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Abstract

A generalized ultrametric space is an ordinary ultrametric space in which the distance need not be symmetric, and where different elements may have distance 0. Our interest in generalized ultrametric spaces is primarily motivated by the following observations: 1. (possibly nondeterministic) transition systems can be naturally endowed with a generalized ultrametric that captures their operational behavior in terms of simulations; 2. the category of generalized ultrametric spaces contains both the categories of preorders and of ordinary ultrametric spaces as full subcategories. A theory of generalized ultrametric spaces is developed along the lines of the work by Smyth and Plotkin (1982) and America and Rutten (1989), such that its restriction to preorders and ordinary ultrametric spaces yields (more and less) familiar facts. Our work has in common with other recent work along the same lines—by Flagg and Kopperman, and Wagner—that it is directly based on Lawvere's $\mathcal{V}$-categorical interpretation of metric spaces, and uses results on quasimetrics by Smyth. It is different in being far less general, and consequently a number of new results, specific for generalized ultrametric spaces, is obtained. In particular, domain equations are solved by means of metric adjoint pairs, and the notions of (generalized) totally-boundedness and bifinite (or 'SFU') domain are introduced and characterized.

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1. Introduction

A *generalized ultrametric* space (gum) is a set X supplied with a distance function $X(-,-): X \times X \to [0,1]$, again denoted by X, satisfying for all x, y, z: $X(x,x) = 0$ and $X(x,z) \leq \max\{X(x,y), X(y,z)\}$. This notion generalizes *ordinary* ultrametric spaces in that the distance need not be symmetric, and different elements may have distance 0. Our interest in generalized ultrametric spaces is primarily motivated by the following observation: (possibly nondeterministic) transition systems can be naturally endowed with a generalized ultrametric that captures their operational behavior in terms of simulations.

A simple example is the following. The set of natural numbers $\omega = \{0, 1, \ldots\}$ is usually described as an (initial) algebra. At the same time, it can be viewed as a transition system by defining a transition relation on ω by $n+1 \to n$, for all n in ω. This induces a generalized ultrametric: for all x and y in ω,

$$\omega(x,y) = \inf\{2^{-k} \mid \text{ if } x \text{ can take } k \text{ subsequent transition steps then so can } y\}.$$

This distance measures the extent to which the transitions of x can be simulated by y ('the smaller the better'). For instance, $\omega(3,8) = 0$ and $\omega(8,3) = 2^{-3}$.

A second observation, on which the present work is based is that the category of all generalized ultrametric spaces contains both the categories of preorders and ordinary ultrametric spaces as full subcategories. As a consequence, our aim has been to develop a domain theory of generalized ultrametric spaces, such that its restriction to these subcategories yields—sometimes more and sometimes less—familiar results for partial orders and ordinary ultrametric spaces. For instance, domain equations over a category of generalized ultrametric spaces will be solved using—metric—adjoint pairs, in a way that generalizes the fixed point theorems of both [SP82] and [AR89]. Another illustration of our theory is the identification of a subcategory of bifinite or 'SFU' domains (Sequences of Finite generalized Ultrametric spaces). Restricted to partial orders, it yields the standard notion of bifinite, also called SFP; for ordinary ultrametric spaces, SFU will turn out to be equivalent to compactness.

The generalized ultrametric on ω introduced above, serves again as an example: it induces both an order and an ordinary ultrametric: $x \leq y$ iff $\omega(x,y) = 0$, and $\omega^s(x,y) = \max\{\omega(x,y), \omega(y,x)\}$, respectively. Clearly these are the standard order and ultrametric on ω, but note that they are here derived from the transition structure on ω. Still more can be illustrated by means of the same example: let $\bar{\omega} = \omega \cup \{\omega\}$ and extend the transition relation with $\omega \to \omega$. The generalized ultrametric induced by the new transition relation on $\bar{\omega}$ is now *complete*: the Cauchy sequence $(0, 1, \ldots)$ has limit ω. All these notions will be defined in such a way that their restrictions look again familiar: the sequence $(0, 1, \ldots)$ is a chain in $\langle \bar{\omega}, \leq \rangle$ with least upperbound ω; and it is Cauchy in the ordinary complete ultrametric space $\bar{\omega}^s$ with limit ω. Furthermore, $\bar{\omega}$ could have been obtained as a solution of the domain equation $X \cong (X)_\perp$, where $(X)_\perp$ is a 'metric lifting' of X (involving shrinking of the metric). In solving this equation, one finds that $\bar{\omega}$ is SFU, which is reflected by the facts that $(\bar{\omega}, \leq)$ is bifinite (SFP) and that $\bar{\omega}^s$ is compact.

Thus we develop a domain theory for generalized ultrametric spaces, essentially along the lines of a combination of [SP82] and [AR89]. The present attempt is by no means the first one in that direction. Much of our work is directly motivated by Lawvere's $\mathcal{V}$-categorical view of metric spaces ([Law73] and [Ken87]). This it has in common with the work of Flagg and

Kopperman ([FK95]) on continuity spaces, and of Wagner ([Wag94]) on abstract preorders, wo aim at a reconciliation of ordered and metric domain theory as well. Furthermore it will similarly depend on some of Smyth' results on quasimetric spaces ([Smy87, Smy91]). Notably our definition of *limit* of a Cauchy sequence is taken from [Smy91].

Nevertheless, there are important differences. Unlike [FK95, Wag94], we have not aimed at generality. The category of generalized ultrametric spaces seems rather to be the smallest category (of sets with structure) that contains both the categories of preorders and ordinary ultrametrics. As a consequence, more can be said about it. What seems to be new, amongst others, is: a proof of its (categorical) completeness and cocompleteness; two fixed point theorems on generalized ultrametric spaces, generalizing the least and unique fixed point theorems of Tarski and Banach, respectively; the definition and characterizations of metric adjoint pairs; two categorical counterparts of the afore mentioned fixed point theorems, based on the use of metric adjoint pairs, and generalizing the ones of [SP82] and [AR89]; and the definition and characterizations of the subcategory SFU of bifinite spaces.

Also in [Wag94] a fixed point theorem for functors is given. Our approach is on one side more restrictive since we deal with generalized ultrametric spaces only, and not with Wagner's more general class of abstract preorders; at the same time, fixed points are in the present paper constructed by means of metric adjoint pairs, which generalize the more standard (metric) embedding-projection pairs. Moreover, two fixed point theorems are given for the families of locally continuous and locally contractive functors.

Then totally-boundedness plays an important role in [Smy91] and [FK95]. Its definition is based on 'symmetrized' metrics (cf. $\bar{\omega}^s$ above). In contrast, our results do not require any assumption on totally-boundedness. Nevertheless, a (different) notion of totally boundedness is introduced here as well, because it turns out to be a characterizing property of SFU domains. Its definition is given in terms of the original, non-symmetric distance and moreover involves the—generalized—notion of finite element. It has the advantage that restricted to ordinary (ultra)metric spaces, it coincides with the standard notion; and for ω-algebraic complete partial orders it is equivalent to Scott compactness.

2. Generalized (ultra)metric spaces

A *generalized metric space* is a set X together with a mapping

$$X(-,-) : X \times X \to [0,1]$$

which satisfies, for all x, y, and z in X:

1. $X(x,x) = 0$,
2. $X(x,z) \leq X(x,y) + X(y,z)$.

The real number $X(x,y)$ will be called the distance from x to y. Since a generalized metric space generally does not satisfy

3. if $X(x,y) = 0$ and $X(y,x) = 0$ then $x = y$,
4. $X(x,y) = X(y,x)$,

which are the additional conditions that hold for an *ordinary* metric space, it is also called a *pseudo-quasi* metric space (cf. [Smy92]). A *generalized ultrametric space* (gum for short), our prime concern in the present paper, is a generalized metric space X that satisfies the so-called strong triangle-inequality:

2'. $X(x,z) \leq \max\{X(x,y), X(y,z)\}$.

The distance between two elements x and y of a generalized (ultra)metric space X is denoted by $X(x,y)$ (rather than by, e.g., $d_X(x,y)$), because we would like to think of the real number $X(x,y)$ as an *abstract hom set*, similar to the hom set $C(a,b)$ (which really is a set) of all arrows between two objects a and b in a category C. The two conditions in the definition of generalized (ultra)metric can be seen to be, in a very precise sense, the analogue of the two conditions that every category C should satisfy: the existence of an identity arrow for every object, and the law of composition of any two composable arrows. Although this analogy will not be treated explicitly here, it plays an important motivating role throughout the present paper. The reader is referred to Lawvere's beautiful account of this so-called enriched-categorical approach ([Law73]). In his terminology, generalized (ultra)metric spaces are $[0,1]$-*categories*.

EXAMPLE 2.1 Some examples of generalized ultrametric spaces are:

1. The set $\bar{\omega} = \{0,1,\ldots\} \cup \{\omega\}$ of natural numbers plus infinity, totally ordered in the usual way, with distance, for x and y in $\bar{\omega}$,

$$\bar{\omega}(x,y) = \begin{cases} 0 & \text{if } x \leq y \\ 1 & \text{if } x > y. \end{cases}$$

2. Again $\bar{\omega} = \{0,1,\ldots\} \cup \{\omega\}$, but now with distance, for x and y in $\bar{\omega}$,

$$\bar{\omega}(x,y) = \begin{cases} 0 & \text{if } x \leq y \\ 2^{-y} & \text{if } x > y. \end{cases}$$

   Note that this is the example from the introduction.

3. The set $[0,1]$ with distance, for x and y in $[0,1]$,

$$[0,1](x,y) = \begin{cases} y & \text{if } x < y \\ 0 & \text{if } x \geq y. \end{cases}$$

4. Let A^∞ be the set of finite and infinite words over some given set (alphabet) A. A generalized ultrametric is defined, for v and w in A^∞, by

$$A^\infty(v,w) = \begin{cases} 0 & \text{if } v \text{ is a prefix of } w \\ 2^{-n} & \text{otherwise,} \end{cases}$$

   where n is the length of the longest common prefix of v and w. Both the familiar prefix ordering $\leq$ and the usual longest-common-prefix ultrametric d on words are encoded in the above generalized ultrametric on A^∞: They can be retrieved by putting

$$v \leq w \text{ if and only if } A^\infty(v,w) = 0$$

and

$$d(v,w) = \max\{A^\infty(v,w), A^\infty(w,v)\},$$

respectively. As we shall see in Section 5, any gum induces in this way both a (pre)order and an ultrametric.

In fact, the examples above are instances of *quasi* ultrametric spaces, which are generalized ultrametric spaces X that satisfy moreover axiom 3 above. Nevertheless, we shall not restrict our attention to quasi ultrametric spaces only, because of the examples in Section 3.

Although generalized *ultra*metrics are our main concern, most of the results of this paper hold for the larger family of generalized *metric* spaces as well. Proofs will be given for generalized ultrametrics only, and it will be indicated when a certain theorem is *not* valid for the larger family of generalized metric spaces. (Where it is possible, it is usually also rather straightforward to adapt the proof to the metric case; replacing 'max' by '+' and being a little more economic in the choices of the epsilons involved, will work wonders.)

We summarize the main definitions of this section. For future reference, also the definition of *pseudo* (ultra)metric space has been included.

DEFINITION 2.2 Let X be a set and $X(-,-) : X \times X \to [0,1]$ a mapping. Consider the following axioms:

1. For all $x \in X$: $X(x,x) = 0$.
2. For all x, y, and z in X: $X(x,z) \leq X(x,y) + X(y,z)$.

2'. For all x, y, and z in X: $X(x,z) \leq \max\{X(x,y), X(y,z)\}$.

3. For all x and y in X: if $X(x,y) = 0$ and $X(y,x) = 0$ then $x = y$.
4. For all x and y in X: $X(x,y) = X(y,x)$.

X is a

generalized ultrametric space	if it satisfies 1 and 2'
quasi ultrametric space	if it satisfies 1, 2', and 3
pseudo ultrametric space	if it satisfies 1, 2', and 4
ordinary ultrametric space	if it satisfies 1, 2', 3, and 4
generalized metric space	if it satisfies 1 and 2
quasi metric space	if it satisfies 1, 2, and 3
pseudo metric space	if it satisfies 1, 2, and 4
ordinary metric space	if it satisfies 1, 2, 3, and 4

□

3. A generalized ultrametric for transition systems

The above definition of a generalized ultrametric structure on A^∞ is in fact an instance of a more general example, stemming from the world of semantics of computation. It is one of our main motivations for having an interest in generalized ultrametric spaces.

Consider a *labelled transition system* $(S, A, \longrightarrow)$, consisting of a set S of states, a set A of action labels, and a transition relation $\longrightarrow \subseteq S \times A \times S$. (As always, $s \stackrel{a}{\longrightarrow} s'$ denotes $(s, a, s') \in \longrightarrow$.) The set S can be given a generalized ultrametric structure by means of the following, inductively defined sequence of relations: let $\leq_0 = S \times S$ and given $\leq_n$, let

$$\leq_{n+1} = \{(s,t) \in S \times S \mid \forall a \in A \forall s' \in S \text{ s.t. } s \stackrel{a}{\longrightarrow} s' \ \exists t' \in S \text{ s.t. } t \stackrel{a}{\longrightarrow} t' \text{ and } (s', t') \in \leq_n\}.$$

We have that $(s,t) \in \leq_n$ if and only if (at least) any first n consecutive transition steps that can be taken starting in state s can all be *simulated* by steps from t. For s and t in S,

$$S(s,t) = \inf \{2^{-n} \mid (s,t) \in \leq_n\}$$

defines a generalized ultrametric on S, which measures the extent to which the transition steps from s can be simulated by steps from t. In general $S(s,t)$ and $S(t,s)$ are different, and $S(s,t)$ may be 0 while $s \neq t$.

It is not difficult to prove that for transition systems that are *image finite* (for any s in S and a in A, the set $\{s' \in S \mid s \stackrel{a}{\longrightarrow} s'\}$ is finite), $S(s,t) = 0$ if and only if there exists a *simulation relation* $R \subseteq S \times S$ with $(s,t) \in R$. (A relation R is a simulation if for all $(s,t) \in R$, a in A, and s' in S: if $s \stackrel{a}{\longrightarrow} s'$ then there is t' in S such that $t \stackrel{a}{\longrightarrow} t'$ and $(s', t') \in R$.)

The set A^∞ can be turned into a transition system $(A^\infty, A, \longrightarrow)$ by putting, for v and w in A^∞,

$$v \stackrel{a}{\longrightarrow} w \text{ if and only if } v = aw.$$

Applying the above definition then yields the following generalized ultrametric:

$$A^\infty(v,w) = \inf \{2^{-n} \mid \forall i,\ 1 \leq i \leq n, \ \text{ if } i \leq |v| \text{ then } i \leq |w| \text{ and } v(i) = w(i)\},$$

where $|v|$ denote the length and $v(i)$ the i-th element (for $i \geq 1$) of a word $v \in A^\infty$. It is not difficult to see that this ultrametric is the same as the one of Example 2.1(4).

4. A category of generalized ultrametric spaces

A category of generalized ultrametric spaces is introduced and a few basic properties discussed: it is complete, cocomplete and Cartesian closed. (The latter fact is proved in [Law73].)

A mapping $f : X \to Y$ between two gums X and Y is *non-expansive* if for all x and x' in X,

$$X(x,x') \geq Y(f(x), f(x')).$$

(Though this definition is completely standard, it can nevertheless be seen to be dictated by the afore mentioned analogy between categories and generalized ultrametric spaces: a non-expansive mapping is the exact counterpart of the notion of functor between categories.) Identity mappings are non-expansive and the composition of two non-expansive mappings is again non-expansive thus the class of all gums, together with all non-expansive mappings between them is a category, called *Gums*. In this category, all limits and colimits exist. Moreover, they are constructed at the level of sets. Formally:

THEOREM 4.1 *Let U : Gums $\rightarrow$ Set be the functor that maps a gum to its underlying set ('forgetting' its metric structure). The functor U creates all limits and all colimits.* □

Rather than proving this theorem, we shall treat two typical examples, from which a general proof can be easily derived. First of all, limits are easy. E.g., the product of two gums X and Y is given by the Cartesian product of their underlying sets together with distance, for x, x' in X and y, y' in Y,

$$X \times Y(\langle x, y\rangle, \langle x', y'\rangle) = \max\{X(x, x'), Y(y, y')\}.$$

The coproduct (sum) of X and Y consists of the disjoint union of their underlying sets:

$$X + Y = (\{0\} \times X) \cup (\{1\} \times Y)$$

with distance

$$X + Y(\langle i, a\rangle, \langle j, b\rangle) = \begin{cases} X(a, b) & \text{if } i = j = 0 \\ Y(a, b) & \text{if } i = j = 1 \\ 1 & \text{otherwise.} \end{cases}$$

In general, though, colimits are somewhat more complicated. In particular, consider two gums X and Y, and two non-expansive mappings $f : X \rightarrow Y$ and $g : X \rightarrow Y$. We shall discuss how the coequalizer of f and g in *Gums* looks like. First, let $q : Y \rightarrow Z$ be the coequalizer of f and g in *Set*:

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{\quad q \quad} Z;$$

thus Z is the quotient of Y with respect to the smallest equivalence relation on Y containing the set $\{(y, y') \in Y \times Y \mid \exists x \in X,\ f(x) = y \text{ and } g(x) = y'\}$, and q maps every element of Y to its equivalence class. Now Z can be supplied with a generalized ultrametric such that q is also the coequalizer of f and g in *Gums*, as follows.

For z and z' in Z, a finite sequence $\pi = \langle y_0, \ldots, y_n\rangle$ of elements in Y (with $n \geq 0$) is called a *path* from z to z', denoted by $z \xrightarrow{\pi} z'$, if $q(y_0) = z$ and $q(y_n) = z'$. The 'cost' $c(\pi)$ of such a path π is defined by

$$c(\pi) = \max\{Y(y_i, y_{i+1}) \mid 0 \leq i \leq n - 1 \text{ and } q(y_i) \neq q(y_{i+1})\}.$$

A path $z \xrightarrow{\pi} z'$ is a finite walk from the equivalence class (inverse image under q) of z to that of z', and its 'cost' is determined by those steps that go from one equivalence class to another. ('Walking within one and the same equivalence class is for free.') The distance from z to z' can now be defined by

$$Z(z, z') = \inf\{c(\pi) \mid z \xrightarrow{\pi} z'\}.$$

It is called 'least-cost distance' ([Law73]) or 'shortest-path distance' ([Smy92]). One can readily prove that it is the unique generalized ultrametric on Z such that q is the coequalizer of f and g in *Gums*.

Furthermore the category *Gums* is Cartesian closed: For two gums A and Y, the set $Y^A = \{f : A \to Y \mid f \text{ is non-expansive }\}$ together with distance

$$Y^A(f, g) = \sup\{Y(f(a), g(a)) \mid a \in A\},$$

for f and g in Y^A, is a generalized ultrametric space. The closedness of *Gums* follows from the bijection

$$\{f : (A \times X) \to Y \mid f \text{ is non-expansive }\} \cong \{f : X \to Y^A \mid f \text{ is non-expansive }\}.$$

Also the category of generalized *metric* spaces and non-expansive mappings has all limits and colimits. Though this category is *not* Cartesian closed, it *is* monoidal closed ([Law73]).

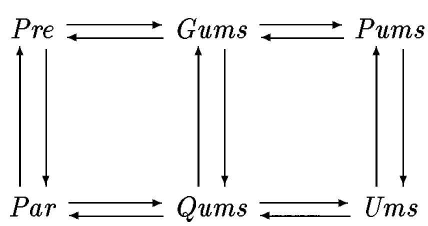
5. Distance and order

It will be shown that both the category of preorders and the category of ordinary ultrametric spaces can be embedded into the category of generalized ultrametric spaces by means of adjunctions.

A *preorder* is a pair $(P, \leq_P)$ consisting of a set P and a relation $\leq_P$ on P satisfying, for all p, q, and r in P, $p \leq_P p$, and if $p \leq_P q$ and $q \leq_P r$ then $p \leq_P r$. For preorders $(P, \leq_P)$ and $(Q, \leq_Q)$, a mapping $f : P \to Q$ is called *monotone* if for all p and p' in P: if $p \leq_P p'$ then $f(p) \leq_Q f(p')$. The collection of all preorders and monotone mappings between them is a category, denoted by *Pre*. A *partial order* is a preorder $(P, \leq_P)$ that moreover is antisymmetric: for all p and q, if $p \leq_P q$ and $q \leq_P p$ then $p = q$. Let *Par* be the category of all partial orders and monotone mappings.

Recall that *Gums* is the category of generalized ultrametric spaces and non-expansive mappings. Similarly, *Qums* is the category of quasi ultrametric spaces, *Pums* the category of pseudo ultrametric spaces, and *Ums* the category of ordinary ultrametric spaces, all with non-expansive mappings.

Theorem 5.1 *There exist the following pairs of adjoint functors:*



In this diagram, the arrows from left to right and from top to bottom are left adjoint to the arrows from right to left, and from bottom to top, respectively. Furthermore, both squares commute, in both directions.

Proof: For a preorder $(P, \leq_P)$ and p and q in P, define

$$P(p,q) = \begin{cases} 0 & \text{if } p \leq_P q \\ 1 & \text{if } p \not\leq_P q. \end{cases}$$

Because $p \leq_P p$, for all p in P, and if $p \not\leq_P r$ then either $p \not\leq_P q$ or $q \not\leq_P r$, for all p, q, and r in P, this defines a generalized ultrametric indeed. Such a generalized ultrametric for which all distances are either 0 or 1, is called *discrete*. If $f : (P, \leq_P) \to (Q, \leq_Q)$ is monotone, then it is also a non-expansive mapping $f : P \to Q$. Hence the above defines a functor $F : \mathit{Pre} \to \mathit{Gums}$, which maps $(P, \leq_P)$ to P, with distance as defined above, and is the identity on arrows. Conversely, any gum X gives rise to a preorder $(X, \leq_X)$ defined by

$$x \leq_X x' \text{ if and only if } X(x,x') = 0,$$

for x and x' in X. (Note that if X is a quasi ultrametric space then the result is a partial order.) Any non-expansive mapping between gums X and Y is also a monotone mapping from $(X, \leq_X)$ to $(Y, \leq_Y)$. This defines a functor: $G : \mathit{Gums} \to \mathit{Pre}$, mapping X to $(X, \leq_X)$ and acting as the identity on arrows. Clearly, $G \circ F = \mathit{Id}$, the identity functor on *Pre*. Moreover, for every preorder $(P, \leq_P)$ and gum X: if $f : (P, \leq_P) \to (X, \leq_X)$ is monotone (in *Pre*) then $f : P \to X$ is non-expansive (in *Gums*). This proves that F is left adjoint to G. The restriction of F to *Par* and of G to *Qums* is again an adjoint pair.

Next we shall compare the categories *Gums* and *Pums*. Let $I : \mathit{Pums} \to \mathit{Gums}$ be the inclusion functor. It has a left adjoint $H : \mathit{Gums} \to \mathit{Pums}$, which is defined as follows. For a generalized ultrametric space X, let $H(X) = X^s$, the 'symmetrized' pseudo ultrametric space consisting of the set X and with distance

$$X^s(x,y) = \max\{X(x,y), X(y,x)\},$$

for x and y in X. Further H is the identity on arrows. Because every non-expansive mapping $f : X \to Y$ between gums X and Y is at the same time a non-expansive mapping $f : X^s \to Y$, it follows that H is left adjoint to I. The restriction of H to *Qums* and of I to *Ums* is again an adjoint pair.

The correspondence between *Gums* and *Qums* is as usual: let X be a generalized ultrametric space, and let $\approx$ be an equivalence relation on X defined by

$$x \approx y \text{ if and only if } X(x,y) = 0.$$

Furthermore let

$$[x] = \{y \in X \mid x \approx y\}, \;\; X/{\approx} = \{[x] \mid x \in X\}.$$

Defining $X/{\approx}([x],[y]) = X(x,y)$ turns $X/{\approx}$ into a quasi ultrametric space. The functor $K : \mathit{Gums} \to \mathit{Qums}$ is defined by $K(X) = X/{\approx}$. If $f : X \to Y$ is a non-expansive mapping in *Gums* then $K(f) : K(X) \to K(Y)$, defined by $K(f)([x]) = [f(x)]$ is non-expansive in *Qums*. The inclusion functor from *Qums* to *Gums* is right adjoint to K. Similarly one defines adjoint pairs between *Pre* and *Par*, and between *Pums* and *Ums*. □

The theorems of this section are straightforward variations of similar theorems in [Ken87], where it is shown that the category of preorders can be embedded into the category of generalized metric spaces. The additional observation made above, that the category of preorders can be embedded into the smaller category of generalized *ultra*metric spaces, is trivial yet of crucial importance for the rest of this paper.

For the generalized ultrametric space A^∞ of Section 2, the induced ordinary ultrametric is the usual longest-common-prefix ultrametric; the induced ordering of $G(A^\infty)$ is the prefix ordering. The set S of states of a transition system $(S, A, \longrightarrow)$, with generalized ultrametric as defined in Section 3, is mapped by G to a preorder $(S, \leq_S)$. If the transition system is image finite then for all s and t in S:

$$s \leq_S t \text{ if and only if there is a simulation } R \subseteq S \times S \text{ with } (s,t) \in R,$$

showing that $\leq_S$ is the so-called *simulation preorder*. This is an immediate consequence of the definition of $\leq_S$ and the observations of Section 3.

6. Convergence, continuity and completeness

Following [Smy91], the notions of Cauchy, convergence, continuity, and completeness are defined for generalized ultrametric spaces. Next two fixed point theorems are proved, generalizing the fact that a continuous function on a complete partial order (with minimal element) has a least fixed point, and Banach's contraction theorem. Finally it is proved that the set of all (non-expansive and continuous) mappings between two complete generalized ultrametric spaces, is again complete.

A sequence $(x_n)_n$ in a gum X is *Cauchy* if

$$\forall \epsilon > 0 \; \exists N \; \forall n \geq m \geq N, \;\; X(x_m, x_n) \leq \epsilon.$$

There are at least two natural variations on this kind of 'forward' Cauchy sequences: 'backward' and 'bi' Cauchy, the definitions of which are obtained by replacing $X(x_m, x_n)$ in the formula above by $X(x_n, x_m)$ and $\max\{X(x_m, x_n), X(x_n, x_m)\}$, respectively; cf. [Smy92].

Because of ultrametricity, a sequence $(x_n)_n$ in a gum X is Cauchy if and only if

$$\forall \epsilon > 0 \; \exists N \; \forall n \geq N, \;\; X(x_n, x_{n+1}) \leq \epsilon.$$

For ordinary ultrametric spaces, the above definition of Cauchy is just the standard one. For preorders it means 'eventually chain': Consider a gum X and suppose that X is a preorder. (What we mean, formally, is that $X = F \circ G(X)$ or, equivalently, that X is discrete. This kind of abuse of language will be frequently indulged in.) A sequence in X is Cauchy if and only if

$$\exists N \forall n \geq N, \; x_n \leq_X x_{n+1}.$$

Here $\leq_X$ is the order associated with X (by the functor G).

A Cauchy sequence $(x_n)_n$ *converges to* x (in X), denoted by $\lim x_n = x$, if

$$\forall \bar{x} \in X, \;\; X(x, \bar{x}) = \lim X(x_n, \bar{x}).$$

As usual, x is called the *limit* of $(x_n)_n$. Note that this definition is circular as long as we have not explained the meaning of the limit in its right-hand side: $\lim X(x_n, \bar{x})$. But that can be readily done by defining it as the usual limit in $[0,1]$ supplied with the ordinary metric

$$[0,1](a,b) = |a-b|,$$

for a and b in $[0,1]$. Now the well-definedness of the above notion of convergence is obtained by the following lemma.

LEMMA 6.1 *If a sequence $(x_n)_n$ in a gum X is Cauchy, then for any $\bar{x}$ in X, the sequence $(X(x_n,\bar{x}))_n$ is Cauchy (in the usual sense) in the ordinary metric space $[0,1]$.*

PROOF: Let $(x_n)_n$ be Cauchy and $\bar{x} \in X$. There is N such that for all $n \geq m \geq N$, $X(x_m,x_n) \leq 1/3$. We claim:

$$\text{either } \{k \mid X(x_k,\bar{x}) \in [0,1/3]\} \text{ is finite, or } \{k \mid X(x_k,\bar{x}) \in [2/3,1]\} \text{ is finite.}$$

Suppose both are infinite. Then there are $n \geq m \geq N$ such that $X(x_m,\bar{x}) \in [2/3,1]$ and $X(x_n,\bar{x}) \in [0,1/3]$. But this is impossible, since

$$X(x_m,\bar{x}) \leq \max\{X(x_m,x_n),\ X(x_n,\bar{x})\} \leq 1/3.$$

Continuing this way, we obtain a chain of strictly shrinking segments, each of which contains all but finitely many elements of the sequence $(X(x_n,\bar{x}))_n$. Hence it is Cauchy in $[0,1]$. □

Note that limits are not unique. But different limits must be close: if $\lim x_n = x$ and $\lim x_n = y$ then $X(x,y) = 0 = X(y,x)$. This implies that in quasi ultrametric spaces, limits are unique.

The above definition of convergence is taken from [Smy91], where $\lim X(x_n,\bar{x})$ is being interpreted in the positive reals extended with infinity.

From an (enriched-)categorical point of view, it would have been cleaner to take in the definition of convergence the generalized ultrametric on $[0,1]$—as defined in Example 2.1(3)—instead of the above ordinary metric. One can prove that for a generalized ultrametric space X; a Cauchy sequence $(x_n)_n$ in X; and an element $\bar{x}$ in X, the sequence $(X(x_n,\bar{x}))_n$ is 'backward' Cauchy with respect to this generalized ultrametric on $[0,1]$. Such sequences are either eventually increasing or converging (in the standard sense) to 0. Therefore, their limit can be defined by means of 'limsup', which would coincide with the standard limit in $[0,1]$. All in all, the two definitions would be equivalent. This explains at the same time the connection with the approach of [Wag94], where the notion of 'limsup convergence' is used.

Though seemingly complicated at first, the above notion of convergence is in fact rather satisfactory, because it is both simple and has the following two properties: if X is an ordinary metric space, the definition coincides with the standard notion of convergence. And if X is a preorder and $(x_n)_n$ is a chain, then $\lim x_n = x$ is equivalent to

$$\forall \bar{x} \in X,\ \ X(x,\bar{x}) = 0 \text{ if and only if } \lim X(x_n,\bar{x}) = 0,$$

which is again equivalent to

$$\forall \bar{x} \in X,\ \ x \leq_X \bar{x} \text{ if and only if } \exists N\ \forall n \geq N,\ x_n \leq_X \bar{x},$$

meaning that x is a *minimal upperbound* of $(x_n)_n$. (In case X is a partial order, x is the least upperbound.)

A mapping $f : X \to Y$ between two gums X and Y is *continuous* if for all Cauchy sequences $(x_n)_n$ and x in X, if $\lim x_n = x$ then $\lim f(x_n) = f(x)$.

For ordinary ultrametric spaces, this is the standard definition of continuity in terms of converging sequences. For preorders, f is continuous if it preserves minimal upperbounds of chains. (And for partial orders, it should preserve least upperbounds.)

In dealing with generalized ultrametric spaces, one should be prepared to reconsider some basic intuitions about ordinary ultrametric spaces. For instance, any non-expansive mapping between ordinary ultrametric spaces is continuous. But:

LEMMA 6.2 *The notions of 'non-expansive' and 'continuous' mapping between generalized (ultra)metric spaces are incomparable.* □

An example of a mapping that is continuous but not non-expansive is $f : \bar{\omega} \to \bar{\omega}$ defined, for x in $\bar{\omega}$, by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x - 1 & \text{if } 0 < x < \omega \\ \omega & \text{if } x = \omega, \end{cases}$$

where $\bar{\omega}$ is supplied with the generalized ultrametric as defined in Example 2.1(2). For instance, $\bar{\omega}(f(2), f(1)) = \bar{\omega}(1, 0) = 1 \not\leq 2^{-1} = \bar{\omega}(2, 1)$. Any mapping between partial orders that is monotone but not continuous (i.e., least-upperbound preserving) yields an example of the converse. In particular, consider again the set $\bar{\omega}$ but now supplied with the discrete generalized ultrametric (as in Example 2.1(1)). The mapping $g : \bar{\omega} \to \bar{\omega}$ defined by

$$g(x) = \begin{cases} 0 & \text{if } x < \omega \\ \omega & \text{if } x = \omega, \end{cases}$$

is non-expansive (because it is monotone) but not continuous.

A gum X is *complete* if all Cauchy sequences in X have a limit in X. Limits are unique in complete *quasi* ultrametric spaces, which therefore are well suited for the construction of fixed points. There are at least two ways:

THEOREM 6.3 *Let X be a complete quasi ultrametric space and $f : X \to X$ non-expansive.*

1. *If f is continuous and if there is x in X with $X(x, f(x)) = 0$, then f has a fixed point.*
2. *If f is continuous and* contractive*:*

$$\exists \epsilon < 1 \, \forall x, y \in X, \;\; X(f(x), f(y)) \leq \epsilon \cdot X(x, y),$$

and X is non-empty, then f has a unique *fixed point. (Note that contractivity does not imply continuity; for an example see below.)*

PROOF:

1. Suppose f is continuous and let x be such that $X(x, f(x)) = 0$. The sequence

$$(x, f(x), f^2(x), \ldots)$$

is trivially Cauchy because f is non-expansive. Since X is complete it has a limit y. By continuity of f, $f(y) = \lim f(f^n(x)) = \lim f^n(x)$. In quasi ultrametric spaces, limits are unique thus $y = f(y)$.

2. Suppose that f is continuous and contractive. Let x be any element in X and consider again the sequence $(x, f(x), f^2(x), \ldots)$. Because f is contractive this sequence is Cauchy: for all $n \geq 0$, $X(f^n(x), f^{n+1}(x)) \leq \epsilon^n \cdot X(x, f(x))$. As in 1, a fixed point y is obtained by completeness of X and continuity of f. Suppose z is another one. Then $X(y, x) = X(f(y), f(z)) \leq \epsilon \cdot X(y, z)$ whence $X(y, z) = 0$. Similarly $X(z, y) = 0$. Because X is a quasi ultrametric space this implies $y = z$. □

Part 1 generalizes the theorem of Knaster-Tarski that continuous functions on a complete partial order with a minimal element, have a least fixed point. Part 2 generalizes Banach's contraction theorem.

Consider the set $\bar{\omega}$ of the natural numbers with infinity with the same distance as defined in Example 2.1.1, but for the value of $\bar{\omega}(1, 0)$, which is now put to $1/2$. Let $f : \bar{\omega} \to \bar{\omega}$ map any $n \geq 0$ to 0, and ω to 1. Then f is contractive but not continuous since $\lim n = \omega$, whereas $\lim f(n) \neq f(\omega)$.

In order to prove that a mapping $f : P \to Q$ between partial orders is continuous (that is, preserves least upperbounds), one usually establishes first that f is monotone, from which then half of the proof follows: if $x = \bigsqcup x_n$ and f is monotone, then $x_n \leq_P x$ implies $f(x_n) \leq_Q f(x)$ whence $\bigsqcup f(x_n) \leq_Q f(x)$. Similarly (and more generally), non-expansiveness of a mapping between generalized ultrametric spaces implies 'half of its continuity'; more precisely:

PROPOSITION 6.4 *Let X and Y be generalized ultrametric spaces and $f : X \to Y$ a non-expansive mapping. Consider a converging sequence* $\lim x_n = x$ *in X. For all y in Y,*

$$\lim Y(f(x_n), y) \leq Y(f(x), y).$$

PROOF: Let $y \in Y$. It follows from $\lim x_n = x$ that $\lim X(x_n, x) = X(x, x) = 0$. Hence there is N such that for all $n \geq N$, $X(x_n, x) \leq Y(f(x), y)$. Thus for all $n \geq N$,

$$\begin{aligned} Y(f(x_n), y) &\leq \max\{Y(f(x_n), f(x)),\ Y(f(x), y)\} \\ &\leq (f \text{ is non-expansive}) \\ &\quad\ \max\{X(x_n, x),\ Y(f(x), y)\} \\ &= Y(f(x), y). \end{aligned}$$

This implies $\lim Y(f(x_n), y) \leq Y(f(x), y)$. □

A first occasion to apply Proposition 6.4 is the following.

THEOREM 6.5
Let X and Y be generalized ultrametric spaces. Let

$$X \longrightarrow Y = \{f \in Y^X \mid f \text{ is both non-expansive and continuous}\},$$

with distance as in Y^X (the supremum distance, cf. Section 4). Then

1. *$X \longrightarrow Y$ is a generalized ultrametric space.*
2. *If Y is complete then $X \longrightarrow Y$ is complete as well.*

PROOF: The proof of 1 is as for ordinary ultrametric spaces. The proof of 2 contains, as it were, both the proofs (of the same statement) for partial orders and ordinary ultrametric spaces, and is somewhat more complicated than both. We list the main steps: consider a Cauchy sequence $(f_n)_n$ in $X \longrightarrow Y$. To show: there is f in $X \longrightarrow Y$ with $\lim f_n = f$.

1. Definition: for any x in X, the sequence $(f_n(x))_n$ is Cauchy in Y. It has a limit, to be called $f(x)$, because Y is complete. This defines a mapping $f : X \to Y$.
2. A useful observation: $\forall \epsilon > 0\ \exists N\ \forall n \geq N\ \forall x \in X,\ \ Y(f_n(x), f(x)) < \epsilon$.
3. From 2, it follows that $f = \lim f_n$.[1]
4. Using 3, one can prove that f is non-expansive.
5. It remains to be shown that f is continuous. Let $\lim x_n = x$ be a converging sequence in X, and let y be in Y. By Proposition 6.4 and 4, $\lim Y(f(x_n), y) \leq Y(f(x), y)$.
6. Using 2 and the fact that the functions f_n are continuous, one can also prove the converse: $Y(f(x), y) \leq \lim Y(f(x_n), y)$. From this and 5, it follows that $\lim f(x_n) = f(x)$. Thus f is continuous. □

A corollary of (clauses 1–4 of the proof of) the theorem above is that the set of all non-expansive mappings from X to Y is complete if Y is.

It is not difficult to prove that function composition is continuous:

THEOREM 6.6 *The composition of functions, viewed for any generalized ultrametric spaces X, Y, and Z, as a function*

$$\circ : (Y \longrightarrow Z) \times (X \longrightarrow Y) \ \to\ (X \longrightarrow Z)$$

is continuous. □

7. Finiteness and totally-boundedness

In this section, finite elements are used to define a generalized version of totally-boundedness, which will be shown to unify the notions of Scott-compactness (of preorders) and totally-boundedness (of ordinary ultrametric spaces). Totally-boundedness will be used in Section 11.

It is an immediate consequence of the definition of convergence that for any $\bar{x}$ in a gum X, the mapping $X(-, \bar{x}) : X \to [0,1]$ is continuous: for any converging sequence $\lim x_n = x$ in X, $\lim X(x_n, \bar{x}) = X(x, \bar{x})$. This does generallly not hold for $X(\bar{x}, -)$. In fact, let us call an element a in X *finite* if $X(a, -)$ is continuous (cf. [Smy87, Wag94]): for any converging sequence $\lim x_n = x$ in X,

[1] Franck van Breugel pointed out that this actually needs to be proved.

$$\lim X(a, x_n) = X(a, x).$$

For a preorder X, this defines the usual notion of finiteness, since for a chain $(x_n)_n$ with minimal upperbound x, $\lim X(a, x_n) = X(a, x)$ if and only if

$$\lim X(a, x_n) = 0 \text{ if and only if } X(a, x) = 0$$

if and only if

$$(\exists n,\ a \leq_X x_n) \text{ if and only if } a \leq_X x,$$

which is the standard definition of a being finite. For an ordinary (ultra)metric space, one easily checks that $X(a, -)$ is continuous for *any* a (hence all elements are finite). This might seem somewhat disappointing at first, but it will shortly turn out to be actually advantageous.

A gum X is *ω-algebraic* if every element is the limit of a Cauchy sequence in $K(X)$, the set of all finite elements in X. For preorders this is the usual definition, and any ordinary (ultra)metric space is algebraic. Of the examples in Section 2, A^∞ and $\bar{\omega}$ are ω-algebraic, and $[0, 1]$ is not.

Let X be a gum, x an element of X, and $\epsilon > 0$. The *closed ϵ-ball* with centre x is defined as usual:

$$\bar{B}_\epsilon(x) = \{y \in X \mid X(x, y) \leq \epsilon\}.$$

An *ϵ-cover* (of closed balls) for X is a set $E \subseteq K(X)$ of finite elements, such that

$$X = \bigcup \{\bar{B}_\epsilon(a) \mid a \in E\}.$$

A generalized ultrametric space X is *totally-bounded*,

$$\text{TB}(X) \text{ iff } \forall \epsilon > 0, \text{ there is a finite } \epsilon\text{-cover for } X.$$

This definition coincides with the usual notion for ordinary (ultra)metric spaces, since there every element is finite. In order to relate it to preorders, the following stronger notion turns out to be more convenient: X is *strongly totally-bounded*,

$$\text{STB}(X) \text{ iff } \forall \epsilon > 0, \text{ every } \epsilon\text{-cover for } X \text{ has a finite subcover.}$$

For ordinary ultrametric spaces, this still coincides with the standard definition of totally-boundedness:

PROPOSITION 7.1 *For an ordinary ultrametric space X, STB(X) if and only if TB(X). (This proposition does* not *hold for ordinary metric spaces.)*

PROOF: Let X be an ordinary ultrametric space. Clearly STB(X) implies TB(X). (In fact, this holds more generally for any ω-algebraic generalized metric space.) Conversely, suppose that X is totally-bounded. Let $\epsilon > 0$ and E be an ϵ-cover for X. Because X is totally-bounded there is a finite ϵ-cover F for X. For every a in F choose b in E such that $a \in \bar{B}_\epsilon(b)$, whence $\bar{B}_\epsilon(a) \cap \bar{B}_\epsilon(b) \neq \emptyset$. Because X is an ordinary ultrametric space this implies $\bar{B}_\epsilon(a) = \bar{B}_\epsilon(b)$. Thus we find finitely many b's in E already covering X. □

An ω-algebraic preorder X is *Scott-compact* if every set $E \subseteq K(X)$ such that

$$\forall x \in X \; \exists a \in E, \quad a \leq_X x,$$

has a finite subset $E' \subseteq E$ with the same property. Scott-compactness is equivalent to STB:

PROPOSITION 7.2 *An ω-algebraic preorder X is Scott-compact if and only if $STB(X)$.*

PROOF: This follows from the fact that, for a preorder X, an element a in X, and $\epsilon > 0$,

$$\begin{aligned}
\bar{B}_\epsilon(a) &= \{x \in X \mid X(a, x) \leq \epsilon\} \\
&= (\text{for } \epsilon < 1) \\
&\quad\; \{x \in X \mid X(a, x) = 0\} \\
&= \{x \in X \mid a \leq_X x\}.
\end{aligned}$$

□

In [Smy91], a quasi metric space X is said to be totally-bounded if the corresponding ordinary metric space X^s (with the 'symmetrized' metric, see Section 5) is totally-bounded in the usual sense. (In such spaces, the various definitions of Cauchy sequence all coincide.) In comparison with the definition of the present paper, two differences are to be mentioned: first, our definition uses the original distance rather than its symmetric version; secondly, the elements of an ϵ-cover are required to be finite. Some interest in the present definition of totally-boundedness (and of its stronger variant) seems to be justified by Propositions 7.1 and 7.2.

8. Metric adjoint pairs

An adjoint pair between preorders is shown to be a special case of a more general metric notion of *'adjoint pair up to ϵ'*, or *ϵ-adjoint* pair for short (for a real number ϵ with $0 \leq \epsilon \leq 1$). As we shall see in Sections 9 and 10, ϵ-adjoint pairs play a central role in the solution of recursive domain equations. Moreover, they will be used in Section 12 to turn the category of generalized ultrametric spaces itself into a large generalized ultrametric space.

The following notation will be used: For real numbers ϵ, r and r' in $[0, 1]$ define

$$r \approx_\epsilon r' \text{ if and only if } |r - r'| \leq \epsilon.$$

Let X and Y be two generalized (ultra)metric spaces. Two non-expansive mappings $f : X \to Y$ and $g : Y \to X$ are *ϵ-adjoint*,

$$f \dashv_\epsilon g \text{ if and only if } \forall x \in X \; \forall y \in Y, \quad Y(f(x), y) \approx_\epsilon X(x, g(y)).$$

The pair $\langle f, g \rangle$ is called an ϵ-adjoint pair. If $f \dashv_0 g$ then $\langle f, g \rangle$ is called a *proper* adjoint pair, denoted by $f \dashv g$; it satisfies by definition:

$$\forall x \in X \; \forall y \in Y, \quad Y(f(x), y) = X(x, g(y)).$$

If X and Y are preorders this is equivalent to

$$\forall x \in X \; \forall y \in Y, \quad Y(f(x), y) = 0 \text{ iff } X(x, g(y)) = 0,$$

that is,

$$\forall x \in X\ \forall y \in Y,\ \ f(x) \leq_Y y \text{ iff } x \leq_X g(y),$$

which is the standard definition of an adjoint pair between preorders. For preorders there exists yet another, equivalent formulation:

$$f \dashv g \text{ if and only if } (1_X \leq g \circ f \text{ and } f \circ g \leq 1_Y).$$

(Here 1_X and 1_Y are the identity mappings on X and Y.) This alternative characterization can be generalized to arbitrary generalized (ultra)metric spaces, by means of the following number. Consider a pair of non-expansive mappings $f : X \to Y$ and $g : Y \to X$ between two generalized (ultra)metric spaces, and define

$$\delta\langle f, g\rangle = \max\{X^X(1_X, g \circ f),\ Y^Y(f \circ g, 1_Y)\}.$$

Intuitively, the number $\delta\langle f, g\rangle$ measures the quality with which Y is approximated by X via f and g. The relationship between ϵ-adjoint pairs and $\delta\langle f, g\rangle$ is stated in the following theorem, which for once we have formulated for generalized *metric* spaces.

THEOREM 8.1 *Let X and Y be generalized metric spaces, and $\epsilon \geq 0$. For all non-expansive mappings $f : X \to Y$ and $g : Y \to X$,*

$$f \dashv_\epsilon g \text{ if and only if } \delta\langle f, g\rangle \approx_\epsilon 0.$$

PROOF: Suppose $f \dashv_\epsilon g$. For any x in X, $X(x, g \circ f(x)) \approx_\epsilon Y(f(x), f(x)) = 0$, thus $X^X(1_X, g \circ f) \approx_\epsilon 0$. Similarly $Y^Y(f \circ g, 1_Y) \approx_\epsilon 0$. It follows that $\delta\langle f, g\rangle \approx_\epsilon 0$. Conversely, suppose $\delta\langle f, g\rangle \approx_\epsilon 0$. For all x in X and y in Y,

$$\begin{aligned} X(x, g(y)) &\leq X(x, g \circ f(x)) + X(g \circ f(x), g(y)) \\ &\leq \epsilon + X(g \circ f(x), g(y)) \\ &\leq \epsilon + Y(f(x), y). \end{aligned}$$

Similarly, $Y(f(x), y) \leq \epsilon + X(x, g(y))$. Thus $Y(f(x), y) \approx_\epsilon X(x, g(y))$. □

Note that for preorders (and $\epsilon = 0$), Theorem 8.1 indeed gives the alternative characterization of adjoints mentioned above. For ordinary metric spaces X and Y, it follows that

$$f \dashv g \text{ iff } (1_X = g \circ f \text{ and } f \circ g = 1_Y);$$

in other words, f is an isomorphism with inverse g.

An immediate corollary of Theorem 8.1 is that $\delta\langle f, g\rangle$ measures the extent to which an arbitrary pair of non-expansive mappings $\langle f, g\rangle$ is adjoint ('the smaller the better'):

COROLLARY 8.2 *For non-expansive mappings $f : X \to Y$ and $g : Y \to X$, $f \dashv_{\delta\langle f,g\rangle} g$.* □

Adjoint pairs have various nice properties. For instance:

PROPOSITION 8.3 *Let $f : X \to Y$ and $g : Y \to X$ be two non-expansive mappings with $f \dashv g$.*

1. *If g is continuous then for all a in X: if a is finite then $f(a)$ is finite.*

2. *f is continuous.*

3. *If X and Y are quasi ultrametric spaces then $f \circ g \circ f = f$ and $g \circ f \circ g = g$.*

PROOF: Suppose g is continuous and let $a \in X$ be finite. We have to show that $Y(f(a), -)$ is continuous. So consider a converging sequence $\lim y_n = y$ in Y:

$$\begin{aligned} Y(f(a), \lim y_n) &= X(a,\ g(\lim y_n)) \\ &= X(a, \lim g(y_n)) \\ &= (a \text{ is finite}) \\ &\quad \lim X(a, g(y_n)) \\ &= \lim Y(f(a), y_n). \end{aligned}$$

The second statement above can be proved similarly. For a proof of the last statement, let X and Y be quasi metric spaces and x in X. On the one hand,

$$Y(f \circ g \circ f(x), f(x)) = X(g \circ f(x), g \circ f(x)) = 0;$$

on the other hand,

$$Y(f(x), f \circ g \circ f(x)) \leq X(x, g \circ f(x)) = Y(f(x), f(x)) = 0,$$

whence $f \circ g \circ f(x) = f(x)$. Similarly, one shows $g \circ f \circ g(x) = g(x)$. □

Much of this section is somehow implicitly suggested by [Law73]. The definition and characterizations of ϵ-adjoint pairs seem to be new.

9. The category of complete quasi ultrametric spaces

Theorem 6.3 shows that complete quasi ultrametric spaces are suitable for finding fixed points of (continuous and contractive) non-expansive mappings. The *category* of complete quasi ultrametric spaces (cqums for short) turns out to be equally suitable for finding fixed points of *functors* (to be discussed in Section 10). As usual, such fixed points are obtained as colimits of certain sequences (chains) of spaces. This section gives a generalization of the standard constructions for partial orders ([SP82]) and ordinary (ultra)metric spaces ([AR89]) to complete quasi ultrametric spaces. In [Wag94], a similar generalization is carried out using embedding-projection pairs. Interestingly, it can be carried out here using (metric) adjoint pairs rather than embedding-projection pairs. Although this is well-known for the special case of ordered spaces, it is new for ordinary (ultra)metric spaces.

As in the case of ordered spaces, the use of adjoint pairs instead of embedding-pairs will not lead to 'more' fixed points of functors. Nevertheless adjoint pairs seem preferable, both because they have all properties that are needed and because their use will lead to a number of additional observations, in Section 12, on the family of all (complete) generalized ultrametric spaces, viewed itself as a large gum.

We shall consider the category $Cqum^{\epsilon}$, which is defined as follows: objects are complete quasi ultrametric spaces; an arrow

$$\langle f, g\rangle : X \to Y$$

is defined as a pair of *non-expansive and continuous* mappings $f : X \to Y$ and $g : Y \to X$. By Corollary 8.2, we know that any such pair is an ϵ-adjoint pair, for $\epsilon = \delta\langle f, g\rangle$. This accounts for the superscript ϵ in $Cqum^\epsilon$. The composition of two arrows $\langle f, g\rangle : X \to Y$ and $\langle h, i\rangle : Y \to Z$ is defined as $\langle h \circ f,\ g \circ i\rangle$. It is straightforward to show that if $f \dashv_\epsilon g$ and $h \dashv_\gamma i$ then $h \circ f \dashv_\lambda g \circ i$ with $\lambda = \max\{\epsilon, \gamma\}$ (for $\epsilon, \gamma, \lambda \geq 0$).

A *chain* in $Cqum^\epsilon$ is a sequence

$$X_0 \overset{\langle f_0, g_0\rangle}{\longrightarrow} X_1 \overset{\langle f_1, g_1\rangle}{\longrightarrow} \cdots$$

of cqums and arrows between them. It is *Cauchy* whenever

$$\forall \epsilon > 0\ \exists N\ \forall n \geq N,\ f_n \dashv_\epsilon g_n,$$

or equivalently,

$$\forall \epsilon > 0\ \exists N\ \forall n \geq N,\ \delta\langle f_n, g_n\rangle \leq \epsilon.$$

In the special case of complete partial orders, the arrows in a Cauchy chain (eventually) are (the standard) adjoint pairs.

We shall see that any Cauchy chain has a (categorical) colimit. The proof makes use of two lemmas, in which the following notation will be of help: for k and l with $0 \leq k < l$, define

$$f_{kl} : X_k \to X_l, \quad f_{kl} = f_{l-1} \circ \cdots \circ f_{k+1} \circ f_k,$$

$$g_{kl} : X_l \to X_k, \quad g_{kl} = g_k \circ g_{k+1} \cdots \circ g_{l-1}.$$

(Note that $f_{k,k+1} = f_k$ and $g_{k,k+1} = g_k$.)

LEMMA 9.1 *Consider a Cauchy chain*

$$X_0 \overset{\langle f_0, g_0\rangle}{\longrightarrow} X_1 \overset{\langle f_1, g_1\rangle}{\longrightarrow} \cdots$$

in $Cqum^\epsilon$. For all $k \geq 0$, the sequence $(g_{kl} \circ f_{kl})_{l>k}$ is Cauchy in $X_k \longrightarrow X_k$. (Consequently, it has a limit since $X_k \longrightarrow X_k$ is complete by Theorem 6.5.) More generally: for every k and l with $0 \leq k < l$, the sequence $(g_{lm} \circ f_{km})_{m>l}$ is Cauchy in $X_k \longrightarrow X_l$, and the sequence $(g_{km} \circ f_{lm})_{m>l}$ is Cauchy in $X_l \longrightarrow X_k$.

PROOF: We prove only the first statement (the other ones not being more difficult). It is an immediate consequence of the Cauchy condition on the chain and the fact that, for all k and l with $0 \leq k < l$,

$$\begin{aligned} X_k \longrightarrow X_k(g_{kl} \circ f_{kl},\ g_{k,l+1} \circ f_{k,l+1}) &= X_k \longrightarrow X_k(g_{kl} \circ f_{kl},\ g_{kl} \circ g_l \circ f_l \circ f_{kl}) \\ &\leq X_k \longrightarrow X_l(f_{kl},\ g_l \circ f_l \circ f_{kl}) \\ &= \sup_{x \in X_k} \{X_l(f_{kl}(x),\ g_l \circ f_l(f_{kl}(x)))\} \\ &\leq \sup_{x \in X_l} \{X_l(x,\ g_l \circ f_l(x))\} \\ &\leq \delta\langle f_l, g_l\rangle. \end{aligned}$$

□

The following lemma states that colimits of Cauchy chains are *locally determined*:

LEMMA 9.2 *Consider a Cauchy chain*

$$\Delta = X_0 \stackrel{\langle f_0,g_0\rangle}{\longrightarrow} X_1 \stackrel{\langle f_1,g_1\rangle}{\longrightarrow} \cdots$$

and let

$$(\langle \alpha_k, \beta_k\rangle : X_k \to X)_k$$

be a cone from Δ *to* X*: i.e., for all* $k \geq 0$, $\langle \alpha_k, \beta_k\rangle = \langle \alpha_{k+1}, \beta_{k+1}\rangle \circ \langle f_k, g_k\rangle$. *If*

1. $\lim \alpha_k \circ \beta_k = 1_X$ *and* 2. $\forall k \geq 0,\ \beta_k \circ \alpha_k = \lim_{l>k} g_{kl} \circ f_{kl}$

then X *is a colimiting cone.*

PROOF: The proof of this lemma combines the proof of the same statement for complete partial orders (cf. [SP82, AJ94]) with the proof of a similar lemma (but for embedding-projection pairs) for ordinary (ultra)metric spaces in [AR89]. We have to show that for any other cone

$$(\langle \bar{\alpha}_k, \bar{\beta}_k\rangle : X_k \to \bar{X})_k$$

from Δ to $\bar{X}$, there exists a unique arrow $\langle f, g\rangle : X \to \bar{X}$ such that, for all $k \geq 0$,

$$\langle f, g\rangle \circ \langle \alpha_k, \beta_k\rangle = \langle \bar{\alpha}_k, \bar{\beta}_k\rangle.$$

The Cauchy condition on Δ implies that the sequence $(\bar{\alpha}_k \circ \beta_k)_k$ is Cauchy in $X \longrightarrow \bar{X}$, since for any $k \geq 0$,

$$\begin{aligned}
(X \longrightarrow \bar{X})(\bar{\alpha}_k \circ \beta_k,\ \bar{\alpha}_{k+1} \circ \beta_{k+1}) &= \sup_{x\in X}\{\bar{X}(\bar{\alpha}_k \circ \beta_k(x),\ \bar{\alpha}_{k+1} \circ \beta_{k+1}(x))\} \\
&= \sup_{x\in X}\{\bar{X}(\bar{\alpha}_{k+1} \circ f_k \circ g_k \circ \beta_{k+1}(x),\ \bar{\alpha}_{k+1} \circ \beta_{k+1}(x))\} \\
&\leq \sup_{x\in X_{k+1}}\{\bar{X}(\bar{\alpha}_{k+1} \circ f_k \circ g_k(x),\ \bar{\alpha}_{k+1}(x))\} \\
&\leq \sup_{x\in X_{k+1}}\{X_{k+1}(f_k \circ g_k(x),\ x)\} \\
&\leq \delta\langle f_k, g_k\rangle.
\end{aligned}$$

By Theorem 6.5, $X \longrightarrow \bar{X}$ is complete so we can define $f = \lim \bar{\alpha}_k \circ \beta_k$. Similarly $\beta : \bar{X} \to X$ is defined as $\beta = \lim \alpha_k \circ \bar{\beta}_k$. Next we show, for $k \geq 0$, one half of $\langle f, g\rangle \circ \langle \alpha_k, \beta_k\rangle = \langle \bar{\alpha}_k, \bar{\beta}_k\rangle$:

$$\begin{aligned}
f \circ \alpha_k &= (\lim_l \bar{\alpha}_l \circ \beta_l) \circ \alpha_k \\
&= \lim_l \bar{\alpha}_l \circ \beta_l \circ \alpha_k \\
&= \lim_{l>k} \bar{\alpha}_l \circ \beta_l \circ \alpha_l \circ f_{kl} \\
&= \text{(by assumption)} \\
&\quad \lim_{l>k} \bar{\alpha}_l \circ (\lim_{m>l} g_{lm} \circ f_{lm}) \circ f_{kl} \\
&= \lim_{l>k}(\lim_{m>l} \bar{\alpha}_l \circ g_{lm} \circ f_{lm}) \circ f_{kl} \\
&= \lim_{l>k}(\lim_{m>l} \bar{\alpha}_m \circ f_{lm} \circ g_{lm} \circ f_{lm}) \circ f_{kl}.
\end{aligned}$$

Note that in the above, the continuity of '$\circ$' (Theorem 6.6) is used. For $\epsilon > 0$ and l and m (with $l < m$) 'big enough', we have $f_{lm} \dashv_\epsilon g_{lm}$, which implies

$$X_l \longrightarrow X_m(f_{lm} \circ g_{lm} \circ f_{lm},\ f_{lm}) \leq \epsilon, \text{ and } X_l \longrightarrow X_m(f_{lm},\ f_{lm} \circ g_{lm} \circ f_{lm}) \leq \epsilon.$$

It follows that the above sequence of equalities can be continued with

$$\begin{aligned}
\lim_{l>k}(\lim_{m>l} \bar{\alpha}_m \circ f_{lm} \circ g_{lm} \circ f_{lm}) \circ f_{kl} &= \lim_{l>k}(\lim_{m>l} \bar{\alpha}_m \circ f_{lm}) \circ f_{kl} \\
&= \lim_{l>k}(\lim_{m>l} \bar{\alpha}_l) \circ f_{kl} \\
&= \lim_{l>k} \bar{\alpha}_l \circ f_{kl} \\
&= \lim_{l>k} \bar{\alpha}_k \\
&= \bar{\alpha}_k.
\end{aligned}$$

Similarly one proves $\beta_k \circ g = \bar{\beta}_k$. This shows that $\langle f, g \rangle$ is a mediating arrow. Furthermore it is unique: if $\langle p, q \rangle : X \to \bar{X}$ is another mediating arrow then

$$\begin{aligned}
p &= p \circ 1_X \\
&= \text{(by assumption)} \\
&\quad\ p \circ (\lim \alpha_k \circ \beta_k) \\
&= \lim p \circ \alpha_k \circ \beta_k \\
&= \lim \bar{\alpha}_k \circ \beta_k \\
&= f,
\end{aligned}$$

and similarly $q = g$. □

Lemma 9.2 plays a crucial role in the following theorem.

Theorem 9.3 *Any Cauchy chain in $Cqum^\epsilon$ has a colimit.*

Proof: Let

$$\Delta = X_0 \stackrel{\langle f_0, g_0 \rangle}{\longrightarrow} X_1 \stackrel{\langle f_1, g_1 \rangle}{\longrightarrow} \cdots$$

be Cauchy. The colimit we are looking for is given, as usual, by the inverse limit:

$$X = \{(x_k)_k \mid \forall k \geq 0,\ x_k \in X_k \text{ and } g_k(x_{k+1}) = x_k\}.$$

On X a distance is defined, for $(x_k)_k, (y_k)_k$ in X, by

$$X((x_k)_k, (y_k)_k) = \sup\{X_k(x_k, y_k)\}.$$

It is a nice little exercise on generalized ultrametrics—which we would not dare to keep from the reader—to prove that X is a complete quasi ultrametric space, in which limits are determined elementwise: that is, for any Cauchy sequence $(x^k)_k$ in X, with $x^k = (x_0^k, x_1^k, \ldots)$,

$$\lim x^k = (\lim x_0^k,\ \lim x_1^k,\ \ldots).$$

Next X is turned into a cone by defining, for every $k \geq 0$, an arrow $\langle \alpha_k, \beta_k \rangle : X_k \to X$ as follows: for x in X_k,

$$\alpha_k(x) = (\lim_{l>k} g_{0,l} \circ f_{kl}(x),\ \lim_{l>k} g_{1,l} \circ f_{kl}(x), \ldots, \lim_{l>k} g_{kl} \circ f_{kl}(x),\ \lim_{l>k+1} g_{k+1,l} \circ f_{kl}(x), \ldots),$$

and for $(x_0, x_1, \ldots) \in X$,

$$\beta_k((x_0, x_1, \ldots)) = x_k.$$

The limits in the definition of α_k exist by Lemma 9.1 and α_k maps indeed into X. Also α_k and β_k are non-expansive and continuous, and $\langle \alpha_{k+1}, \beta_{k+1} \rangle \circ \langle f_k, g_k \rangle = \langle \alpha_k, \beta_k \rangle$. Furthermore,

$$\forall \epsilon > 0 \; \exists N \; \forall k \geq N, \; \alpha_k \dashv_\epsilon \beta_k,$$

which is an immediate consequence of

1. $\lim X \longrightarrow X(\alpha_k \circ \beta_k, \; 1_X) = 0$, and
2. $\lim X_k \longrightarrow X_k(1_{X_k}, \; \alpha_k \circ \beta_k) = 0$.

We prove only statement 1 (the latter is easy): for $k \geq 0$ and $(x_n)_n$ in X,

$$X(\alpha_k \circ \beta_k((x_n)_n), (x_n)_n) = \sup_{m>k} \{X_m(\lim_{l>m} g_{ml} \circ f_{kl}(x_k), \; x_m)\},$$

by definition of the metric on X. Because for all $m > k$,

$$\begin{array}{rcl} X_m(\lim_{l>m} g_{ml} \circ f_{kl}(x_k), \; x_m) & = & (X_m(-, x_m) \text{ is continuous, cf. Section 7}) \\ & & \lim_{l>m} X_m(g_{ml} \circ f_{kl}(x_k), \; x_m) \\ & = & ((x_n)_n \text{ is an element of } X) \\ & & \lim_{l>m} X_m(g_{ml} \circ f_{kl} \circ g_{kl}(x_l), \; g_{ml}(x_l)) \\ & \leq & \lim_{l} X_l(f_{kl} \circ g_{kl}(x_l), \; x_l) \\ & \leq & \delta\langle f_{kl}, g_{kl} \rangle \\ & \leq & \delta\langle f_k, g_k \rangle, \end{array}$$

statement 1 follows from the fact that our chain is Cauchy. The proof of the present lemma is concluded by the verification of the conditions of Lemma 9.2, from which it follows that

$$(\langle \alpha_k, \beta_k \rangle : X_k \to X)_k$$

is a colimiting cone. Firstly, for all $k \geq 0$,

$$\beta_k \circ \alpha_k = \lim_{l>k} g_{kl} \circ f_{kl},$$

by definition of α_k and β_k. Secondly, $\lim \alpha_k \circ \beta_k = 1_X$ because, for x and $\bar{x}$ in X, $\epsilon > 0$, and k 'big enough',

$$\begin{array}{rcl} X(x, \bar{x}) & = & \sup\{X_n(\beta_n(x), \beta_n(\bar{x}))\} \\ & \approx_\epsilon & X_k(\beta_k(x), \beta_k(\bar{x})) \\ & \approx_\epsilon & X(\alpha_k \circ \beta_k(x), \bar{x}). \end{array}$$

The last step uses the fact proven above, that for big k, $\alpha_k \dashv_\epsilon \beta_k$. □

It can be deduced from the proof above that the converse of Lemma 9.2 holds as well. Thus:

THEOREM 9.4 *Consider a Cauchy chain*

$$\Delta = X_0 \stackrel{\langle f_0, g_0 \rangle}{\longrightarrow} X_1 \stackrel{\langle f_1, g_1 \rangle}{\longrightarrow} \cdots$$

and let $(\langle \alpha_k, \beta_k \rangle : X_k \to X)_k$ *be a cone from* Δ *to* X. *Then* X *is a colimit* if and only if

$$1.\ \lim \alpha_k \circ \beta_k = 1_X \text{ and } \ 2.\ \forall k \geq 0,\ \beta_k \circ \alpha_k = \lim_{l > k} g_{kl} \circ f_{kl}.$$

□

COROLLARY 9.5 *Let* Δ *and* $(\langle \alpha_k, \beta_k \rangle : X_k \to X)_k$ *be as in Theorem 9.4. If* X *is a colimit then*

$$\forall \epsilon > 0\ \exists N\ \forall n \geq N,\ \alpha_n \dashv_\epsilon \beta_n.$$

The converse does not hold: take for Δ the constant chain consisting of the ordered space $\{0,1\}$ with $0 \leq 1$, and for X the space $\{1\}$.

10. FIXED POINTS OF FUNCTORS

Two theorems will be formulated on the existence of fixed points of functors, which can be seen as categorical versions of parts 1 and 2 of Theorem 6.3. These theorems generalize the standard order-theoretic and (ultra)metric solutions (of [SP82] and [AR89, RT93], respectively).

As usual, we shall concentrate on functors with so-called *local* properties (cf. [SP82]): returning for a moment to the category *Gums* of all generalized ultrametric spaces, a functor F : *Gums* $\to$ *Gums* is *locally-non-expansive* if, for all gums X and Y the mapping

$$F_{XY} : Y^X \to F(Y)^{F(X)},$$

which maps $f : X \to Y$ to $F(f) : F(X) \to F(Y)$, is non-expansive. Similarly one defines the notions of *locally-continuous* and *locally-contractive*. (In the formulation of the latter, one should be careful with the order of the quantification: there should exist $\epsilon < 1$ such that for all X and Y, F_{XY} is contractive 'with factor ϵ'.)

As announced in Section 9, fixed points of functors will be constructed using complete quasi ultrametric spaces. Recall that $Cqum^\epsilon$ is the category of such spaces together with pairs of non-expansive and continuous mappings between them. We shall concentrate on functors $F^\epsilon : Cqum^\epsilon \to Cqum^\epsilon$ that are 'stemming from' functors F : *Cqum* $\to$ *Cqum*, where *Cqum* is the category of complete quasi ultrametric spaces with (single) non-expansive and continuous mappings as arrows. More precisely, any functor F : *Cqum* $\to$ *Cqum* defines a functor $F^\epsilon : Cqum^\epsilon \to Cqum^\epsilon$ which acts on objects as F does, and maps an arrow $\langle f, g \rangle : X \to Y$ to $\langle F(f), F(g) \rangle : F(X) \to F(Y)$. We shall use the following lemma, which can be readily verified.

LEMMA 10.1 *Consider a functor* F : *Cqum* $\to$ *Cqum, and an arrow* $\langle f, g \rangle : X \to Y$ *in* $Cqum^\epsilon$.

1. *If F is locally-non-expansive then $\delta(F^\epsilon(\langle f,g\rangle)) = \delta\langle F(f),F(g)\rangle \le \delta\langle f,g\rangle$. (Consequently, if $f \dashv_\epsilon g$ then $F(f) \dashv_\epsilon F(g)$.)*

2. *If F is locally-contractive, say with factor ϵ with $0 \le \epsilon < 1$, then $\delta(F^\epsilon(\langle f,g\rangle)) = \delta\langle F(f),F(g)\rangle \le \epsilon \cdot \delta\langle f,g\rangle$.*

□

We are ready for the first fixed point theorem, which is the categorical version of part 1 of Theorem 6.3.

THEOREM 10.2 *Let $F : Cqum \to Cqum$ be locally-non-expansive. If F is locally-continuous and if there exists X and $\langle f,g\rangle : X \to F(X)$ such that $f \dashv g$, then F has a fixed point.*

PROOF: Consider the following chain in $Cqum^\epsilon$,

$$\Delta = X_0 \stackrel{\langle f_0,g_0\rangle}{\longrightarrow} X_1 \stackrel{\langle f_1,g_1\rangle}{\longrightarrow} \cdots,$$

which is inductively defined by $X_0 = X$, $X_{n+1} = F^\epsilon(X_n) = F(X_n)$, $\langle f_0,g_0\rangle = \langle f,g\rangle$, and

$$\langle f_{n+1},g_{n+1}\rangle = F^\epsilon(\langle f_n,g_n\rangle) = \langle F(f_n),F(g_n)\rangle.$$

Because F is locally-non-expansive, the chain is trivially Cauchy by Lemma 10.1: for all $n \ge 0$, $f_n \dashv g_n$. By (the proof of) Theorem 9.3, it has a colimit

$$(\langle \alpha_n,\beta_n\rangle : X_n \to X)_n,$$

satisfying

1. $\lim \alpha_n \circ \beta_n = 1_X$, and
2. $\forall k \ge 0,\ \beta_k \circ \alpha_k = \lim_{l>k} g_{kl} \circ f_{kl}$.

Because F is locally-continuous, this implies

$$\begin{aligned} 1.\ \lim F(\alpha_n)\circ F(\beta_n) &= \lim F(\alpha_n \circ \beta_n) \\ &= F(\lim \alpha_n \circ \beta_n) \\ &= F(1_X) \\ &= 1_{F(X)}, \end{aligned}$$

and, for all $k \ge 0$,

$$\begin{aligned} 2.\ F(\beta_k)\circ F(\alpha_k) &= F(\beta_k \circ \alpha_k) \\ &= F(\lim_{l>k} g_{kl} \circ f_{kl}) \\ &= \lim_{l>k} F(g_{kl} \circ f_{kl}) \\ &= \lim_{l>k} F(g_{kl}) \circ F(f_{kl}) \\ &= \lim_{l>k} g_{k+1,l+1} \circ f_{k+1,l+1}. \end{aligned}$$

By Lemma 9.2, it follows that

$$(F^\epsilon(\langle \alpha_n, \beta_n \rangle) : F^\epsilon(X_n) \to F^\epsilon(X))_n,$$

which is equal to

$$(\langle F(\alpha_n), F(\beta_n) \rangle) : X_{n+1} \to F(X))_n,$$

is a colimit of

$$F^\epsilon(\Delta) = X_1 \xrightarrow{\langle f_1, g_1 \rangle} X_2 \xrightarrow{\langle f_2, g_2 \rangle} \cdots .$$

Since Δ and $F^\epsilon(\Delta)$ are the same but for the first element, the fact that both X and $F(X)$ are colimits implies that they are isomorphic. □

A simple example is the following. Let $(-)_\perp : Cqum \to Cqum$ be defined, for any cqum X, as follows: $(X)_\perp$ is the disjoint union of $\{\perp\}$ and X, with distance, for a and b in $(X)_\perp$,

$$(X)_\perp(a,b) = \begin{cases} 0 & \text{if } a = \perp \\ 1 & \text{if } a \in X \text{ and } b = \perp \\ X(a,b) & \text{if } a \in X \text{ and } b \in X. \end{cases}$$

On arrows $(-)_\perp$ is defined as one would expect. This defines a functor that is both locally-non-expansive and locally-continuous, and applying Theorem 10.2 with $X = \{\perp\}$ yields a fixed point, which is actually a complete partial order: it is (isomorphic to) $\bar{\omega}$, the set of natural numbers plus infinity, with distance as in Example 2.1(1).

If X is a partial order then $(X)_\perp$ is the usual 'lifting' of X. It is a special case of what could be called 'ϵ-lifting', which is defined as follows. For ϵ with $0 \leq \epsilon \leq 1$, let the set $(X)_\perp$ be as before but now with distance, for a and b in $(X)_\perp$,

$$(X)_\perp(a,b) = \begin{cases} 0 & \text{if } a = \perp \\ 1 & \text{if } a \in X \text{ and } b = \perp \\ \epsilon \cdot X(a,b) & \text{if } a \in X \text{ and } b \in X. \end{cases}$$

Again Theorem 10.2 applies. For $X = \{\perp\}$ and $\epsilon = 1/2$, the resulting fixed point is again $\bar{\omega}$ but now with metric as in Example 2.1(2).

The second fixed point theorem is the categorical version of part 2 of Theorem 6.3.

THEOREM 10.3 *If $F : Cqum \to Cqum$ is locally-contractive and locally-continuous then F has a fixed point, which is unique (up to isomorphism). This fixed point is (both an initial F-algebra and) a final F-coalgebra.*

PROOF: Let X_0 be an arbitrary complete quasi ultrametric space, and let $\langle f_0, g_0 \rangle : X_0 \to F(X_0)$ be an arbitrary arrow. As in the proof of Theorem 10.2, we can inductively define a chain $\Delta = (\langle f_n, g_n \rangle : X_n \to X_{n+1})_n$. Part 2 of Lemma 10.1 implies that it is Cauchy. As before this leads to the existence of a fixed point. Suppose there are two such fixed points, X and Y with isomorphisms $k : X \to F(X)$ and $l : Y \to F(Y)$. It follows from the local properties of F that

$$\Phi : (X \longrightarrow Y) \to (X \longrightarrow Y),$$

defined, for h in $X \longrightarrow Y$, by $\Phi(h) = l^{-1} \circ F(h) \circ k$, is continuous and contractive. Therefore it has by Theorem 6.3 a unique fixed point $\pi : X \to Y$ with $\pi = l^{-1} \circ F(\pi) \circ k$, or equivalently, $l \circ \pi = F(\pi) \circ k$. Similarly one can prove that there is a unique mapping $\rho : Y \to X$ such that $k \circ \rho = F(\rho) \circ l$; that 1_X is the unique mapping in $X \longrightarrow X$ such that $k \circ 1_X = F(1_X) \circ k$; and that 1_Y is the unique mapping in $Y \to Y$ such that $l \circ 1_Y = F(1_Y) \circ l$. Because also $k \circ (\rho \circ \pi) = F(\rho \circ \pi) \circ k$ and $l \circ (\pi \circ \rho) = F(\pi \circ \rho) \circ l$, it follows that $1_X = \rho \circ \pi$ and $1_Y = \pi \circ \rho$. Thus $X \cong Y$. Alternatively, uniqueness follows from the fact that any fixed point is a final F-coalgebra, which can be proved by a similar argument, and the fact that any two final F-coalgebras are isomorphic. Cf. [RT93] for a proof of the latter; see also [BW94]. □

An example: let 1 be a one element set and ϵ such that $0 < \epsilon < 1$. Consider the functor that maps a cqum X to $1 + (\epsilon \cdot X)$, where $\epsilon \cdot X$ is like X but with all distances multiplied by ϵ. This functor is both locally-continuous and locally-contractive. For $\epsilon = 1/2$, its unique fixed point is again the set $\bar{\omega}$, now with the (ordinary) metric, for x and y in $\bar{\omega}$,

$$\bar{\omega}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 2^{-\min\{x,y\}} & \text{if } x \neq y. \end{cases}$$

Note that this metric is the symmetric version of that of Example 2.1(2).

11. SFU: sequences of finite ultrametrics

A complete quasi ultrametric space is called *SFU* if it is the colimit in the category $Cqum^\epsilon$ of a (Cauchy) 'Sequence of Finite quasi Ultrametrics'. (Another name could be 'bifinite'.) Clearly this definition is in analogy to Plotkin's definition of *SFP* objects as colimits of sequences of finite partial orders ([Plo76]). It is a little different in that (metric) adjoint pairs are used instead of embedding-projection pairs, but one can show that both definitions would be equivalent. Moreover complete partial orders that are SFP are pointed: they have a least element. It would be straightforward to show that a complete quasi ultrametric space which actually is a pointed complete partial order, is SFU if and only if it is SFP. Less trivial will be the observation that an ordinary complete ultrametric space is SFU if and only if it is compact. First the following properties are established.

Lemma 11.1 *Let X be SFU: that is, there exists a Cauchy chain $\Delta = (\langle f_n, g_n \rangle : X_n \to X_{n+1})_n$ in $Cqum^\epsilon$, with X_n finite for all $n \geq 0$, and arrows $(\langle \alpha_n, \beta_n \rangle : X_n \to X)_n$ in $Cqum^\epsilon$ such that X is a colimit of Δ. For every $n \geq 0$ and every a in X_n, $\alpha_n(a)$ is finite in X.*

Proof: Let $n \geq 0$ and a in X_n. We have to show that $X(\alpha_n(a), -)$ is continuous, i.e., for every limit $x = \lim_k x_k$ in X,

$$X(\alpha_n(a), x) = \lim_k X(\alpha_n(a), x_k).$$

Let $x = \lim_k x_k$ in X and $\epsilon > 0$. By Corollary 9.5, there exists $N > n$ such that $\alpha_N \dashv_\epsilon \beta_N$. Let $f : X_n \to X_N$ is defined by $f = f_{N-1} \circ \cdots \circ f_n$. Note that because X_N is finite, all its elements are finite; in particular, $f(a)$ is. Now

$$X(\alpha_n(a),\ x) \quad = \quad X(\alpha_N(f(a)),\ x)$$

$$
\begin{aligned}
&\approx_\epsilon \quad X_N(f(a),\ \beta_N(x)) \\
&= \quad (\beta_N \text{ is continuous and } x = \lim_k x_k) \\
&\quad\quad X_N(f(a),\ \lim_k \beta_N(x_k)) \\
&= \quad (f(a) \text{ is finite in } X_N) \\
&\quad\quad \lim_k X_N(f(a),\ \beta_N(x_k)) \\
&\approx_\epsilon \quad \lim_k X(\alpha_N(f(a)),\ x_k) \\
&= \quad \lim_k X(\alpha_n(a),\ x_k).
\end{aligned}
$$

Since ϵ was arbitrary it follows that $X(\alpha_n(a), x) = \lim_k X(\alpha_n(a), x_k)$. □

Let X be as above and $x \in X$. For all $n \geq 0$, $\alpha_n \circ \beta_n(x)$ is finite in X according to the above lemma. Because $x = \lim \alpha_n \circ \beta_n(x)$, it follows that X is ω-algebraic.

THEOREM 11.2 *If a complete quasi ultrametric space is SFU then it is totally-bounded.*

PROOF: Consider a Cauchy chain $(\langle f_n, g_n\rangle : X_n \to X_{n+1})_n$ in $Cqum^\epsilon$, with X_n finite for all $n \geq 0$, and arrows $(\langle \alpha_n, \beta_n\rangle : X_n \to X)_n$ in $Cqum^\epsilon$ such that X is a colimit of Δ. Let $\epsilon > 0$. Let N be such that $\alpha_N \dashv_\epsilon \beta_N$. Define $E = \{\alpha_N(a) \in X \mid a \in X_N\}$. By Lemma 11.1, all these elements are finite. Let x be any element of X. Then $\alpha_N \circ \beta_N(x) \in E$ and

$$X(\alpha_N \circ \beta_N(x),\ x) \approx_\epsilon X_N(\beta_N(x),\ \beta_N(x)) = 0,$$

thus $x \in \bar{B}_\epsilon(\alpha_N \circ \beta_N(x))$. This proves that E is a finite ϵ-cover for X. □

THEOREM 11.3 *Let X be a complete generalized ultrametric space. If X is a complete ordinary ultrametric space, then*

X is SFU iff X is compact.

PROOF: Let X be a complete ordinary ultrametric space, and suppose X is SFU. By Theorem 11.2, X is totally-bounded. Since any ordinary (ultra)metric space is compact if and only if it is complete and totally bounded, this implies that X is compact.

Conversely, let X be a compact ordinary ultrametric space. Thus X is complete and totally-bounded. Let $(\epsilon_n)_n$ be a decreasing sequence of real numbers with $\lim \epsilon_n = 0$. Because X is totally-bounded there are finite subsets $(X_n)_n$ of X such that, for every $n \geq 0$, X_n is an ϵ_n-cover for X. Every X_n is a finite complete quasi ultrametric space with the metric inherited from X. By ultrametricity, the collection

$$P_n = \{\bar{B}_{\epsilon_n}(b) \mid b \in X_n\}$$

is a partitioning of X, for every $n \geq 0$, and P_n is refined by P_{n+1}. Let $f_n : X_n \to X_{n+1}$ send $a \in X_n$ to the (uniquely determined) element b in X_{n+1} with $a \in \bar{B}_{\epsilon_{n+1}}(b)$. In the other direction, let $g_n : X_{n+1} \to X_n$ map an element b of X_{n+1} to the (uniquely determined) element a in X_n with $b \in \bar{B}_{\epsilon_n}(a)$. It follows from the ultrametricity of X that f_n and g_n are non-expansive (which—for ordinary metric spaces—implies continuity). Moreover $f_n \dashv_{\epsilon_n} g_n$, since $g_n \circ f_n = 1_{X_n}$ and $X_{n+1} \longrightarrow X_{n+1}(f_n \circ g_n, 1_{X_{n+1}}) \leq \epsilon_n$. Thus we have defined a chain $(\langle f_n, g_n \rangle : X_n \to X_{n+1})_n$ in $Cqum^\epsilon$. It is Cauchy because $\lim \epsilon_n = 0$. The space X can be turned into a colimiting cone of this chain as follows. For $n \geq 0$ let $\alpha_n : X_n \to X$ be the inclusion. In the other direction, let β_n map x in X to the (uniquely determined) element a in X_n with $x \in \bar{B}_{\epsilon_n}(a)$. This defines a cone $(\langle \alpha_n, \beta_n \rangle : X_n \to X)_n$ in $Cqum^\epsilon$. It is colimiting because $\lim \alpha_n \circ \beta_n = 1_X$ and $\beta_n \circ \alpha_n = 1_{X_n}$, for every $n \geq 0$. This proves that X is SFU. □

The last part of the proof above refines a similar topological fact stating that any compact ordinary ultrametric space is the inverse limit of a sequence of finite discrete spaces (cf. [Smy92]).

12. A large generalized ultrametric space

We shall see that the class $\mathcal{G}$ of all generalized ultrametric spaces, which can be obtained from the *category Gums* by 'forgetting' the arrows, can be turned into a large generalized ultrametric space. A number of categorical definitions and facts of the previous sections will be rephrased in terms of this metric. For the special case of the class of *compact ordinary* (ultra)metric spaces, this will lead to a non-categorical fixed point theorem. The latter result, which has been independently obtained by F. Alessi, P. Baldan and G. Bellè, is only mentioned here. For a proof we refer to [ABBR95].

A generalized ultrametric on $\mathcal{G}$ is defined, for gums X and Y, by

$$\mathcal{G}(X,Y) = \inf\{\epsilon \mid \exists \langle f, g \rangle : X \to Y, \; f \vdash_\epsilon g\}.$$

(As in Section 9, $\langle f, g \rangle$ is here a pair of non-expansive and continuous mappings $f : X \to Y$ and $g : Y \to X$.) The proof that this defines a generalized ultrametric is not difficult and therefore omitted.

The metric structure on $\mathcal{G}$ gives rise to the following observations.

1. Cauchy *chains* (as in the category $Cqum^\epsilon$) are simply Cauchy *sequences* in $\mathcal{G}$.
2. A *locally non-expansive functor* on the category *Gums* is a *non-expansive mapping* on $\mathcal{G}$. Similarly, a *locally contractive functor* is a *contractive mapping* on $\mathcal{G}$.
3. Question: Is the subclass $\mathcal{C}$ of $\mathcal{G}$, consisting of all complete quasi ultrametric spaces, with the metric inherited from $\mathcal{G}$, complete (in the metric sense of the word, that is)? We don't know. Nor do we have an answer to the following.
4. Question: are *locally-continuous functors* on the category $Cqum^\epsilon$ *continuous* mappings on $\mathcal{C}$?

For complete *ordinary* ultrametric spaces, the answer to both questions 3 and 4 is affirmative. Completeness follows from the observation that for any Cauchy sequence of complete ordinary ultrametric spaces, a (categorical) colimit can be constructed as in Theorem 9.3, which is then readily seen to be a (metric) limit. For the subclass $\mathcal{K}$ of compact ordinary (ultra)metric spaces, this leads to a non-categorical fixed point theorem: any contractive (large) mapping (which need not be functorial) from $\mathcal{K}$ to itself has a fixed point which is unique up to isomorphism. (This follows from the fact that $\mathcal{K}$ itself is a large complete pseudo ultrametric space, with the additional property: if two compact spaces have distance 0 then they are isomorphic. Hence Banach's theorem can be applied as usual. For a full proof see [ABBR95].)

The idea of viewing the category of quasi metric spaces as a (large) quasi metric space is already present in [Ken87], though the metric above, based on ϵ-adjoint pairs, is new.

13. Future research

Together with Marcello Bonsangue and Franck van Breugel, we are at present studying completion and powerdomains for generalized (ultra)metric spaces ([BBR95]). This involves amongst others an investigation of suitable topologies to characterize powerdomains as collections of subsets that are closed (or compact) with respect to these topologies. This might also shed some light on the somewhat ad hoc notion of finitely-boundedness introduced here. Once functors for powerdomains will have been defined and shown to posses the usual desirable properties (such as local continuity), we intend to return to the study of nondeterministic transition systems and simulations, our initial source of inspiration.

Furthermore, we feel that the category of all SFU domains and ϵ-adjoint pairs between them deserves further study. For instance, it is to be investigated whether it is Cartesian closed, and whether it is will be closed under the powerdomain constructions (in preparation) of [BBR95]. There is reason to be optimistic about this, since both the subcategories of SFP domains and of compact ordinary ultrametric spaces have these properties.

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