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M. Hazewinkel

Department of Analysis, Algebra and Geometry

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CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

# Linked Balanced Designs are Symmetric BIBD's

Michiel Hazewinkel
*CWI*
*P.O. Box 94079*
*1090 GB Amsterdam*
*The Netherlands*
mich@cwi.nl

**Abstract**

Let *A* be a reduced incidence relation between *n* lines and *m* points. Suppose that

(a) Through each two points there pass $\lambda$ lines

(b) Each two lines intersect in $\mu$ points.

If $\lambda = \mu = 1$ assume, moreover, that there are four points no three of which are on one line. Then $n = m$, $\lambda = \mu$, and there is a number *r* such that all lines have *r* points and through each point there pass *r* lines. The number *r* is given by $r(r-1) = (n-1)\lambda$.

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## 1. Introduction and statement of the main theorem

Let *X* be a finite set (of points) equipped with a finite collection of subsets called lines or blocks. I shall refer to such a structure as a *design.* Other words are incidence structure or relation. The design is called *reduced* if no two lines consist of the same points (i.e. the points distinguish lines) and, dually, no two points have exactly the same set of lines passing through them (the lines distinguish points). The main theorem is now as follows.

1.1. Theorem. Let *A* be a reduced design with *n* lines and *m* points. Suppose that

(a) Through each distinct two points there pass $\lambda > 0$ lines

(b) Each two distinct lines intersect in $\mu > 0$ points.

If $\lambda = \mu = 1$ assume, moreover, that there are four points no three of which are on one line.
Then $n = m$, $\lambda = \mu$, and there is a number *r* such that all lines have *r* points and through each point there pass *r* lines. The number *r* is given by $r(r-1) = (n-1)\lambda$.

In design theory, [1, 3] condition (a) is referred to as *balance*, *pairwise balance,* or *2-balance*, and such a design is often called a *2-design.* A design that satisfies condition (b) is called *linked*, [3].

A balance incomplete block design (BIBD) is a reduced design such that each line (block) has the same number $k < m$ of points, through each point there pass the same number *r* of lines and the design is 2-balanced. A BIBD is symmetric if $n = m$. It then follows that the design is also linked with $\lambda = \mu$ (and, trivially, that $k = r$). Inversely a linked BIBD is symmetric, loc. cit.

In this terminology the theorem above is reformulated as

1.2. Theorem. A linked, balanced, reduced design is a symmetric BIBD or it is given by a matrix of the form (2.3.1) below (which is a fan of $(n-1) \geq 1$ lines through a point intersecting one additional line).

This formulation fits with the title of this paper.

In case $\lambda = \mu = 1$, theorem 1.1 is a well known and elementary result from finite projective geometry.

**2. Proof of the main theorem**.

Let *A* be a reduced balanced linked design as in the formulation of the theorem. We shall work with the incidence matrix of the design which will be denoted *B*. The columns of *B* are indexed by the lines of the design and the rows are indexed by the points of the design. The entry at spot $(i, j)$ of *B* is 1 if and only if line *j* passes through point *i* (or point *i* is on line *j*). All other entries are zero.

The condition that *A* be reduced is equivalent to the condition that the incidence matrix *B* has no identical columns and no identical rows. Condition (a) of theorem 1.1 means that that each two rows of *B* have $\lambda$ 1's in common, and condition (b) means that each two columns have $\mu$ 1's in common.

2.1. *Proof of the theorem.* The case that $\lambda = \mu = 1$, is taken care of by a very well known and elementary fact in finite projective geometry, cf e.g. [2]. Thus we can from now on assume that at least one of $\lambda, \mu$ is larger than 1.

For each row *i* let $r_i$ be the number of 1's in it. Then the product of the $m \times n$ matrix *B* with its transpose is equal to

$$BB^T = \begin{pmatrix} r_1 & \lambda & \cdots & \lambda \\ \lambda & r_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \lambda \\ \lambda & \cdots & \lambda & r_m \end{pmatrix} \tag{2.1.1}$$

with $r_i \geq \lambda > 0$ for each *i*. Suppose that $r_i = r_j = \lambda$ for two different indices *i* and *j*. Then these two rows would be identical which is not the case by assumption. By lemma 2.4 below, it follows that the matrix (2.1.1) is nonsingular. It follows that $n \geq m$. Similarly, by considering $B^T B$ we see that $m \geq n$, and so $m = n$.

So we have:

$$BB^T = \begin{pmatrix} r_1 & \lambda & \cdots & \lambda \\ \lambda & r_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \lambda \\ \lambda & \cdots & \lambda & r_n \end{pmatrix} = \begin{pmatrix} r_1 - \lambda & & \\ & \ddots & \\ & & r_n - \lambda \end{pmatrix} + \begin{pmatrix} \lambda & \cdots & \cdots & \lambda \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \lambda & \cdots & \cdots & \lambda \end{pmatrix} \tag{2.1.2}$$

$$B^T B = \begin{pmatrix} s_1 & \lambda & \cdots & \lambda \\ \lambda & s_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \lambda \\ \lambda & \cdots & \lambda & s_n \end{pmatrix} = \begin{pmatrix} s_1 - \mu & & \\ & \ddots & \\ & & s_n - \mu \end{pmatrix} + \begin{pmatrix} \mu & \cdots & \cdots & \mu \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \mu & \cdots & \cdots & \mu \end{pmatrix} \tag{2.1.3}$$

The next step is to count the number of pairs of 1's in the rows. Each row *i* has $r_i$ 1's in it. So in total there are

$$\frac{1}{2} r_1 (r_1 - 1) + \cdots + \frac{1}{2} r_n (r_n - 1)$$

pairs of 1's in the rows. On the other hand each two colums have $\mu$ 1's in common. There are $n(n-1)/2$ pairs of columns. Thus from this point of view there are $\mu(n-1)n/2$ row pairs of 1's. Hence,

$$n(n-1)\mu = \sum r_i^2 - \sum r_i \tag{2.1.4}$$

Doing the same for pairs of 1's in the columns we get similarly

$$n(n-1)\lambda = \sum s_i^2 - \sum s_i \tag{2.1.5}$$

Further, $\mathrm{Tr}(BB^T) = \mathrm{Tr}(B^T B)$, so

$$\sum r_i = \sum s_i \tag{2.1.6}$$

(which is of course just the total number of 1's in *B*).

Now consider the product $BB^T BB^T$ and its trace. The diagonal elements of this product are

$$r_1^2 + (n-1)\lambda^2, \ldots, r_n^2 + (n-1)\lambda^2$$

So,

$$\mathrm{Tr}(BB^T BB^T) = n(n-1)\lambda^2 + \sum r_i^2 \tag{2.1.7}$$

Similarly,

$$\mathrm{Tr}(B^T BB^T B) = n(n-1)\mu^2 + \sum s_i^2 \tag{2.1.8}$$

However, $\mathrm{Tr}(B^T BB^T B) = \mathrm{Tr}(BB^T BB^T)$, and thus

$$n(n-1)\lambda^2 + \sum r_i^2 = n(n-1)\mu^2 + \sum s_i^2 \tag{2.1.9}$$

From (2.1.4) - (2.1.6) it folows that

$$n(n-1)(\lambda - \mu) = \sum s_i^2 - \sum r_i^2 \tag{2.1.10}$$

Now combine (2.1.9) and (2.1.10) to see that

$$\mu = \lambda \tag{2.1.11}$$

$$\sum s_i^2 = \sum r_i^2 \tag{2.1.12}$$

Now consider $\mathrm{Tr}((BB^T)^3)$. This is equal to an expression of the form

$$r_1^3 + \cdots + r_n^3 + a_2(r_1, \ldots, r_n)\lambda + a_1(r_1, \ldots, r_n)\lambda^2 + a_0(\lambda) \tag{2.1.13}$$

where the $a_2$ and $a_1$ are symmetric functions in the $r_i$ of degree 2 and 1 respectively. Similarly

$$\mathrm{Tr}((B^T B)^3) = s_1^3 + \cdots + s_n^3 + a_2(s_1, \ldots, s_n)\mu + a_1(s_1, \ldots, s_n)\mu^2 + a_0(\mu) \tag{2.1.14}$$

For the same functions $a_i$ (because the matrices $BB^T$ and $B^TB$ are of the same size and shape). Now $\lambda = \mu$, $\sum s_i = \sum r_i$, and $\sum s_i^2 = \sum r_i^2$, so, as the $a_i$, $i = 1,2$, are symmetric functions, $a_i(r_1, \ldots, r_n) = a_i(s_1, \ldots, s_n)$, $i = 1,2$. Moreover, $\mathrm{Tr}((BB^T)^3) = \mathrm{Tr}((B^TB)^3)$, and so

$$\sum s_i^3 = \sum r_i^3 \tag{2.1.15}$$

More generally, for $\lambda = 0$

$$BB^T = \begin{pmatrix} r_1 & & \\ & 0 & \\ & & r_n \end{pmatrix}$$

So

$$\mathrm{Tr}((BB^T)^k)\big|_{\lambda=0} = r_1^k + \cdots + r_n^k$$

and it follows that

$$\mathrm{Tr}((BB^T)^k) = \sum r_i^k + a_{k-1}^k(r)\lambda + \cdots + a_1^k(r)\lambda^{k-1} + a_0(\lambda) \tag{2.1.16}$$

and similarly

$$\mathrm{Tr}((B^TB)^k) = \sum s_i^k + a_{k-1}^k(s)\mu + \cdots + a_1^k(s)\mu^{k-1} + a_0(\mu) \tag{2.1.17}$$

where the $a_i^k(r)$ are symmetric functions in the $r_1, \ldots, r_n$ of degree $i$, $i = 1, \ldots, k-1$. By induction $\sum s_i^l = \sum r_i^l$, $l = 1, \ldots, k-1$, and hence, $a_i^k(r) = a_i^k(s)$, $i = 1, \ldots, k-1$, and so, as $\mathrm{Tr}((BB^T)^k) = \mathrm{Tr}((B^TB)^k)$, by (2.1.16) and (2.1.17)

$$\sum s_i^k = \sum r_i^k$$

Thus, with induction, all the symmetric functions in the $r$'s are equal to those in the $s$'s. Therefore the $s_1, \ldots, s_n$ are a permutation of the $r_1, \ldots, r_n$. Permuting columns of $B$ does not change its defining properties so we can assume that $s_1 = r_1, \ldots, s_n = r_n$.

Let $d_i = r_i - \lambda = s_i - \mu$, and let $D$ be the diagonal matrix with diagonal entries $d_1, \ldots, d_n$. Then multiplying (2.1.2) on the right with $B$, and (2.1.3) on the left with $B$ we get

$$BB^TB = DB + \lambda EB \tag{2.1.18}$$

$$BB^TB = BD + \lambda BE \tag{2.1.19}$$

where $E$ is the $n \times n$ matrix consisting completely of 1's. Choose any $i, j \in \{1, \ldots, n\}$. Because $\lambda \geq 1$ there is a $k$ such that $b_{ik} = b_{jk} = 1$. Look at the $(i,k)$-entry of (2.1.18) and (2.1.19). This gives

$$d_i b_{ik} + \lambda r_k = b_{ik} d_k + \lambda r_i$$

and hence $(\lambda-1)r_i=(\lambda-1)r_k$, so that $r_i=r_k$, because as remarked in the beginning of the proof we can assume that at least one of $\lambda,\mu$ is $>1$. Similarly $r_j=r_k$. Thus $r_1=\cdots=r_n=s_1=\cdots=s_n$ and that makes $B$ (the incidence matrix of) a BIBD. That $r$ is determined by $r(r-1)=(n-1)\lambda$ follows immediately from (2.1.4).

2.3. Remark. If, in the case $\lambda=\mu=1$, the four point condition is left out in the statement of the theorem, one additional exceptional class of designs arises. They are given by the $n\times n$, $n\neq 1,3$, incidence matrices

$$\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{2.3.1}$$

So there is one exceptional point $P_1$ and one exceptional line $l_1$ which consists of all the other points $P_2,\cdots,P_n$. The other lines all have two points, $l_i=\{P_1,P_i\}$, $i=2,\cdots,n$.

2.3. Lemma. Let $r_i\geq\lambda\geq 0$, $i=1,\cdots,m$. Then

$$\det\begin{pmatrix} r_1 & \lambda & \cdots & \lambda \\ \lambda & r_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \lambda \\ \lambda & \cdots & \lambda & r_m \end{pmatrix}\geq 0$$

Proof. Write $D_m(r_1,\cdots,r_m)$ for the determinant above. Then $D_m(\lambda,\cdots,\lambda)=0$, and with induction, $D_m(r_1,\cdots,\hat{r}_i,\cdots,r_m)\geq 0$, where as usual a hat over a symbol means that it is to be deleted. Now $\dfrac{\partial D_m}{\partial r_i}=D_m(r_1,\cdots,\hat{r}_i,\cdots,r_m)\geq 0$, and we are through.

2.4. Lemma. Let $r_i\geq\lambda>0$, and suppose that at most one $r_i$ is equal to $\lambda$. Then $D_m(r_1,\cdots,r_m)>0$.

Proof. Because the determinant is symmetric in the $r_i$ we can assume $r_i\geq\lambda+\varepsilon$ for $i=1,\cdots,m-1$ for a certain $\varepsilon>0$. Now $D_m(\lambda+\varepsilon,\cdots,\lambda+\varepsilon,\lambda)=\varepsilon^{m-1}\lambda>0$, and, by lemma 2.3,

$$\frac{\partial D_m}{\partial r_i}=D_m(r_1,\cdots,\hat{r}_i,\cdots,r_m)\geq 0.$$

This proves the lemma.

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