 | 1.9 – 2.2 | 2.7 – 3.2 | 3.3 – 4.0 |
| 20 | 3.1 – 3.5 | 3.3 – 3.8 | 3.7 – 4.3 | 3.9 – 4.5 |
| 40 | 1.3 – 1.4 | 1.8 – 2.0 | 2.3 – 2.5 | 2.5 – 2.8 |
| 60 | 1.0 – 1.1 | 1.4 – 1.5 | 1.8 – 1.9 | 1.9 – 2.1 |

Table 26: Squared coefficient of variation busy period length, $K = 5$

| Squared coefficient of variation of busy period length | | | | |
|---|---|---|---|---|
| $E_u$ | D | E2 | M | H2 |
| 5 | 0.3 – 0.4 | 1.0 – 1.1 | 1.6 – 1.8 | 2.2 – 2.5 |
| 10 | 0.9 – 1.1 | 1.9 – 2.2 | 2.8 – 3.3 | 3.6 – 4.4 |
| 20 | 6.1 – 7.3 | 6.4 – 7.8 | 6.5 – 7.8 | 6.8 – 8.4 |
| 40 | 1.4 – 1.6 | 2.1 – 2.3 | 2.7 – 3.0 | 3.1 – 3.5 |
| 60 | 1.1 – 1.2 | 1.4 – 1.6 | 1.9 – 2.1 | 2.1 – 2.3 |

Table 27: Squared coefficient of variation busy period length, $K = 10$

| System Ineffectiveness | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | without PM | | with Preventive Maintenance | | | | | |
| | | | Erlang-2 | | | | K-2 | |
| | | | $T = 12$ | | $T = 18$ | | $T = 12$ | $T = 18$ |
| $K$ | App.1A | Simul. | App. | Sim. | App. | Sim. | App. | App |
| 0.00 | .048 | .048 | .035 – .035 | .039 | .038 – .038 | .041 | .044 | .044 |
| 0.20 | .039 | .039 | .022 – .029 | .027 | .026 – .031 | .030 | .029 | .031 |
| 0.40 | .032 | .033 | .017 – .025 | .022 | .020 – .026 | .025 | .022 | .024 |
| 0.60 | .026 | .028 | .013 – .021 | .019 | .016 – .022 | .021 | .017 | .019 |
| 0.80 | .021 | .024 | .010 – .019 | .016 | .013 – .019 | .018 | .014 | .015 |
| 1.00 | .018 | .021 | .008 – .017 | .014 | .010 – .016 | .016 | .011 | .012 |
| 2.00 | .006 | .014 | .003 – .011 | .008 | .004 – .009 | .010 | .004 | .004 |

Table 28: System effectiveness under block replacement
Exp. down times, $E_c = 1$, $E_p = 0.1$, $E_u = 20$.

| System Ineffectiveness | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | without PM | | with Preventive Maintenance | | | | | |
| | | | Erlang-2 | | | | K-2 | |
| | | | $T = 30$ | | $T = 36$ | | $T = 30$ | $T = 36$ |
| $K$ | App.1A | Simul. | App. | Sim. | App. | Sim. | App. | App |
| 0.0 | .024 | .024 | .019 – .019 | .020 | .020 – .021 | .021 | .023 | .023 |
| 0.2 | .020 | .020 | .013 – .016 | .014 | .014 – .016 | .015 | .016 | .016 |
| 0.4 | .016 | .016 | .010 – .013 | .012 | .011 – .014 | .012 | .012 | .013 |
| 0.6 | .013 | .014 | .008 – .012 | .010 | .009 – .012 | .010 | .010 | .010 |
| 0.8 | .011 | .011 | .007 – .010 | .008 | .007 – .010 | .009 | .008 | .008 |
| 1.0 | .009 | .010 | .005 – .009 | .007 | .006 – .009 | .007 | .007 | .007 |
| 2.0 | .003 | .004 | .002 – .005 | .003 | .002 – .005 | .003 | .002 | .002 |

Table 29: System effectiveness under block replacement
Exp. down times, $E_c = 1$, $E_p = 0.1$, $E_u = 40$.

| System Ineffectiveness | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | without PM | | with Preventive Maintenance | | | | | |
| | | | Erlang-2 | | | | K-2 | |
| | | | $T = 40$ | | $T = 54$ | | $T = 40$ | $T = 54$ |
| $K$ | App.1A | Simul. | App. | Sim. | App. | Sim. | App. | App |
| 0.0 | .016 | .016 | .013 – .013 | .013 | .014 – .014 | .014 | .016 | .016 |
| 0.2 | .013 | .013 | .009 – .011 | .009 | .010 – .011 | .010 | .011 | .011 |
| 0.4 | .011 | .011 | .007 – .009 | .007 | .007 – .009 | .008 | .008 | .009 |
| 0.6 | .009 | .009 | .005 – .008 | .006 | .006 – .008 | .007 | .007 | .007 |
| 0.8 | .007 | .008 | .004 – .007 | .005 | .005 – .007 | .006 | .005 | .006 |
| 1.0 | .006 | .006 | .004 – .006 | .004 | .004 – .006 | .005 | .004 | .005 |
| 2.0 | .002 | .003 | .001 – .004 | .002 | .001 – .003 | .002 | .002 | .002 |

Table 30: System effectiveness under block replacement
Exp. down times, $E_c = 1$, $E_p = 0.1$, $E_u = 60$.

| System Ineffectiveness | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | without PM | | with Preventive Maintenance | | | | | |
| | | | Erlang-2 | | | | K-2 | |
| | | | $T = 12$ | | $T = 18$ | | $T = 12$ | $T = 18$ |
| $K$ | App.1A | Simul. | App. | Sim. | App. | Sim. | App. | App. |
| 0.00 | .048 | .048 | .037 | .037 | .041 | .041 | .046 | .046 |
| 0.20 | .039 | .039 | .026 | .026 | .031 | .031 | .033 | .035 |
| 0.40 | .032 | .033 | .021 | .022 | .025 | .026 | .027 | .029 |
| 0.60 | .026 | .028 | .017 | .019 | .021 | .022 | .022 | .023 |
| 0.80 | .021 | .024 | .014 | .016 | .017 | .019 | .018 | .019 |
| 1.00 | .018 | .021 | .012 | .014 | .014 | .017 | .015 | .016 |
| 2.00 | .006 | .014 | .004 | .008 | .005 | .010 | .005 | .005 |

Table 31: System effectiveness under age replacement
Exp. down times, $E_c = 1$, $E_p = 0.1$, $E_u = 20$.

| System Ineffectiveness | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | without PM | | with Preventive Maintenance | | | | | |
| | | | Erlang-2 | | | | K-2 | |
| | | | $T = 30$ | | $T = 36$ | | $T = 30$ | $T = 36$ |
| $K$ | App.1A | Simul. | App. | Sim. | App. | Sim. | App. | App |
| 0.0 | .024 | .024 | .019 | .020 | .021 | .021 | .023 | .023 |
| 0.2 | .020 | .020 | .015 | .015 | .016 | .016 | .018 | .018 |
| 0.4 | .016 | .016 | .012 | .012 | .013 | .013 | .014 | .015 |
| 0.6 | .013 | .014 | .010 | .010 | .011 | .011 | .012 | .012 |
| 0.8 | .011 | .011 | .008 | .008 | .009 | .009 | .009 | .010 |
| 1.0 | .009 | .010 | .006 | .007 | .007 | .008 | .008 | .008 |
| 2.0 | .003 | .004 | .002 | .003 | .003 | .004 | .003 | .003 |

Table 32: System effectiveness under age replacement
Exp. down times, $E_c = 1$, $E_p = 0.1$, $E_u = 40$.

| System Ineffectiveness | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | without PM | | with Preventive Maintenance | | | | | |
| | | | Erlang-2 | | | | K-2 | |
| | | | $T = 40$ | | $T = 54$ | | $T = 40$ | $T = 54$ |
| $K$ | App.1A | Simul. | App. | Sim. | App. | Sim. | App. | App |
| 0.0 | .016 | .016 | .013 | .013 | .014 | .014 | .016 | .016 |
| 0.2 | .013 | .013 | .009 | .009 | .011 | .011 | .012 | .012 |
| 0.4 | .011 | .011 | .008 | .008 | .009 | .009 | .009 | .010 |
| 0.6 | .009 | .009 | .006 | .006 | .007 | .007 | .008 | .008 |
| 0.8 | .007 | .008 | .005 | .005 | .006 | .006 | .006 | .007 |
| 1.0 | .006 | .006 | .004 | .004 | .005 | .005 | .005 | .005 |
| 2.0 | .002 | .003 | .002 | .002 | .002 | .002 | .002 | .002 |

Table 33: System effectiveness under age replacement
Exp. down times, $E_c = 1$, $E_p = 0.1$, $E_u = 60$.

| System Ineffectiveness | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Erlang-2 | | | | | | K-2 | | | |
| | $T = 40$ | | | $T = 54$ | | | $T = 40$ | | $T = 54$ | |
| $K$ | App. | MM1. | Sim. | App. | MM1. | Sim. | App. | MM1 | App | MM1. |
| 0.0 | .013 | .013 | .013 | .014 | .014 | .014 | .016 | .016 | .016 | .016 |
| 0.2 | .009 | .009 | .009 | .011 | .010 | .011 | .012 | .011 | .012 | .011 |
| 0.4 | .008 | .006 | .008 | .009 | .007 | .009 | .009 | .008 | .010 | .009 |
| 0.6 | .006 | .004 | .006 | .007 | .006 | .007 | .008 | .006 | .008 | .007 |
| 0.8 | .005 | .003 | .005 | .006 | .004 | .006 | .006 | .004 | .007 | .005 |
| 1.0 | .004 | .000 | .004 | .005 | .003 | .005 | .005 | .003 | .005 | .004 |
| 2.0 | .002 | .000 | .002 | .002 | .001 | .002 | .002 | .001 | .002 | .001 |

Table 34: System effectiveness under age replacement
Exp. down times, $E_c = 1$, $E_p = 0.1$, $E_u = 60$.