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# An $H_\infty$-Parameter Estimator and its Interpretation

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**Abstract**

In this paper we study the $H_\infty$ estimator used in the recently developed $H_\infty$ control theory. We study stochastic interpretations related to a weighted maximum-likelihood and the Kullback-Leibler distance and show the specific modifications compared to the standard Kalman filter.

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## 1 Introduction

Most of the identification methods currently used both in practice and in theory are based upon least-squares estimation [8]. Of course, least-squares estimation is directly connected to Kalman filtering. [1]. In turn, Kalman filtering is the main ingredient in $H_2$ or linear quadratic Gaussian (LQG) control.

Recently $H_\infty$ control theory has been developed. This was mainly motivated by the requirement to guard against model uncertainty and it yields a stable and robust design if the model uncertainty is clearly structured and bounded by a priori known bounds. This theory was developed during the last decade [4, 11]. It showed that identification did not deliver a major ingredient needed for the successful application of $H_\infty$ control theory, namely a way to derive via your identification algorithms

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knowledge of the structure as well as bounds on the model uncertainty. Recently people have started to investigate identification techniques which are specifically designed to deliver this structural information and/or quantitative bounds of the model uncertainty [12, 7, 9]

Our objective in this paper is different. We note that an $H_\infty$ controller, like in LQG, has the form of a state observer interconnected with a state feedback. However this state observer is essentially different from the Kalman filter used in LQG. The Kalman filter has several interpretations (besides the obvious determination of the conditional expectation). It has an interpretation as least squares, maximum likelihood and finally the minimal distance solution using the Kullback-Leibler distance measure [3, 10].

In order to understand the specific features of the $H_\infty$ state observer, we will show that this observer also enables these different interpretations but with clear and very intuitive adaptations.

## 2 Classical least squares estimation

Suppose we have the following discrete-time system:

$$y(k) = \varphi^{\mathrm{T}}(k)\theta + w(k) \tag{1}$$

where, at time $k$, $\varphi(k)$ is a known vector consisting of past inputs and outputs (the specific form of $\varphi$ is not relevant in this paper) and $y$ a scalar measurement. Moreover, $w$ is a standard Gaussian white noise process (standard refers to the fact that the variance is equal to the identity) and $\theta$ is an unknown parameter vector. It is our objective to determine an estimate of $\theta$ on the basis of past observations. In order to make a clear distinction between the random variable $y(k)$ and our actual measurements, we will denote the latter by $\bar{y}(k)$.

The least squares estimate is a non stochastic approach where we minimize

$$\sum_{k=0}^{N} [\bar{y}(k) - \varphi^{\mathrm{T}}(k)\theta]^2$$

over $\theta$ and take that as our estimate of $\theta$ at time $N$.

Another non-Bayesian approach is maximum likelihood. Here we derive a density function of $y(0), \dots, y(N)$ which is parameterized by $\theta$. The maximum likelihood estimate is obtained by replacing the variables of the density function by the measurements and then maximizing the resulting function over $\theta$. The resulting maximizing $\theta$, we take as our estimate.

Another interpretation of the above is based on the Kullback-Leibler distance e.g.[13]. Based on our measurements of $y$ we first build an experimental distribution for $y$ in the general class of Gaussian distributions with expectation the actual measurement $\bar{y}$ and variance $I$ (the choice $I$ is arbitrary but does not influence the final estimate). We also have a Gaussian distribution for $y$ based on the system description (1) with expectation $\phi^{\mathrm{T}}\theta$ and covariance $I$. We then find the closest element in this parameterized family of distributions to the experimental distribution. As a distance concept we use the Kullback-Leibler distance or Kullback-Leibler divergence. Suppose two distributions have density functions $f_1$ and $f_2$ with respect to some arbitrary measure $\mu$. Then the Kullback-Leibler distance is defined by

$$d_{KL}(f_1, f_2) := \int f_1(y) \log \frac{f_1(y)}{f_2(y)} d\mu(y)$$

which is a well-defined nonnegative and real valued functional. However, this distance measure does not satisfy the triangle inequality. It is known that maximizing the likelihood function is equivalent to minimizing the Kullback-Leibler distance to the maximum likelihood estimate. For another way to relate maximum likelihood estimation and the Kullback-Leibler distance see [10].

In a Bayesian setting we assume an a priori (Gaussian) distribution on $\theta$ with expectation 0 and covariance $R$. Then a natural estimate for $y$ is given by the conditional expectation of $y$ given past observations $y(0), \ldots, y(T)$. This conditional expectation can e.g. be determined using the Kalman filter:

$$\begin{aligned} \hat{\theta}(k) = \hat{\theta}(k-1) + [1 + Q(k)\phi(k)\phi^{\mathrm{T}}(k)]^{-1} \\ \cdot Q(k)\phi(k) \left[ y(k) - \phi^{\mathrm{T}}(k)\hat{\theta}(k-1) \right] \end{aligned} \tag{2}$$

where $Q$ is given by:

$$Q^{-1}(k) = R^{-1} + \sum_{j=0}^{k-1} \phi(j)\phi^{\mathrm{T}}(j) \tag{3}$$

The amazing fact is that all the above non-Bayesian approaches yield the same estimator for $\theta$. Moreover, if we use the Bayesian approach and take the limit as $R \to \infty$ (in the limit we have no a priori information) then we obtain again the same estimator. Most identification methods are based on this estimator.

# 3 Parameter estimation

We again work with the system (1). However, in the $H_\infty$ control problem we use a deterministic approach and derive a state observer [11] which is not equal to the standard Kalman filter. Given the model (1), we can apply standard $H_\infty$ theory to the estimation of $\theta$ based on measurements $y$ [2]. We have a system of the form:

$$\Sigma : \begin{cases} \theta(k+1) = \theta(k) \\ y(k) = \phi^{\mathrm{T}}(k)\theta(k) + w(k) \\ z(k) = \theta(k) - \hat{\theta}(k) \end{cases} \tag{4}$$

We look for a compensator from $y$ to $\hat{\theta}$ which minimizes the $H_\infty$ norm from $w$ to $z$. We find an (causal) estimator $K$ such that $\hat{\theta} = Ky$ minimizes the following criterion:

$$\sup_{w \in \ell_2} \sup_{\theta} \gamma^{-2} \|\theta - \hat{\theta}\|^2 - \|w\|^2 - \theta^{\mathrm{T}} R^{-1} \theta \tag{5}$$

and an optimal estimator of the form

$$\hat{\theta}(k) = \hat{\theta}(k-1) + [1 + Q(k)\phi(k)\phi^{\mathrm{T}}(k)]^{-1} \cdot Q(k)\phi(k)\left[y(k) - \phi^{\mathrm{T}}(k)\hat{\theta}(k-1)\right] \tag{6}$$

where $Q$ is determined by the following recursion:

$$\begin{aligned} Q^{-1}(k+1) &= Q^{-1}(k) + \phi(k)\phi^{\mathrm{T}}(k) - \gamma^{-2}I, \\ Q(0) &= R. \end{aligned}$$

and $\gamma$ is large enough such that $Q$ is well-defined and positive-definite.

In order to compare this new estimator with the estimator in section 2, we first have to derive a stochastic interpretation for the $H_\infty$ estimator. This can be done using LEQG [14] for the system (4). We use a Bayesian approach where $\theta$ has an a priori Gaussian distribution with expectation 0 and covariance $R$. Moreover, $w$ is a standard Gaussian white noise process. We define the following criterion:

$$\min_{\hat{\theta}} \mathcal{E} \left[\exp\left(\gamma^{-2}\sum_{k=0}^{N} \|\theta - \hat{\theta}(k)\|^2\right)\right] \tag{7}$$

where $\hat{\theta}(k)$ is constrained to be $\mathcal{F}_k^y$ measurable, i.e. $\hat{\theta}(k)$ should be a function measurements up to time $k$ only. It can be shown that the $H_\infty$ estimator given above is actually an optimal controller with respect to the criterion (7).

Note that it is essential to have the summation in the above criterion. If we would define the following criterion for a fixed $k \in \mathbb{Z}_+$:

$$\min_{\hat{\theta}(k)} \mathcal{E}\ \exp\left(\gamma^{-2}\|\theta - \hat{\theta}(k)\|^2\right)$$

then we obtain as a solution the classical Kalman filter.

Using the criterion (7) as a starting point, we can derive maximum likelihood and Kullback-Leibler interpretations for $H_\infty$ parameter estimators. Note that (5) already yields an indefinite least squares interpretation. From now on we will assume that our estimate $\hat{\theta}(k)$ has been chosen for all $k < N$. Our objective is to determine the mechanism which yields the new estimate $\hat{\theta}(N)$.

Let $f$ be the density function of $y := [y(0), \ldots, y(N)]^{\mathrm{T}}$ as a function of the uncertain parameter $\theta$. We define the following optimization problem:

$$\sup_{\theta} f(\bar{y}(0), \ldots, \bar{y}(N), \theta) \times \exp\left(\gamma^{-2}\sum_{k=0}^{N-1} \|\theta - \hat{\theta}(k)\|^2\right)$$

where $\bar{y}(k)$ denotes our actual measurements. The $\theta$ maximizing the above criterion will be taken as our estimate of $\theta$ at time $N$. This optimization yields a weighted maximum likelihood estimation and it is not very difficult to show that as $R \to \infty$

the $H_\infty$ estimator converges to the estimator obtained via this weighted maximum likelihood estimation.

Next, we focus on an interpretation related to the Kullback-Leibler distance measure.

We define $\tilde{\theta}$ as a Gaussian variable with expectation $\theta$ and variance $\gamma^2 I$. We also have the actual Gaussian distribution of $y$ with expectation $\phi^{\mathrm{T}}\theta$ and covariance $I$.

Next, we define an experimental Gaussian distribution for the measurements (vector) $y$ with expectation the measurements $\bar{y}$ and variance $I$. Then we construct an experimental distribution of $\tilde{\theta}$ with expectation $\bar{\theta}$ and variance $\bar{W}$ on the basis of previous estimates $\hat{\theta}$ where

$$\begin{aligned} \bar{\theta} &:= \frac{1}{N}\sum_{k=0}^{N-1}\hat{\theta}(k) \\ \bar{W} &:= \frac{1}{N}\sum_{k=0}^{N-1}(\hat{\theta}(k)-\bar{\theta})^2. \end{aligned}$$

Let $f_\theta^y$ and $f_{exp}^y$ be density functions of the actual and experimental distribution of $y$ with respect to some measure $\mu^y$. Similarly, let $f_\theta^{\tilde{\theta}}$ and $f_{exp}^{\tilde{\theta}}$ be density functions of the actual and experimental distribution of $\tilde{\theta}$ with respect to some measure $\mu$. Then we can define the following criterion:

$$\inf_\theta \frac{1}{N} d_{KL}(f_{exp}^y, f_\theta^y) - d_{KL}(f_{exp}^{\tilde{\theta}}, f_\theta^{\tilde{\theta}})$$

The $\theta$ which minimizes the above criterion is actually equal to the estimator obtained via our weighted maximum likelihood estimator discussed above.

A related though different approach is followed in [5, 6] where a linear estimation problem is formulated in Krein spaces.

In the above Kullback-Leibler distance we have not used a specific form for $\varphi$. Hence the measurement $y$ might not be stationary. In case $y$ is stationary we can adapt the above distance concept to make explicit use of this fact.

## 4 Conclusion

In this paper we showed least-squares, maximum likelihood and Kullback-Leibler interpretations of the $H_\infty$ estimator. This gives us a starting point for a detailed analysis of the differences and the similarities between the classic Kalman filter and the newly developed $H_\infty$ estimator.

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