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The Workload in the M/G/1 Queue with Work Removal

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Abstract

We consider an M/G/1 queue with the special feature of additional negative customers, who arrive according to a Poisson process. Negative customers require no service, but at their arrival a stochastic amount of work is instantaneously removed from the system. We show that the workload distribution in this M/G/1 queue with negative customers equals the waiting time distribution in a GI/G/1 queue with ordinary customers only; the effect of the negative customers is incorporated in the new arrival process.

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1. Introduction

Queueing systems with negative customers were recently introduced by Gelenbe (1991). In contrast with the ordinary customers, upon arrival to a queue a negative customer removes one ordinary customer from the queue. Negative customers can, for example, be interpreted as work removal signals in production networks, as inhibitor signals in neural networks, and as synchronisation signals in parallel computation. In the latter application, negative customers may also indicate the breakdown of a processor and the resulting destruction of work.

Since their introduction, negative customers have been studied by many authors. Product form results for the equilibrium distribution of the number of ordinary customers in networks of queues with exponential service and negative customers were,

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amongst others, derived by Gelenbe (1991), Boucherie and van Dijk (1994), and Henderson (1993). Ergodicity is addressed in Gelenbe *et al.* (1991) for the M/G/1 queue, and for product form networks with exponential service in Gelenbe and Schassberger (1992). For the M/G/1 queue with negative customers the queue length distribution is analysed by Harrison and Pitel (1994). Their analysis of the generating function for the equilibrium queue length distribution leads to a Fredholm integral equation of the first kind, that must be solved numerically. This integral equation gives rise to intricate numerical difficulties, as its numerical solution is a so-called ill-posed problem. A more tractable M/G/1 system results when a negative customer removes all the work (hence all the customers) from the system; this model is analysed by Jain and Sigman (1994).

In the present paper we analyse a generalization of the model of Jain and Sigman (1994): the M/G/1 queue with negative customers, in which a negative customer removes a *random amount of work* that does not necessarily correspond to an integer number of customers. The analysis leads to a Wiener-Hopf equation for the Laplace-Stieltjes transform of the equilibrium distribution of the workload in the queue. The Wiener-Hopf equation can be analysed using standard methods. In addition to the direct analysis leading to a Wiener-Hopf equation we also show that, for the analysis of the workload, the M/G/1 queue with negative customers can be transformed into a GI/G/1 queue with only ordinary customers. The effect of the negative customers is incorporated in the new *arrival* process. This is rather surprising, since intuitively the effect of negative customers corresponds to a change in *service* effort.

The mathematical accessibility of our model, compared with that of Harrison and Pitel (1994), and the resulting new insight into the latter model, represents part of the motivation for the study of the amount of *work* in the system. Furthermore, the model is related to, and of relevance for, inventory theory and risk theory (cf. Prabhu (1980)). In inventory systems an instantaneous removal of inventory occurs, e.g., when perishable goods are stored; such a removal usually depends on the length of time that the goods have been in the system. In risk theory the net gain of an insurance company increases linearly (contrary to the linearly *decreasing* workload in a queue), with downward jumps due to claims and upward jumps due to the death of a life-insurance policyholder. The ruin problem, which is central to risk theory, also gives rise to a Wiener-Hopf problem, cf. Cramér (1955).

The model with removal of a random amount of work according to a Poisson process is a special case of the M/G/1 queue, or dam, with as output process a process with stationary independent increments. The latter M/G/1 queue was studied by Grinstein and Rubinovitch (1974); see also Gani and Pyke (1960), and the fundamental studies of Rogozin concerning processes with independent increments (see for example Rogozin (1966), which was kindly mentioned to us by A.A. Borovkov). Boxma (1975) considered the generalization of the model of Grinstein and Rubinovitch (1974) to the GI/G/1 case. The model presently under consideration is sufficiently simple to allow a much more detailed analysis than the above references.

The paper is organised as follows. In Section 2 we present the model, and in Section 3 we use level crossing arguments to derive a Wiener-Hopf equation for the Laplace-Stieltjes transform of the work in the system. In Section 4 we introduce the transformation of the M/G/1 queue with negative customers into a standard GI/G/1 queue. Via PASTA the workload at arrival epochs for this GI/G/1 queue determines the workload at arbitrary times in the M/G/1 queue. Section 5 contains several examples, including the case in which a negative customer removes *all* the work from the queue (Jain and Sigman (1994)), and the case in which a negative customer removes an amount of work corresponding to the service requirement of a customer (Harrison and Pitel (1994)). The sojourn time distribution is also analysed in Section 5.

2. The model

Consider the M/G/1 queue. Customers arrive to the queue according to a Poisson process with rate λ^+. We shall refer to these customers as ordinary or positive customers. Their service requirement has an absolutely continuous distribution $B(\cdot)$, with finite mean β, $B(0+) = 0$, and Laplace-Stieltjes transform $\beta(s) := \int_0^\infty \mathrm{e}^{-sx} dB(x)$, which exists for $Re\ s \geq 0$.

The server works at unit rate when customers are present. In addition to the ordinary customers, negative customers arrive to the queue with Poisson(λ^-) arrival rate. These negative customers reduce the amount of work in the system according to an absolutely continuous distribution $C(\cdot)$, with mean γ, $C(0+) = 0$, and Laplace-Stieltjes transform $\gamma(s) := \int_0^\infty \mathrm{e}^{-sx} dC(x)$, which exists for $Re\ s \geq 0$. Assume that $\lambda^+\beta < 1 + \lambda^-\gamma$; this is the necessary and sufficient *ergodicity condition* for the queue, as will be shown in Lemma 4.1 below.

Let v_t denote the workload in the system at time t, i.e., the sum of the remaining service time of the customer in service and the (possibly reduced) service times of the waiting customers. Clearly, v_t is independent of the service discipline, as well as of the discipline used to reduce the amount of work upon arrival of a negative customer. Let $V_t(x, v) := \mathbf{P}(v_t < x | v_0 = v)$, $x \geq 0$, $v \geq 0$, be the distribution of the workload in the queue at time t. As is shown in Lemma 4.2, if $\lambda^+\beta < 1 + \lambda^-\gamma$ then the workload distribution $V_t(x, v)$ approaches the - unique - equilibrium version $V(x)$ for $t \to \infty$.

In this paper we establish a relation for $V(\cdot)$. To this end, we use two methods. The first method uses level crossing arguments (cf. Brill and Posner (1977), Cohen (1977)) to establish a Wiener-Hopf equation for $V(\cdot)$. This Wiener-Hopf equation can be analysed following standard methods (cf. Zabreyko *et al.* (1975)). The second method is based on an interpretation of the arrival process of positive and negative customers. The M/G/1 queue with negative customers is transformed into an equivalent GI/G/1 queue with only positive customers. This allows us to analyse the M/G/1 queue with negative customers via standard arguments for the GI/G/1 queue as presented in Cohen (1982).

3. Derivation of the Wiener-Hopf equation using level crossings

We shall derive a Wiener-Hopf integral equation for the workload distribution, using a level crossing argument (cf. Brill and Posner (1977)). Upcrossings of level x occur when a positive customer arrives at the queue. Due to the assumption of Poisson arrivals and stationarity, the long run average rate of jumps starting at level y is $\lambda^+ dV(y)$. The proportion of jumps starting at level $y < x$ that end above level x is $1 - B(x - y)$, the probability that the required service exceeds $x - y$. Thus, the average rate of upcrossings of level x is $\lambda^+ \int_{0-}^{x} (1 - B(x - y))dV(y)$. Similarly, the long run average rate of downward jumps starting at level y is $\lambda^- dV(y)$, and the proportion of jumps starting at level $y > x$ that end below level x is $1 - C(y - x)$, the probability that the amount of removed work exceeds $y - x$. Thus, the average rate of downcrossings of level x due to jumps is $\lambda^- \int_{x}^{\infty} (1 - C(y - x))dV(y)$. Let $v(x)$ denote the stationary density function of the workload ($v(x)$ can be shown to exist for $x > 0$). Then $v(x)$ equals the average rate of downcrossings of the level x at points of continuity of the sample functions of v_t. Collecting terms, and equating the rate of up- and downcrossings, gives the following Wiener-Hopf integral equation for the workload:

$$v(x) = \int_{0-}^{\infty} k(x - y)dV(y) \tag{3.1}$$

$$= \lambda^+ V(0+)(1 - B(x)) + \int_{0}^{\infty} k(x - y)v(y)dy, \quad x > 0, \tag{3.2}$$

with

$$k(x) := \lambda^+(1 - B(x))\mathbf{1}(x > 0) - \lambda^-(1 - C(-x))\mathbf{1}(x \le 0), \quad -\infty < x < \infty,$$

($\mathbf{1}(\cdot)$ denoting the indicator function), and with normalisation condition

$$\int_{0-}^{\infty} dV(y) = V(0+) + \int_{0}^{\infty} v(y)dy = 1.$$

Equation (3.1), or (3.2), is the Wiener-Hopf integral equation for the workload. It is in the standard Wiener-Hopf form, that can be analysed following methods as described in chapter VIII of Zabreyko *et al.* (1975). The usual approach is to extend (3.1) to $x < 0$, and to subsequently apply Fourier or Laplace-Stieltjes transforms.

Define new variables $V_+(x)$, $v_+(x)$, and $v_-(x)$, as

$$\begin{aligned}
V_+(x) &:= V(x), & x \ge 0, \quad & V_+(x) := 0, & x < 0,\\
v_+(x) &:= v(x), & x > 0, \quad & v_+(x) := 0, & x \le 0,\\
v_-(x) &:= 0, & x > 0, \quad & v_-(x) := \int_{0-}^{\infty} k(x - y)dV(y), & x \le 0.
\end{aligned}$$

Then (3.1) can be written as

$$v_+(x) + v_-(x) = \int_{-\infty}^{\infty} k(x-y)dV_+(y), \quad -\infty < x < \infty. \tag{3.3}$$

Define

$$\begin{aligned} \varphi_+(s) &:= \int_{0-}^{\infty} \mathrm{e}^{-sx} dV_+(x), \quad Re\ s \geq 0, \\ \varphi_-(s) &:= \int_{-\infty}^{0} \mathrm{e}^{-sx} v_-(x)dx, \quad Re\ s \leq 0. \end{aligned}$$

It can easily be verified that $\varphi_+(s)$ and $\varphi_-(-s)$ are bounded and analytic for $Re\ s > 0$ and continuous for $Re\ s \geq 0$. From (3.1) we obtain that

$$(1 - K(s))\varphi_+(s) = V(0+) - \varphi_-(s), \quad Re\ s = 0, \tag{3.4}$$

where

$$K(s) := \int_{-\infty}^{\infty} k(x)\mathrm{e}^{-sx}dx, \quad Re\ s = 0.$$

From the definition of $k(\cdot)$ it follows that

$$K(s) = \lambda^+ \frac{(1-\beta(s))}{s} - \lambda^- \frac{(1-\gamma(-s))}{-s}, \quad Re\ s = 0. \tag{3.5}$$

We have to construct two functions $\varphi_+(s)$ and $\varphi_-(s)$ that satisfy (3.4) as well as the boundedness and analyticity conditions formulated above, and

$$\lim_{s \downarrow 0} \varphi_+(s) = 1, \quad \lim_{\substack{|s|\to\infty \\ \arg s = 0}} \varphi_+(s) = V(0+), \quad \lim_{\substack{|s|\to\infty \\ \arg s = \pi}} \varphi_-(s) = 0.$$

Thus, we have to consider the Wiener-Hopf factorisation of the kernel $1 - K(s)$. This problem can be solved formally using the Wiener-Hopf theory, cf. Cohen (1982), Section I.6.6 or II.6.3. The idea is to factorize: $1 - K(s) = G_+(s)/G_-(s)$, where $G_+(s)$ is analytic in the right half plane and $G_-(s)$ is analytic in the left half plane. Then

$$G_+(s)\varphi_+(s) = G_-(s)[V(0+) - \varphi_-(s)], \quad Re\ s = 0, \tag{3.6}$$

where the left-hand side is analytic and bounded for $Re\ s > 0$ and the right-hand side is analytic and bounded for $Re\ s < 0$, and the sides are equal for $Re\ s = 0$. Thus the left-hand side and right-hand side are analytic continuations of each other. From Liouville's theorem and the behaviour of both sides at infinity, both sides are determined, and hence so is $\varphi_+(s)$. Explicit expressions for the above factorization, and hence for $\varphi_+(s)$ and $\varphi_-(s)$, are obtained when either $\beta(s)$ or $\gamma(s)$ is a rational function of s. In Section 5.1 we work out the details for the latter case. But first we show, in Section 4, that - for the analysis of the workload - the M/G/1 queue with negative customers can be transformed into a GI/G/1 queue with only ordinary customers.

4. Transformation into an equivalent GI/G/1 queue

For the M/G/1 queue Poisson arrivals see time averages. Since the arrival process of positive customers is a Poisson process that is independent of the state of the queue and of the arrival process of negative customers, this property remains valid for the M/G/1 queue with negative customers (cf. Wolff (1989), p. 294). As a consequence, the amount of work found by a positive customer arriving to the queue equals in distribution the steady-state amount of work in the system. The amount of work found by such customers can be analysed via a transformation of the M/G/1 queue into a GI/G/1 queue with positive customers only. This transformation is based on the observation that the influence of negative customers can be seen as a lengthening of the interarrival times for positive customers.

Let τ_n denote the interarrival time between the nth and $(n+1)$st positive customers in the original M/G/1 queue, and let σ_n be the required amount of service of positive customer n. Then $\{\tau_n\}_n$ and $\{\sigma_n\}_n$ are independent sequences of random variables (r.v.'s), and each of these sequences consists of independent and identically distributed (iid) random variables. Let w_n denote the amount of work in the system found by customer n. The amount of work, w_{n+1}, found by the next arriving positive customer, is then given by

$$w_{n+1} = \max(w_n + \sigma_n - \tau_n - d_n, 0), \tag{4.1}$$

where d_n is the amount of work destroyed by the negative customers which arrive during τ_n.

The arrival processes of the positive and negative customers are independent Poisson processes with parameters λ^+ and λ^- respectively. Thus, the probability that exactly k negative customers arrive during an interarrival time of positive customers is $(1-p)p^k$, where $p := \lambda^-/(\lambda^+ + \lambda^-)$, $k = 0, 1, 2, \ldots$. Let K_n denote the number of negative customers arriving during the interarrival time τ_n. The amount of work destroyed by the jth negative customer arriving during this interarrival time is denoted by C_j^n, and has distribution $C(\cdot)$. As a consequence $\{d_n\}_n$ is an iid sequence, where d_n has the same distribution as $\sum_{j=1}^{K_n} C_j^n$. It is obvious that $\{d_n\}_n$ is independent of $\{\sigma_n\}_n$, but that τ_n and d_n are dependent.

From (4.1) we obtain that the amount of work found in the queue by an arriving positive customer in the M/G/1 queue with negative customers corresponds to the waiting time of a customer arriving in a GI/G/1 queue with required service times $\{\sigma_n\}_n$ and interarrival times $\{\tau_n^*\}_n$, where

$$\tau_n^* := \tau_n + d_n = \tau_n + \sum_{j=1}^{K_n} C_j^n. \tag{4.2}$$

The following lemma states the equivalence formally. Furthermore, this lemma states the ergodicity condition for the GI/G/1 queue, and therefore also for the M/G/1 queue with negative customers.

Lemma 4.1 *The amount of work found by arriving customers in the M/G/1 queue with Poisson(λ^+) arrival rate of positive customers with service requirements σ_n and Poisson(λ^-) arrival rate of negative customers, which destroy an amount of work according to the distribution $C(\cdot)$, has the same distribution as the amount of work found by arrivals in the GI/G/1 queue with interarrival times τ_n^* described in (4.2) and service requirements σ_n. This queue is ergodic if and only if $\lambda^+\beta < 1 + \lambda^-\gamma$.*

Proof The equivalence between the M/G/1 queue with negative customers and the GI/G/1 queue with interarrival times τ_n^* and service times σ_n is obvious from the preceding discussion.
The r.v.'s K_n and C_j^n are independent. Therefore, $\mathbf{E}\tau_n^* = \mathbf{E}\tau_n + \mathbf{E}K_n\mathbf{E}C_j^n = \frac{1}{\lambda^+} + \frac{\lambda^-}{\lambda^+}\gamma$.
The ergodicity condition for the GI/G/1 queue is $\mathbf{E}\tau_n^* < \mathbf{E}\sigma_n$. □

Lemma 4.1 shows that the number of customers served in a busy cycle for the M/G/1 queue with negative customers is distributed as the same quantity for the GI/G/1 queue with interarrival times τ_n^* and service requirements σ_n. Cohen (1982), Section II.5.5, analyses the generating function of the number of customers served in a busy cycle for the GI/G/1 queue. The stability condition $\lambda^+\beta < 1 + \lambda^-\gamma$ guarantees that the number of customers served in a busy cycle is finite a.s. and has finite first moment (Cohen (1982)). It also follows that the length of a busy cycle for the M/G/1 queue with negative customers is finite a.s. and has finite first moment.

The following result justifies using the analysis of the workload found by arrivals to the GI/G/1 queue to determine the steady-state workload for the M/G/1 queue with negative customers.

Lemma 4.2 *Under the ergodicity condition $\lambda^+\beta < 1+\lambda^-\gamma$ the workload v_t converges in distribution to the a.s. finite r.v. that is distributed as the stationary workload found by customers arriving to the GI/G/1 queue with interarrival times τ_n^* and service requirements σ_n.*

Proof Under the ergodicity condition the workload process for the M/G/1 queue with negative customers is a regenerative process with regeneration points the time points at which an ordinary customer arrives to an empty system. The cycle length has non-lattice distribution with finite mean. From standard results on regenerative processes (e.g. Asmussen (1987), Section V.1) we obtain that v_t converges in distribution to a r.v. v. The PASTA property (cf. Wolff (1989), p. 294) subsequently implies that v is distributed as the stationary workload found by customers arriving to the GI/G/1 queue with interarrival times τ_n^* and service requirements σ_n. □

The waiting time in the stationary GI/G/1 queue can be analysed via the Wiener-Hopf technique, as described in Cohen (1982), p. 338. Recall that w_n is distributed as the waiting time of customer n arriving to the GI/G/1 queue with interarrival times $\{\tau_n^*\}_n$ and service times $\{\sigma_n\}_n$.

Following Cohen, we define $W(x) := \lim_{n\to\infty} \mathbf{P}(w_n < x | w_1 = w)$, which is independent of w, and

$$\begin{aligned} &W_+(x) := W(x), \quad W_-(x) := 0, && x \geq 0, \\ &W_+(x) := 0, \qquad W_-(x) := \int_{-\infty}^{x} W(x-u)dU(u), && x < 0, \end{aligned}$$

where $U(\cdot)$ is the distribution of $\sigma_n - \tau_n^*$. The Laplace-Stieltjes transforms

$$\begin{aligned} \omega_+(s) &:= \int_{0-}^{\infty} \mathrm{e}^{-sx} dW_+(x), \quad Re\ s \geq 0, \\ \omega_-(s) &:= \int_{-\infty}^{0-} \mathrm{e}^{-sx} dW_-(x), \quad Re\ s \leq 0, \end{aligned}$$

exist for $Re\ s = 0$, and $\omega_+(s)$ and $\omega_-(-s)$ are bounded and analytic for $Re\ s > 0$ and continuous for $Re\ s \geq 0$. The problem is reduced to the construction of two such functions $\omega_+(s)$, $\omega_-(s)$ that satisfy

$$\omega_+(s)\{1 - \beta(s)\alpha(-s)\} = -\omega_-(s), \quad Re\ s = 0, \tag{4.3}$$

and

$$\lim_{s\downarrow 0} \omega_+(s) = 1, \quad \lim_{\substack{|s|\to\infty \\ \arg s = 0}} \omega_+(s) = W_+(0), \quad \lim_{\substack{|s|\to\infty \\ \arg s = \pi}} \omega_-(s) = 0, \tag{4.4}$$

where

$$\alpha(s) := \int_{0-}^{\infty} \mathrm{e}^{-st} d\mathbf{P}(\tau_n^* < t), \quad Re\ s \geq 0.$$

As a consequence, we have to consider the Wiener-Hopf decomposition of the kernel $1 - \beta(s)\alpha(-s)$.

For *generally* distributed interarrival times τ_n we now proceed with the computation of $\alpha(\cdot)$. To this end, let $F(\cdot)$ be the distribution of the interarrival times, with Laplace-Stieltjes transform $\tau(\cdot)$. Then

$$\begin{aligned} \alpha(s) &= \sum_{k=0}^{\infty} \int_{t=0-}^{\infty} \mathrm{e}^{-st} \int_{x=0}^{t} dF(x) d\mathbf{P}(\tau_n^* < t, K_n = k | \tau_n = x) \\ &= \sum_{k=0}^{\infty} \int_{x=0-}^{\infty} \mathrm{e}^{-sx} dF(x) \int_{t=x}^{\infty} \mathrm{e}^{-s(t-x)} d\mathbf{P}(\sum_{j=1}^{k} C_j^n < t - x) \mathbf{P}(K_n = k | \tau_n = x) \\ &= \tau(s + \lambda^-(1 - \gamma(s))), \quad Re\ s \geq 0. \end{aligned} \tag{4.5}$$

For the M/G/1 queue with Poisson(λ^+) arrivals of positive customers, we have $\tau(s) = \lambda^+/(\lambda^+ + s)$. This gives

$$\alpha(s) = \frac{\lambda^+}{\lambda^+ + s + \lambda^-(1-\gamma(s))}, \quad Re\ s \geq 0. \tag{4.6}$$

The explicit expression for $\alpha(s)$ allows us to obtain the Wiener-Hopf decomposition of the kernel $1-\beta(s)\alpha(-s)$ explicitly under the assumption that either $\gamma(\cdot)$ or $\beta(\cdot)$ is rational. This is illustrated in the examples in the next section.

We end the present section with some remarks.

Remark 4.1 The amount of work found by customers arriving to a GI/G/1 queue is not affected by a change of the length of the idle times as long as this change is independent of the service requirements and the interarrival times of customers in the subsequent busy period. This observation allows us to further generalise the result. For example, a negative customer that reduces the amount of work to zero may temporarily prevent the arrival of positive customers to the queue. Let customer N_i be the first customer arriving after the ith time that the amount of work in the system is reduced to 0 due to a negative customer removing all work from the queue. Let the r.v. D_i denote the length of time that the arrival process is stopped. Assume that $\{D_i\}_i$ is independent of $\{\tau_n, \sigma_n\}_n$. Then the interarrival time τ_{N_i} of positive customer N_i is changed into $\tau^\dagger_{N_i} = \tau_{N_i} + D_i$, and the interarrival times of the other customers are unaffected. It is a simple matter to verify that the distributions of the amounts of work found by customers with interarrival times $\{\tau_n\}_n$ and $\{\tau^\dagger_n\}_n$ are identical.

Remark 4.2 The transformation to a standard GI/G/1 queue only makes use of the memoryless property of the interarrival times of *negative* customers. As a consequence, this transformation can be extended to analyse the amount of work found by ordinary customers arriving to a GI/G/1 queue where additional negative customers arrive according to a Poisson(λ^-) process.
Let $\{\tau_n\}_n$ be the interarrival times of positive customers. The r.v.'s K_n and C^n_j can be defined as before, and the amount of work d_n removed by negative customers during τ_n can be computed for given τ_n. Furthermore, equation (4.2) remains valid. The Laplace-Stieltjes transform of the distribution of the new interarrival times τ^*_n is given in (4.5) since the derivation of this result was given for generally distributed interarrival times τ_n. We have the following generalisation of Lemma 4.1:
The amount of work found by arriving customers in the GI/G/1 queue with interarrival times τ_n for positive customers with service requirements σ_n and Poisson(λ^-) arrival rate of negative customers, which destroy an amount of work according to the distribution $C(\cdot)$, has the same distribution as the amount of work found by arrivals (the waiting time) in the GI/G/1 queue with interarrival times τ^*_n described in (4.2) and service requirements σ_n. This queue is ergodic if and only if $\lambda^+\beta < 1 + \lambda^-\gamma$.
The result of Lemma 4.2 cannot be extended to the GI/G/1 queue with negative customers, since PASTA does not hold for this model.

Remark 4.3 In the extension of Remark 4.2 we have used the memoryless property of the interarrival times of negative customers. The following extension does not make use of this memoryless property. Let customers arrive to the queue with iid interarrival times $\{\tau_n\}_n$. Upon arrival to the queue a customer is declared to be positive with probability p and negative with probability $1-p$. A positive customer has service requirement σ_n with distribution $B(\cdot)$, and a negative customer removes an amount of work with distribution $C(\cdot)$. The amount of work found by a customer that is declared positive can again be related to the amount of work found by a customer arriving to a GI/G/1 queue. To this end, let customer N be declared positive. Let K_N be the number of customers that is declared negative before a customer is declared positive. Then $\mathbf{P}(K_N = k) = (1-p)^k p$, $k = 0, 1, 2, \ldots$. Let C_j^N be the amount of work that is removed by the jth negative customer, then the next positive customer finds an amount of work $w_{N+1} = \max(w_N + \sigma_N - \tau_N^*, 0)$, where the distribution of τ_N^* has Laplace-Stieltjes transform $(1-p)\tau(s)/[1-p\tau(s)\gamma(s)]$ (in the case of a Poisson$(\lambda^+ + \lambda^-)$ arrival process, this coincides with formula (4.6)). The analysis proceeds as before, but the result of Lemma 4.2 cannot be obtained here.

Remark 4.4 The model of Remark 4.3 can be slightly reformulated, by considering just *one* type of customer, the nth customer having service requirement $\hat{\sigma}_n$. With probability p customer n has a positive service requirement with distribution $B(\cdot)$, and with probability $1-p$ a negative service requirement with distribution $C(\cdot)$. Clearly $\hat{\beta}(s) := \mathbf{E}[e^{-s\hat{\sigma}_n}] = p\beta(s) + (1-p)\gamma(-s)$, $Re\ s = 0$. Let $\{\tau_n\}_n$ be the iid interarrival times, and let $\tau(s)$ denote the Laplace-Stieltjes transform of their distribution. The workload w_{n+1} found by customer $n+1$ is then given by $w_{n+1} = \max(w_n + \hat{\sigma}_n - \tau_n, 0)$. Now follow the derivation of the Laplace-Stieltjes transform for the workload in the GI/G/1 queue (cf. Cohen (1982); there service times are positive). Using the identity

$$e^{-s[x]^+} + e^{-s[x]^-} = e^{-sx} + 1,$$

where $[x]^+ := \max(0, x)$ and $[x]^- := \min(0, x)$, we find:

$$e^{-s\omega_{n+1}} = e^{-s(\omega_n + \hat{\sigma}_n - \tau_n)} + 1 - e^{-s[\omega_n + \hat{\sigma}_n - \tau_n]^-}.$$

Taking expectations one finds in the stationary case:

$$\mathbf{E}[e^{-sw_n}][1 - \hat{\beta}(s)\tau(-s)] = 1 - \mathbf{E}[e^{-s[w_n + \hat{\sigma}_n - \tau_n]^-}]. \tag{4.7}$$

Hence we have to consider the Wiener-Hopf factorisation of

$$1 - \hat{\beta}(s)\tau(-s) = 1 - \tau(-s)\{p\beta(s) + (1-p)\gamma(-s)\}, \quad Re\ s = 0.$$

Let us now restrict ourselves to the case of exponentially distributed interarrival times, and relate the above kernel to the kernels appearing in (3.4) and (4.3). We have $\tau(s) = (\lambda^+ + \lambda^-)/(\lambda^+ + \lambda^- + s)$, and $p = \lambda^+/(\lambda^+ + \lambda^-)$. This gives, with $Re\ s = 0$:

$$1-\hat{\beta}(s)\tau(-s)=\frac{\lambda^+(1-\beta(s))+\lambda^-(1-\gamma(-s))-s}{\lambda^++\lambda^--s}=-\frac{s(1-K(s))}{\lambda^++\lambda^--s}.$$

Now note that a factorization $1-K(s)=G_+(s)/G_-(s)$, as briefly discussed at the end of Section 3, immediately yields the factorization

$$1-\hat{\beta}(s)\tau(-s)=\frac{sG_+(s)}{G_-(s)(s-\lambda^+-\lambda^-)},$$

and hence, see (4.7),

$$sG_+(s)\mathbf{E}[e^{-sw_n}]=G_-(s)(s-\lambda^+-\lambda^-)(1-\mathbf{E}[e^{-s[w_n+\hat{\sigma}_n-\tau_n]^-}]).$$

We have essentially the same Wiener-Hopf problem as at the end of Section 3; application of Liouville's theorem will confirm the equality of $\mathbf{E}[e^{-sw_n}]$ and $\varphi_+(s)$. Hence the workload at arrival epochs for the model of the present remark (with positive and negative service requirements) has the same equilibrium distribution as the workload in the model of Section 3.
Similarly, the kernel $1-K(s)$ appearing in (3.4) can be related to the kernel $1-\beta(s)\alpha(-s)$ that appears in (4.3) for the GI/G/1 model with extended interarrival times:

$$1-\beta(s)\alpha(-s)=\frac{s(1-K(s))}{s-\lambda^+-\lambda^-(1-\gamma(-s))}=-\frac{s(1-K(s))\alpha(-s)}{\lambda^+}. \tag{4.8}$$

This leads to the factorization

$$sG_+(s)\omega_+(s)=\lambda^+\frac{G_-(s)}{\alpha(-s)}\omega_-(s), \quad Re\ s=0.$$

Note that $\alpha(-s)$, as given in (4.6), cannot become zero for finite s, $Re\ s<0$, since $\gamma(s)$ is the Laplace-Stieltjes transform of a probability distribution. Solving the Wiener-Hopf problem then confirms what has already been shown in Lemma 4.2: $\varphi_+(s)=\omega_+(s)$, $Re\ s\geq 0$, and thus the equilibrium distribution $V(\cdot)$ of the workload analysed in Section 3 equals the equilibrium distribution $W(\cdot)$ of the waiting time of a customer in the equivalent GI/G/1 queue.

5. Examples

The above-indicated equivalence between $\varphi_+(s)$, $\omega_+(s)$ and $\mathbf{E}[e^{-sw_n}]$ allows us to arbitrarily consider any one of them. Let us concentrate for the moment on $\omega_+(s)$, hence on the GI/G/1 queue with extended interarrival times. For rational $\beta(\cdot)$ the analysis follows the standard lines for the G/K$_n$/1 queue as presented in Cohen (1982), Section II.5.10. Below we present several examples for which the zeros of the kernel of (4.3) can be found. It is shown in Section 5.1 that for work removal ('killing') distributions with rational Laplace-Stieltjes transform the analysis of the M/G/1 queue with negative customers follows the lines of the K_m/G/1 queue. Section 5.2 studies systems with complete breakdowns that remove all the work from the system. Here the results of Jain and Sigman (1994) are reproduced. In Section 5.3 removal of customer equivalents is discussed. It is shown that the choice $\gamma(\cdot) = \beta(\cdot)$, (killing = service) generally does *not* lead to a good correspondence between the model with work removal and the model with customer removal. This example also allows us to make some comments on the relation between the workload and waiting times of positive customers in queues with negative customers. The investigation of the sojourn time is continued in Section 5.4, where we study the time it takes before an amount of work is removed from the system by service and killing.

5.1 Killing distributions with rational transform

Rationality of $\gamma(s)$ implies rationality of $\alpha(s)$, cf. (4.6). Suppose that $\gamma(s) = \gamma_1(s)/\gamma_2(s)$, where $\gamma_2(\cdot)$ is a polynomial of degree m, and $\gamma_1(\cdot)$ a polynomial of degree less than m, such that $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ have no common zeros; normalize $\gamma_2(s)$ such that $\gamma_2(0) = 1$. Then

$$\alpha(s) = \frac{\lambda^+\gamma_2(s)}{(\lambda^+ + s)\gamma_2(s) + \lambda^-(\gamma_2(s) - \gamma_1(s))}, \quad Re\ s \geq 0.$$

Thus $\alpha(s) = \alpha_1(s)/\alpha_2(s)$, where $\alpha_1(\cdot)$ is a polynomial of degree m and $\alpha_2(\cdot)$ is a polynomial of degree $m+1$. $\alpha(-s)$ is analytic for $Re\ s < 0$. This implies that $\alpha_2(-s)$ has no zeros for $Re\ s < 0$, i.e., that $\alpha_2(-s)$ has $m+1$ zeros, counted according to their multiplicity, for $Re\ s \geq 0$. The kernel can be written as

$$1 - \beta(s)\alpha(-s) = \frac{1}{\alpha_2(-s)}\left[\alpha_2(-s) - \beta(s)\alpha_1(-s)\right], \quad Re\ s = 0.$$

Apply Rouché's theorem (Cohen (1982), app. 6) to this kernel, on the closed contour Ω consisting of the line from $-\epsilon - i\infty$ to $-\epsilon + i\infty$ and the closing semicircle in the right half plane, for ϵ small enough. It then follows that, when the ergodicity condition holds, $\alpha_2(-s) - \beta(s)\alpha_1(-s)$ has exactly m zeros δ_i, $i = 1, \ldots, m$, with $Re\ \delta_i > 0$, and one zero $\delta_0 = 0$. This implies (Cohen (1982), pp. 329-330) that (4.3) can be written as

$$\omega_+(s)\frac{[\alpha_2(-s) - \beta(s)\alpha_1(-s)]}{s\prod_{i=1}^m(\delta_i - s)} = -\omega_-(s)\frac{\alpha_2(-s)}{s\prod_{i=1}^m(\delta_i - s)}, \quad Re\ s = 0,$$

where the left-hand side is analytic and bounded for $Re\ s > 0$ and the right-hand side is analytic and bounded for $Re\ s < 0$, and the sides are equal for $Re\ s = 0$. Thus the left-hand side and right-hand side are analytic continuations of each other. From Liouville's theorem and the bounded behaviour of both sides at infinity, cf. also (4.4), it follows that both sides are equal to the same constant c; hence in particular

$$\omega_+(s) = c\frac{s\prod_{i=1}^{m}(\delta_i - s)}{\alpha_2(-s) - \beta(s)\alpha_1(-s)}, \quad Re\ s \geq 0. \tag{5.1}$$

The constant c is determined from (4.4) by taking the limit $s \downarrow 0$ in (5.1). This gives

$$\omega_+(s) = \frac{\lambda^+\beta - 1 - \lambda^-\gamma}{\alpha_2(-s) - \beta(s)\alpha_1(-s)}\, s\prod_{i=1}^{m}\frac{\delta_i - s}{\delta_i}, \quad Re\ s \geq 0.$$

The case of exponential killing is a special case of the result above. We now have $\gamma(s) = 1/(1+\gamma s)$. The kernel has one zero $\delta_0 = 0$, and one zero δ_1 in the right half plane. If, in addition, we assume that the service requirements are exponentially distributed, then we can compute the value of δ_1: it is the unique positive root (use the ergodicity condition) of

$$\gamma\beta\delta_1^2 + (\gamma - \beta - \lambda^+\gamma\beta - \lambda^-\gamma\beta)\delta_1 + (-1 - \lambda^-\gamma + \beta\lambda^+) = 0. \tag{5.2}$$

In this example

$$\omega_+(s) = (1+\beta s)\frac{\zeta}{\zeta + s}, \quad Re\ s \geq 0,$$

where $\zeta = (1+\lambda^-\gamma - \lambda^+\beta)/(\beta\gamma\delta_1)$ is minus the (real) negative root of (5.2). Note that the Laplace-Stieltjes transform of the workload, $\varphi_+(s)$, equals $\omega_+(s)$. Hence

$$V(x) * B(x) = 1 - \mathrm{e}^{-\zeta x}, \quad x \geq 0. \tag{5.3}$$

5.2 System breakdown

In this example we assume that a negative customer removes all work in the system upon arrival. A negative customer can be seen as a complete breakdown of the system removing all work, or as a reset of the system. In addition, it is possible for the breakdown to add a subsequent repair period during which all arriving positive customers are lost. This will not affect the analysis.

For this example we have that $C(x) = 0$, $x < \infty$, with a point mass at infinity; hence $\gamma = \infty$. Following the analysis presented in Section 4, we obtain that the queue is ergodic for all values of $\lambda^+\beta$ if $\lambda^- > 0$. The Laplace-Stieltjes transform of $C(\cdot)$ is $\gamma(s) = 0$ for all $s < \infty$. Insertion of this expression into (4.6) gives

$$\alpha(s) = \lambda^+/(\lambda^+ + \lambda^- + s), \quad Re\ s \geq 0,$$

which corresponds to the expression obtained for rational $\gamma(\cdot)$ with $\gamma_1(\cdot) \equiv 0$, $\gamma_2(\cdot) \equiv 1$. The analysis for rational $\gamma(\cdot)$ of the previous subsection cannot be followed further.

The kernel now reads

$$1-\beta(s)\alpha(-s)=\frac{\lambda^{+}+\lambda^{-}-s-\beta(s)\lambda^{+}}{\lambda^{+}+\lambda^{-}-s},\quad Re\ s=0.$$

The numerator of this expression has one zero δ with $Re\ \delta>0$, the denominator has one zero. Formula (4.3) can be written as

$$\omega_{+}(s)\frac{\lambda^{+}+\lambda^{-}-s-\beta(s)\lambda^{+}}{s-\delta}=-\omega_{-}(s)\frac{\lambda^{+}+\lambda^{-}-s}{s-\delta},\quad Re\ s=0.$$

From Liouville's theorem and the bounded behaviour of both sides at infinity, it follows that both sides are equal to the same constant c; we thus obtain that

$$\omega_{+}(s)=c\frac{s-\delta}{\lambda^{+}+\lambda^{-}-s-\beta(s)\lambda^{+}},\quad Re\ s\geq 0, \tag{5.4}$$

where c can be determined from (4.4). Taking $s=0$ gives $c=-\lambda^{-}/\delta$. Taking the limit $s\to\infty$ we obtain that $\delta W(0)=\lambda^{-}$.

For the case of total removal of all work the Wiener-Hopf equation (3.1) can be analysed directly via Laplace-Stieltjes transforms. In that case we do not have to use the Wiener-Hopf technique, but obtain immediately a closed form expression for $\varphi_{+}(s)=\int_{0-}^{\infty}\mathrm{e}^{-sx}dV(x)$, the Laplace-Stieltjes transform of the work in the system.

The model with complete breakdown corresponds to the system with disasters analysed in Jain and Sigman (1994). In particular, equation (5.4) is obtained via a rate conservation approach in Proposition 1 of that reference.

5.3 Removal of customer equivalents

A direct relation between the results for work removal as presented above, and customer removal as presented in Harrison and Pitel (1994), seems to be rather involved when generally distributed service requirements are allowed. For some special cases a direct comparison is possible.

An easy example for which both models can be made equivalent is given by the M/D/1 queue with negative customers removing *customers* at the end of the queue. In the model of the present paper we can set $C(\cdot)=B(\cdot)$, which obviously gives an equivalent model. However, this example seems to be the only example of such simplicity. The problem with comparing the two models already becomes apparent when we compare the stability conditions. Harrison and Pitel (1994) present the stability conditions for their model with several disciplines for customer removals. These - for some disciplines quite involved - conditions do not resemble our simple stability condition, except in special cases. One such case is the M/M/1 queue with negative customers. For that case, we shall analyse the average amount of work in the system for both models. Before proceeding to this analysis, let us first present some comments on the relation between the amount of work in the system and the waiting time of positive customers.

In Section 2 we have defined the workload distribution $V(\cdot)$ as the sum of the residual service time of the customer in service plus the (possibly reduced) service times of the customers in the waiting room. For the FIFO queue with negative customers *that remove the customer in service* there is an explicit expression for the workload distribution $V_c(\cdot)$:

$$V_c(x) = \pi(0) + \sum_{j=1}^{\infty} B_e(x) * B^{(j-1)*}(x)\pi(j),$$

where $B_e(\cdot)$ is the residual life time distribution of the service requirements, $B^{j*}(\cdot)$ is the j-fold convolution of $B(\cdot)$ with itself, and $\pi(j)$ is the probability that an arriving customer meets j customers in the system. In the definition of the workload distribution $V_c(\cdot)$ we count the full (remaining) service requirement of all customers in the system. The service requirement is worked off at unit rate and makes downward jumps upon arrival of a negative customer.

For the case of exponential service time distribution the equilibrium queue length distribution in the model with customer-in-service removal is $\pi(n) = (1-\rho)\rho^n$, where $\rho = \lambda^+\beta/(1+\lambda^-\beta)$, as can easily be seen when the effect of negative customers is incorporated in the service requirements: the resulting effectively obtained service time is exponentially distributed with mean $\beta/(1+\lambda^-\beta)$. The stability condition is $\rho < 1$. We find

$$\mathbf{E}V_c = \frac{\lambda^+\beta^2}{1+\lambda^-\beta-\lambda^+\beta}, \tag{5.5}$$

and (cf. Harrison and Pitel (1993)):

$$\mathbf{E}W_c = \frac{\lambda^+\beta^2}{(1+\lambda^-\beta-\lambda^+\beta)(1+\lambda^-\beta)}, \tag{5.6}$$

$\mathbf{E}W_c$ denoting the mean waiting time in the model with customer-in-service removal. These expressions are equal if $\lambda^- = 0$, i.e., no negative customers.

For the M/M/1 queue with work removal we have shown in Section 4 that the amount of work in the system is distributed as the amount of work found by customers arriving to a GI/M/1 queue, i.e., their waiting time W. From Cohen (1982), p. 230, and Lemma 4.2 we obtain that

$$\mathbf{E}V = \mathbf{E}W = \frac{\nu\beta}{1-\nu}, \tag{5.7}$$

where $\nu = 1 - \zeta\beta$, and ζ is minus the unique negative root of (5.2). Comparison of the average workload $\mathbf{E}V$ for the model with work removal (cf. (5.7)), and $\mathbf{E}V_c$ for the model with customer-in-service removal (cf. (5.5)) shows that equality occurs iff $\nu = \rho$. Formula (5.2) shows that $\nu = \rho$ iff

$$\gamma = (1 + \lambda^- \beta)/\lambda^+ = \beta/\rho. \tag{5.8}$$

This value of γ is such that the average number of negative customers needed to remove one positive customer equals 1, which can be seen as follows. The probability that an arriving negative customer does not fully remove a customer present in the system is $1/(1+\gamma/\beta)$. As a consequence of the memoryless property of the negative exponential distribution the negative customer did not affect the state of the system. Hence the number of negative customers required to remove one positive customer is geometrically distributed with parameter $1/(1+\gamma/\beta)$, and the average number of negative customers required to kill one positive customer is γ/β. Thus, to remove a positive customer on the average more than one negative customer is required. However, the probability that at least one positive customer is present equals ρ. Combining these results shows that the average *effect* of a negative customer is that it removes one positive customer when γ is chosen as in (5.8).

Thus, for the M/M/1 queue, by a choice of γ that is slightly larger than the average service requirement β we obtain that on the average a negative customer removes one positive customer from the queue, and $\mathbf{E}V_c = \mathbf{E}V$. Observe that $\gamma > \beta$ is not required for the general model, as can be seen from the M/D/1 queue. This suggests that a general rule for the comparison of the model with customer removal and the model with work removal might be difficult.

5.4 The sojourn time

In the previous example we analysed the mean workload in the system, and studied the relation between the mean workload and the mean waiting time for the model of Harrison and Pitel (1994). The sojourn time of a customer in our model with FIFO service and work removal from the head of the queue will now be studied in more detail. To this end, define $\mathbf{T}(x)$ to be the time it takes before an amount of work x has disappeared from the system.

Obviously $\mathbf{P}(\mathbf{T}(x) > x) = 0$, while

$$\mathbf{P}(\mathbf{T}(x) = x) = e^{-\lambda^- x}, \quad x \geq 0;$$

this corresponds to no negative arrival during x. When there *are* negative arrivals, then the amount of work x will disappear in less than x time units. Conditioning on the number of negative arrivals we can write:

$$\mathbf{P}(\mathbf{T}(x) > z) = \sum_{j=0}^{\infty} e^{-\lambda^- z} \frac{(\lambda^- z)^j}{j!} C^{j*}(x - z), \qquad 0 \leq z < x. \tag{5.9}$$

Hence

$$\mathbf{ET}(x) = \int_{z=0}^{x} \mathbf{P}(\mathbf{T}(x) > z)dz = \int_{z=0}^{x} \sum_{j=0}^{\infty} e^{-\lambda^- z} \frac{(\lambda^- z)^j}{j!} C^{j*}(x-z)dz.$$

The Laplace-Stieltjes transform of $\mathbf{ET}(x)$ readily follows from this relation:

$$\psi(s) := \int_{x=0}^{\infty} e^{-sx} d[\mathbf{ET}(x)] = \frac{1}{s + \lambda^- - \lambda^- \gamma(s)}, \qquad Re\ s \geq 0. \tag{5.10}$$

In the case of exponential killing, so $\gamma(s) = 1/(1+s\gamma)$, we have

$$\psi(s) = \frac{s + 1/\gamma}{s(s + \lambda^- + 1/\gamma)}, \quad Re\ s \geq 0,$$

and hence

$$\mathbf{ET}(x) = \frac{x}{1+\lambda^-\gamma} + \frac{\lambda^-\gamma^2}{(1+\lambda^-\gamma)^2}(1 - e^{-(\lambda^- + 1/\gamma)x}), \qquad x \geq 0. \tag{5.11}$$

For large x the factor $x/(1+\lambda^-\gamma)$ dominates. For a general killing distribution the application of a Tauber theorem for Laplace-Stieltjes transforms (cf. Cohen (1982), app. 4; note that $\mathbf{ET}(x)$ is non-decreasing in x) to (5.10) also shows that, for $x \to \infty$, $\mathbf{ET}(x) \approx x/(1+\lambda^-\gamma)$. The obvious interpretation is that on the average $\lambda^-\gamma$ work is killed per unit of time, when there is a large enough amount of work present.

Note that the study of the decrement of the workload from x to 0 amounts to the study of the increment, from 0 to x, of a process with stationary independent increments. The nice structure of that process in our model leads to relatively simple expressions. It also enables us to analyse the sojourn time $\mathbf{T}$ of an arbitrary customer K, for the case that work is removed from the *head* of the queue. Because of the PASTA property, the amount of work found by K upon its arrival has distribution the steady-state workload distribution $V(\cdot)$. $\mathbf{T}$ equals the time it takes before that workload, plus K's service request, has disappeared. So

$$E[e^{-s\mathbf{T}}] = \int_{z=0}^{\infty} e^{-sz} \int_{x=z}^{\infty} d(V * B)(x)\ d_z\mathbf{P}(\mathbf{T}(x) < z), \quad Re\ s \geq 0; \tag{5.12}$$

partial integration and (5.9) lead for $Re\ s \geq 0$ to

$$E[e^{-s\mathbf{T}}] = 1 - s\int_{x=0}^{\infty} d(V * B)(x) \int_{z=0}^{x} e^{-sz} \sum_{j=0}^{\infty} e^{-\lambda^- z} \frac{(\lambda^- z)^j}{j!} C^{j*}(x-z)dz. \tag{5.13}$$

We now restrict ourselves to the case of exponential service and work removal distributions. It follows from (5.3) that $V(x) * B(x) = 1 - \exp[-\zeta x]$, $x \geq 0$. After some arithmetic we find:

$$E[e^{-s\mathbf{T}}] = \frac{\zeta[1+\gamma\zeta+\gamma\lambda^-]}{s[1+\gamma\zeta]+\zeta[1+\gamma\zeta+\gamma\lambda^-]}, \qquad Re\ s \geq 0. \tag{5.14}$$

Apparently, for this case in which all four interarrival-, service- and work removal distributions are memoryless, the sojourn time $\mathbf{T}$ is also exponentially distributed; while $V(\cdot) * B(\cdot)$ has mean $1/\zeta$, the mean sojourn time equals

$$\mathbf{ET} = \frac{1}{\zeta}\frac{1+\gamma\zeta}{1+\gamma\zeta+\gamma\lambda^-}. \tag{5.15}$$

For the case of an exponential killing distribution but general service time distribution, it follows from (5.11) that the mean sojourn time equals

$$\int_{x=0}^{\infty} \mathbf{ET}(x) d(V*B)(x) = \frac{\mathbf{E}V+\beta}{1+\lambda^-\gamma} + \frac{\lambda^-\gamma^2}{(1+\lambda^-\gamma)^2}[1-\omega_+(\lambda^-+1/\gamma)\beta(\lambda^-+1/\gamma)].$$

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