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The Real Positive Semidefinite Completion Problem for Series-Parallel Graphs

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Abstract

We consider the partial real symmetric matrices X whose diagonal entries are equal to 1 and whose off-diagonal entries are specified only on a subset of the positions. The question is to determine whether X can be completed to a positive semidefinite matrix. Extending a result of [BJT93], we give a set of necessary conditions for X to be completable and show that these conditions are also sufficient if and only if the graph coresponding to the positions of the specified entries is series-parallel (i.e., has no K_4-minor).

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1 Introduction

A positive semidefinite matrix whose diagonal entries are all equal to 1 is called a **correlation matrix**. Let $\mathcal{E}_{n\times n}$ denotes the set of $n \times n$ correlation matrices, i.e.,

$$\mathcal{E}_{n\times n} := \{X = (x_{ij}) \text{ symmetric } n \times n \mid X \succeq 0, x_{ii} = 1 \text{ for all } i = 1, \ldots, n\}.$$

The notation: $X \succeq 0$ means that X is positive semidefinite, i.e., $x^T X x \geq 0$ for all $x \in \mathbb{R}^n$. Let $G = (V, E)$ be a graph, where $V = \{1, \ldots, n\}$. (All the graphs considered here are **simple**, i.e., have no loops, nor parallel edges.) Then, the set $\mathcal{E}(G)$ is defined as the projection of $\mathcal{E}_{n\times n}$ on the subspace $\mathbb{R}^E$ indexed by the edge set of G, i.e.,

$$\mathcal{E}(G) := \{x \in \mathbb{R}^E \mid \exists A = (a_{ij}) \in \mathcal{E}_{n\times n} \text{ such that } a_{ij} = x_{ij} \text{ for all } ij \in E\}.$$

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In particular, $\mathcal{E}(K_n)$ consists of the projections of the correlation matrices on their upper triangular part. The convex sets $\mathcal{E}_{n\times n}$ and its projection $\mathcal{E}(G)$ are called **elliptopes.**

The problem of characterizing the elliptope $\mathcal{E}(G)$ has been studied in [GJSW84] for chordal graphs, and in [Fie66, BJT93] for cycles. Our work is, in a sense, a continuation of these papers. Using the result of [BJT93], we derive a set of necessary conditions for membership in the elliptope $\mathcal{E}(G)$ of an arbitrary graph G. Furthermore, we characterize the graphs for which these conditions are also sufficient as the class of graphs with no K_4-minor (i.e., the simple series-parallel graphs). In fact, a much stronger set of necessary conditions for membership in the elliptope $\mathcal{E}(G)$ can be derived from a result of [GW94]; it turns out that these conditions are sufficient for the same class of graphs. We show, moreover, that the elliptope $\mathcal{E}(G)$ coincides with the convex hull of its rank one matrices if and only if the graph G is acyclic.

The problem of characterizing the members of the elliptope $\mathcal{E}(G)$ is also known in the literature as the **positive semidefinite completion problem**, which is defined as follows.

Consider a partial real symmetric matrix X whose entries are specified on the diagonal and on a certain subset E of the off-diagonal positions, while the remaining entries of X are free. The question is to determine whether the free entries can be chosen so as to make X positive semidefinite. If this is the case, we say that X is **completable.**

An easy observation is that it suffices to consider the positive semidefinite completion problem for matrices whose diagonal entries are all equal to 1. (Indeed, if X is completable, then its digonal entries are nonnegative. Moreover, we can suppose that all diagonal entries are positive as, otherwise, the problem reduces to considering the submatrix of X with positive diagonal entries. Finally, if D denotes the diagonal matrix whose i-th diagonal entry is $\frac{1}{\sqrt{x_{ii}}}$ then the matrix $X' := DXD$ has diagonal entries 1 and is completable if and only if X is completable.)

Suppose X has diagonal entries 1 and let $x := (x_{ij})_{ij\in E} \in \mathbb{R}^E$ denote the vector whose components are the specified entries of X. Moreover, let G denote the graph with edge set E. Then, by definition of the elliptope $\mathcal{E}(G)$, the following equivalence holds:

$$x \in \mathcal{E}(G) \Longleftrightarrow X \text{ is completable.}$$

The set $\mathcal{E}_{n\times n}$ of correlation matrices has also been studied in [CM79, Loe80, GPW90, LT94], where is mainly considered the question of determining the possible ranks for extreme elements of $\mathcal{E}_{n\times n}$. The set $\mathcal{E}_{n\times n}$ has been recently reintroduced in [PR92, LP93, GW94] as a nonlinear relaxation for a hard combinatorial optimization problem, namely, the max-cut problem. Indeed, the rank one matrices of $\mathcal{E}_{n\times n}$, which are of the form aa^T for $a \in \{-1,1\}^n$, play a special role in discrete optimization as they corespond to the cuts of the complete graph. A result of [GW94] shows, moreover, that by optimizing over the elliptope one obtains a very good approximation for the max-cut problem. Several results are given in [LP93, LP94] on the faces of $\mathcal{E}_{n\times n}$. In particular, the vertices of $\mathcal{E}_{n\times n}$ are described in [LP93]; they are precisely the rank one matrices. The possible dimensions for the faces (and the polyhedral faces) of $\mathcal{E}_{n\times n}$ are described in [LP94]. For the graph K_4, which is the smallest graph for which the parametric description does not apply, a description of the faces of its elliptope $\mathcal{E}_{4\times 4}$ can be found in [LP94].

The paper is organized as follows. In Section 2, we introduce some polytopes related with the elliptope, that we will need in the sequel. In Section 3, we recall the characterization of [GJSW84] for the elliptope of chordal graphs and we give a short proof for one of the key lemmas needed for establishing the result. In Section 4, we present some necessary conditions for membership in the elliptope $\mathcal{E}(G)$ and show that they are sufficient if and only if the graph G is series-parallel. In Section 5, we show that the elliptope $\mathcal{E}(G)$ coincides with the cut polytope if and only if the graph G is acyclic. We group in Section 6 several additional remarks. In particular, we formulate a result of [GW94] on the inequalities that hold for the pairwise angles between any set of unit vectors.

2 Related polytopes

We introduce here several polytopes related with the elliptope $\mathcal{E}(G)$ and with the max-cut problem. Let $G = (V, E)$ be a graph with node set $V := \{1, \ldots, n\}$. Let π_E denote the projection from the space $\mathrm{SYM}_{n\times n}$ of the symmetric $n \times n$ matrices to the subspace $\mathbb{R}^E$ indexed by the edge set of G. We consider the following polytopes:

$$\mathrm{CUT}^{\pm 1}_{n\times n} := \mathrm{Conv}(xx^T \mid x \in \{0,1\}^n),$$

$$\begin{array}{lll} \mathrm{MET}^{\pm 1}_{n\times n} := \{X \in \mathrm{SYM}_{n\times n} \mid & X_{ii} = 1 & \text{for } i = 1, \ldots, n, \\ & X_{ij} - X_{ik} - X_{jk} \geq -1 & \text{for } 1 \leq i, j, k \leq n, \\ & X_{ij} + X_{ik} + X_{jk} \geq -1 & \text{for } 1 \leq i, j, k \leq n\}, \end{array}$$

which are called, respectively, the **cut polytope** and the **metric polytope**. The vertices of the cut polytope $\mathrm{CUT}^{\pm 1}_{n\times n}$ are the matrices xx^T for $x \in \{-1,1\}^n$, which are called **cut matrices**. We also consider the projections of $\mathrm{CUT}^{\pm 1}_{n\times n}$ and $\mathrm{MET}^{\pm 1}_{n\times n}$ on $\mathbb{R}^E$:

$$\mathrm{CUT}^{\pm 1}(G) := \pi_E(\mathrm{CUT}^{\pm 1}_{n\times n}), \;\; \mathrm{MET}^{\pm 1}(G) := \pi_E(\mathrm{MET}^{\pm 1}_{n\times n}).$$

In fact, using a result of [Bar93], one can give an explicit description of the polytope $\mathrm{MET}^{\pm 1}(G)$ by linear inequalities. Namely, $\mathrm{MET}^{\pm 1}(G)$ consists of the vectors $x \in \mathbb{R}^E$ satisfying the inequalities:

$$
(2.1) \qquad \begin{cases} -1 \le x_e \le 1 & \text{for } e \in E, \\ x(F) - x(C \setminus F) \ge 2 - |C| & \text{for } F \subseteq C,\ C \text{ cycle of } G,\ |F| \text{ odd.} \end{cases}
$$

It is easy to check that

$$\mathrm{CUT}^{\pm 1}(G) \subseteq \mathrm{MET}^{\pm 1}(G).$$

Hence, the metric polytope $\mathrm{MET}^{\pm 1}(G)$ is a linear relaxation of the cut polytope $\mathrm{CUT}^{\pm 1}(G)$. Moreover, equality: $\mathrm{CUT}^{\pm 1}(G) = \mathrm{MET}^{\pm 1}(G)$ holds if and only if the graph G has no K_5-minor [BM86].

Every matrix xx^T obviously belongs to the elliptope $\mathcal{E}_{n \times n}$ for each vector $x \in \{-1,1\}^n$. Therefore,

$$\mathrm{CUT}^{\pm 1}(G) \subseteq \mathcal{E}(G).$$

In other words, the elliptope $\mathcal{E}(G)$ is also a (in general, nonpolyhedral) relaxation of the cut polytope $\mathrm{CUT}^{\pm 1}(G)$. This fact (combined with the additional property that one can optimize a linear function over the elliptope in polynomial time) was the essential motivation for considering the elliptope in the papers [PR92, LP93, GW94]. We will characterize in Section 5 the graphs G for which equality: $\mathrm{CUT}^{\pm 1}(G) = \mathcal{E}(G)$ holds.

We also need in the paper the analogues of the polytopes $\mathrm{CUT}^{\pm 1}(G)$ and $\mathrm{MET}^{\pm 1}(G)$ in the 0-1 variables. For this, let $f : \mathbb{R}^E \longrightarrow \mathbb{R}^E$ denote the linear mapping defined by $f(x) = y$, where

$$y_e = \frac{1 - x_e}{2} \text{ for } e \in E.$$

Hence, f maps $(1,-1)$-vectors to $(0,1)$-vectors. Set

$$\mathrm{CUT}^{01}(G) := f(\mathrm{CUT}^{\pm 1}(G)),\ \mathrm{MET}^{01}(G) := f(\mathrm{MET}^{\pm 1}(G)).$$

Therefore, $\mathrm{MET}^{01}(G)$ consists of the vectors $y \in \mathbb{R}^E$ satisfying the inequalities:

$$
(2.2) \qquad \begin{cases} 0 \le y_e \le 1 & \text{for } e \in E, \\ y(F) - y(C \setminus F) \le |F| - 1 & \text{for } F \subseteq C,\ C \text{ cycle of } G,\ |F| \text{ odd.} \end{cases}
$$

3 The elliptope for chordal graphs

Let X be a partial real symmetric matrix with ones on the diagonal. An obvious necessary condition for X to be completable to a positive semidefinite matrix is that every principal minor of X composed of specified entries be nonnegative. In fact, as shown in [GJSW84], this condition is also sufficient if the graph corresponding to the specified entries of X is chordal. Theorem 3.1 below is a reformulation of this result. We recall that a **chord** of a cycle C is any edge between two nodes of C which is not an edge of C. Then, a graph $G = (V, E)$ is said to be **chordal** if every cycle of G of length ≥ 4 has a chord.

Theorem 3.1 [GJSW84] *Let $G = (V, E)$ be a chordal graph and let $x \in \mathbb{R}^E$. The following assertions are equivalent.*
(i) x belongs to $\mathcal{E}(G)$.
(ii) For each clique $K = (V(K), E(K))$ of G, the projection $x_K := (x_e)_{e \in E(K)}$ of x on $\mathbb{R}^{E(K)}$ belongs to $\mathcal{E}(K)$.

The proof of Theorem 3.1 is by induction on the number of edges; it is based on the following three lemmas.

Lemma 3.2 *Let G be a chordal graph and let u, v be two nonadjacent nodes of G. Then, the graph $G + (u, v)$ has a unique maximal clique containing both nodes u and v.*

Lemma 3.3 *Let $G = (V, E)$ be a chordal graph. Then, there exists a sequence of chordal graphs $G_i = (V, E_i)$ $(0 \leq i \leq s)$ such that $G_0 = G$, G_s is the complete graph, and G_i is obtained by adding one edge to G_{i-1} for $i = 1, \ldots, s$.*

Lemma 3.4 *Theorem 3.1 holds for the complete graph with one deleted edge.*

Lemma 3.2 is given in [GJSW84], Lemma 3.3 follows from [LRT76], while Lemma 3.4 follows from [DG81]. Dym and Gohberg show, in fact, that Theorem 3.1 holds for all band graphs; a graph with node set $V = \{1, \ldots, n\}$ is a **band** graph if, up to permutation of the nodes, its edges are the pairs (i, j) with $1 \leq i < j \leq \min(i + p, n)$, for some $1 \leq p \leq n$. Dym and Gohberg's result has a quite technically involved proof. For this reason, we give here a short and easy proof for Lemma 3.4. It uses the following well known result.

Lemma 3.5 *Let $M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ be a symmetric matrix where A is nonsingular. Then,* $\det(M) = \det(A)\det(C - B^T A^{-1} B)$.

Proof. The proof follows from the identity:

$$M = \begin{pmatrix} I & 0 \\ B^T A^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & C - B^T A^{-1} B \end{pmatrix} \begin{pmatrix} I & A^{-1} B \\ 0 & I \end{pmatrix}.$$

∎

Proof of Lemma 3.4. Let $G = (V, E)$ denote the graph on $V = \{1, \ldots, n\}$ whose edges are all pairs of nodes except the pair $(n-1, n)$. Hence, G has two maximal cliques K_1 and K_2 with respective node sets $\{1, \ldots, n-2, n-1\}$ and $\{1, \ldots, n-2, n\}$. Let $x \in \mathbb{R}^E$ such that its projection x_{K_i} belongs to $\mathcal{E}(K_i)$ for

$i = 1, 2$. We show that $x \in \mathcal{E}(G)$. Let X denote the partial symmetric matrix corresponding to x; hence, X is of the form

$$X = \left(\begin{array}{c|c|c} A & a & b \\ \hline a^T & 1 & z \\ \hline b^T & z & 1 \end{array} \right)$$

where A is a symmetric $(n-2) \times (n-2)$ matrix, $a, b \in \mathbb{R}^{n-2}$ and z is a free entry of X to be determined. Then, for $i = 1, 2$, the matrix X_i corresponding to x_{K_i} is given by

$$X_1 := \left(\begin{array}{c|c} A & a \\ \hline a^T & 1 \end{array} \right), \ X_2 := \left(\begin{array}{c|c} A & b \\ \hline b^T & 1 \end{array} \right).$$

By assumption, $X_1 \succeq 0$ and $X_2 \succeq 0$; we have to show the existence of a scalar z for which $X \succeq 0$. As shown in [GJSW84], it suffices to show this statement under the stronger assumption that both X_1 and X_2 are positive definite. So, we supose that $X_1 \succ 0$ and $X_2 \succ 0$; we show the existence of a scalar z such that $X \succ 0$. For this, it is enough to find z such that $\det(X) > 0$. By assumption, $\det(A) > 0$ and, from Lemma 3.5, we obtain:

$$\det(X_1) = \det(A)(1 - a^T A^{-1} a) > 0, \ \det(X_2) = \det(A)(1 - b^T A^{-1} b) > 0,$$

$$\begin{aligned} \det(X) &= \det(A) \cdot \det(\begin{pmatrix} 1 & z \\ z & 1 \end{pmatrix} - (a\ b)^T A^{-1} (a\ b)) \\ &= \det(A) \det \begin{pmatrix} 1 - a^T A^{-1} a & z - a^T A^{-1} b \\ z - b^T A^{-1} a & 1 - b^T A^{-1} b \end{pmatrix} \\ &= \det(A) \left((1 - a^T A^{-1} a)(1 - b^T A^{-1} b) - (z - a^T A^{-1} b)^2 \right). \end{aligned}$$

Hence, for $z := a^T A^{-1} b$, $\det(X) > 0$ holds. ∎

As observed in [GJSW84], Theorem 3.1 does not hold for nonchordal graphs. For instance, consider the cycle $C_n = (1, 2, \ldots, n)$ of length $n \geq 4$. Let $x \in \mathbb{R}^{E(C_n)}$ be defined by $x_{12} = x_{23} = \ldots = x_{n-1,n} = 1$ and $x_{1n} = -1$. Then, x does not belong to the elliptope $\mathcal{E}(C_n)$, as there exists no matrix of $\mathcal{E}_{n \times n}$ having ones on the positions $(1, 2), (2, 3), \ldots, (n-1, n)$ and value -1 at position $(1, n)$. On the other hand, the projection of x on each edge of C_n belongs trivially to $\mathcal{E}(K_2)$. Hence, stronger conditions are necessary for characterizing membership in the elliptope of a cycle. Such conditions are discussed in the next section.

4 The elliptope for series-parallel graphs

For the characterization of the elliptope for series-parallel graphs, we use as an essential tool the following parametrization for members of the elliptope; it was introduced in [BJT93]. Let $x \in \mathcal{E}(G)$. As each component x_e of x must satisfy $-1 \leq x_e \leq 1$, we can parametrize it as

$$x_e = \cos(\pi a_e)$$

for some scalar a_e, $0 \leq a_e \leq 1$. For short, we write

$$x = \cos(\pi a), \tag{4.1}$$

which means that the relation holds componentwise.

The elliptope of a cycle has been characterized in [BJT93], using the parametrization from (4.1). An equivalent result is given in [Fie66], but the formulation of [BJT93] turns out to be more convenient for our purpose of finding a generalization to a larger class of graphs. The result of [BJT93] basically says that the elliptope of a cycle C is the image of the metric polytope $\mathrm{MET}^{01}(C)$ (scaled by the factor π) of C under the cosine mapping.

THEOREM 4.2 [BJT93] *Let $C = (V, E)$ be a cycle. Then,*

$$\mathcal{E}(C) = \{\cos(\pi a) \mid a \in \mathrm{MET}^{01}(C)\}.$$

∎

An immediate consequence of Theorem 4.2 is:

PROPOSITION 4.3 *Let G be a graph. Then, we have the inclusion:*

$$\mathcal{E}(G) \subseteq \{\cos(\pi a) \mid a \in \mathrm{MET}^{01}(G)\}.$$

∎

In fact, the following stronger result can be derived from [GW94]. We give the proof in Section 6 as it is very simple and beautiful.

THEOREM 4.4 *Let G be a graph. We have the inclusion:*

$$\mathcal{E}(G) \subseteq \{\cos(\pi a) \mid a \in \mathrm{CUT}^{01}(G)\}.$$

∎

Therefore, we have the following chain of inclusions:

$$\mathrm{CUT}^{\pm 1}(G) \subseteq \mathcal{E}(G) \subseteq \{\cos(\pi a) \mid a \in \mathrm{CUT}^{01}(G)\} \subseteq \{\cos(\pi a) \mid a \in \mathrm{MET}^{01}(G)\}.$$

We shall see in Section 5 that equality holds in the left most inclusion for acyclic graphs. Equality is known to hold in the right most inclusion for graphs with no K_5-minor. Let $\mathcal{G}_{met}$ denote the class of graphs G for which

$$\mathcal{E}(G) = \{\cos(\pi a) \mid a \in \mathrm{MET}^{01}(G)\}.$$

Similarly, let $\mathcal{G}_{cut}$ denote the class of graphs for which

$$\mathcal{E}(G) = \{\cos(\pi a) \mid a \in \mathrm{CUT}^{01}(G)\}.$$

Clearly,

$$\mathcal{G}_{met} \subseteq \mathcal{G}_{cut}.$$

In fact, we show below that both classes coincide, with the class of graphs with no K_4-minor.

By Theorem 4.2, we already know that cycles belong to the class $\mathcal{G}_{met}$. Note that K_4 does not belong to $\mathcal{G}_{cut}$. For this, consider the vector $x \in \mathbb{R}^{E(K_4)}$ defined by $x = \cos(\pi a) = (-\frac{1}{2}, \ldots, -\frac{1}{2})$, where $a = (\frac{2}{3}, \ldots, \frac{2}{3})$. Hence, $a \in \mathrm{MET}^{01}(K_4) = \mathrm{CUT}^{01}(K_4)$. But x does not belong to $\mathcal{E}(K_4)$ as the matrix

$$X := \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$$

is not positive semidefinite. (Indeed, $Xe = -\frac{1}{2}e$, where $e = (1,1,1,1)^T$.)

Before proceeding further with the description of the classes $\mathcal{G}_{met}$ and $\mathcal{G}_{cut}$, we recall some definitions. A graph H is said to be a **minor** of a graph G if H can be obtained from G by repeatedly deleting and/or contracting edges. **Deleting** an edge e in G means simply discarding it from the edge set of G. **Contracting** an edge $e = uv$ means identifying both endnodes of e and discarding multiple edges if some are created during the identification of nodes u and v.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that the set $K := V_1 \cap V_2$ induces a clique (possibly empty) in both G_1 and G_2 and there is no edge between a node of $V_1 \setminus K$ and a node of $V_2 \setminus K$. Then, the graph $G := (V_1 \cup V_2, E_1 \cup E_2)$ is called the **clique sum** of G_1 and G_2. We also say that G is their **k-clique sum** if $k = |K|$.

We will use the following well known characterization for graphs with no K_4-minor (it can be derived from [Duf65]). Let G be a graph. Then, G has no

K_4-minor if and only if $G = K_3$, or G is a subgraph of a k-clique sum ($k = 0, 1, 2$) of two smaller (i.e., with less nodes than G) graphs each having no K_4-minor. Such graphs are also known as the (simple) **series-parallel** graphs. (We stress "simple" as series parallel graphs are, in general, allowed to have loops or multiple edges. But, here, we consider only simple graphs.)

We show now that the classes $\mathcal{G}_{met}$ and $\mathcal{G}_{cut}$ are composed precisely of the graphs with no K_4-minor. In view of the above result, the key steps consist of showing that $\mathcal{G}_{met}$ and $\mathcal{G}_{cut}$ are closed under taking minors and clique sums.

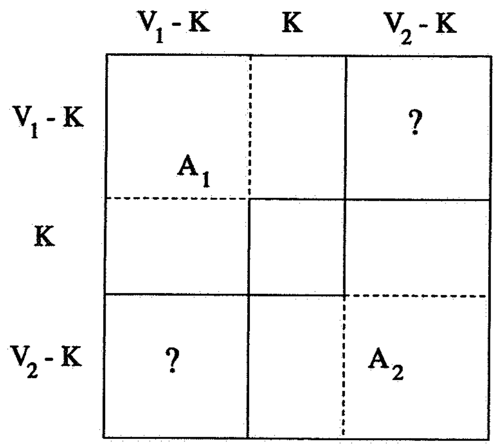
PROPOSITION 4.5 *Each of the classes $\mathcal{G}_{met}$ and $\mathcal{G}_{cut}$ is closed under taking minors.*

PROOF. Let G be a graph in $\mathcal{G}_{met}$. Let $e = uv$ be an edge of G and let G' denote the graph obtained from G by deleting or contracting the edge e. We show that $G' \in \mathcal{G}_{met}$. For this, let $a \in \mathrm{MET}^{01}(G')$. We show that $\cos(\pi a) \in \mathcal{E}(G')$. As $a \in \mathrm{MET}^{01}(G')$, it is not difficult to construct $b \in \mathrm{MET}^{01}(G)$ whose projection on the edge set of G' is a (see [LP92]). Then, $\cos(\pi b) \in \mathcal{E}(G)$ as $G \in \mathcal{G}$. This implies that $\cos(\pi a) \in \mathcal{E}(G')$.

Suppose now that $G \in \mathcal{G}_{cut}$. We show that the graph G' obtained from G by deleting or contracting the edge e belongs to $\mathcal{G}_{cut}$. Let $a \in \mathrm{CUT}^{01}(G')$. Again, it is easy to construct $b \in \mathrm{CUT}^{01}(G)$ whose projection on the edge set of G' is a. Hence, $\cos(\pi b) \in \mathcal{E}(G)$, which implies that $\cos(\pi a) \in \mathcal{E}(G')$. This shows that $G' \in \mathcal{G}_{cut}$. ∎

PROPOSITION 4.6 *The class $\mathcal{G}_{met}$ is closed under taking clique sums.*

PROOF. Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs in $\mathcal{G}_{met}$ such that $K := V_1 \cap V_2$ induces a clique in both G_1 and G_2 and there are no edges between a node from $V_1 \setminus K$ and a node from $V_2 \setminus K$. Let $G = (V := V_1 \cup V_2, E := E_1 \cup E_2)$ denote their clique sum. We show that $G \in \mathcal{G}_{met}$. For this, let $a \in \mathrm{MET}^{01}(G)$; we show that $\cos(\pi a) \in \mathcal{E}(G)$. Let a_i denote the projection of a on $\mathbb{R}^{E_i}$ for $i = 1, 2$. So, $a_i \in \mathrm{MET}^{01}(G_i)$, which implies that $\cos(\pi a_i) \in \mathcal{E}(G_i)$. Hence, there exists a matrix $A_i \in \mathcal{L}_{n_i}$ ($n_i := |V_i|$) whose entries indexed by the edges $e \in E_i$ are precisely $\cos(\pi a_e)$. Consider the following partial $n \times n$ ($n = |V|$) matrix M, where the entries (u, v), for $u \in V_1 \setminus K, v \in V_2 \setminus K$, remain to be specified.



Let H denote the graph on V whose edges are all pairs contained either in V_1 or in V_2. So the entries of M are determined on all edges of the graph H. As H is a chordal graph, we deduce from Theorem 3.1 that M can be completed to a matrix of $\mathcal{E}_{n \times n}$. In other words, values can be found for the unspecified entries of M that make M positive semidefinite. This shows that $\cos(\pi\alpha)$ belongs to $\mathcal{E}(G)$. ∎

Theorem 4.7 *Let G be a graph. The following assertions are equivalent.*
(i) $G \in \mathcal{G}_{met}$.
(ii) $G \in \mathcal{G}_{cut}$.
(iii) G has no K_4-minor.

Proof. Clearly, $(i) \Longrightarrow (ii)$. The implication $(ii) \Longrightarrow (iii)$ follows from the fact that $\mathcal{G}_{cut}$ is closed under minors and $K_4 \notin \mathcal{G}_{cut}$. We show $(iii) \Longrightarrow (i)$. Suppose G is a graph with no K_4-minor. We show that $G \in \mathcal{G}_{met}$ by induction on the number of nodes. If $G = K_3$ then $G \in \mathcal{G}_{met}$ by Theorem 4.2. Otherwise, G can be obtained as a subgraph of a clique sum of two smaller graphs G_1 and G_2 having no K_4-minor. By the induction assumption, G_1 and G_2 belong to $\mathcal{G}_{met}$. Therefore, $G \in \mathcal{G}_{met}$ by Proposition 4.6. ∎

As an example of application, we have the following result.

Corollary 4.8 *Suppose $G = (V, E)$ has no K_4-minor. Let $x \in \mathbb{R}^E$ such that $x_e = \cos(\pi\alpha)$ for all $e \in E$, for some scalar α.*
(i) If G is bipartite, then $x \in \mathcal{E}(G)$ if and only if $0 \leq \alpha \leq 1$.
(ii) If G is not bipartite and if k denotes the smallest length of an odd cycle in G, then $x \in \mathcal{E}(G)$ if and only if $0 \leq \alpha \leq \frac{k-1}{k}$.

Proof. By Theorem 4.7, $x \in \mathcal{E}(G)$ if and only if α satisfies (2.2), i.e., $0 \le \alpha \le 1$ and $\alpha \le \min(\frac{|F|-1}{2|F|-|C|} \mid F \subseteq C, C \text{ cycle}, |F| \text{ odd}, 2|F| - |C| > 0)$. The result follows. ∎

5 The elliptope for acyclic graphs

As mentioned in Section 2, the elliptope $\mathcal{E}(G)$ is a (in general, nonpolyhedral) relaxation of the cut polytope $\mathrm{CUT}^{\pm 1}(G)$, i.e.,

$$\mathrm{CUT}^{\pm 1}(G) \subseteq \mathcal{E}(G).$$

This inclusion is strict, for instance, for $G = K_3$; indeed, the vector $x := (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ belongs to $\mathcal{E}(K_3) \setminus \mathrm{CUT}^{\pm 1}(K_3)$. We show that equality holds in the above inclusion precisely for the acyclic graphs. A graph is **acyclic** if it contains no cycle, i.e., it is a forest or, equivalently, it has no K_3-minor.

Theorem 5.1 *Let $G = (V, E)$ be a graph. Then, $\mathcal{E}(G) = \mathrm{CUT}^{\pm 1}(G)$ if and only if G is acyclic, i.e., G is a forest.*

Proof. If G is acyclic, then G has no K_4-minor and, thus, by Theorem 4.7, $\mathcal{E}(G) = \{\cos(\pi a) \mid a \in \mathrm{MET}^{01}(G)\}$. But, $\mathrm{MET}^{01}(G) = [0,1]^E = \mathrm{CUT}^{01}(G)$. Hence, $\mathcal{E}(G) = [-1,1]^E = \mathrm{CUT}^{\pm 1}(G)$. Conversely, suppose that $\mathcal{E}(G) = \mathrm{CUT}^{\pm 1}(G)$. We show that G is acyclic. For this, it suffices to show that the property: $\mathcal{E}(G) = \mathrm{CUT}^{\pm 1}(G)$ is closed under taking minors, as this will indeed imply that G has no K_3-minor. So, let G be a graph such that $\mathcal{E}(G) = \mathrm{CUT}^{\pm 1}(G)$ and let e be an edge of G. Let us first consider the graph G' obtained from G by deleting the edge e; we show that $\mathcal{E}(G') \subseteq \mathrm{CUT}^{\pm 1}(G')$. For $x \in \mathcal{E}(G')$ there exists a matrix $A \in \mathcal{E}_{n \times n}$ ($n = |V|$) whose ij-th entries are x_{ij} for $ij \in E \setminus \{e\}$. Let $y \in \mathbb{R}^E$ whose ij-th coordinate is a_{ij} for $ij \in E$. Hence, $y \in \mathcal{E}(G) = \mathrm{CUT}^{\pm 1}(G)$. This implies that its projection x on $\mathbb{R}^{E \setminus \{e\}}$ belongs to $\mathrm{CUT}^{\pm 1}(G')$. Let now G' denote the graph obtained from G by contracting the edge e; we again show that $\mathcal{E}(G') \subseteq \mathrm{CUT}^{\pm 1}(G')$. Say, the endnodes of e are v_{n-1} and v_n and the node set of G' is $V \setminus \{v_n\}$. For $x \in \mathcal{E}(G')$ there exists a matrix $A \in \mathcal{E}_{(n-1)\times(n-1)}$ whose ij-th entries are x_{ij} for $ij \in E(G')$. Let B denote the $n \times n$ matrix obtained from A by duplicating its last column and its last row and setting the $(n-1, n), (n, n-1), (n, n)$-entries equal to 1. Clearly, $B \in \mathcal{E}_{n \times n}$. Let $y \in \mathbb{R}^E$ whose ij-th coordinate is b_{ij} for $ij \in E$. Then, $y \in \mathcal{E}(G) = \mathrm{CUT}^{\pm 1}(G)$. This implies easily that $x \in \mathrm{CUT}^{\pm 1}(G')$. ∎

6 A geometrical result

Let G be a graph. By Theorem 4.4, we know that

$$\{\frac{1}{\pi}\arccos(x) \mid x \in \mathcal{E}(G)\} \subseteq \mathrm{CUT}^{01}(G).$$

In fact, the polytope $\mathrm{CUT}^{01}(G)$ is the smallest convex set containing the set $\{\frac{1}{\pi}\arccos(x) \mid x \in \mathcal{E}(G)\}$. In other words,

$$\mathrm{CUT}^{01}(G) = \mathrm{Conv}(\{\frac{1}{\pi}\arccos(x) \mid x \in \mathcal{E}(G)\}).$$

(Here, "Conv" denotes the operation of taking the convex hull.) This follows from the fact that the mapping $a \mapsto \cos(\pi a)$ maps every vertex of $\mathrm{CUT}^{01}(G)$ to an element of $\mathcal{E}(G)$. In particular, by Theorem 4.7, the set $\{\frac{1}{\pi}\arccos(x) \mid x \in \mathcal{E}(G)\}$ is convex if and only if the graph G has no K_4-minor.

For any graph G, we have the following situation: The elliptope $\mathcal{E}(G)$ contains the cut polytope $\mathrm{CUT}^{\pm 1}(G)$ (in the ± 1-variable) and is contained in the image of the cut polytope $\mathrm{CUT}^{01}(G)$ (in the 01-variable) - scaled by the factor π - under the cosine mapping. Recall that $\mathrm{CUT}^{01}(G)$ is the image of $\mathrm{CUT}^{\pm 1}(G)$ under the mapping $x \mapsto \frac{1-x}{2}$. This permits to derive that

$$\{\cos(\pi a) \mid a \in \mathrm{CUT}^{01}(G)\} = \{\sin(\frac{\pi}{2}b) \mid b \in \mathrm{CUT}^{\pm 1}(G)\}.$$

Therefore, we have the inclusions:

$$\mathrm{CUT}^{\pm 1}(G) \subseteq \mathcal{E}(G) \subseteq \{\sin(\frac{\pi}{2}b) \mid b \in \mathrm{CUT}^{\pm 1}(G)\},$$

with equality in the right most inclusion if and only if G has no K_4-minor. As an illustration, compare the polytope $\mathrm{CUT}^{\pm 1}(K_3)$ (which is a 3-dimensional simplex) and the elliptope $\mathcal{E}(K_3)$ (whose picture can be found in [LP93]).

We now state a result of geometrical flavour, which shows how to derive valid relations for the pairwise angles between any set of unit vectors.

THEOREM 6.1 [GW94] *Let $v_1, \ldots, v_n$ be unit vectors in $\mathbb{R}^n$. Let $a \in \mathbb{R}^{E(K_n)}$, $a_0 \in \mathbb{R}$ such that the inequality $a^T x \leq a_0$ is valid for the cut polytope $\mathrm{CUT}^{01}(K_n)$ (i.e., $a^T x \leq a_0$ holds for all $x \in \mathrm{CUT}^{01}(K_n)$). Then,*

$$\sum_{1 \leq i < j \leq n} a_{ij} \frac{\arccos(v_i^T v_j)}{\pi} \leq a_0.$$

PROOF. The proof is based on the following randomized procedure, described in [GW94]:
- Select a random unit vector $r \in \mathbb{R}^n$.
- Set $S_r := \{i \in \{1, \ldots, n\} \mid v_i^T r \geq 0\}$.
Then, the expected weight $E(S_r)$ (with respect to the weights a_{ij}) of the cut in K_n determined by S_r is equal to

$$E(S_r) = \sum_{1 \leq i < j \leq n} a_{ij} \frac{\arccos(v_i^T v_j)}{\pi}.$$

Indeed, the probability that the edge ij belongs to the cut determined by S_r is equal to the probability that a random hyperplane separates the two vectors v_i and v_j, which is equal to $\frac{\arccos(v_i^T v_j)}{\pi}$. But, $E(S_r)$ is less than or equal to the maximum weight of a cut, which is less than or equal to a_0 by assumption. This shows that $\sum_{1 \leq i < j \leq n} a_{ij} \frac{\arccos(v_i^T v_j)}{\pi} \leq a_0$. ∎

Theorem 4.4 can now be derived in the following way. Let $x \in \mathcal{E}(G)$. We show that $\frac{1}{\pi} \arccos(x) \in \mathrm{CUT}^{01}(G)$. Let $X \in \mathcal{E}_{n \times n}$ whose projection on $\mathbb{R}^E$ is x. As $X \succeq 0$ with diagonal entries 1, it is the Gram matrix of a set of unit vectors $v_1, \ldots, v_n$, i.e., $X_{ij} = v_i^T v_j$ for all $i, j = 1, \ldots, n$. By Theorem 6.1, the vector $(\frac{1}{\pi} \arccos(v_i^T v_j))_{1 \leq i < j \leq n}$ belongs to the cut polytope $\mathrm{CUT}^{01}(K_n)$. Therefore, its projection $(\frac{1}{\pi} \arccos(v_i^T v_j))_{ij \in E(G)}$ on the edge set of G belongs to the polytope $\mathrm{CUT}^{01}(G)$. This shows that $\frac{1}{\pi} \arccos(x) \in \mathrm{CUT}^{01}(G)$.

Theorem 6.1 contains as a special case the well known relations:

$$\sum_{1 \leq i < j \leq 3} \arccos(v_i^T v_j) \leq 2\pi,$$

$$\arccos(v_1^T v_2) \leq \arccos(v_1^T v_3) + \arccos(v_2^T v_3)$$

which hold for any three unit vectors v_1, v_2, v_3 in the 3-dimensional space (see [Ber87],Corollary 18.6.12.3). They follow from the valid inequalities:

$$\sum_{1 \leq i < j \leq 3} x_{ij} \leq 2, \; x_{12} \leq x_{13} + x_{23}$$

for the polytope $\mathrm{CUT}^{01}(K_3)$. But Theorem 6.1 gives a whole wealth of other inequalities. Indeed, *every* valid inequality for the cut polytope $\mathrm{CUT}^{01}(K_n)$ yields some inequality for the pairwise angles among any set of n vectors.

For instance, the inequality

$$\sum_{1 \leq i < j \leq 2k+1} x_{ij} \leq k(k+1)$$

is valid for $\mathrm{CUT}^{01}(K_{2k+1})$ $(k \geq 1)$. This implies that

$$\sum_{1\leq i<j\leq 2k+1} \arccos(v_i^T v_j) \leq k(k+1)\pi$$

holds for any $2k+1$ unit vectors $v_1, \ldots, v_{2k+1}$. Similarly, the inequality

$$\sum_{1\leq i<j\leq 2k} \arccos(v_i^T v_j) \leq k^2\pi$$

holds for any $2k$ unit vectors. As another example, let $b_1, \ldots, b_n$ be integers whose sum $\sigma := \sum_{1\leq i\leq n} b_i$ is odd. Then, the inequality

$$\sum_{1\leq i<j\leq n} b_i b_j x_{ij} \leq \frac{\sigma^2 - 1}{4}$$

is valid for $\mathrm{CUT}^{01}(K_n)$. Therefore,

$$\sum_{1\leq i<j\leq n} b_i b_j \arccos(v_i^T v_j) \leq \pi\frac{\sigma^2 - 1}{4}$$

holds for any n unit vectors.

Many other inequalities valid for the cut polytope are known; see, e.g., [DL92a, DL92b]. Most of them have, in fact, a quite complicated form. As a last example, let us mention the following relation (which follows from a valid inequality given in [Gri90]) which holds for any seven unit vectors $v_1, \ldots, v_7$:

$$\begin{gathered}\textstyle\sum_{1\leq i<j\leq 4} \arccos(v_i^T v_j) - 2\sum_{1\leq i\leq 4} \arccos(v_5^T v_i) \\ - \arccos(v_1^T v_6) - \arccos(v_3^T v_6) - \arccos(v_2^T v_7) - \arccos(v_4^T v_7) \\ + \arccos(v_5^T v_6) + \arccos(v_5^T v_7) - \arccos(v_6^T v_7) \leq 0.\end{gathered}$$

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