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Analysis of a two-dimensional algebraic nearest-neighbour random walk (Queue with paired services)

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Abstract

In the present study the generating function of the stationary distribution of a positive recurrent nearest-neighbour random walk is analysed. The approach as developed by FLATTO [6] appears also to be applicable for the construction of the solution of the inherent functional equation. It is shown that the generating function is a fairly simple algebraic function, its derivation is, however, rather intricate. In the final section a slightly different nearest-neighbour random walk is considered. It turns out that its generating function is not algebraic. Its character is exposed, and it is concluded that a slightly different approach, viz. as discussed in [7], is required to construct the generating function. Together with the already available results in the literature it may be concluded that presently quite some insight has been obtained concerning the analytical techniques to be used in deriving explicit expressions for the generating function of the stationary distribution of a positive recurrent nearest-neighbour random walk.

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1. Introduction

In the present study we consider the two-dimensional stochastic process $\{(\mathbf{x}_n\mathbf{y}_n), n = 0, 1, 2, \ldots\}$ with structure described by: for $n = 0, 1, 2, \ldots,$

$$\begin{aligned} \mathbf{x}_{n+1} &= [\mathbf{x}_n - 1]^+ + \boldsymbol{\xi}_n, \\ \mathbf{y}_{n+1} &= [\mathbf{y}_n - 1]^+ + \boldsymbol{\eta}_n. \end{aligned} \tag{1.1}$$

Here

$$\{(\boldsymbol{\xi}_n, \boldsymbol{\eta}_n), n = 0, 1, 2, \ldots\} \text{ with } (\boldsymbol{\xi}_n, \boldsymbol{\eta}_n) \in \{0, 1, 2, \ldots\}^{2\times},$$

is a sequence of i.i.d. stochastic vectors. The bivariate generating function of the joint distribution of $(\boldsymbol{\xi}_n, \boldsymbol{\eta}_0)$ is given by

$$\phi(p,q) := \mathrm{E}\{p^{\mathbf{x}_n} q^{\mathbf{y}_n}\} = \frac{ar_1}{1+a} p^2 q + \frac{ar_2}{1+a} pq^2 + \frac{1}{1+a}, \tag{1.2}$$

$$a > 0, \quad r_1 + r_2 = 1, \quad 0 < r_1 < r_2 < 1.$$

From (1.1) and (1.2) it is seen that the $(\mathbf{x}_n, \mathbf{y}_n)$-process is a two-dimensional nearest-neighbour random walk with one-step transition probabilities at an interior point of the state space all zero, except for

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those to the North, to the East and to the South-West see fig. 1.1.

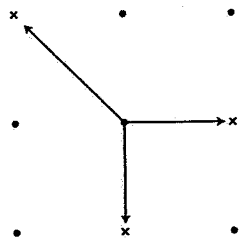


FIGURE 1.1.

It is readily verified, cf.[1], p. 95, that the $(\mathbf{x}_n, \mathbf{y}_n)$-process is positive recurrent if and only if

$$0 < a < \min(\frac{1}{r_1}, \frac{1}{r_2}); \tag{1.3}$$

note that the $(\mathbf{x}_n, \mathbf{y}_n)$-process is aperiodic and that its state space is irreducible.
Henceforth (1.3) is assumed to apply, and so the $(\mathbf{x}_n, \mathbf{y}_n)$-process has a unique stationary distribution. With $(\mathbf{x}, \mathbf{y})$ a stochastic vector with distribution this stationary distribution, put

$$\Phi(p,q) := \mathrm{E}\{p^{\mathbf{x}} q^{\mathbf{y}}\}, \quad |p| \le 1, \, |q| \le 1. \tag{1.4}$$

From (1.1), (1.2) and (1.4) it is readily derived that for $\Phi(p,q)$ holds:

i. for fixed q with $|q| \le 1, \Phi(p,q)$ is regular in $|p| < 1$, continuous in $|p| \le 1$, and similary with p and q interchanged; (1.5)

ii. for $|p| \le 1, |q| \le 1$,

$$\frac{\phi(p,q) - pq}{\phi(p,q)} \Phi(p,q) = (1-p)(1-q)[\frac{\Phi(p,0)}{1-p} + \frac{\Phi(0,q)}{1-q} - \Phi(0,0)],$$

iii. $\Phi(1,1) = 1,$

iv. the coefficients in the series expanssion of $\Phi(p,q), |p| \le 1, |q| \le 1$ in terms of $p^x q^y$, $x, y \in \{0,1,2,\ldots\}$ are all positive.

The present study concerns the construction of an explicit representation for a function $\Phi(p,q)$ satisfying the four conditions in (1.5); it will be shown that these conditions (1.5) determine $\Phi(p,q)$ uniquely.
Random walks characterised by (1.1) occur frequently in the analysis of queueing systems. For the case that $\phi(p,q)$ is given by (1.2) the random walk models a single server system with paired services. In this model there are two queues in front of the server. At each of these queues customers arrive according to a Poisson process; these arrival processes are independent and have generally different arrival rates. Customers are served in pairs, one from each queue, whenever, at a moment that the server becomes available for the next service both queues are not empty. Paired customers have equal service times. Whenever one of the queues is empty at a moment that the server becomes available he takes a customer from the non-empty queue and serves him individually; whenever both queues are empty the server waits for the next arrival. Service times of paired as well as of individual customers are all negatively exponentially distributed with the same mean and all service times are independent and also independent of the arrival process. The $(\mathbf{x}_n, \mathbf{y}_n)$-process introduced above is actually the joint queue length process at the successive epochs immediately after a service completion.

The paired-service queueing model with a non-specified service time distribution leads also to the conditions (1.5) but with $\phi(p,q)$ not explicitly specified. This problem has been extensively studied by

BLANC [2]. Here the problem is transformed into a Boundary Value Problem. Its solution requires the construction of a conformal mapping which in general cannot be obtained explicitly and so has to be determined numerically if $\phi(p,q)$ is known, cf.COHEN, BOXMA [3]. Only for the case that the service time distribution is negative exponential a nearest-neighbour random walk can be used to model the joint queue length process; if it is not negative exponential a more complicated random walk is needed in the modeling.

From an analytic viewpoint the nearest-neighbour random walks hold a special position among the two-dimensional random walks on the lattice with integer coordinates in the first quarter plane; this being due to the fact that for nearest-neighbour random walks $\phi(p,q)$, cf.(1.2), is a biquadratic polynomial in p and q. Concerning the character of the bivariate generating function $\Phi(p,q)$, cf.(1.4), of positive recurrent nearest-neighbour random walks already quite some information is available. For these random walks with the one-step transition probabilities to the North, to the North-East and to the East all zero it is known that the functions $\Phi(p,0)$ and $\Phi(0,q)$ in (1.5)ii are meromorphic functions which can be explicitly determined, COHEN [4]. FAYOLLE, IASNOGORODSKI and MALYSHEV [5] investigate conditions for the functions $\Phi(p,q)$ to be algebraic. FLATTO and HAHN [6] and WRIGHT [7] construct explicit solutions for nearest-neighbour random walks with non-zero one-step transition probabilities to the North, to the North-East and to the East. The functional equations in their studies differ from (1.5)ii. However, their techniques are of a quite general character and in the present study we can use the approach developed by FLATTO and HAHN. A general analysis of nearest-neighbour random walks is described in the study of MALYSHEV [11].
We continue this introduction with a review of the sections of this paper.

The determination of $\Phi(p,q)$ requires the construction of the functions $\Phi(p,0), |p| \leq 1$, and $\Phi(0,q), |q| \leq 1$, cf.(1.5)ii. In section 2 the functional equation for these functions is formulated, cf. (2.1). This functional equation holds on the surface $K(p,q) = 0, |p| \leq 1, |q| \leq 1$, here $K(p,q)$ is the so-called kernel, cf.(3.1).

In section 3 various properties of the zero-tuples $(\hat{p},\hat{q})$ of $K(p,q)$ are described, note that $\hat{q}$ is a two-valued function of $\hat{p}$. The four branch points $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ of this function are analysed. In fig.3.1 the curve $K(p,q) = 0$ for real p and q is plotted together with the sequence of zeros of $K(p,q)$ generated by the zero-tuple (1,1) which is a finite sequence, see the marked points on the circuit described on the oval of the curve. These zeros play a prominent role and together with the branch points just mentioned they determine the expression for $\Phi(p,0)$ completely, cf.(8.12) and (8.13).

Section 4 starts with the introduction of the conformal mapping $\xi(p)$ of the upper semi-p-plane onto the interior of a rectangle; the branch points $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ being mapped onto the corner points of the rectangle. The inverse mapping $p(\xi)$ which is also a conformal mapping is a double periodic meromorphic function with a zero of multiplicitly two, and as such it is an elliptic function of order two. Next to $p(\xi)$ a function $q(\xi)$ is defined in (4.13), it is also an elliptic function of order two. The so constructed $p(\xi), q(\xi)$ constitute for every ξ a zero-tuple of $K(p,q) = 0$, i.e. with ξ as the uniformising parameter the functions $p(\xi), q(\xi)$ represent a uniformisation of the zero-tuples of the kernel.

In section 5 properties of $p(\xi)$ and $q(\xi)$ are described, in particular the location of their zeros and poles is indicated, and further the domain $\{\xi : |p(\xi)| < 1, |q(\xi)| < 1\}$ is characterised. In the closure of this domain the functional equation for $\Phi(p(\xi),0)$ and $\Phi(0,q(\xi))$ is formulated, viz. with

$$P(\xi) := \frac{\Phi(p(\xi),0)}{\Phi(0,0)} \quad , \quad Q(\xi) := \frac{\Phi(0,q(\xi))}{\Phi(0,0)}, \tag{1.6}$$

it reads, cf.(5.4),

$$\frac{1}{1-p(\xi)}P(\xi) + \frac{1}{1-q(\xi)}Q(\xi) = 1 \, , \; \xi \in \bar{\Gamma}_p \cap \bar{\Gamma}_q. \tag{1.7}$$

In section 6 it is shown by starting from the functional equation (2.1)iii that $\Phi(p,0)$ can be continued meromorphically as a one-valued function on a double sheeted Riemann surface R_P with the sheets crosswise connected along the slit formed by the branch points α_2, α_3, here $\alpha_2 > 1, \alpha_3 > 1$. This meromorphic continuation, which is indicated by the same symbol $\Phi(p,0)$ has on R_P exactly two poles and two zeros. Similar results are obtained for $\Phi(0,q)$. The poles of these functions are uniquely determined by the zeros of $K(p,q)=0$ which are generated by the zero $(1,1)$. The main point in the construction of $\Phi(p,0)$ concerns the allocation of the poles and zeros on the two sheets of R_P, analogously for $\Phi(0,q)$. This question is solved in section 8 by using results derived in section 7.

In section 7 the functions $P(\xi)-1$ and $Q(\xi)-1$, cf.(1.6), are shown to be elliptic functions of order 4, their periods are calculated, and asymptotic relations for their behaviour in the vicinity of their zeros are derived, see theorem 7.1. With the information so obtained explicit expressions for $P(\xi)$ and $Q(\xi)$ are derived from (1.7). These expressions are given in section 8, cf.(8.9).

Section 8 starts with the introduction of two elliptic functions $\lambda_p(\xi)$ and $\lambda_q(\xi)$ both of order 4. It is shown that the function $P(\xi)-1$ can be expressed in terms of $\lambda_p(\xi)$. Since $\lambda_p(\xi)$ and also $P(\xi)$ is a function of $p(\xi)$ the relation between $P(\xi)$ and $\lambda_p(\xi)$ leads by using (1.6) to an expression for $\Phi(p,0)$, analogously for $\Phi(0,q)$, see (8.12). The only remaining unknown in these results is $\Phi(0,0)$, it is determined by using the relations (2.1)iii, which result from the norming condition, see (8.13). The functions $\Phi(p,0)$ and $\Phi(0,q)$ appear to be the unique solution which possesses the properties of lemma 2.1 and as such lead uniquely via (1.5)ii to the determination of the bivariate generating function $\Phi(p,q)$.

The analysis of sections 3,...,8 applies for the case $r_2 > r_1$, cf. (1.2). In section 9 the case with $r_1 = r_2 = \frac{1}{2}$ is discussed. For this case the analysis of the proceeding sections requires only minor changes. However, it is not difficult to see that by an appriopriate limiting procedure the results for the present case can be obtained from those for $r_2 > r_1$.

In section 10 a nearest-neighbour random walk is considered with the only non-zero one-step transition probabilities those to the N, NE, E and SW; so this random walk differs only from that in the previous sections by a non-zero one-step transition probability to the NE. The main point in this section concerns the question whether the meromorphic continuation of $\Phi(p,0)$, and similarly that of $\Phi(0,q)$, is an algebraic function, as it is the case for the random walk with no NE-transition. It is shown that for the case discussed in section 10 this continuation of $\Phi(p,0)$ is not an algebraic function. Actually, its pole set has at least one accumulation point. This observation leads to the conclusion that the technique as described by WRIGHT [7] should be used to determine $\Phi(p,0)$ and $\Phi(0,q)$.

From the results obtained in the studies [4], [5], [6], [7], [14], [15], [16] and the present one quite some insight in the analytic structure of the bivariate generating function of the stationary distribution for a positive recurrent nearest-neighbour random walk is obtained. The structure is fairly diverse, but for analytical as well as numerical purposes the generating function is quite applicable, see [6], [7], [17].

2. THE FUNCTIONAL EQUATION

In this section we formulate the functional equation of which the solution leads to the determination of $\Phi(p,q)$.

LEMMA 2.1

i. $\Phi(p,0)$, *and also* $\Phi(0,p)$, *is regular for* $|p|<1$, *continuous for* $|p|\le 1$; (2.1)

ii. $(1-\hat{q})\Phi(\hat{p},0)+(1-\hat{p})\Phi(0,\hat{q})=(1-\hat{p})(1-\hat{q})\Phi(0,0)$,

for every zero-tuple $(\hat{p},\hat{q})$ *of*

$$\phi(p,q)-pq \equiv \frac{ar_1}{1+a}p^2q+\frac{ar_2}{1+a}pq^2-pq+\frac{1}{1+a}, \quad |p|\le 1, |q|\le 1,$$

iii. $$\Phi(1,0)=\frac{1-ar_2}{1+a},\ \Phi(0,1)=\frac{1-ar_1}{1+a}.$$

iv. *the coefficients of the series expansion of* $\Phi(p,0)$ *in* $|p|<1$ *are positive, analogously for* $\Phi(0,p)$.

PROOF. (2.1)i follows from (1.5)i. From (1.4) it follows that $|\Phi(p,q)| \leq 1$ for $|p| \leq 1, |q| \leq 1$, and so the righthand side of (1.5)ii should be zero if $\phi(\hat{p},\hat{q}) - \hat{p}\hat{q} = 0$, $|\hat{p}| \leq 1$, $|q| \leq 1$, i.e. (2.1)ii should hold. Take $p = 1$ in (1.5)ii, then divide the resulting expression by $1 - q$ and let $q \to 1$, then from (1.5)iii the first relation in (2.1)iii follows, the second is obtained by symmetry. The last statement follows from (1.4). □

REMARK 2.1 From (1.5)ii and (2.1)iii it is readily derived that for $|p| \leq 1, |q| \leq 1$,

$$\Phi(p,1) = \frac{1 - ar_1}{1 - ar_1 p}\phi(p,1), \; \Phi(1,q) = \frac{1 - ar_2}{1 - ar_2 q}\phi(1,q). \tag{2.2}$$

Note that by dividing the functional equation in (2.1)ii by $1 - \hat{q}$ with $|\hat{q}| = 1, \hat{q} \neq 1$, and letting $\hat{q} \to 1, \hat{p} \to 1$, it is readily seen that

$$\frac{\Phi(1,0)}{\Phi(0,1)} = \frac{1 - ar_2}{1 - ar_1}. \tag{2.3}$$

Further it is observed that the Kolmogorov equations for the stationary probabilities for the $(\mathbf{x}_n, \mathbf{y}_n)$-process have a unique solution except for a constant factor which is determined by the norming condition and hence $\Phi(0,0) > 0$ implies that (2.1) iv holds. □

3. On the zero-tuples of the kernel

In this section we investigate several properties of the zero-tuples of the kernel

$$K(p,q) = \frac{ar_1}{1+a}p^2 q + \frac{ar_2}{1+a}pq^2 - pq + \frac{1}{1+a}. \tag{3.1}$$

LEMMA 3.1. *For fixed q with $|q| = 1, q \neq 1$, the kernel $K(p,q)$ has two zeros, one $p(q)$, say, inside the unit disk $|p| < 1$, the other $p_2(q)$ in $|p| > 1$; similarly with p and q interchanged.*

PROOF. For $|q| = 1, q \neq 1$, we have from(3.1),

$$K(p,q) = 0 \Leftrightarrow p = \frac{1}{q}\left[\frac{ar_1}{1+a}p^2 q + \frac{ar_2}{1+a}pq^2 + \frac{1}{1+a}\right].$$

Because for $|p| \leq 1, \; |q| = 1, \; q \neq 1$,

$$\left| \frac{1}{q}\left[\frac{ar_1}{1+a}p^2 q + \frac{ar_2}{1+a}pq^2 + \frac{1}{1+a}\right] \right| < 1,$$

and $K(p,q)$ is regular in $|p| \leq 1$ for every fixed q, it follows by applying Rouché's theorem with contour the unit circle $|p| = 1$ that $K(p,q) = 0$ has exactly one zero $p_1(q)$ in $|p| < 1$. Since $K(p,q) = 0$ has two zeros, the other zero is in $|p| \geq 1$. For the product of the zeros it follows from (1.2) and (1.3) that,

$$|p_1(1)p_2(q)| = \frac{1}{ar_1}\frac{1}{|q|} > 1, \quad |q| = 1,$$

and so $|p_2(q)| > 1$. Analogously with p and q interchanged. □

REMARK 3.1. For $(\hat{p},\hat{q})$ a zero-tuple of $K(p,q)$ define

i. $p_1 = \hat{p}, \; q_1 = \hat{q}$, (3.2)

ii. for $n = 1, 2, \ldots$,
$q_{n+1} := \frac{1}{ar_2 p_n q_n}, \; p_{n+1} := \frac{1}{ar_1 p_n q_n}$.

Because $K(p,q)$ is a biquadratic form in p as well as in q it follows readily that

$$(p_n, q_n), (p_n, q_{n+1}), (p_{n+1}, q_{n+1}),\ n = 1, 2, , \ldots, \tag{3.3}$$

are all zero-tuples of $K(p, q)$. For the present case it is readily verified by direct computation that

$$p_4 = \hat{p}_1,\ q_4 = \hat{q}_1.$$

Note that a simple calculation shows that with $\hat{p} = 1, \hat{q} = 1$ the generated zero-tuples are given in the following table 3.1.

p	1	1	$\frac{r_2}{r_1}$	$\frac{r_2}{r_1}$	$\frac{1}{ar_1}$	$\frac{1}{ar_1}$
q	1	$\frac{1}{ar_2}$	$\frac{1}{ar_2}$	$\frac{r_1}{r_2}$	$\frac{r_1}{r_2}$	1

TABLE 3.1

In fig. 3.2, see below, the circuit on the oval shows geometrically the set of zeros generated by $(\hat{p}, \hat{q}) = (1, 1)$; follow the arrows in this figure. □

From (3.1) it is seen that the zeros of $K(p, q)$ are given by:

$$\text{i. } p_1(q) = \frac{1}{2ar_1q}[(1 + a - ar_2q)\, q - \sqrt{D_q(q)}],$$
$$p_2(q) = \frac{1}{2ar_1q}[(1 + a - ar_2q)q + \sqrt{D_q(q)};$$
$$\text{ii. } q_1(p) = \frac{1}{2ar_2p}[(1 + a - ar_1p)p - \sqrt{D_p(p)}],$$
$$q_2(p) = \frac{1}{2ar_2p}[(1 + a - ar_1p)p + \sqrt{D_p(p)}], \tag{3.4}$$

with

$$D_q(q) := (1 + a - ar_2q)^2q^2 - 4ar_1q,$$
$$D_p(p) := (1 + a - ar_1p)^2p^2 - 4ar_2p. \tag{3.5}$$

It follows from (3.5) that

$$D_q(q) = \frac{r_1^2}{r_2^2}D_p(\frac{r_2}{r_1}q). \tag{3.6}$$

Denote by $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ the zeros of $D_p(p)$ and by $\beta_0, \beta_1, \beta_2, \beta_3$ those of $D_q(q)$. It is readily seen from

$$D_p(-\infty) = \infty,\ D_p(0) = 0,\ D_p(1) = (1 - ar_2)^2 < 1,\ D_p(\frac{r_2}{r_1}) = (1 - ar_1)^2(\frac{r_2}{r_1})^2, \tag{3.7}$$

$$D_p(\frac{1}{ar_1}) = \frac{1 - 4a^2r_1r_2}{(ar_1)^2} > 0,\ D_p(\frac{1+a}{ar_1}) = -4(1 + a)\frac{r_2}{r_1} < 0,\ D_p(\infty) = \infty,$$

that the $\alpha_i, i = 0, 1, 2, 3$, are all real and that the indices may be chosen so that

$$\text{i. } \alpha_0 = 0 < \alpha_1 < 1 < \frac{r_2}{r_1} < \frac{1}{ar_1} < \alpha_2 < \frac{1+a}{ar_1} < \alpha_3 < \infty;$$

$$\text{ii. } \beta_0 = 0 < \beta_1 < \frac{r_1}{r_2} < 1 < \frac{1}{ar_2} < \beta_2 < \frac{1+a}{ar_2} < \beta_3 < \infty; \tag{3.8}$$

iii. $\beta_i = \frac{r_1}{r_2}\alpha_i\ ,\ i = 0,1,2,3;$

here (3.8)ii follows by symmetry and (3.8)iii follows from (3.7). Further we have from (3.8),

$$\begin{array}{llll} D_p > 0 \text{ for } & p < 0, & D_q > 0 \text{ for } & q < 0, \\ < 0 \text{ '' } & 0 < p < \alpha_1, & < 0 \text{ '' } & 0 < q < \beta_1, \\ > 0 \text{ '' } & \alpha_1 < p < \alpha_2, & > 0 \text{ '' } & \beta_1 < q < \beta_2, \\ < 0 \text{ '' } & \alpha_2 < p < \alpha_3, & < 0 \text{ '' } & \beta_2 < q < \beta_3, \\ > 0 \text{ '' } & \alpha_3 < p, & > 0 \text{ '' } & \beta_3 < q. \end{array} \quad (3.9)$$

In fig. 3.1 the inequalities (3.8) and (3.9) are graphically shown.

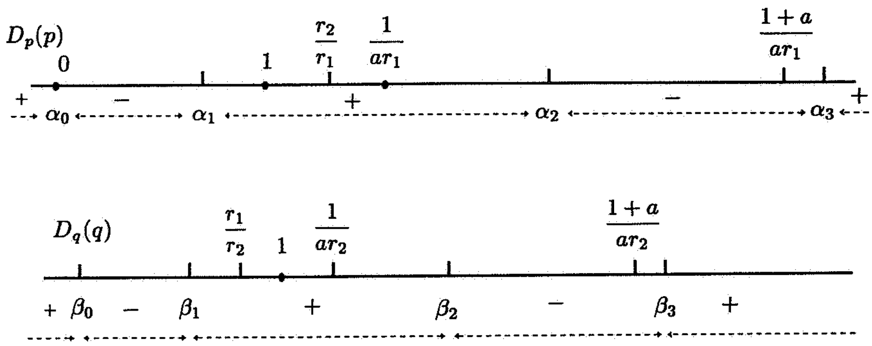


FIGURE 3.1

Next we consider for real p and q the curve $K(p,q) = 0$. By writing

$$(1+a)K(p,q) = 0 = pq(ar_1p + ar_2q - (1+a)) + 1,$$

It is readily sen that the asymptotes of this curve are given by the lines

i. $p = 0$, (3.10)

ii. $q = 0$,

iii. $ar_1p + ar_2q = 1 + a$.

The curve has been traced in fig 3.2, note that it has three inflexion points at infinity.

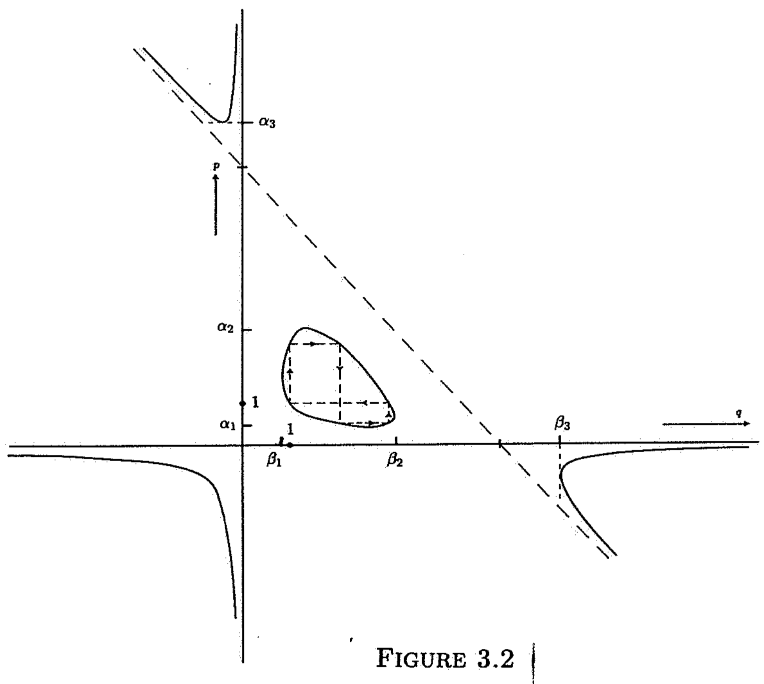


FIGURE 3.2

In section 7 we need an asymptotic relation for $q_1(p)$ with $p \to \infty$. We derive it here. From (3.4)ii and (3.5) we have for $p \to \infty$,

$$\begin{aligned} q_1(p) &= \frac{1}{2ar_2}[1+a-ar_1p-\{(1+a-ar_1p)^2-\frac{4ar_2}{p}\}^{\frac{1}{2}}] \\ &= \frac{1+a-ar_1p}{2ar_2}[1-\{1-\frac{4ar_2}{p}(1+a-ar_1p)^{-2}\}^{\frac{1}{2}}] \\ &= \frac{1+a-ar_1p}{2ar_2}[1-\{1-\frac{2ar_2}{p}(1+a-ar_1p)^{-2}+O(\frac{1}{p^6})\} \\ &= \frac{1}{p}\frac{1}{1+a-a_1rp}+O(\frac{1}{p^5}) = -\frac{1}{ar_1}\frac{1}{p^2}\{1+O(\frac{1}{p^3})\}. \end{aligned} \tag{3.11}$$

Analogously for $q \to \infty$,

$$p_1(q) = -\frac{1}{ar_2}\frac{1}{q^2}\{1+O(\frac{1}{q^3}\}. \tag{3.12}$$

4. Conformal mappings and uniformization

Put

$$\begin{aligned} H_p^+ &:= \{p:\ \mathrm{Im}p>0\},\quad H_p^- := \{p:\ \mathrm{Im}p<0\}, \\ H_q^+ &:= \{q:\ \mathrm{Im}q>0\},\quad H_q^- := \{q:\ \mathrm{Im}q<0\}. \end{aligned} \tag{4.1}$$

From the theory of conformal mappings, cf.[9], it is known that a unique conformal mapping $\xi(p), p \in H_p^+$, exists such that H_p^+ is mapped onto a rectangle $\mathcal{R}_p$,

$$\mathcal{R}_p := \{\xi:\ 0< \mathrm{Re}\,\xi < \omega_1,\ 0< \mathrm{Im}\,\xi < \mathrm{Im}\,\omega_3\},\ \mathrm{Im}\,\omega_1 = 0,\ \mathrm{Re}\,\omega_3 = 0, \tag{4.2}$$

with the corresponding boundaries given by

$$\begin{aligned} &\alpha_0 \to 0,\ \alpha_1 \to \omega_1,\quad \alpha_2 \to \omega_1+\omega_3,\ \alpha_3 \to \omega_3, \\ &|p| = \infty \to \delta,\qquad \mathrm{Re}\,\delta = 0,\ 0< \mathrm{Im}\,\delta < \mathrm{Im}\,\omega_3, \\ &(\alpha_0,\alpha_1) \to (0,\omega_1),\quad (\alpha_1,\alpha_2) \to (\omega_1,\omega_1+\omega_3), \\ &(\alpha_2,\alpha_3) \to (\omega_1+\omega_3,\omega_3),\ (\alpha_3,\infty) \to (\omega_3,\delta),\ (-\infty,\alpha_0) \to (\delta,0); \end{aligned} \tag{4.3}$$

here (x, y) is the line segment between x and y and directed from x to y, x and y points of the complex plane $\mathbb{C}$. This mapping is known as the Schwarz-Christoffel transformation, see for details, in particular for the relations between ω_1, ω_3 and the branch points α_i of $q(p), i = 0, 1, 2, 3$, the book [9]. The mapping $\xi(p)$ is continuous on the closure $\bar{H}_p^+$ of H_p^+.

Denote by

$$p(\xi) : \mathcal{R}_p \to H_p^+,$$

the inverse mapping of $\xi(p)$, it is also a conformal map, continuous on the closure $\bar{\mathcal{R}}_p$ of $\mathcal{R}_p$, cf.[9]. By using the reflection principle $p(\xi)$ can be continued meromorphically to the whole ξ-plane. This continuation $p(\xi)$ is an even, elliptic function of order 2 with periods $2\omega_1, 2\omega_3$, cf.[9] and [10], so that

$$\begin{aligned} p(\xi) &= p(\xi+2m\omega_1+2n\omega_3),\quad m,n \in \{\ldots,-2,-1,0,1,2,\ldots\}, \\ p(\xi) &= p(-\xi), \\ p(\bar{\xi}) &= \overline{p(\xi)}. \end{aligned} \tag{4.4}$$

Further $p(\xi)$ maps conformally:

$$\begin{aligned} &\{\xi:\ 0< \mathrm{Re}\,\xi < \mathrm{Re}\,\omega_1,\quad 0> \mathrm{Im}\,\xi > -\mathrm{Im}\,\omega_3\} \to H_p^-, \\ &\{\xi:\ 0> \mathrm{Re}\,\xi > -\mathrm{Re}\,\omega_1,\quad 0< \mathrm{Im}\,\xi < \mathrm{Im}\,\omega_3\} \to H_p^-, \\ &\{\xi:\ 0> \mathrm{Re}\,\xi > -\mathrm{Re}\,\omega_1,\quad 0> \mathrm{Im}\,\xi > -\mathrm{Im}\,\omega_3\} \to H_p^+. \end{aligned} \tag{4.5}$$

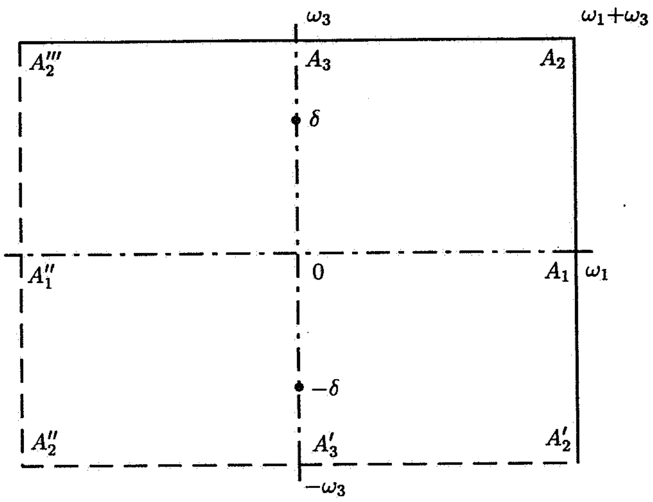


FIGURE 4.1.

REMARK 4.1. Note that H_p^+ is to the left of the axis $\operatorname{Im} p = 0$ if $p : -\infty \to +\infty$ and so $\xi(p)$ traverses the boundary of $\mathcal{R}_p$ from $\delta \to 0 \to \omega_1 \to \omega_1 + \omega_3 \to \omega_3 \to \delta$. □

In fig 4.1. the fundamental rectangle

$$\mathcal{R}_p := \{\xi : -\omega_1 < \operatorname{Re}\xi \le \omega_1, \ -\operatorname{Im}\omega_3 \le \operatorname{Im}\xi \le \operatorname{Im}\omega_3\} \tag{4.6}$$

of the elliptic function $p(\xi)$ is shown, and

$$p(\xi) \text{ maps } \alpha_1 \to A_1, A_1''; \ \alpha_2 \to A_2, A_2', A_2'', A_2'''; \ \alpha_3 \to A_3, A_3'.$$

An elliptic function of order N has N zeros and also N poles in its fundamental rectangle, zeros and poles counted according to their multiplicities. For these zeros z_j and poles $p_j, j = 1, \ldots, N$, a so called complete set of zeros and poles, holds, cf.[10],

$$\sum_{j=1}^{N} z_j - \sum_{j=1}^{N} p_j = 0 \bmod (2m\tilde{\omega}_1 + 2n\tilde{\omega}_3), \tag{4.7}$$

with $2\tilde{\omega}_1, 2\tilde{\omega}_3$ the periods of the elliptic function.

Put

$$G_1 := \{p : 0 \le p \le \alpha_1\}, \ \{G_2 := \{p : \alpha_2 \le p \le \alpha_3\}. \tag{4.8}$$

For

$$p \in \mathcal{H} := \{\bar{H}_p^+ \cup \bar{H}_p^-\} \setminus \{G_1 \cup G_2\},$$

denote by $\sqrt{D_p(p)}$ that square root of $D_p(p)$ which is regular for $p \in \mathcal{H}$ and for which, cf.(1.2) and (1.4),

$$\sqrt{D_p(1)} = 1 - ar_2 > 0. \tag{4.9}$$

Put further for $k = \ldots, -2, -1, 0, 1, 2, \ldots$,

$$\mathcal{S}_k := \{\xi : k \operatorname{Im}\omega_3 < \operatorname{Im}\xi < (k+1)\operatorname{Im}\omega_3\}, \tag{4.10}$$

so that, cf.(4.3), (4.4) and (4.5),

$$p(\xi) \text{ maps } \mathcal{S}_k \text{ onto } \operatorname{int} \mathcal{H} \text{ for each } k. \tag{4.11}$$

Define: for $\xi \in \mathcal{S}_k$,

$$\begin{aligned} d(\xi) &:= -\sqrt{D_p(p(\xi))} \quad \text{for} \quad k \text{ even},\\ &:= \sqrt{D_p(p(\xi))} \quad \text{,,} \quad k \text{ odd};\\ \text{for } \xi \in \bar{\mathcal{S}}_k,&\\ d^-(\xi) &:= \lim_{\substack{\eta\to\xi\\ \eta\in\mathcal{S}_{k-1}}} d(\eta);\\ d^+(\xi) &:= \lim_{\substack{\eta\to\xi\\ \eta\in\mathcal{S}_k}} d(\eta); \end{aligned} \tag{4.12}$$

and put, cf.(3.4),

$$q(\xi) := \frac{1}{2ar_2p(\xi)}[(1+a-ar_1p(\xi))p(\xi)+d(\xi)]. \tag{4.13}$$

It follows from (4.11) that $d(\xi)$ is elliptic with periods $2\omega_1, 2\omega_3$, and from (4.12) that

$$q(\xi) \text{ is elliptic with periods } 2\omega_1, 2\omega_3, \tag{4.14}$$

since a double periodic function, which is meromorphic in the entire plane, is by definition an elliptic function, cf.[10].

From the discussion above it is seen that $(p(\xi), q(\xi))$ is a zero-tuple of the kernel $K(p,q)$ for every ξ, and

$$\hat{p} = p(\xi),\ \hat{q} = q(\xi),$$

represents a uniformisation of the manifold $K(p,q) = 0, \xi$ is here the uniformisation variable.

REMARK 4.2. $p(\xi)$ and $q(\xi)$ as defined above may be expressed in terms of the Weierstrasz function $\mathcal{P}(\xi)$, cf.[7] and [8], p. 453. □

5. PROPERTIES OF $p(\xi)$ AND $q(\xi)$

In this section we derive some properties of the functions $p(\xi)$ and $q(\xi)$, which have been defined in the preceeding section, cf.(4.4) and (4.13).

Define for $0 < \operatorname{Re}\xi \le \omega_1, 0 < \operatorname{Im}\xi \le \operatorname{Im}\omega_3$,

$$\begin{aligned} p(\xi) &= 1 \quad \text{for} \quad \xi = \sigma,\\ &= \frac{r_2}{r_1} \quad \text{''} \quad \xi = \chi,\\ &= \frac{1}{ar_1} \quad \text{''} \quad \xi = \tau; \end{aligned} \tag{5.1}$$

it will be shown below that σ, χ and τ are uniquely defined.

THEOREM 5.1.

i. $p(\xi)$ has in $\mathcal{R}_p$, cf.(4.6), a zero of multiplicity two at $\xi = 0$, and single poles at $\xi = \pm\delta$; (5.2)

ii. $q(\xi)$ has in $\mathcal{R}_p$, a zero of multiplicity two at $\xi = \delta$ and single poles at $\xi = 0, \xi = -\delta$;

iii. $\delta = \frac{2}{3}\omega_3$,

iv. $q(\xi) = q(\frac{4}{3}\omega_3 - \xi) = q(2\delta - \xi)$, *v.* $q(\xi) = \frac{r_1}{r_2} p(\xi - \frac{2}{3}\omega_3)$,

vi. $\tau - \sigma = \frac{2}{3}\omega_3$,

vii. $\frac{1}{2}(\sigma + \chi) = \frac{1}{3}\omega_3 + \omega_1$,

viii. $\operatorname{Re}\sigma = \operatorname{Re}\tau = \operatorname{Re}\chi = \omega_1, \quad 0 < \operatorname{Im}\sigma < \frac{1}{3}\operatorname{Im}\omega_3 < \operatorname{Im}\chi < \frac{2}{3}\operatorname{Im}\omega_3 < \operatorname{Im}\tau < \operatorname{Im}\omega_3$.

PROOF OF (5.2)i. From the principle of corresponding boundaries, cf.[9], it follows, cf.(4.3), that δ is a simple pole of $p(\xi)$, and hence, since $\operatorname{Im}\delta = 0$ and $p(\xi)$ is even, $\delta = -\xi$ is also a simple pole of $p(\xi)$. Because $\xi = 0$ is the only zero of $p(\xi)$ in $\mathcal{R}_p$, note that $p(\xi)$ maps the domains mentioned in (4.3) and (4.5) conformally onto H_p^+ and H_p^-, respectively, and because $p(\xi)$ is elliptic of order 2 it is seen that $\xi = 0$ should be a double zero of $p(\xi)$, cf.(4.7).

PROOF OF (5.2)ii. Since $p(\xi)$ has at $\xi = 0$ a double zero it is seen from (cf.(3.4))

$$q_{1,2}(p) = \frac{1}{2ar_2}[1 + a - ar_1 p \pm \{(1 + a - ar_1 p)^2 - \frac{4ar_2}{p}\}^{\frac{1}{2}},$$

that $q(\xi)$ has at $\xi = 0$ a simple pole. Next consider $\xi = -\delta \in \mathcal{S}_{-1}$ cf.(4.10), so that from (4.12) and (4.13): for $|\epsilon| << 1$,

$$2ar_2 q(-\delta + \epsilon) = 1 + a - ar_1 p(-\delta + \epsilon) + \{[1 + a - ar_1 p(-\delta + \epsilon)]^2 - \frac{4ar_2}{p(-\delta + \epsilon)}\}^{\frac{1}{2}},$$

from which it follows by letting $\epsilon \to 0$ that $q(\xi)$ has at $\xi = -\delta$ a simple pole since $p(\xi)$ has a simple pole at $\xi = -\delta$.

To continue the proof of (5.2)ii consider

$$\xi = \delta + \epsilon \in \mathcal{S}_0 \ , \ |\epsilon| << 1,$$

so that, since $p(\xi)$ has a pole at $\xi = \delta$, it is seen from (3.12) that: for $\epsilon \to 0$,

$$q(\delta + \epsilon) = -\frac{1}{ar_1}[\frac{1}{p(\delta + \epsilon)}]^2\{1 + O([\frac{1}{p(\delta + \epsilon)}]^3)\}.$$

Because the pole $\xi = \delta$ of $p(\xi)$ is a simple pole it is seen that $q(\xi)$ has a zero of multiplicity two at $\xi = \delta$. Hence $q(\xi)$ has no other zeros and poles in $\mathcal{R}_p$ since it is elliptic and of order 2.

PROOF OF (5.2)iii.
By applying the property (4.7) to the zeros and poles of $q(\xi), \xi \in \mathcal{R}_p$, it follows that

$$\delta + \delta - (-\delta) = 2m\omega_1 + 2n\omega_3;$$

so that, since $\operatorname{Re}\delta = 0, 0 < \operatorname{Im}\delta < \operatorname{Im}\omega_3$, cf.(4.3), we have $m = 0$ and $n = 1$, so $\delta = \frac{2}{3}\omega_3$.

PROOF OF (5.2)iv.
Since an elliptic function of order 2 is completely determined by its two zeros and two poles in $\mathcal{R}_p$, apart from a constant factor, it is seen by using (5.2)ii and (5.2)iii that (5.2)iv holds.

PROOF OF (5.2)v.
By using (5.2)i, ii, iii it is readily verified that $q(\xi)$ and $p(\frac{2}{3}\omega_3 - \xi)$ have the same zeros and poles in $\mathcal{R}_p$ taking into account their multiplicity, so that

$$q(\xi) = bp(\tfrac{2}{3}\omega_3 - \xi),$$

with b a constant. It follows by using the periodicity in $2\omega_3$ that

$$q(-\frac{2}{3}\omega_3+\xi) = bp(\frac{4}{3}\omega_3 - 3) = bp(-\frac{2}{3}\omega_3 - \xi).$$

From the asymptotic relation in the proof of (5.2)ii we have with $\delta = \frac{2}{3}\omega_3$, cf.(5.2)iii, for $\epsilon \to 0$ that

$$q(-\frac{2}{3}\omega_3+\epsilon) = -\frac{r_1}{r_2}p(-\frac{2}{3}\omega_3+\epsilon)+o(\epsilon).$$

Because $p(\xi)$ has a simple pole at $\xi = -\frac{2}{3}\omega_3$ the last but one relation implies that

$$bp(-\frac{2}{3}\omega_3+\epsilon) = -bp(-\frac{2}{3}\omega_3-\epsilon) = -q(-\frac{2}{3}\omega_3+\epsilon),$$

and so $b = r_1/r_2$, i.e. (5.2)v. has been verified, since $p(\xi) = p(-\xi)$, cf.(4.4).

PROOF OF (5.2)vi.
From the properties of the conformal mapping $p(\xi), \xi \in \bar{\mathcal{R}}_p$, it follows, cf.(3.8)i and (4.3), that each of the three equations in (5.1) has a unique solution and that $0 < \operatorname{Im}\sigma < \operatorname{Im}\chi < \operatorname{Im}\tau < \operatorname{Im}\omega_3$. So from $\sigma, \tau, \chi \in \mathcal{S}_0$ and (4.13), cf.also table 3.1, it follows that $q(\sigma) = 1, q(\chi) = \frac{r_1}{r_2}, q(\bar{\tau}) = 1$. Hence the elliptic function $q(\xi) - 1$, which is of order 2, has in $\mathcal{R}_p$ two zeros, viz. $\xi = \sigma$ and $\xi = \bar{\tau}$, and two poles, cf.(5.2)ii, at $\xi = 0$ and $\xi = -\frac{2}{3}\omega_3$. So by applying (4.7) we have

$$\sigma+\bar{\tau}+\frac{2}{3}\omega_3 = \operatorname{Re}(\sigma+\tau) - \operatorname{Im}(\tau-\sigma)+\frac{2}{3}\omega_3 = 2m\omega_1+2n\omega_3.$$

From $\operatorname{Re}\sigma = \operatorname{Re}\tau = \omega_1$, and $0 < \operatorname{Im}\sigma < \operatorname{Im}\tau < \operatorname{Im}\omega_3$ it is seen that we have to take $n = 0$ and $m = 1$, and so (5.2)vi follows.

PROOF OF (5.2)vii.
From table 3.1 we have since $\bar{\sigma}, \bar{\chi} \in \mathcal{S}_{-1}$ that $q(\bar{\sigma}) = q(\bar{\chi}) = \frac{1}{ar_2}$. Hence $q(\xi) - \frac{1}{ar_2}$ has simple zeros at $\xi = \bar{\sigma}$ and $\xi = \bar{\chi}$ and poles at $\xi = 0$ and $\xi = -\frac{2}{3}\omega_3$ so that by applying (4.7) we have

$$\bar{\sigma}+\bar{\chi}+\frac{2}{3}\omega_3 = 2m\omega_1+2n\omega_3,$$

or

$$\sigma+\chi-\frac{2}{3}\omega_3 = 2m\omega_1-2n\omega_3.$$

As in the proof of (5.2)vii it follows that $\operatorname{Re}\sigma = \operatorname{Re}\chi = \omega_1, 0 < \operatorname{Im}\sigma < \operatorname{Im}\chi < \operatorname{Im}\omega_3$, so we have to take $m = 1, n = 0$, and (5.2)vii follows.

PROOF OF (5.2)viii.
The inequalities in (5.2)viii follow from (5.2)vi, vii and from the inequalities derived in the proofs of (5.2)vi, vii. □

REMARK 5.1. The relation (5.2)iv can be also obtained from a result in [5], cf.[5] form. (4.2), since the group of our random walk is finite and of order 6. □

REMARK 5.2. In table 5.1, cf. also table 3.1, the values of $p(\xi)$ and $q(\xi)$ at the points occurring in the theorem above are listed.

ξ	σ	$\bar{\sigma}$	$\bar{\chi}$	χ	τ	$\bar{\tau}$
p	1	1	$\frac{r_2}{r_1}$	$\frac{r_2}{r_1}$	$\frac{1}{ar_1}$	$\frac{1}{ar_1}$
q	1	$\frac{1}{ar_2}$	$\frac{1}{ar_2}$	$\frac{r_1}{r_2}$	$\frac{r_1}{r_2}$	1

TABLE 5.1 □

THEOREM 5.2. *The function $q(\xi)$ maps:*

$$\text{i.}\quad \frac{2}{3}\omega_3 \to 0\,,\ \omega_1+\frac{2}{3}\omega_3 \to \beta_1\,,\ \omega_1+\frac{5}{3}\omega_3 \to \beta_2\,,\ \frac{5}{3}\omega_3 \to \beta_3, \tag{5.3}$$

$$(\frac{2}{3}\omega_3,\ \omega_1+\frac{2}{3}\omega_3) \to (0,\beta_1),\ (\omega_1+\frac{2}{3}\omega_3,\ \omega_1+\frac{5}{3}\omega_3) \to (\beta_1,\ \beta_2),$$

$$(\omega_1+\frac{5}{3}\omega_3, \frac{5}{3}\omega_3) \to (\beta_2,\beta_3)\,,\ (\frac{5}{3}\omega_3\,,\ \frac{4}{3}\omega_3) \to (\beta_3,\infty),$$

$$(\frac{4}{3}\omega_3\,,\ \frac{2}{3}\omega_3) \to (-\infty, 0);$$

$$\text{ii.}\quad \mathcal{R}_q := \{\xi : 0 < \operatorname{Re}\xi < \omega_1\,,\ \frac{2}{3}\operatorname{Im}\omega_3 < \operatorname{Im}\xi < \frac{5}{3}\operatorname{Im}\omega_3\}$$

conformally onto H_q^+ and
$\{\xi : 0 < \operatorname{Re}\xi < \omega_1, -\frac{1}{3}\operatorname{Im}\omega_3 < \operatorname{Im}\xi < \frac{2}{3}\operatorname{Im}\omega_3\}$
conformally onto H_q^-.

PROOF. The statement in (5.3)i follows immediately from $q(\xi) = \frac{r_1}{r_2}p(\xi - \frac{2}{3}\omega_3)$, cf.(5.2)v, (3.8)iii and (4.3), note further that $|q(\frac{4}{3}\omega_3)| = |q(-\frac{2}{3}\omega_3)| = \infty$, cf.(5.2)ii. Because $p(\xi)$ maps $\mathcal{R}_p$ conformally onto H_p^+, it is seen from (5.2)v that $q(\xi)$ maps $\mathcal{R}_q$ conformally onto H_q^+ cf. remark 4.1. Using the reflection principle with respect to the line with $\operatorname{Im}\xi = \frac{2}{3}\operatorname{Im}\omega_3$ the conformal mapping onto H_q^- follows. □

REMARK 5.3 The property that every zero-tuple $(\hat{p},\hat{q})$ of $K(p,q)$ generates a periodic sequence with period 6, cf. remark 3.1, can be also shown by using (4.4) and (5.2)iv. This is seen as follows. For every ξ we know that $p_1 = p(\xi), q_1 = q(\xi)$ is a zero-tuple of $K(p,q) = 0$. From (4.4) and (5.2)iv it is now readily seen that

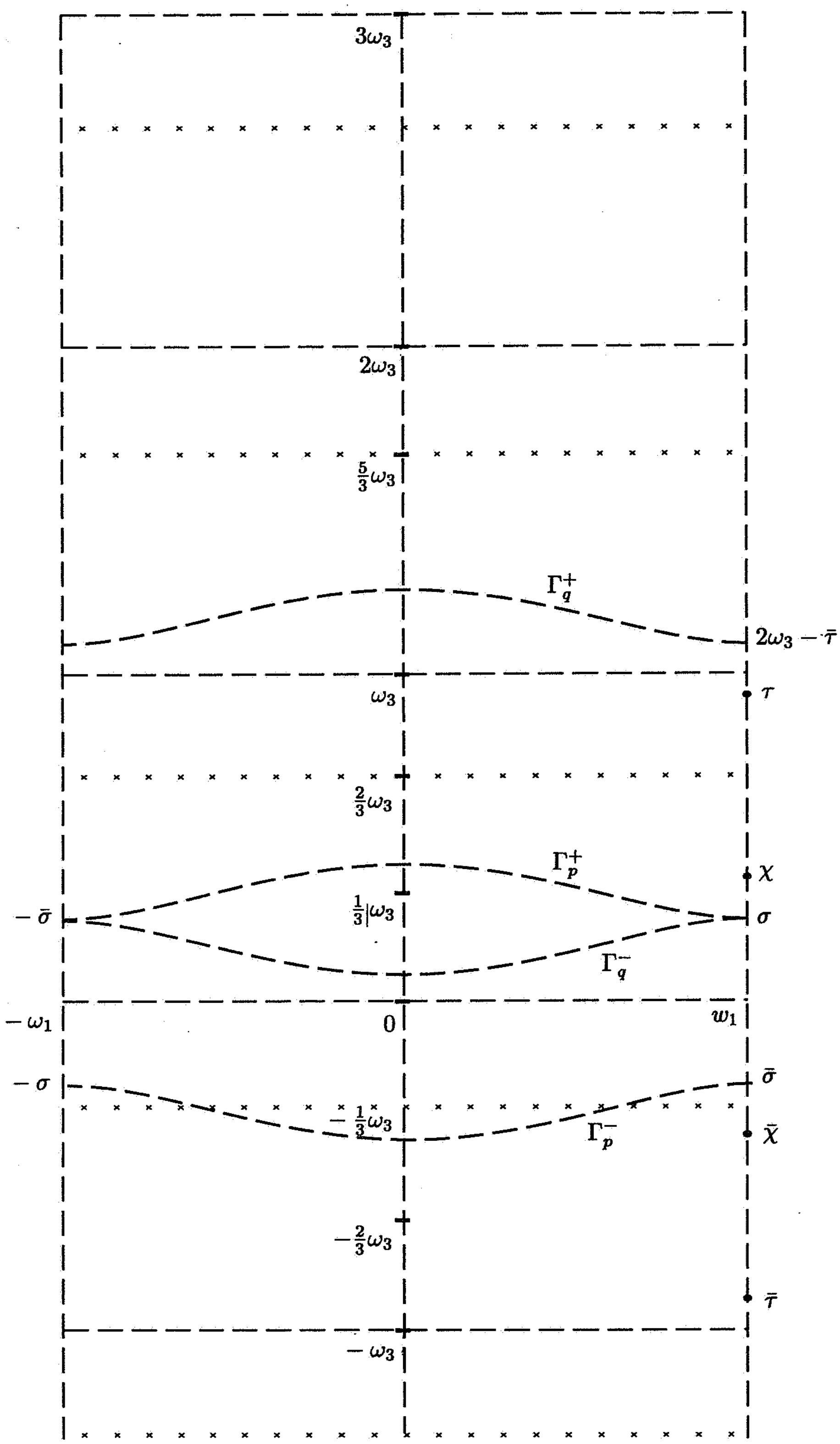
$$p(\xi),q(\xi)\ ;\ p(-\xi),\ q(\xi);\ p(2\delta+\xi),q(2\delta+\xi);\ p(-2\delta+\xi),\ q(-2\delta+\xi); \tag{5.4}$$

$$p(4\delta+\xi),\ q(4\delta+\xi); p(-4\delta+\xi),\ q(-4\delta+\xi);\ p(6\delta+\xi),\ q(6\delta+\xi);$$

are all zero-tuples of $K(p,q)$. The construction of this sequence is equivalent with that of (3.2)ii. Because δ/ω_3 is a rational number, actual equal to 2/3, it is seen since $2\omega_3$ is a period of $p(\xi)$ as well as of $q(\xi)$, that $p(6\delta+\xi) = p(\xi)\ q(6\delta+\xi) = q(\xi)$.
Hence it has again been shown that every zero-tuple generates via (4.4) and (5.2)iv a periodical sequence of zero-tuples. Note that also the points at infinity of the curve $K(p,q) = 0$ form such a periodic sequence, cf. fig. 3.2. □

Define, see also fig. 5.1,

$$\begin{aligned}
\Gamma_p &:= \{\xi : |p(\xi)| < 1, |\operatorname{Re}\xi| < \omega_1, -\operatorname{Im}\omega_3 < \operatorname{Im}\xi < \operatorname{Im}\omega_3\},\\
\Gamma_q &:= \{\xi : |q(\xi)| < 1, |\operatorname{Re}\xi| < \omega_1, \tfrac{2}{3}\operatorname{Im}\omega_3 < \operatorname{Im}\xi < 2\tfrac{2}{3}\operatorname{Im}\omega_3\},\\
\Gamma_p^+ &:= \{\xi : |p(\xi)| = 1, |\operatorname{Re}\xi| \le \omega_1, 0 \le \operatorname{Im}\xi \le \operatorname{Im}\omega_3\},\\
\Gamma_p^- &:= \{\xi : |p(\xi)| = 1, |\operatorname{Re}\xi| \le \omega_1, -\operatorname{Im}\omega_3 \le \operatorname{Im}\xi \le 0\},\\
\Gamma_q^+ &:= \{\xi : |q(\xi)| = 1, |\operatorname{Re}\xi| \le \omega_1, \tfrac{2}{3}\operatorname{Im}\omega_3 \le \operatorname{Im}\xi \le \tfrac{5}{3}\operatorname{Im}\omega_3\},\\
\Gamma_p^- &:= \{\xi : |q(\xi)| = 1, |\operatorname{Re}\xi| \le \omega_1, -\tfrac{1}{3}\operatorname{Im}\omega_3 \le \operatorname{Im}\xi \le \tfrac{2}{3}\operatorname{Im}\omega_3\}.
\end{aligned} \tag{5.5}$$



$3\omega_3$
$2\omega_3$
$\frac{5}{3}\omega_3$
Γ_q^+
$2\omega_3 - \bar{\tau}$
ω_3
τ
$\frac{2}{3}\omega_3$
Γ_p^+
χ
$-\bar{\sigma}$
$\frac{1}{3}\omega_3$
σ
Γ_q^-
$-\omega_1$
0
w_1
$-\sigma$
$\bar{\sigma}$
$-\frac{1}{3}\omega_3$
$\bar{\chi}$
Γ_p^-
$-\frac{2}{3}\omega_3$
$\bar{\tau}$
$-\omega_3$

FIGURE 5.1

From the conformal mapping properties of $p(\xi)$, cf.(4.2), (4.4) and (4.5), it is seen that Γ_p is a simply connected domain with boundary $\Gamma_p^+ \cup \Gamma_p^-$, that Γ_p^+ is a simple curve, symmetric with respect to $\operatorname{Re}\xi = 0$ and that it intersects the lines $\operatorname{Re}\xi = \omega_1$, $\operatorname{Re}\xi = 0$ and $\operatorname{Re}\xi = -\omega_1$ perpendicularly. From $|p(\frac{2}{3}\omega_3)| = \infty, p(0) = 0$, it is seen that Γ_p^+ intersects the line $\operatorname{Re}\xi = 0$ between $\xi = 0$ and $\xi = \frac{2}{3}\omega_3$. From (4.2) and (4.5) it further follows that Γ_p^+ when reflected with respect to the real axis coincides with Γ_p^-. From theorem 5.2 it is seen that analogous properties hold for Γ_q, Γ_q^+ and Γ_q^-, here Γ_q^+ and Γ_q^- mirror with respect to the line $\operatorname{Im}\xi = \frac{2}{3}\operatorname{Im}\omega_3$. From lemma 3.1 it follows that Γ_q and Γ_p have a nonempty intersection

$$\Gamma_p \cap \Gamma_q \neq \emptyset. \tag{5.6}$$

6. Meromorphic continuations

The functional equation for $\Phi(p,0), |p| \leq 1$, and $\Phi(0,q), |q| \leq 1$, has been formulated in section 2, cf.(2.1)ii. By using (3.4) and lemma 3.1 we obtain from (2.1)ii: for $|p| = 1$, $p \neq 1$,

$$\frac{1}{1-p}\frac{\Phi(p,0)}{\Phi(0,0)} + \frac{1}{1-q_1(p)}\,\Phi(0,q_1(p)) = 1. \tag{6.1}$$

The first term in (6.1) is a regular function of p for $|p| < 1$, cf.(2.1)i, and also for $|p| = 1, p \neq 1$, because here $q_1(p)$ is differentiable. Consequently, it follows from (6.1) that the second term in (6.1) has an analytic continuation in $|p| < 1$. The function $q_1(p)$ has two branch points in $|p| < 1$, viz. $p = 0$ and $p = \alpha_1$, cf.(3.8)i. Consider a simple curve in the p-plane starting at a point p_0 with $|p_0| = 1, p_0 \neq 1$, which crosses the interval $(0, \alpha_1)$ only once and returns to p_0. Then $q_1(p)$, when p crosses the interval $(0,\alpha_1)$, has to be replaced by $q_2(p)$, cf.(3.4)i. Hence the analytic continuation leads to: for $|p| = 1, p \neq 1$,

$$\frac{1}{1-p}\frac{\Phi(p,0)}{\Phi(0,0)} + \frac{1}{1-q_2(p)}\frac{\Phi(0,q_2(p))}{\Phi(0,0)} = 1. \tag{6.2}$$

Analogously we have: for $|q| = 1$, $q \neq 1$,

$$\text{i.}\quad \frac{1}{1-p_1(q)}\frac{\Phi(p_1(q),0)}{\Phi(0,0)} + \frac{1}{1-q}\frac{\Phi(0,q)}{\Phi(0,0)} = 1, \tag{6.3}$$

$$\text{ii.}\quad \frac{1}{1-p_2(q)}\frac{\Phi(p_2(q),0)}{\Phi(0,0)} + \frac{1}{1-q}\frac{\Phi(0,q)}{\Phi(0,0)} = 1.$$

Because $|q_2(p)| > 1$ for $|p| = 1$, cf. lemma 3.1, it is seen that $\Phi(0,q)$ is regular for all q with

$$q \in \{q : |q| < \sup_{|p|=1} |q_2(p)|\}.$$

Hence there exists a domain in $|q| \geq 1$ into which $\Phi(0,q)$ can be continued analytically out from $|q| \leq 1$. Analogously for $\Phi(p,0)$. Starting out from these domains and by using repeatedly (6.1), (6.2) and (6.3) it is seen that $\Phi(p,0)$ possesses analytic continuations for all $p, |p| > 1$, except for those p where $p = \alpha_2$ or $p = \alpha_3$ or $|\Phi(p,0)| = \infty$. Analogously for $\Phi(0,q)$. Note that α_2, α_3 are branch points of these continuations of $\Phi(p,0)$. Put

$$R_P \equiv (K_p^+, K_p^-) \quad , \quad R_Q \equiv (K_q^+, K_q^-), \tag{6.4}$$

with R_P the double sheeted Riemann surface formed by the lower plane K_p^- and the upper plane K_p^+, both slitted along (α_2, α_3) and crosswise connected along these slits. Analogously R_Q is defined with the planes K_q^- and K_q^+ slitted along (β_2, β_3). The above analytic continuation of $\Phi(p,0)$ leads to a one-valued representation of $\Phi(p,0)$ on R_P, similarly for $\Phi(0,q)$ on R_Q. We define it so that

$$\Phi(0,p) = \Phi(0,0) \quad \text{for} \quad p = 0 \in K_p^-,$$
$$\Phi(0,q) = \Phi(0,0) \quad ,, \quad q = 0 \in K_q^-, \tag{6.5}$$

so that, cf.(2.1)i,

$$\Phi(p,0) \quad \text{is} \quad \text{regular} \quad \text{for} \quad \{p = |p| \leq 1\} \subset K_p^-,$$
$$\Phi 0,q) \quad ,, \quad ,, \quad ,, \quad \{q = |q \leq 1\} \subset K_q^-. \tag{6.6}$$

To determine the values of p for which $|\Phi(p,0)| = \infty$ note that $\hat{p} = 1, \hat{q} = 1$, and $\hat{p} = \frac{1}{ar_1} > 1, \hat{q} = 1$, are both zero-tuples of $K(p,q)$, see table 3.1. Consequently, it is seen from (2.1)ii that $p = \frac{1}{ar_1}$ is a pole of $\Phi(p,0)$. Further $\hat{p} = \frac{1}{ar_1}, \hat{q} = \frac{r_1}{r_2}$ is also a zero-tuple so that from (2.1)ii and $|\Phi(\frac{1}{ar_1}, 0)| = \infty$ it is seen that r_1/r_2 is a pole of $\Phi(0,q)$. Applying this argument again it is seen that $\Phi(p,0)$ has poles at $p = \frac{1}{ar_1}$ and $p = \frac{r_2}{r_1}$, $\Phi(0,q)$ has poles at $\frac{1}{ar_2}$ and $\frac{r_1}{r_2} < 1$. From (2.1)ii and $K(\hat{p},\hat{q}) = 0$ it is readily verified that (see table 6.1 below).

$$\Phi(p,0) \quad \text{has} \quad \text{simple} \quad \text{poles} \quad \text{at} \quad p = \tfrac{1}{ar_1},\ p = \tfrac{r_2}{r_1},$$
$$\Phi(0,q) \quad \text{has} \quad \text{simple} \quad \text{poles} \quad \text{at} \quad q = \tfrac{1}{ar_2},\ q = \tfrac{r_1}{r_2}. \tag{6.7}$$

REMARK 6.1. Because $r_2 > r_1$, cf.(1.2), it follows from (6.6) and (6.7) that

$$|\Phi(0,q)| = \infty \quad \text{for} \quad q = \frac{r_1}{r_2} \in K_q^+. \tag{6.8}$$

□

From (1.5) and (6.6) it follows that

$$|\Phi(p,0)| \leq 1 \quad \text{for} \quad |p| = 1, \quad p \in K_p^-,$$
$$|\Phi(0,q)| \leq 1 \quad ,, \quad |q| = 1, \quad q \in K_q^-,$$

and consequently, it is seen that the above analytic continuation of $\Phi(p,0)$ starting out from a point p with $|p| = 1$, $p \neq 1$, and avoiding the points $p = \frac{1}{ar_1}$, $p = \frac{r_2}{r_1}$ can never have a limiting point at which $\Phi(p,0)$ becomes infinite, analogously for $\Phi(0,q)$, so that $\Phi(p,0)$ and $\Phi(0,q)$ have only isolated poles, cf.(6.7), i.e.

$$p = \tfrac{1}{ar_1},\ p = \tfrac{r_2}{r_1} \quad \text{are} \quad \text{the} \quad \text{only} \quad \text{poles} \quad \text{of} \quad \Phi(p,0) \text{ on } R_P,$$
$$q = \tfrac{1}{ar_2},\ q = \tfrac{r_1}{r_2} \quad ,, \quad ,, \quad ,, \quad ,, \quad ,, \quad \Phi(0,q) \text{ on } R_Q. \tag{6.9}$$

□

The poles are listed in table 6.1.

ξ	σ	$\bar{\sigma}$	$\bar{\chi}$	χ	τ	$\bar{\tau}$
p	1	1	$\frac{r_2}{r_1}$	$\frac{r_2}{r_1}$	$\frac{1}{ar_1}$	$\frac{1}{ar_1}$
q	1	$\frac{1}{ar_2}$	$\frac{1}{ar_2}$	$\frac{r_1}{r_2}$	$\frac{r_1}{r_2}$	1
$\lvert\Phi(p,0)\rvert$			∞	∞	∞	∞
$\lvert\Phi(0,q)\rvert$		∞	∞	∞	∞	

TABLE 6.1

Put with $p(\xi)$ and $q(\xi)$ as defined in (4.4) and (4.13), respectively,

$$P(\xi) := \frac{\Phi(p(\xi),0)}{\Phi(0,0)} \quad , \quad Q(\xi) := \frac{\Phi(0,q(\xi))}{\Phi(0,0)}. \tag{6.10}$$

It then follows from (2.1)ii, (5.4) and (5.5), that: for $\xi \in \bar{\Gamma}_p \cap \bar{\Gamma}_q$,

$$\frac{1}{1-p(\xi)}P(\xi) + \frac{1}{1-q(\xi)}\, Q(\xi) = 1, \tag{6.11}$$

with $\bar{\Gamma}_p$ the closure of Γ_p, $\bar{\Gamma}_q$ that of Γ_q, since, cf.(5.5),

$$|p(\xi)| \leq 1 \text{ for } \xi \in \bar{\Gamma}_p, \ |q(\xi)| \leq 1 \text{ for } \xi \in \bar{\Gamma}_q.$$

From $|p(\xi)| < 1$, $\xi \in \Gamma_p$, it follows that $|P(\xi)| < \infty$ for $\xi \in \Gamma_p$. Further, $\Phi(p,0)$ is regular for $|p| < 1$, and $p(\xi)$ is meromorph for $\xi \in \mathcal{R}_p$. Hence the second term in (6.11) may be continued analytically into Γ_p out from $\bar{\Gamma}_p \cap \bar{\Gamma}_q$. It follows from (4.4), (6.10) and (6.11) that

$$\frac{1}{1-q(-\xi)}\, Q(-\xi) = \frac{1}{1-q(\xi)}\, Q(\xi) \text{ for } \xi \in \Gamma_p. \tag{6.12}$$

Analogously, it is seen that the first term in (6.8) can be continued analytically into Γ_q and by using (4.4) and (5.2)iv it follows that

$$\frac{1}{1-p(\xi-\frac{4}{3}\omega_3)}\, P(\xi - \frac{4}{3}\omega_3) = \frac{1}{1-p(\xi)}\, P(\xi), \text{ for } \xi \in \Gamma_q. \tag{6.13}$$

REMARK 6.2. When comparing (6.1) and (6.2) with (6.10) and (6.11), respectively, it is seen that the transformation $\xi \to -\xi$ corresponds to interchanging the two semi-planes H_p^+ and H_p^-, similarly the transformation $\xi \to \xi - \frac{4}{3}\,\omega_3$ corresponds to interchanging H_q^+ and H_q^-. □

By starting from, cf.(6.12) and (6.13),

$$Q(-\xi) = \frac{1-q(-\xi)}{1-q(\xi)}\, Q(\xi), \quad \xi \in \Gamma_p,$$

$$P(\xi - \frac{4}{3}\omega_3) = \frac{1-p(\xi-\frac{4}{3}\omega_3)}{1-p(\xi)}\, P(\xi), \quad \xi \in \Gamma_q,$$

it may be seen that $Q(\xi)$ and $P(\xi)$ can be continued meromorphically to the whole ξ-plane, and the principle of permanence implies that for these continuations (6.11) remains valid, for a detailed proof see [6]. where a similar meromorphic continuation is discussed in great detail.

7. PROPERTIES OF THE FUNCTIONS $P(\xi)$ AND $Q(\xi)$.

In this section we describe some further properties of the functions $P(\xi)$ and $Q(\xi)$ defined in (6.10).

Define

$$\hat{\mathcal{R}}_P := \{\xi : -\omega_1 < \operatorname{Re}\xi \le \omega_1,\ -\operatorname{Im}\omega_3 < \operatorname{Im}\xi \le 3\operatorname{Im}\omega_3\},$$
$$\hat{\mathcal{R}}_Q := \{\xi : -\omega_1 < \operatorname{Re}\xi \le \omega_1,\ -\tfrac{1}{3}\operatorname{Im}\omega_3 < \operatorname{Im}\xi \le 3\tfrac{2}{3}\operatorname{Im}\omega_3\} \tag{7.1}$$

THEOREM 7.1

i. The functions $P(\xi)$ and $Q(\xi)$ with fundamental rectangles $\hat{\mathcal{R}}_P$ and $\hat{\mathcal{R}}_Q$, respectively, are elliptic, of order 4, have periods $2\omega_1$, $4\omega_3$, satisfy

$$\frac{P(\xi)}{1-p(\xi)} + \frac{Q(\xi)}{1-q(\xi)} = 1, \tag{7.2}$$

ii. $P(\xi)-1$ has in $\hat{\mathcal{R}}_P$

double zeros at $\xi = 0,\ 2\omega_3,$

and single poles at $\xi = \bar{\chi},\ \chi,\ \tau+2\omega_3,\ \bar{\tau}+2\omega_3;$

$Q(\xi)-1$ has in $\hat{\mathcal{R}}_Q$

double zeros at $\xi = \tfrac{2}{3}\omega_3,\ 2\tfrac{2}{3}\omega_3,$

and single poles at $\xi = \bar{\chi},\ \bar{\sigma},\ \chi,\ \tau+2\omega_3;$

$$\begin{aligned} P(\tfrac{2}{3}\omega_3+\epsilon) &= c_p\,\epsilon + O(\epsilon^2) \quad \text{for} \quad \epsilon \to 0,\\ Q(\epsilon) &= c_q\,\epsilon + O(\epsilon^2) \quad ,, \quad \epsilon \to 0, \end{aligned} \tag{7.3}$$

with

$$0 < |c_p| < \infty, \qquad 0 < |c_q| < \infty.$$

PROOF. From (6.13) it is seen that $P(\xi)/(1-p(\xi))$ has periods $2\omega_1$, $\frac{4}{3}\omega_3$. Because $p(\xi)$ has periods $2\omega_1$, $2\omega_3$, cf.(4.4), it is seen from (6.13) that $P(\xi)$ has period $2\omega_1$, and that the other period is the smallest common multiple of $2\omega_3$ and $\frac{4}{3}\omega_3$, which is $4\omega_3$. The relation (7.2) follows from (6.11). From (5.2)i and (6.10) it is seen that $P(\xi)-1$ has double zeros at $\xi=0$ and $\xi=2\omega_3$; from (5.2)viii, (6.10) and table 6.1 the statement concerning the poles of $P(\xi)-1$ follows, note $p(\tau)=p(\tau+2\omega_3)$. Further note that in $\hat{\mathcal{R}}_P$ the sum of the zeros of $P(\xi)-1$ is $4\omega_3$ and that of its poles is $\bar{\chi}+\chi+\tau+2\omega_3+\bar{\tau}+2\omega_3 = 4\omega+4\omega_3$. Hence the difference of these sums is equal to 0 mod $(2m\omega_1+4n\omega_3)$. Because $P(\xi)$ has a meromorphic continuation out from $\Gamma_q \cap \Gamma_p$, see section 6, it follows that $P(\xi)$ is elliptic and of order 4, so the statements (7.2)i, ii for $P(\xi)$ have been proved.

That $Q(\xi)$ has periods $2\omega_1$, $4\omega_3$ follows from (7.2) and the just proved periodicity of $P(\xi)$; the other statements of i for $Q(\xi)$ are proved analogously to those of $P(\xi)$. Note that for $Q(\xi)-1$, $\xi\in\hat{\mathcal{R}}_Q$, the sum of its zeros $= 6\frac{2}{3}\omega_3$ and that of its poles $= 4\omega_1+2\omega_3+\operatorname{Im}\tau-\operatorname{Im}\sigma = 4\omega_1+2\frac{2}{3}\omega_3$, use (5.2)iv, so that their difference $-4\omega_1+4\omega_3 = 0 \bmod (2m\omega_1+4n\omega_3)$.

To prove (7.3) note that we have from (3.11) with $p=p(\xi)$, $q(\xi)=q(p(\xi))$, $\xi=\frac{2}{3}\omega_3+\epsilon$ and $|\epsilon| << 1$, that $p(\xi)\to\infty$, $q(\xi)\to 0$ for $\epsilon\to 0$, cf.(5.2)i, ii, and

$$q(\frac{2}{3}\omega_3+\epsilon) = -\frac{1}{ar_1}[p(\frac{2}{3}\omega_3+\epsilon)]^{-2}\{1+O([p(\frac{2}{3}\omega_3+\epsilon)]^{-3})\} \text{ for } \epsilon\to 0.$$

Because $p(\xi)$ has a simple pole at $\xi=\frac{2}{3}\omega_3$, we may write for $\epsilon\to 0$,

$$[p(\frac{2}{3}\omega_3+\epsilon)]^{-1} = \epsilon f_p(1+O\,(\epsilon)),$$

with f_p a finite, nonzero constant. It follows from the asymptotic relation for $q(\frac{2}{3}\omega_3+\epsilon)$ that: for $\epsilon \to 0$,

$$q(\frac{2}{3}\omega_3+\epsilon) = -\frac{1}{ar_1}\epsilon^2 f_p\{1+O(\epsilon^3)\}. \tag{7.4}$$

From (2.1) we have for $p = p(\xi)$, $q = q(\xi)$,

$$\frac{-1/p}{1-1/p}\frac{\Phi(p,0)}{\Phi(0,0)} + \frac{1}{1-q}[1+q\{\frac{\mathrm{d}}{\mathrm{dt}}\frac{\Phi(0,t)}{\Phi(0,0)}\}_{t=0}] + O(q^2) = 1, \quad q \to 0, \tag{7.5}$$

because, cf.(1.4),

$$0 < \frac{\mathrm{d}}{\mathrm{dt}}\frac{\Phi(0,t)}{\Phi(0,0)}|_{t=0} = \frac{\Pr\{\mathbf{x}=1\}}{\Pr\{\mathbf{x}=0,\mathbf{y}=0\}} < \infty, \tag{7.6}$$

here the inequalties follow from the positive recurrence of the $(\mathbf{x}_n.\mathbf{y}_n)$-process, cf. section 1. From (6.11), (7.4) and (7.5), it readily follows that: for $\epsilon \to 0$,

$$P(\frac{2}{3}\omega_3+\epsilon) = -\frac{1}{ar_1}f_p\epsilon[1+\{\frac{\mathrm{d}}{\mathrm{dt}}\frac{\Phi(0,t)}{\Phi(0,0)}\}_{t=0}] + O(\epsilon^2),$$

and so with

$$c_p := -\frac{1}{ar_1}f_p[1+\{\frac{\mathrm{d}}{\mathrm{dt}}\frac{\Phi(0,t)}{\Phi(0,0)}\}_{t=0}],$$

the first relation of (7.3) follows. The proof of the asymptotic relation for $Q(\epsilon)$, $\epsilon \to 0$ is similar. Note that $q(\epsilon)$ has at $\epsilon = 0$ a simple pole, cf.(5.2)ii, so that we may write

$$q(\epsilon) = \epsilon f_q(1+O(\epsilon)), \quad 0 < |f_q| < \infty.$$

Further $p(\epsilon)$ has at $\epsilon = 0$ a double zero, cf.(5.2)i, and so by the same arguments as used above we obtain the asymptotic relation for $Q(\epsilon)$ with

$$c_q := -\frac{1}{ar_2}f_q[1+\{\frac{\mathrm{d}}{\mathrm{dt}}\frac{\Phi(t,0)}{\Phi(0,0)}\}_{t=0}].$$

□

8. Solution of the functional equation

In this section we shall describe the explicit expressions for $\Phi(p,0)$ and $\Phi(0,q)$. Herefore we introduce the functions $\lambda_p(\xi)$ and $\lambda_q(\xi)$ defined by

i. $\lambda_p(\xi)$ is elliptic with periods $2\omega_1$, $4\omega_3$ and of order 4, (8.1)

ii. $\lambda_p(\xi)$ has in $\hat{\mathcal{R}}_P$, cf.(7.1),
simple zeros at $\xi = \omega_1+\omega_3,\ \omega_1+3\omega_3,\ \omega_3,\ 3\omega_3$,
,, poles ,, $\xi = \pm\frac{2}{3}\omega_3,\ \frac{4}{3}\omega_3,\ \frac{8}{3}\omega_3$,

iii. $\lambda_p(\xi) := -[(\alpha_2-p(\xi))(\alpha_3-p(\xi))]^{1/2}$ for $\mathrm{Re}\,\xi = \omega_1$, $-\mathrm{Im}\,\omega_3 < \mathrm{Im}\,\xi \le \mathrm{Im}\,\omega_3$,
$:= [(\alpha_2-p(\xi))(\alpha_3-p(\xi))]^{1/2}$,, $\mathrm{Re}\,\xi = \omega_1$, $\mathrm{Im}\,\omega_3 < \mathrm{Im}\,\xi \le 3\,\mathrm{Im}\,\omega_3$,

iv. $\lambda_p^2(\xi) = (\alpha_3-p(\xi))(\alpha_3-p(\xi))$;

i. $\lambda_q(\xi)$ is elliptic with periods $2\omega_1$, $4\omega_3$ and of order 4; (8.2)

ii. $\lambda_q(\xi)$ has in $\hat{\mathcal{R}}_Q$, cf.(7.1),
simple zeros at $\xi = \omega_1 + \frac{5}{3}\omega_3,\ \frac{5}{3}\omega_3,\ \omega_1 + \frac{11}{3}\omega_3,\ \frac{11}{3}\omega_3,$
,, poles ,, $\xi = 0,\ 2\omega_3,\ \frac{4}{3},\ \omega_3,\ \frac{10}{3}\omega_3;$

iii. $\lambda_q(\xi) := -[(\beta_2 - q(\xi))(\beta_3 - q(\xi))]^{1/2}$ for $\text{Re}\ \xi = \omega_1,\ -\frac{2}{3}\text{Im}\ \omega_3 < \text{Im}\ \xi \leq \frac{4}{3}\omega_3\ \text{Im}\ \omega_3$,

$:= [(\beta_2 - q(\xi))(\beta_3 - q(\xi))]^{1/2}$,, $\text{Re}\ \xi = \omega_1,\ \frac{4}{3}\text{Im}\omega_3 \leq \text{Im}\ \xi \leq \frac{10}{3}\ \text{Im}\ \omega_3$,

iv. $\lambda_q^2(\xi) = (\beta_2 - q(\xi))(\beta_3 - q(\xi)).$

By noting that $\lambda_p^2(\xi)$ and $(\alpha_2 - p(\xi))(\alpha_3 - p(\xi))$ have the same zeros and also the same poles in $\hat{\mathcal{R}}_P$ and similarly for $\lambda_q^2(\xi)$ and $(\beta_2 - q(\xi))(\beta_3 - q(\xi))$ in $\hat{\mathcal{R}}_Q$ it is seen that the definitions in (8.8) and also those in (8.2) are consistent. It is readily verified from (3.8)iii and (5.2)v that

$$\lambda_q^2(\xi) = \frac{r_1^2}{r_2^2}\lambda_p^2(\xi - \frac{2}{3}\omega_3). \tag{8.3}$$

Next we consider $\lambda_p(\xi)$ and $\lambda_q(\xi)$ for various values of ξ, viz. those of the zeros and poles of $P(\xi)$ and $Q(\xi)$, cf. theorem 7.1.

i. $\lambda_p(0) = -(\alpha_2\alpha_3)^{1/2}$, cf.(8.1)iii, (8.4)
and
$\lambda_p(\xi) + (\alpha_2\alpha_3)^{1/2}$ has at $\xi = 0$ a double zero;

ii. $\lambda_p(2\omega_3) = (\alpha_2\alpha_3)^{1/2}$, cf. (8.1)iii,
and
$\lambda_p(\xi) - (\alpha_2\alpha_3)^{1/2}$, has at $\xi = 2\omega_3$ a double zero;

iii. $\lambda_p(\xi) + [(\alpha_2 - \frac{r_2}{r_1})^{1/2}(\alpha_3 - \frac{r_2}{r_1})^{1/2}]$
has simple zeros at $\xi = \chi$ and $\chi = \bar{\chi}$, cf.(8.1)iii, and table 6.1;

iv. $\lambda_p(\xi) = [(\alpha_2 - \frac{1}{ar_1})(\alpha_3 - \frac{1}{ar_1})]^{1/2}$
has simple zeros at $\xi = \tau + 2\omega_3$ and $\xi = \bar{\tau} + 2\omega_3$.

i. $\lambda_q(\frac{2}{3}\omega_3) = -(\beta_2\beta_3)^{1/2}, \quad \lambda_q(2\frac{2}{3}\omega_3) = (\beta_1\beta_0)^{1/2},$ (8.5)

$\lambda_q(\xi) + (\beta_2\beta_3)^{1/2}$ has a double zero at $\xi = \frac{2}{3}\omega_3,$

$\lambda_q(\xi) - (\beta_2\beta_3)^{1/2}$,, ,, ,, ,, ,, $\xi = 2\frac{2}{3}\omega_3;$

ii. $\lambda_q(\chi) = -[(\beta_2 - \frac{r_1}{r_2})(\beta_3 - \frac{r_1}{r_2})]^{1/2}.$

Because, cf.(5.2)vi, vii, $\tau + \chi = \frac{4}{3}\omega_3 + 2\omega_1$, we have

$$\tau + 2\omega_3 = \omega_1 - \ \text{Im}\ \chi + \frac{4}{3}\omega_3 + 2\omega_3,$$

or

$$\tau = 3\frac{1}{3}\omega_3 + \bar{\chi},$$

and so

$$\lambda_q(\tau + 2\omega_3) = \lambda(3\frac{1}{3}\omega_3 + \bar{\chi}) = -(\beta_2 - \frac{r_1}{r_2})(\beta_3 - \frac{r_1}{r_2}) = \lambda_q(\chi),$$

i.e.

$$\lambda_q(\chi) + [(\beta_2 - \frac{r_1}{r_2})(\beta_3 - \frac{r_1}{r_2})]^{1/2}$$

has single zeros at $\xi = \tau + 2\omega_3$ and $\xi = \chi$;

iii. $$\lambda_q(\bar{\chi}) = -[(\beta_2 - \frac{1}{ar_2}))\beta_3 - \frac{1}{ar_2})]^{1/2},$$

because, cf.(5.2)vii, $\sigma + \chi = \frac{2}{3}\omega_3 + 2\omega_1 \Rightarrow \bar{\sigma} + \bar{\chi} = -\frac{2}{3}\omega_3 + 2\omega_1 \Rightarrow \bar{\sigma} = -\frac{2}{3}\omega_3 + \chi$, we have

$$\lambda_q(\bar{\sigma}) = \lambda_q(-\frac{2}{3}\omega_3 + \chi) = -[(\beta_2 - \frac{1}{ar_2})(\beta_3 - \frac{1}{ar_2})]^{1/2} = \lambda_q(\bar{\chi}),$$

and so

$$\lambda_q + [(\beta_2 - \frac{1}{ar_2})\ (\beta_3 - \frac{1}{ar_2})]^{1/2}$$

has single zeros at $\xi = \bar{\sigma}$ and $\xi = \bar{\chi}$.

Next we define for $\xi \in \hat{\mathcal{R}}_P$,

$$\mu_p(\xi) := \frac{\lambda_p^2(\xi) - \alpha_2\alpha_3}{\{\lambda_p(\xi) - [(\alpha_2 - \frac{1}{ar_1})(\alpha_3 - \frac{1}{ar_1})]^{1/2}\}\{\lambda_p(\xi) + [(\alpha_2 - \frac{r_2}{r_1})(\alpha_3 - \frac{r_2}{r_1})]^{1/2}\}}, \tag{8.6}$$

and for $\xi \in \hat{R}_Q$,

$$\mu_q(\xi) = \frac{\lambda_q^2(\xi) - \beta_2\beta_3}{\{\lambda_q(\xi) + [(\beta_2 - \frac{1}{ar_2})(\beta_3 - \frac{1}{ar_2})]^{1/2}\}\{\lambda_q(\xi) + [(\beta_2 - \frac{r_1}{r_2})(\beta_3 - \frac{r_1}{r_2})]^{1/2}\}}. \tag{8.7}$$

It follows from theorem 7.1, (8.1), (8.4) and (8.6) that $\mu_p(\xi)$ and $P(\xi) - 1$ which are both elliptic, and which have the same order, the same zeros and the same poles, can only differ by a constant factor; analogously for $\mu_q(\xi)$ and $Q(\xi) - 1$. So we may write with γ_p and γ_q finite constants:

$$P(\xi) = 1 + \gamma_p\mu_p(\xi), \tag{8.8}$$

$$Q(\xi) = 1 + \gamma_q\mu_q(\xi).$$

To determine γ_p and γ_q we use (7.3). By noting that $|\lambda_p(\xi)| \to \infty$ for $\xi \to \frac{2}{3}\omega_3$, it is seen from (7.3) and (8.6) that $1 + \gamma_p = 0$. Analogously it follows from (7.3) and (8.7), since $|\lambda_q(\xi)| \to \infty$ for $\xi \to 0$, that $1 + \gamma_q = 0$. Consequently, from (8.8),

i. $$P(\xi) = 1 - \mu_p(\xi), \tag{8.9}$$

ii. $$Q(\xi) = 1 - \mu_q(\xi).$$

Observe that $p(\xi)$ is the inverse of the conformal mapping $\xi(p)$ of H_p^+ onto $\mathcal{R}_p$, cf.(4.2). Hence by using (8.1) we define

$$X(p) := -\lambda_p(\xi(p)) := \sqrt{(\alpha_2 - p)(\alpha_3 - p)}. \tag{8.10}$$

with

$X(0) > 0$ and $X(p)$ regular in $|p| \le 1$.

Similarly, define with $\xi(q) \in H_q^+$ the inverse of $q(\xi)$, cf.(5.3)ii,

$$Y(q) := -\lambda_q(\xi(q)) = \sqrt{(\beta_2 - q)(\beta_3 - q)}, \tag{8.11}$$

with

$Y(0) > 0$ and Y(q) regular in |q| ≤ 1.

By replacing ξ in (8.9)i by $\xi(p)$, it is seen by using (6.10) and (8.6) that for $|p| \le 1$, $|q| \le 1$,

$$\text{i.} \quad \frac{\Phi(p,0)}{\Phi(0,0)} = 1 - \frac{X^2(p) - \alpha_2\alpha_3}{\{X(p) - [(\alpha_2 - \frac{1}{ar_1})(\alpha_3 - \frac{1}{ar_1})]^{1/2}\}\{X(p) + [(\alpha_2 - \frac{r_2}{r_1})(\alpha_3 - \frac{r_2}{r_1})]^{1/2}\}}, \tag{8.12}$$

$$\text{ii.} \quad \frac{\Phi(0,q)}{\Phi(0,0)} = 1 - \frac{Y^2(q) - \beta_2\beta_3}{\{Y(q) + [(\beta_2 - \frac{1}{ar_2})(\beta_3 - \frac{1}{ar_2})]^{1/2}\}\{Y(q) + [(\beta_2 - \frac{r_1}{r_2})(\beta_3 - \frac{r_1}{r_2})]^{1/2}\}},$$

where the derivation of (8.12)ii is analogous to (8.12)i. From (8.10) and (8.12)i it is seen that $\Phi(p,0)$ is regular in $|p| \le 1$, similarly (8.11) and (8.12)ii show that $\Phi(0,q)$ is regular in $|q| \le 1$.

It remains to determine $\Phi(0,0)$. From (2.1)iii and (8.12) with $p = 1$ and $q = 1$ it follows that

$$\Phi^{-1}(0,0) = \frac{1 - ar_1}{1 + a}\{1 - \frac{X^2(1) - \alpha_2\alpha_3}{[X(1) - X(\frac{1}{ar_1})][X(1) + X(\frac{r_2}{r_1})]}\} \tag{8.13}$$

$$= \frac{1 - ar_1}{1 + a}\{1 - \frac{Y^2(1) - \beta_2\beta_3}{[Y(1) + Y(\frac{1}{ar_2})][Y(1) + Y(\frac{r_1}{r_2})]}\} > 1;$$

note that the second equality in (8.13) also follows from (2.3) and from the fact that $\Phi(\hat{p},0)$ and $\Phi(0,\hat{q})$, as given by (8.12) with $K(\hat{p},\ \hat{q}) = 0$, $|\hat{p}| \le 1$, $|\hat{q}| \le 1$, satisfy (2.1)ii, as it follows from the analysis which has led to (8.12). The inequality in (8.13) follows easily from (8.10), (8.11) and $\frac{r_2}{r_1} < \frac{1}{ar_1} < \alpha_2 < \alpha_3$, $\frac{r_1}{r_2} < \frac{1}{ar_2} < \beta_2 < \beta_3$.

The expression for $\Phi(p,q)$, $|p| \le 1$, $|q| \le 1$, can now be obtained from (1.5)ii, (8.12) and (8.13) if $K(p,q) \ne 0$. If $K(p,q) = 0$ the expression for $\Phi(p,q)$ is obtained by an appropriate limiting procedure, i.e. start from $\Phi(p_0,q_1)$, $K(p_0,q_1) \ne 0$ and $q_1 \to q_1(p_0)$, cf.(3.4)i.

Since (8.13) implies that $1 > \Phi(0,0) > 0$ it follows by using remark 2.1 that the expressions for $\Phi(p,0)$ and $\Phi(0,q)$ in (8.12) also satisfy (2.1)iv. From the analysis above it is further seen that the construction of these expressions is unique, and that they possess the properties in (2.1). Because the Markov process $\{(\mathbf{x}_n, \mathbf{y}_n),\ n = 0,1,2,\ldots\}$ possesses a unique stationary distribution it follows that for $\Phi(p,0)$ and $\Phi(0,q)$, as given by (8.12) and (8.13) holds, cf.(1.4), with $|p| \le 1$, $|q| \le 1$,

$$\Phi(p,0) = \mathrm{E}\,\{p^{\mathbf{x}}(\mathbf{y} = 0)\}, \quad \Phi(0,q) = \mathrm{E}\,q^{\mathbf{y}}(\mathbf{x} = 0)\},$$

and so $\mathrm{E}\{p^{\mathbf{x}}q^{\mathbf{x}}\}$, $|p| \le 1$, $|q| \le 1$, follows from (1.5)ii.

REMARK 8.1. The function $X(p)$, cf.(8.10), can be continued analytically on the Riemann surface R_P, cf.(6.4), similarly $Y(p)$ on R_Q, with

$$X(p) = -(\alpha_2\alpha_3)^{1/2} \quad \text{for} \quad p = 0 \in K_p^-, \tag{8.14}$$

$$Y(q) = -(\beta_2\beta_3)^{1/2} \quad ,, \quad q = 0 \in K_q^-.$$

With $X(p)$ and $Y(q)$ so defined on R_P and R_Q, respectively, define the meromorphic continuations of $\Phi(p,0)$ and $\Phi(0,q)$ according to (8.12) on R_p and R_Q. It is readily seen that this continuation of $\Phi(p,0)$ as a one-valued function on R_P and that of $\Phi(0,q)$ as a one-valued function on R_Q agree with those in section 6. □

9. THE CASE $r_1 = r_2 = \frac{1}{2}$.
In the preceding sections all derivations and results concern the case that $r_2 > r_1$, cf.(1.2). The case $r_1 < r_2$ can be obtained by using the symmetry. In this section we shall consider the case

$$r_1 = r_2 = \frac{1}{2}. \tag{9.1}$$

Hence, cf.(3.5).

$$D_p(p) = D_q(p) = [1 + a - \frac{1}{2}ap]^2 p^2 - 2ap. \tag{9.2}$$

It is readily verified that $D_p(p)$ and $D_q(q)$ have the same zeros; they can be simply calculated, viz.

$$\text{i.} \qquad \alpha_i = \beta_i, \qquad i = 0, 1, 2, \ldots, , \tag{9.3}$$

$$\text{ii.} \qquad \alpha_0 = 0, \qquad \alpha_2 = \frac{2}{a} > 1 \,,$$

$$\alpha_1 = \frac{1}{a^2}[1 - 2a - \sqrt{(1-2a)^2 - 4a^3}] < 1,$$

$$\alpha_3 = \frac{1}{a^2}[1 - 2a + \sqrt{(1-2a)^2 - 4a^3}] > 1.$$

Note that for the following zero-tuples $(\hat{p}, \hat{q})$ of $K(p, q)$ holds:

i. $\hat{p} = 1 \Rightarrow \hat{q}_1 = 1$ and $\hat{q}_2 = \frac{2}{a}$ are both single zeros, (9.4)

$\hat{q} = 1 \Rightarrow \hat{p}_1 = 1$,, $\hat{p}_2 = \frac{2}{a}$,, ,, ,, ,,;

ii. $\hat{p} = \frac{2}{a} \Rightarrow \hat{q} = 1$ is a double zero,

$\hat{q} = \frac{2}{a} \Rightarrow \hat{p} = 1$,, ,, ,, ,, .

It is further readily verified that with $0 < -\epsilon << 1$ we have for $\epsilon \uparrow 0$, note $0 < a < 2$,

i. $\hat{p} = 1 + \epsilon \Rightarrow \hat{q}_2 = \frac{2}{a}(1 + O(\epsilon^2))$ and $\hat{q}_1 = 1 - \epsilon + O(\epsilon^2)$, (9.5)

ii. $\hat{p} = \frac{2}{a} + \epsilon \Rightarrow \hat{q} = 1 \pm \sqrt{\epsilon(\frac{1}{2}a - 1)\,(a + O(\sqrt{|\epsilon|})}$.

From, cf.(2.1)ii,

$$\frac{1}{1-\hat{p}} \frac{\Phi(\hat{p}, 0)}{\Phi(0,0)} + \frac{1}{1-q} \frac{\Phi(0, \hat{q})}{\Phi(0,0)} = 1, \tag{9.6}$$

it is seen that for $\hat{p} = 1 + \epsilon$, $\hat{q} = 1 + \epsilon$, since by symmetry $\Phi(1,0) = \Phi(0,1)$, the relation (9.6) can be satisfied with $|\Phi(1,0)| < \infty$.

By taking in (9.6) $\hat{p} = 1 + \epsilon$, $\hat{q} = \frac{2}{a}(1 + O(\epsilon^2))$, we obtain from (9.6): for $\epsilon \to 0$,

$$-\frac{1}{\epsilon} \frac{\Phi(1+\epsilon, 0)}{\Phi(0,0)} + \frac{1}{1 - \frac{2}{a}} \frac{\Phi(0, \frac{2}{a}(1 + O(\epsilon^2)))}{\Phi(0,0)} = 1, \tag{9.7}$$

from which it follows with $|\Phi(1,0)| < \infty$ that $\Phi(0,q)$ has at $q = \frac{2}{a}$ a simple pole.

By taking (cf.(9.5)ii and use the symmetry) $\hat{q} = \frac{2}{a} + \epsilon$, $\hat{p} = 1 \pm \sqrt{\epsilon(\frac{1}{2}a - 1)(1 + \mathrm{O}(\sqrt{|\epsilon|})}$ we obtain from (9.6): for $\epsilon \to 0$,

$$\pm \frac{1}{\sqrt{\epsilon(\frac{a}{2} - 1)(1 + \mathrm{O}(\sqrt{|\epsilon|}))}} \frac{\Phi(1 \pm \sqrt{\epsilon(\frac{1}{2}a - 1)(1 + \mathrm{O}(\sqrt{|\epsilon|}))}, 0)}{\Phi(0,0)} + \frac{1}{1 - \frac{2}{a} - \epsilon} \frac{\Phi(0, \frac{2}{a} + \epsilon)}{\Phi(0,0)} = 1. \quad (9.8)$$

Hence from (9.7) and (9.8) it follows that

$$\frac{1}{\sqrt{|\epsilon|}} \frac{\Phi(1 \pm \sqrt{\epsilon(\frac{1}{2}a - 1)(1 + \mathrm{O}(\sqrt{|\epsilon|}))}, 0)}{\Phi(0,0)} = \frac{1}{\epsilon} \frac{\Phi(1+\epsilon, 0)}{\Phi(0,0)} + \mathrm{O}(\epsilon). \quad (9.9)$$

By noting that $\Phi(p,0)$ and $\Phi(0,q)$ are two-valued functions, cf. remark 8.1, it is seen from (9.8) that if one value of $\Phi(p,0)$ is finite at $p = 1$ then the other value of $\Phi(p,0)$ has a single pole at $p = 1$. Analogous results hold for $\Phi(0,q)$ at $q = 1$.

With the information obtained so far for the case $r_1 = r_2 = 1/2$ we can repeat the analysis of the preceeding sections for the present case. However, it is very easy to prove that we can obtain for the present case the expressions for $\Phi(p,0)$ and $\Phi(0,q)$ by letting in (8.12) $r_1/r_2 \to 1$. It results from (8.12),

$$\frac{\Phi(p,0)}{\Phi(0,0)} = \frac{\Phi(0,p)}{\Phi(0,0)} = \quad (9.10)$$

$$1 - \frac{(\alpha_2 - p)(\alpha_3 - p) - \alpha_2\alpha_3}{[(\alpha_2 - p)(\alpha_3 - p)]^{1/2}\{[(\alpha_2 - p)(\alpha_3 - p)]^{1/2} + [(\alpha_2 - 1)(\alpha_3 - 1)]^{1/2}\}},$$

with, cf.(8.10),

$$[(\alpha_2 - p)(\alpha_3 - p)]^{1/2} \Big|_{p=0} = (\alpha_2\alpha_3)^{1/2},$$

and α_2, α_3 given by (9.3)ii. Further $\Phi(0,0)$ follows from (9.11) and (2.1)iii, i.e.

$$\Phi^{-1}(0,0) = \frac{1+a}{1 - \frac{1}{2}a} \left[1 + \frac{\alpha_2 + \alpha_3 - 1}{2[(\alpha_2 - 1)(\alpha_3 - 1)]^{3/2}}\right] > 1. \quad (9.11)$$

REMARK 9.1. With the notation of remark 8.1 it is seen that $\Phi(p,0)$ has a single pole at $p = \alpha_2 = \frac{2}{a} \in K_p^-$ and a single pole at $p = 1 \in K_p^+$, similarly for $\Phi(0,q)$; this agreees with the analysis above, which has been based on (9.5). □

10. COMMENT'S ON A RELATED RANDOM WALK.

Consider the $(\tilde{\mathbf{x}}_n, \tilde{\mathbf{y}}_n)$-random walk with $\phi(p,q)$, cf.(1.2), replaced by

$$\tilde{\phi}(p,q) := \frac{a\tilde{r}_3}{1+a} p^2q^2 + \frac{a\tilde{r}_1}{1+a} p^2q + \frac{a\tilde{r}_2}{1+a} pq^2 + \frac{1}{1+a}, \quad (10.1)$$

$$a > 0, \ \tilde{r}_1 + \tilde{r}_2 + \tilde{r}_0 = 1, \ \tilde{r}_i > 0, \ i = 1,2,3.$$

This random walk is also a nearest-neighbour random walk. By using the conditions formulated in [3], cf.p.95, it is readily verified that this random walk is positive recurrent if and only if

$$a < \min((\tilde{r}_1 + \tilde{r}_3)^{-1}, (\tilde{r}_2 + \tilde{r}_3)^{-1}), \quad (10.2)$$

henceforth (10.2) is assumed to apply.

REMARK 10.1. In this section symbols covered by a " $\sim$" are defined as the corresponding symbols defined for the $(\mathbf{x}_n, \mathbf{y}_n)$- random walk studied in the previous sections. □

The analysis in the preceding sections has shown that $\Phi(p,0)$, $|p| \leq 1$, has a meromorphic continuation on the two-sheeted Riemann surface R_P, cf.(6.4), and that this extended function has only two poles, similarly for $\Phi(0,q)$. Further, it was seen that these poles stem from the (periodical) sequence of zero-tuples generated by the zero-tuple (1,1) of $K(p,q)$ cf. remark 3.1 and (6.7).

The question arises whether analogous results hold for the $(\tilde{\mathbf{x}}_n, \tilde{\mathbf{y}}_n)$-random walk with kernel $\tilde{K}(p,q)$ cf.(3.1) and (10.1), given by

$$\tilde{K}(p,q) := \tilde{\phi}(p,q) - pq. \tag{10.3}$$

As in (3.2) zero-tuples (p_n, q_n), $n = 1,2,\ldots$; generated by the zero-tuple $(\hat{p},\hat{q})$ follow recursively from: for $n = 1,2,\ldots$,

$$p_1 = \hat{p},\ q_1 = \hat{q}, \tag{10.4}$$

$$q_{n+1} = \frac{1}{p_n q_n} \frac{1}{ar_3 p_n + ar_2},$$

$$p_{n+1} = \frac{1}{p_n q_{n+1}} \frac{1}{ar_3 q_{n+1} + ar_1};$$

the successive zero-tuples are

$$(p_1, q_1) = (\hat{p},\hat{q}),\ (p_1,q_2),\ (p_2,q_2),\ (p_3,q_3),\ (p_4,q_3),\ldots. \tag{10.5}$$

So we have to investigate the question whether this sequence of zero-tuples is periodic, i.e. whether a finite M exists such that

i. $\quad p_{2M+1} = \hat{p},\ q_{2M+1} = \hat{q},$ (10.6)

ii. $\quad \hat{p} = 1,\ \hat{q} = 1.$

If such an M exists then (10.6) also applies for every multiple of M, so we consider only the smallest $N > 1$ for which (10.6) holds; $2N$ is then called the period of the sequences (10.5).

In the following considerations we first show that:

no finite M exists for which (10.6) *holds* , (10.7)

and then discuss the consequences of (10.7) for the analysis concerning the determination of the bivariate generating function $\tilde{\Phi}(p,q)$ of the stationary distribution of the positive recurrent nearest-neighbour random walk $(\tilde{\mathbf{x}}_n, \tilde{\mathbf{y}}_n)$.

The proof of (10.7) requires the proofs of several statements.

The zeros $\tilde{q}_{1,2}(p)$ of $\tilde{K}(p,q) = 0$ for fixed p, cf.(3.4) and remark 10.1, have four branch points $\tilde{\alpha}_0$, $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $\tilde{\alpha}_3$, with

$$0 = \tilde{\alpha}_0 < \tilde{\alpha}_1 < 1 < \tilde{\alpha}_2 < \tilde{\alpha}_3, \tag{10.8}$$

i.e. these $\tilde{\alpha}_i$, $i = 0,1,2,3$, are the zeros of $\tilde{D}_p(p)$, cf.(3.5), similarly the zeros $\tilde{\beta}_0$, $\tilde{\beta}_1$, $\tilde{\beta}_2$, $\tilde{\beta}_3$ of $\tilde{D}_q(q)$ satisfy

$$0 = \tilde{\beta}_0 < \tilde{\beta}_1 < 1 < \tilde{\beta}_2 < \tilde{\beta}_3.$$

Further

$$\tilde{D}_p(p) > 0 \quad \text{for} \quad p \in (-\infty, 0) \cup (\alpha_1, \alpha_2) \cup (\alpha_3, \infty), \qquad (10.9)$$
$$< 0 \quad ,, \quad p \in (0, \tilde{\alpha}_1) \cup (\tilde{\alpha}_2, \tilde{\alpha}_3).$$

The proofs of (10.8) and (10.9) are omitted, they are similar to those for $D_p(p)$.

For the present case the function $\tilde{p}(\xi)$ is defined completely analogous to $p(\xi)$, see section 4, and $\tilde{q}(\xi)$ is analogous to $q(\xi)$, cf.(4.13). It may again be shown that the analogon of (4.4) holds for $\tilde{p}(\xi)$ and similarly that of (5.2)iv for $\tilde{q}(\xi)$, i.e.

$$\tilde{p}(\xi) = \tilde{p}(-\xi), \; \hat{q}(\xi) = \tilde{q}(2\tilde{\delta} - \xi); \qquad (10.10)$$

here $\tilde{\delta}$ is the pole of $\tilde{p}(\xi)$ and further

$$\text{Re}\,\tilde{\delta} = 0, \; 0 < \text{Im}\,\tilde{\delta} < \text{Im}\,\tilde{\omega}_3, \; \tilde{\alpha}_1 = \tilde{p}(\tilde{\omega}_1), \; \tilde{\alpha}_3 = \tilde{p}(\tilde{\omega}_3), \qquad (10.11)$$

$\tilde{p}(\xi)$ and $\tilde{q}(\xi)$ are both elliptic, of order two and with periods $2\tilde{\omega}$, $2\tilde{\omega}_3$; $\tilde{p}(\xi)$ has a double zero at $\xi = 0$, simple poles at $\xi = \pm\tilde{\delta}$; $\tilde{q}(\xi)$ has simple poles at $\xi = 0, \tilde{\gamma}$, a double zero at $\xi = \tilde{\delta}$ and $\text{Re}\,\tilde{\gamma} = 0$, $-\text{Im}\,\tilde{\delta} < \tilde{\gamma} < 0$, $2\tilde{\delta} - \tilde{\gamma} = 2\tilde{\omega}_3$.

By applying the same arguments as used in remark 5.3 it is shown that if one zero-tuple $(\hat{p}_1, \hat{q}_1)$ of $\tilde{K}(p,q) = 0$ generates via (10.4) a periodic sequence, which is only possible if $\tilde{\delta}/\tilde{\omega}_3$ is rational, then every zero-tuple $(\hat{p}_1, \hat{q}_1)$ of $\tilde{K}(p,q) = 0$ generates via (10.4) a periodic sequence and they all have the same period. Further, it is readily seen that such a periodic sequence is also generated by the recursive scheme

$$p_{n+1} = \frac{1}{p_n q_n} \frac{1}{ar_3 q_n + ar_1}, \qquad (10.12)$$
$$q_{n+1} = \frac{1}{p_{n+1} q_n} \frac{1}{ar_3 p_{n+1} + ar_2},$$

with the same starting point as in (10.4). Moreover periodic sequences with the same starting point but one constructed via (10.4), the other via (10.12) contains the same set of zero-tuples.

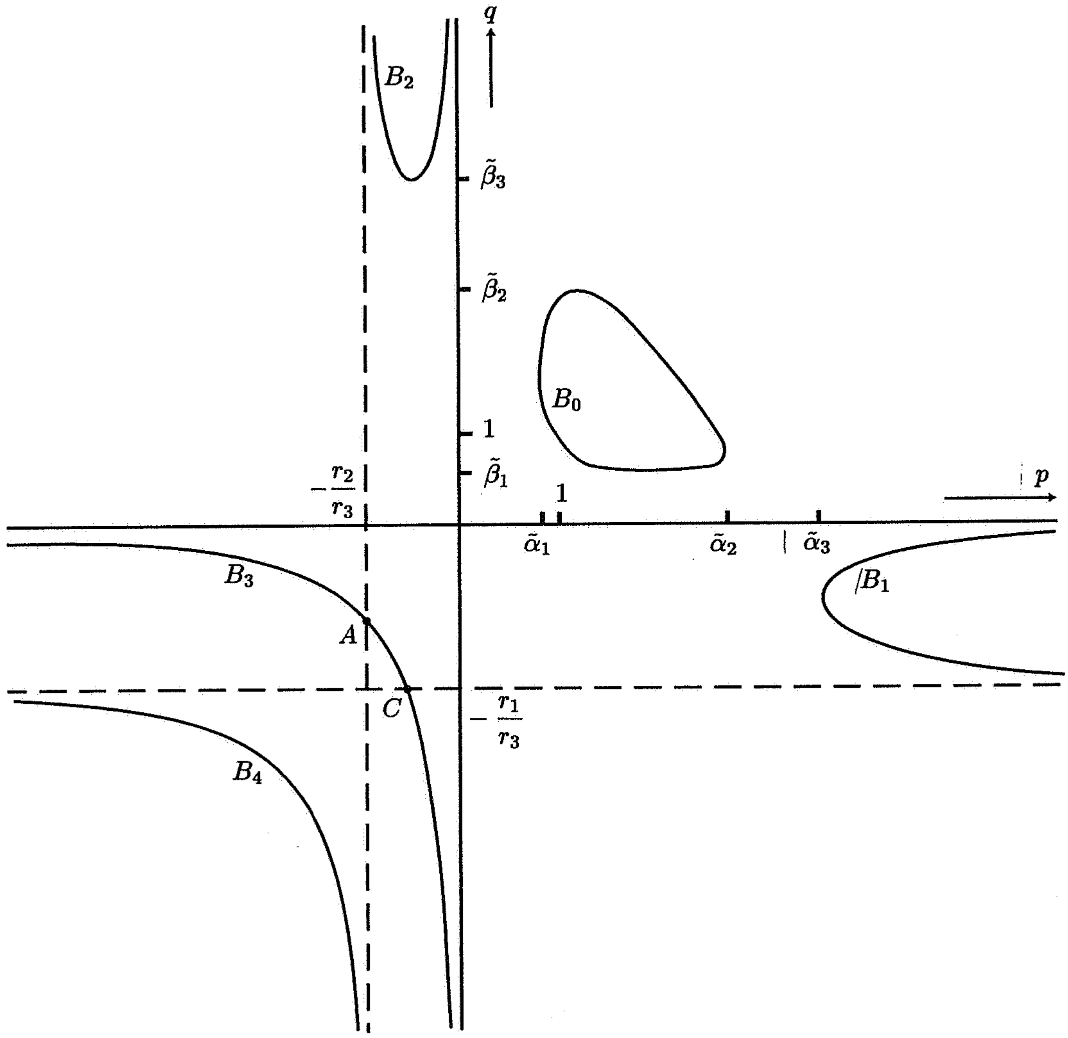
REMARK 10.2. The property just formulated concerning the existence of periodic sequences of zero-tuples of the quartic $\tilde{K}(p,q)$ is not unknown in the theory of higher plane curves, see [12], p. 253, and [13].

Hence to prove (10.7) it suffices to consider only one zero-tuple (p_A, q_A), say, of $\tilde{K}(p,q) = 0$ and to show that the sequence of zero-tuples generated by this point via (10.4) is a periodic.

Herefor we have traced in fig. 10.1 the graph of the curve $\tilde{K}(p,q)$ for real p and q. Because $\tilde{K}(1,1) = 0$, cf.(10.1) and (10.3), it is readily seen from (10.8) and (10.9) that $(1,1)$ is not an isolated point of this graph and that it lies on a simple closed contour B_0 in the first quadrant. Further it is not difficult to show that the curve has four asymptotes, viz. the lines

$$q = 0, \; q = -\frac{r_1}{r_3}, \; p = 0, \; p = -\frac{r_2}{r_3}. \qquad (10.13)$$

The curve has four infinite branches B_1, B_2, B_3 and B_4. The points at infinity of the asymptotes $q = 0$ and $p = 0$ are inflexion points of the curve. That of the asymptote $p = -r_2/r_3$ is at infinity a simple point of the curve, similarly for the asymptotic $q = -r_1/r_3$.



q
p
B_2
$\tilde{\beta}_3$
$\tilde{\beta}_2$
B_0
1
$\tilde{\beta}_1$
$-\frac{r_2}{r_3}$
1
$\tilde{\alpha}_1$
$\tilde{\alpha}_2$
$\tilde{\alpha}_3$
B_1
B_3
A
C
$-\frac{r_1}{r_3}$
B_4

FIGURE 10.1.

The asymptotic $p = -r_2/r_3$ intersects the curve in a second (real) finite point A.

The coordinates (p_A, q_A) of the point A are used as the zero-tuple which generates via (10.4) a sequence of zero-tuples. It is readily seen that we then obtain the sequence

$$(p_A, q_A),\ (p_A, \infty),\ (0, \infty),\ (0, -\infty),\ (p_A, -\infty),\ (p_A, q_A). \tag{10.14}$$

However, for the present case the sequence generated by (10.12) with starting point (p_A, q_A) contains a finite zero-tuple situated at the branch B_1. Hence in order that this sequence contains the zero-tuples (10.4), returns to A and consists of a finite number of zero-tuples, it should contain the finite point C, i.e. the intersection of the asymptote $q = -\frac{r_1}{r_3}$ and the curve. But these two sequences both generated by (p_A, q_A), one via (10.4) the other via (10.12), are not identical and do not have the same period. They also have not the same period if the sequence generated by (p_A, q_A) via (10.12) does not contain the point C. Because this latter sequence can then only contain the points (10.14) in the limit, and so, necessarily, it contains a countable infinite number of zero-tuples. Consequently for the sequence (10.5) with $(\hat{p}, \hat{q})$ given by (10.6)ii there exists no finite M for which (10.6)i applies, since if M is finite then every sequence is periodic and has the same period, i.e. (10.7) has been proved.

From the results so far obtained it follows that the sequence of zero-tuples generated by $\hat{p} = 1$, $\hat{q} = 1$, via (10.4), is not finite and that every zero-tuple of this sequence lies on the oval B_0. Because this oval B_0 has no points at infinity it follows that this sequence contains at least one accumulation point. Consequently it follows from, cf. lemma 2.1,

$$\frac{1}{1-p}\,\tilde{\Phi}(p,0) + \frac{1}{1-q}\,\tilde{\Phi}(0,q) = \tilde{\Phi}(0,0),$$

with (p,q) a zero-tuple of $\tilde{K}(p,q) = 0$ and by using the zero-tuples of the sequences generated by $\hat{p} = 1$, $\hat{q} = 1$, that the meromorphic continuation of $\tilde{\Phi}(p,0)$ on the two-sheeted Riemann surface $\tilde{R}_P = (\tilde{K}_p^-, \tilde{K}_p^+)$ with slit $(\tilde{\alpha}_2, \tilde{\alpha}_3)$, cf.(6.4), has a denumerable set of poles with at least one accumulation point. Hence this extension of $\tilde{\Phi}(p,0)$ is not an algebraic meromorphic function, since it has not a finite number of poles in every finite domain, similarly for $\tilde{\Phi}(0,q)$: So it will be evident that the construction of the expression for $\Phi(p,0)$ along the lines as exposed in section 8 cannot be applied in the present case. Instead of it the approach of WRIGHT [7] should be followed to derive the expressions for $\tilde{\Phi}(p,0)$ and $\tilde{\Phi}(0,q)$. Although in [7] the functional equation is homogeneous, as it is in [6], the required changes are of the same character as those by which the analysis in [6] differs from that in section 8, cf. the determination of the constants γ_p and γ_q in (8.8). In the present study we shall refrain from a further analysis of $\tilde{\Phi}(p,0)$ and $\tilde{\Phi}(0,q)$ along the lines indicated in [7]. As in [7] the result of such an analysis will be fairly explicit expressions for $\tilde{\Phi}(\tilde{p}(\xi),0)$ and $\tilde{\Phi}(0,\tilde{q}(\xi))$ from which the relations for $\tilde{\Phi}(p,0)$ can be obtained by using the inverse of the mapping $\tilde{p}(\xi) : \tilde{\mathcal{R}}_p \to \tilde{H}_p^+$, cf.(4.2)), similarly for $\tilde{\Phi}(0,q)$. These relations are quite intricate when compared to those for the case studied in the previous sections.

To conclude it may be said that from the already available results in literature, see [4], [5], [6], [7], [14], [15], [16] and the present study quite same insight is available concerning the analytical techniques to be used in deriving explicit expressions for the generating function of the stationary distribution of a positive recurrent nearest-neighbour random walk.

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