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Analysis of the Asymmetrical Shortest Two-Server Queueing Model

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Abstract

This study presents the analytic solution for the asymmetrical two-server queueing model with arriving customers joining the shorter queue for the case with Poisson arrivals and negative exponentially distributed service times. The bivariate generating function of the stationary joint distribution of the queue lengths is explicity determined by the results obtained.

The determination of this bivariate generating function requires the construction of four generating functions. It is shown that each of these functions is the sum of a polynomial and a meromorphic function. The poles and residues at the poles of the meromorphic functions can be simply calculated recursively; the coefficients of the polynomials are easily found, in particular if the asymmetry in the model parameters is not excessively large. The starting point for the asymptotic analysis for the queue lengths is obtained. The approach developed in the present study is applicable to a larger class of random walks modelling asymmetrical two-dimensional queueing processes.

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1. Introduction

The "two-server shortest queueing" model also known as the "two-queues in parallel" model has obtained quite some attention in Queueing Theory literature, the greater part of it concerning the symmetrical model. For a short overview of the studies on the symmetrical model see [2]. The asymmetrical model presents an analytic problem which appeared inaccessible for quite a long time. In the present study the solution of this problem will be given.

The model concerns a two-server system with a Poisson arrival stream of the customers,rate λ, and an arriving customer joins the shorter queue if the queues are unequal; if they are equal he joins the queue in front of server i with probability $\pi_i, i = 1,2$. The server provides customers exponentially distributed service times with service rate $1/\beta_i, i = 1,2$. The symmetrical model concerns the case with $\pi_1 = \pi_2 = 1/2, \beta_1 = \beta_2$. The analysis of the queueing model requires the investigation of the stochastic process $\{\mathbf{x}_1(t), \mathbf{x}_2(t), t > 0\}$ with $\mathbf{x}_i(t)$ the number of customers present with server i at time $t, i = 1,2$. The analysis of this stochastic process can be reduced to that of a random walk with state space the set of lattice points in $\mathbb{R}_2$ with integer-valued, nonnegative coordinates.

In the applications of the two-server, shortest queueing model the stationary joint distribution of the queue lengths, or its bivariate generating function, usually contains all the information required to calulate the various performance characteristics of the model. Approximative techniques have been studied to obtain this information. Usually, they concern the replacement of the infinite state space of the random walk by a finite one. For the so resulting process the Kolmogorov equations for the statioinary probabilities are then solved numerically.

Blanc [5], [10], applies the power-series algorithm. Here it is assumed that the stationary probabilities can be expressed as a power series in powers of some suitable chosen function of the traffic to be handled by the servers. Substitution of these series into the Kolmogorov equations then leads to a set of equations from which the coefficients of these series can be recursively calculated. The results obtained by this approach are quite satisfactory when compared to those obtained by simulation. Actually, this approach is also based on a special truncation of the state space. Unfortunately, a sufficient mathematical justification of this approach is still not available. ADAN, WESSELS and ZIJM [4], and ADAN [7] present an iterative approach which is claimed in [7] to converge to the exact solution. The results of the present study show, however, that this claim is not justified. For a discussion of this point see the comments at the end of this introduction.

Random walks on the lattice in $\mathbb{R}_2$ with integer valued, nonnegative coordinates are instrumental in the analytical investigation of a large class of present day queueing models. For such random walks and their application to Queueing Theory quite some information is presently available concerning a fairly general approach of their analysis, cf.[1], [11]. For the subclass of semi-homogeneous, nearest-neighbour random walks a more effective analytical approach came recently available, cf.[12], [13], and in particular for nearest-neighbour random walks with no one-step transition probabilities to the North, the North-East and the East, cf. [2], [8], [9]. For this latter case it appeared that the bivariate generating function of the stationary joint distribution of the random walk, if it exists, can be explicitly expressed in terms of meromorphic functions. A meromorphic function is a function which is regular in the whole complex plane, except for at most a finite number of poles in any finite domain.

The random walk to be used in the analysis of the asymmetrical shortest queue is indeed a nearest-neighbour random walk. However, it is not semi-homogeneous because the one-step transition probabilities at points above the diagonal of the first quadrant differ from those below this diagonal. In fact this random walk consists of two semi-homogeneous random walks, one at the points above, the other at the points below the diagonal, and they are coupled at the points on the diagonal. These two random walks do not have one-step transition probabilities to the North, the North-East and the East and so it was conjectured that the bivariate generating function of its stationary distribution can be indeed described in terms of meromorphic functions. This conjecture was the starting point of our analysis and it appeared to be true.

We continue this introduction with an overview of the sections of the present study.

In Section 2 we formulate the functional equations for the bivariate generating function of the stationary joint distribution of the queue length process, cf. (2.4). From these equations the following two equations are derived, cf. (2.9):

$$\begin{aligned} &i. \quad \Omega_2(\rho) + \frac{a_1\tau}{\rho-\tau}B(\tau) + k_1(\rho,\tau)\Phi_0(\tau) = 0, \\ &\qquad \text{for } (\rho,\tau) \text{ a zero-tuple of } h_1(\rho,\tau)=0, \quad |\rho|\le 1, \quad |\tau|\le 1; \\ &ii. \quad \Omega_1(r) - \frac{a_2 t}{r-t}B(t) + k_2(r,t)\Phi_0(t) = 0, \\ &\qquad \text{for } (r,t) \text{ a zero-tuple of } h_2(r,t)=0, \quad |r|\le 1, \quad |t|\le 1; \end{aligned} \tag{1.1}$$

here $k_i(\cdot,\cdot)$ is a polynomial of the first degree, and $h_i(\cdot,\cdot)$ a polynomial of the second degree, they are given; the functions $B(\cdot), \Phi_0(\cdot), \Omega_1(\cdot)$ and $\Omega_2(\cdot)$ are all generating functions which are regular inside the unit disk and continuous in the closure of the unit disk. Next to these conditions there is the norming condition, which is equivalent with

$$\frac{1}{a_1}\Omega_2(1) + \frac{1}{a_2}\Omega_1(1) = \frac{1}{a_1} + \frac{1}{a_2} - 1, \quad a_1 = \lambda\beta_1,\ a_2 = \lambda\beta_2. \tag{1.2}$$

From the conditions mentioned so far the functions $B(\cdot), \Phi_0(\cdot), \Omega_i(\cdot), i = 1,2,$ have to be determined, cf. (2.10).

In Section 3 properties of the zeros of $h_1(\rho,\tau) = 0$ and of $h_2(r,t) = 0$ are described. These zeros are denoted by, cf. (3.9),

$$\begin{aligned} &\rho^{\pm}(\tau) \quad \text{or} \quad \tau^{\pm}(\rho) \quad \text{for} \quad h_1(\rho,\tau)=0, \\ &r^{\pm}(t) \quad \text{or} \quad t^{\pm}(r) \quad \text{for} \quad h_2(r,t)=0. \end{aligned} \tag{1.3}$$

The curve $h_1(\rho,\tau)=0$ when traced for real ρ and τ is a hyperbola which has one of its branches in the first quadrant. A graph with succesive vertical and horizontal edges is inscribed in the branch in the first quadrant, see Figure 3.2. The corner points of such a graph correspond to zeros of $h_1(\rho,\tau)$. Analogously such a graph is inscribed in the branch of $h_2(r,t)=0$. Any point of $h_1(\rho,\tau)=0$ induces also such a graph on $h_2(r,t)=0$ and vice versa. In Figure 3.4, a binary tree is traced, starting at $\tau_0^{(0)}$. Out from $\tau_0^{(0)}$ a graph as mentioned above is constructed on $h_1(\rho,\tau)=0$ and on $h_2(r,t)=0$. Each corner point of such a graph induces on the other hyperbola again such a graph, only ascending graphs are used here. The set of corner points so obtained constitutes the tree generated by $\tau_0^{(0)}$, see for details Section 3. Such a tree with a properly chosen top $\tau_0^{(0)}$ is instrumental in the construction of the functions $B(\cdot),\Phi_0(\cdot),\Omega_i(\cdot),i=1,2$.

Section 4 starts with the formulation of a lemma. It states that the functions $B(s),\Phi_0(s),\Omega_i(s),i=1,2$, can all be continued meromorphically into $|s|>1$. For those meromorphic continuations a set of relations is derived, cf.(4.2),..., (4.5). We mention here two of those relations, viz.

$$\begin{aligned}
&\Omega_2(\rho^-(\tau)) + \frac{a_1\tau}{\rho^-(\tau)-\tau}B(\tau)+k_1(\rho^-(\tau),\tau)\Phi_0(\tau)=0 \quad \text{for} \quad h_1(\rho^-(\tau),\tau)=0,\\
&\Omega_1(r^-(\tau)) - \frac{a_2 t}{r^-(t)-t}B(t)+k_2(r^-(t),t)\Phi_0(t)=0 \quad \text{for} \quad h_2(r^-(\tau),\tau)=0.
\end{aligned}\tag{1.4}$$

It is assumed, cf. assumption 4.2, that all poles of $B(s),\Phi_0(s),\Omega_i(s),i=1,2$, in $|s|>1$ are simple; at a pole of $B(\cdot)$ its residue is indicated by $b(\cdot)$; $\phi_0(\cdot),\omega_i(\cdot)$, stand for residues of $\Phi_0(\cdot),\Omega_i(\cdot),i=1,2$, respectively. Note that $b(s)=0$ implies that s is not a pole of $B(\cdot)$. From the set of relations (4.2),...,(4.5), a set of relations for the residues is obtained; e.g. if τ is a pole of $B(\cdot)$ and also of $\Phi_0(\cdot)$, then it is seen from (1.4) that in general $\rho^-(\tau)$ is a pole of $\Omega_2(\cdot)$ and $r^-(\tau)$ one of $\Omega_1(\cdot)$. The essential point of the analysis is to construct from the set of equations for the residues a nonnull solution such that $B(\cdot),\Phi_0(\cdot),\Omega_i(\cdot),i=1,2$, are true meromorphic functions, i.e. the pole set of each of these functions has not a finite accumulation point. It turns out that there is a non-trivial solution to this problem. The solution of this problem is discussed in Sections 5, 6 and 7.

In Section 5 it is shown that a unique T in $\tau>1$ exists such that the equations

$$\begin{aligned}
&\frac{a_1 T}{\rho^-(T)-T}b(T)+k_1(\rho^-(T),T)\phi_0(T)=0 \quad , \quad h_1(\rho^-(T),T)=0,\\
&\frac{-a_2 T}{r^-(T)-T}b(T)+k_2(r^-(T),T)\phi_0(T)=0 \quad , \quad h_2(r^-(T),T)=0,
\end{aligned}\tag{1.5}$$

for the residues $b(T)$ and $\phi_0(T)$ of $B(\cdot)$ and $\Phi_0(\cdot)$, possess a nonnull solution. Actually, T is then determined by that zero of τ in $\tau>1$ for which the main determinant of the set of equations (1.5) is zero. Note that (1.4) and (1.5) imply that $\rho^-(T)$ is not a pole of $\Omega_2(\cdot)$, and $r^-(T)$ not a pole of $\Omega_1(\cdot)$.

In Section 6 it is shown that the set of nodes of the tree generated by $\tau_0^{(0)}=T$ is pole set of $B(\cdot)$ as well as of $\Phi_0(\cdot)$. The poles of $\Omega_i(\cdot),i=1,2$, can be deduced from those of $B(\cdot)$ and $\Phi_0(\cdot)$. Further a recursive set of equations is derived for the residues at these poles, see Lemmas 6.1 and 6.2. In order to decide whether the pole set of $B(\cdot)$, say, can be used to define a meromorphic function information is required concerning the asymptotic behaviour of the poles of $B(\cdot)$ and of their residues $b(\cdot)$. This asymptotic behavior is studied in Section 7. It is shown that this asymptotic behaviour is such that for the pole set of $B(\cdot)$ indeed a class of meromorphic functions can be constructed, similarly for $\Phi_0(\cdot),\Omega_i(\cdot),i=1,2$. The elements of these classes of meromorphic functions are parametrised by nonnegative integers $m_b,m_\phi,m_i,i=1,2$, bounded below by $\tilde{M}-1$, $\tilde{M}$ being defined by the asymptotic behaviour of the residues, cf.(8.6).

In Section 8 the meromorphic functions $\tilde{B}(\cdot),\tilde{\Phi}_0(\cdot),\tilde{\Omega}_i(\cdot),i=1,2$, are introduced by using the pole sets obtained in Section 6 for the functions $B(\cdot),\Phi_0(\cdot),\Omega_i(\cdot),i=1,2$; they contain the parameters $m_b,m_\phi,m_i,i=1,2$, cf.(8.2). Next to these functions polynomials $\hat{B}(s),\hat{\Phi}_0(s),\hat{\Omega}_i(s),i=1,2$ are introduced, their degrees are indicated by $\hat{N}_b, \hat{N}_\phi, \hat{N}_i, i = 1, 2$. With the functions so introduced the functions

$$B(\cdot) = \hat{B}(\cdot) + \tilde{B}(\cdot), \quad \Phi_0(\cdot) = \hat{\Phi}(\cdot) + \tilde{\Phi}(\cdot),$$
$$\Omega_i(\cdot) = \hat{\Omega}_i(\cdot) + \tilde{\Omega}_i(\cdot), \quad i = 1, 2, \tag{1.6}$$

are considered, cf.(8.7). These functions are substituted into the equations (1.4), cf.(8.8). The asymptotic behaviour at infinity of the relations so obtained is investigated. It leads to relations between $m_{...}$ and the degrees $\hat{N}_{...}$ of the polynomials. An appropriate choice of $m_{...}$ given $\tilde{M}$ determines the degrees $\hat{N}_{...}$. With the degrees so determined the functions (1.6) are again substituted in the equations (1.4). By considering the resulting equations for properly chosen zero-tuples of $h_1(\rho, \tau)$ and of $h_2(r, t)$ a system of linear, inhomogeneous equations for the unknown coefficients of the polynomials $\hat{B}(\cdot), \hat{\Phi}_0(\cdot), \hat{\Omega}_i(\cdot), i = 1, 2$ is obtained. It is shown that this system has a unique solution and so the functions of (1.6) are all known and satisfy the equations, moreover they are all regular in the closed unit disk. Theorem 8.1 states that for $\tilde{M} \geq 1, a_1 \neq a_2$, the functions so constructed, whenever taking into account the norming condition, determine the bivariate generating function of the stationary distribution uniquely if and only if $\frac{1}{a_1} + \frac{1}{a_2} > 1$, the case $\tilde{M} = 1$ is an important practical case. Next the construction of the solution for the case $\tilde{M} = 2$ is discussed whereas that for $\tilde{M} \geq 3$ is exposed. In Remark 8.2 the meaning of T for the asymptotic behaviour of the constructed solution is discussed.

In Section 9 the case $a_1 = a_2, \pi_1 \neq \pi_2$, is discussed. For this case the construction of the solution of (1.4) is essentially simpler, because here the pole sets do not have a tree structure. In Section 10 relations are derived for some characteristic probabilities and moments of the model.

The approach developed in the present study is applicable to a larger class of asymmetrical two-server models, e.g. the asymmetrical variant of the model in [9]. This class is characterised by zero one-step transition probabilities to the N, NE and E in the upper as well as in the lower triangle of the first quadrant.

We conclude this introduction with some comments on the analysis of ADAN, WESSELS and ZIJM [4] and ADAN [7]. They avoid the use of generating functions and claim that the stationary probabilities for the queue lengths $(\mathbf{x}_1, \mathbf{x}_2)$ can be represented as:

$$\begin{aligned} &i. \quad \Pr\{\mathbf{x}_1 = s, \mathbf{x}_2 = s + t\} = C^{-1} \sum_{i \in L} d_i(\alpha^s_{p(i)} + c_i d^s_i)\beta^t_i, \qquad t \geq 1, \\ &ii. \quad \Pr\{\mathbf{x}_1 = s + t, \mathbf{x}_2 = s\} = C^{-1} \sum_{i \in R} d_i(\alpha^s_{p(i)} + c_i d^s_i)\beta^t_i, \qquad t \geq 1, \end{aligned} \tag{1.7}$$

for $s + t > N$ with N sufficiently large. Here C is a constant, L is the set of the odd positive integers, R that of the even positive integers, d_i and c_i are determined recursively and α_i, β_i are the nodes of a binary tree at level i, the tree being generated by α_0. Of this tree in [4], the subtree formed by the α-nodes is simply related to the tree of Figure 3.4 of the present paper. Actually, the nodes of the latter tree are the inverses of the corresponding α-nodes of the tree in [7], where (β_i, α_i) are zero-tuples of $h_1(\frac{1}{\rho}, \frac{1}{\tau})$ or $h_2(\frac{1}{\rho}, \frac{1}{\tau})$, cf.(2.8), depending on $i \in L$ or $i \in R$. In [4], [7], a recursive set of equations for the coefficients d_i and c_i is derived from the Kolmogorov equations. The recursion is obtained by splitting these equations in two subsets. Partial sums of the righthand sides in (1.7) from $i = 1, \ldots, K$, say, satisfy the equations of one subset, those from $i = 1, \ldots, K + 1$, satisfy the equations of the other subset, $K = 1, 2, \ldots$. All terms in the righthand sides of (1.7) can be calculated once α_0 is known.

From the results of the present study it can be seen that the lefthand sides of (1.7) can be expressed indeed by sums of the types as in the righthand sides of (1.7), α_i^{-1} are the poles of the functions $B(\cdot)$ and $\Phi_0(\cdot), \beta_i^{-1}$ those of $\Omega_i(\cdot)$, cf. (8.7), below. Because the α_i are strictly decreasing it follows from (1.7) that for fixed $t > 1$,

$$\frac{\Pr\{\mathbf{x}_1 = s + 1, \mathbf{x}_2 = s + 1 + t\}}{\Pr\{\mathbf{x}_1 = s, \mathbf{x}_2 = s + t\}} \sim \alpha_0 \quad \text{for} \quad s \to \infty. \tag{1.8}$$

In [4], [7] the determination of α_0 is based on a heuristic asymptotic argument, it is sustained by numerical results for the truncated model with large truncation parameter. The results of the present study prove that the value of α_0 taken in [4], [7], i.e. $\alpha_0 = (a_1^{-1} + a_2^{-1})^2$, is the correct one, see Lemma 5.1 and Remark 8.2 below.

Whether the iterative technique in [4] leads in the limit to the correct solution of the Kolmogorov equations may be checked by comparing the asymptotic results for $d_i, c_i, i \to \infty$, cf. Section 5.5 of [4], with those for the residues at the poles in the present study. It appears that these asymptotic results differ, and so leads to the conclusion that the limiting results in [4], [7], do not satisfy all the Kolmogorov equations. This conclusion, however, does not imply that the approach in [4] necessarily leads to numerical results, which are not accurate in quite a few digits, because usually only the smaller poles contribute essentially to the numerical values of the probabilities in (1.7).

2. The functional equation

The functional equations for the bivariate generating function of the joint distribution of the queue lengths $\mathbf{x}_1, \mathbf{x}_2$, of the asymmetrical shortest queue model have been derived in [1], see Section III.1.2, p. 245. Below we recall these functional equations using mainly the same notation as in [1].

Put for $|r| \le 1$,

$$
\begin{aligned}
&i. && \Phi_0(r) := \mathrm{E}\{r^{\mathbf{x}_1}(\mathbf{x}_1 = \mathbf{x}_2)\},\\
&ii. && \Omega_2(r) := \mathrm{E}\{r^{\mathbf{x}_2}(\mathbf{x}_1 = 0)\},\\
&iii. && \Omega_1(r) := \mathrm{E}\{r^{\mathbf{x}_1}(\mathbf{x}_2 = 0)\},\\
&iv. && B(r) := \Phi_0(r)(\frac{1}{a_2} + \pi_1) - \frac{1}{a_2}\Phi_0(0) - (1 + \frac{1}{a_1 r})\,\mathrm{E}\{r^{\mathbf{x}_1}(\mathbf{x}_1 = \mathbf{x}_2 + 1)\} =\\
& && -\Phi_0(r)(\frac{1}{a_1} + \pi_2) + \frac{1}{a_1}\Phi_0(0) + (1 + \frac{1}{a_2 r})\,\mathrm{E}\{r^{\mathbf{x}_2}(\mathbf{x}_2 = \mathbf{x}_1)\};
\end{aligned}
\tag{2.1}
$$

$$
K_1(r_1, r_2) := r_1 + \frac{1}{a_1 r_1} + \frac{1}{a_2 r_2} - \frac{b}{a_1 a_2}, \tag{2.2}
$$

$$
K_2(r_1, r_2) := r_2 + \frac{1}{a_1 r_1} + \frac{1}{a_2 r_2} - \frac{b}{a_1 a_2};
$$

with

$$
\begin{aligned}
&a_1 > 0,\ a_2 > 0, \quad b := a_1 + a_2 + a_1 a_2,\\
&1 > \pi_1 > 0,\ 1 > \pi_2 > 0,\ \pi_1 + \pi_2 = 1,\\
&\quad \pi_1 \ne \pi_2 \quad \text{if } a_1 = a_2, \text{cf. Remark 2.3, below.}
\end{aligned}
\tag{2.3}
$$

The functional equations read: for $|r_1| \le 1, |r_2| \le 1$,

$$
\begin{aligned}
&i. && \mathrm{E}\{r_1^{\mathbf{x}_1} r_2^{\mathbf{x}_2}(\mathbf{x}_2 > \mathbf{x}_1)\}K_1(r_1, r_2) + \Omega_2(r_2)\frac{1}{a_1}(1 - \frac{1}{r_1}) +\\
& && \Phi_0(r_1 r_2)[\pi_2 r_2 + \frac{1}{a_1 r_1} - \frac{1}{a_1} - \pi_2] - B(r_1 r_2) = 0,\\
&ii. && \mathrm{E}\{r_1^{\mathbf{x}_1} r_2^{\mathbf{x}_2}(\mathbf{x}_1 > \mathbf{x}_2)\}K_2(r_1, r_2) + \Omega_1(r)\frac{1}{a_2}(1 - \frac{1}{r_2}) +\\
& && \Phi_0(r_1 r_2)[\pi_1 r_1 + \frac{1}{a_2 r_2} - \frac{1}{a_2} - \pi_1] + B(r_1 r_2) = 0.
\end{aligned}
\tag{2.4}
$$

From these equations it follows by using the norming condition, i.e. $\Pr\{\mathbf{x}_1 \ge 0, \mathbf{x}_2 \ge 0\} = 1$, that, cf.[1], p. 242,

$$\frac{1}{a_1}\Pr\{\mathbf{x}_1=0\}+\frac{1}{a_2}\Pr\{\mathbf{x}_2=0\}=\frac{1}{a_1}+\frac{1}{a_2}-1. \tag{2.5}$$

It follows that the condition

$$\frac{1}{a_1}+\frac{1}{a_2}>1, \tag{2.6}$$

is a necessary condition for the existence of a stationary distribution.

REMARK 2.1. It will be shown, cf. Theorem 8.1, that (2.6) is also a sufficient condition. □

ASSUMPTION 2.1. In the present analysis it will be assumed that

$$\frac{1}{a_1}+\frac{1}{a_2}>1.$$ □

For $(\hat{r}_1,\hat{r}_2)$ a zero-tuple of $K_1(r_1,r_2), |r_1|\le 1, |r_2|\le 1$, we have

$$0<|\mathrm{E}\{\hat{r}_1^{\mathbf{x}_1}\hat{r}_2^{\mathbf{x}_2}(\mathbf{x}_2>\mathbf{x}_1)\}|\le 1,$$

and so it follows from (2.4)i,

$$\Omega_2(\hat{r}_2)\frac{1}{a_2}(1-\frac{1}{\hat{r}_1})+\Phi_0(\hat{r}_1\hat{r}_2)[\pi_2\hat{r}_2+\frac{1}{a_1\hat{r}_1}-\frac{1}{a_1}-\pi_2]-B(\hat{r}_1\hat{r}_2)=0. \tag{2.7}$$

Put

$$\begin{aligned} h_1(\rho,\tau) &:= a_1a_2\tau^2+(a_1-b\rho)\tau+a_2\rho^2,\\ h_2(r,t) &:= a_1a_2t^2+(a_2-br)t+a_1r^2,\\ k_1(\rho,\tau) &:= -1+a_2\pi_2(a_1\tau-\rho),\\ k_2(r,t) &:= -1+a_1\pi_1(a_2t-r). \end{aligned} \tag{2.8}$$

By using the properties of the zeros of $K_1(r_1,r_2)$ in simplifying the term between square brackets in (2.7) and taking $\rho=\hat{r}_2, \tau=\hat{r}_1\hat{r}_2$ it is readily verified that (2.7) is equivalent with

$$\begin{aligned} &i.\quad \Omega_2(\rho)+\frac{a_1\tau}{\rho-\tau}B(\tau)+k_1(\rho,\tau)\Phi_0(\tau)=0,\\ &\qquad \text{for } (\rho,\tau) \text{ a zero-tuple of } h_1(\rho,\tau); |\rho|\le 1, |\tau|\le 1;\\ &ii.\quad \Omega_1(r)-\frac{a_2t}{r-t}B(t)+k_2(r,t)\Phi_0(t)=0,\\ &\qquad \text{for } (r,t) \text{ a zero-tuple of } h_2(r,t); |r|\le 1,\ |t|\le 1; \end{aligned} \tag{2.9}$$

note that the derivation of (2.9)ii is analogous to that of (2.9)i.

From the above it is seen that the determination of the bivariate generating function $\mathrm{E}\{r_1^{\mathbf{x}_1}r_2^{\mathbf{x}_2}\}$ of the stationary joint distribution of the queue lengths requires the construction of the functions $\Omega_1(\rho), \Omega_2(\rho), \Phi_0(\rho)$ and $B(\rho)$ which should satisfy the following conditions:

i. they are regular for $|\rho|<1$ and the sum of the coefficients in their series expansions in powers of ρ^n converges absolutely; (2.10)

ii. they satisfy the relations (2.1)iv, (2.5), (2.9)i, ii and, cf.(2.1), $\Omega_i(0)=\Phi_0(0), i=1,2$.

REMARK 2.2. Once functions $B(\cdot), \Phi_0(\cdot), \Omega_1(\cdot)$ and $\Omega_2(\cdot)$ have been constructed which are not identically zero and satisfy the conditions (2.10), so that $\mathrm{E}\{r_1^{\mathbf{x}_1}r_2^{\mathbf{x}_2}\}, |r_1|\le 1,\ |r_2|\le 1$, is then determined via (2.4), then it follows that the Kolmogorov equations for the stationary probabilities possess an absolute convergent nonnull solution. By applying a wellknown Foster criterion, cf.[3], p.25, it follows that the queue length process $(\mathbf{x}_1(t), \mathbf{x}_2(t))$, of which the state space is irreducible, is positive recurrent and further that there is only one solution which satisfies (2.10) and the norming condition. Hence it suffices to construct functions $B(\cdot), \Omega_1(\cdot)$ and $\Omega_1(\cdot)$ which satisfy (2.10). □

REMARK 2.3. The analysis of the problem formulated in (2.10) for the case $a_1 = a_2, \pi_1 \neq \pi_2$, differs from that for the case $a_1 \neq a_2$. The analysis in Sections 6, 7 and 8 concerns the case $a_1 \neq a_2$. In Section 9 the case $a_1 = a_2, \pi_1 \neq \pi_2$ is discussed. For the case $a_1 = a_2, \pi_1 = \pi_2$, see [2]. □

3. ON THE ZEROS OF $h_1(\rho, \tau)$ AND $h_2(r, t)$.

In this section we shall describe several properties of the zeros of $h_1(\rho, \tau)$ and of $h_2(r, t)$, and introduce several functions of these zeros; these functions are needed to describe the functions $\Omega_1(r), \Omega_2(\rho), \Phi_0(\tau)$ and $B(\tau)$. Because of the symmetry between $h_1(\rho, \tau)$ and $h_2(r, t)$ we mainly restrict the discussion to $h_1(\rho, \tau)$, those for $h_2(r, t)$ follow by interchanging a_1 and a_2.

From (2.8) we have

$$\begin{aligned} &i. \quad h_1(\rho,\tau) = 0 \quad \Rightarrow \quad \tau(\rho) = \frac{1}{2a_1a_2}[-a_1 + b\rho \pm \sqrt{(a_1 - b\rho)^2 - 4a_1a_2^2\rho^2}], \\ &ii. \quad h_1(\rho,\tau) = 0 \quad \Rightarrow \quad \rho(\tau) = \frac{1}{2a_2}[b\tau \pm \sqrt{b^2\tau^2 - 4a_1a_2(\tau + a_2\tau^2)}]. \end{aligned} \tag{3.1}$$

LEMMA 3.1. *For every ρ with $|\rho| \geq 1, \rho \neq 1$, the two zeros $\tau^{\pm}(\rho)$ of $h_1(\rho, \tau)$ may be defined so that*

$$|\tau^-(\rho)| < |\rho| < |\tau^+(\rho)|, \tag{3.2}$$

and similarly for the two zeros $\rho(\tau)$ with $|\tau| \geq 1, \tau \neq 1$, i.e.

$$|\rho^-(\tau)| < |\tau| < |\rho^+(\tau)|.$$

Analogously for $h_2(r, t) = 0$:

$$\begin{aligned} &|t^-(r)| < |r| < |t^+(r)| \quad for \quad |r| \geq 1,\ r \neq 1, \\ &|r^-(t)| < |t| < |r^+(t)| \quad for \quad |t| \geq 1,\ t \neq 1. \end{aligned} \tag{3.3}$$

For the proof of Lemma 3.1 see Appendix $\mathcal{A}$.

From (3.1) it is seen that $\tau(\rho)$ has two branch points ρ^- and ρ^+ and that $\rho(\tau)$ has also two branch points τ^- and τ^+. It is readily verified that

$$\begin{aligned} &\rho^{\pm} = [1 + \frac{a_2}{a_1}\,(1 \mp \sqrt{a_1})^2]^{-1}\ ,\ 0 < \rho^- < \rho^+ \leq 1, \\ &\tau^- = 0\ ,\ 0 < \frac{4a_1a_2}{b^2 - 4a_1a_2^2} \ = \tau^+ < 1; \end{aligned} \tag{3.4}$$

analogous relations hold for the branch points $r^{\pm}$ and $t^{\pm}$ of the zeros of $h_2(r, t)$.

The curve $h_1(\rho, \tau) = 0$ is for real ρ and τ a hyperbola with center $(\check{\rho}_m, \check{\tau}_m)$ given by

$$\check{\rho}_m = \frac{a_1 b}{b^2 - 4a_1a_2^2} > 0 \quad , \quad \check{\tau}_m = \frac{2a_1a_2}{b^2 - 4a_1a_2^2} > 0, \tag{3.5}$$

and asymptotes given by

$$\rho - \check{\rho}_m = \frac{1}{2a_2}[b \pm \sqrt{b^2 - 4a_1a_2^2}](\tau - \check{\tau}_m). \tag{3.6}$$

Some special zeros of $h_1(\rho, \tau)$ are listed below.

$$
\begin{aligned}
&\tau^+(0) = -\frac{1}{a_2}\,, && \tau^-(0) = 0\,, \quad \rho^\pm(0) = 0, \\
&\tau^-(1) = \min(1, \frac{1}{a_1}), && \tau^+(1) = \max(1, \frac{1}{a_1}), \\
&\rho^-(1) = \min(1, a_1 + \frac{a_1}{a_2}), && \rho^+(1) = \max(1, a_1 + \frac{a_1}{a_2}), \\
&\frac{\mathrm{d}\rho(\tau)}{\mathrm{d}\tau} = \frac{a_2(a_1-1)}{a_1a_2 + a_1 - a_2} \quad \text{for} && (\rho(\tau), \tau) = (1,1) \text{ and } a_2a_1 + a_2 - 1 \neq 0.
\end{aligned}
\tag{3.7}
$$

In Figure 3.1 the hyperbola $h_1(\rho, \tau) = 0$ is traced, the line $\tau = 0$ is a tangent at $\rho = 0, \tau = 0$.

FIGURE 3.1.

Let (ρ_n, τ_n) be a zero-tuple of $h_1(\rho, \tau)$ with $\rho_n = \rho^-(\tau_n), \tau_n = \tau^+(\rho_n)$ and $\tau_n > 1$, see Figure 3.2.

FIGURE 3.2.

Starting from (ρ_n, τ_n) we construct a series of zero-tuples of $h_1(\rho, \tau)$, cf.(3.2) and (3.3):

$$
(\rho_\nu, \tau_\nu), \ldots, (\rho_{n-1}, \tau_{n-1}), (\rho_n, \tau_n), (\rho_{n+1}, \tau_{n+1}), \ldots;, \tag{3.8}
$$

they are recursively defined by:

$$\begin{aligned} \tau_{n-1} &= \tau^-(\rho_n)\ , & \rho_{n+1} &= \rho^+(\tau_n), \\ \rho_{n-1} &= \rho^-(\tau_{n-1})\ , & \tau_{n+1} &= \tau^+(\rho_{n+1}). \end{aligned} \tag{3.9}$$

The sequence in (3.8) will be called *the ladder generated by* τ_n *on the hyperbola* $h_1(\rho,\tau)$. Its "up-ladder" is unbounded, its "down-ladder" is finite and stopped at that index ν for which $0 < \tau_\nu < 1$ or $\rho_\nu < 1$.
Whenever $a_1 \neq a_2$, cf. Remark 2.1, the zero-tuple (ρ_n, τ_n) of $h_1(\rho,\tau)$ *induces also a ladder* viz.

$$,\ldots,(\tilde{r}_{n-1},\tilde{t}_{n-1}),\ (\tilde{r}_n,\tilde{t}_n),\ (\tilde{r}_{n+1},\tilde{t}_{n+1}),\ldots, \tag{3.10}$$

on the hyperbola $h_2(r,t) = 0$, and $\tilde{t}_{n\mp i} \neq \tau_{n\mp i}, i \neq 0$. It is recursively defined by

$$\begin{aligned} \tilde{t}_n &= \tau_n\ , & \tilde{r}_n &= r^-(\tilde{t}_n), \\ \tilde{t}_{n-1} &= t^-(\tilde{r}_n)\ , & \tilde{r}_{n+1} &= r^+(\tilde{t}_n), \\ \tilde{r}_{n-1} &= r^-(\tilde{t}_{n-1}), & \tilde{t}_{n+1} &= t^+(\tilde{r}_{n+1}); \end{aligned} \tag{3.11}$$

again the "down-ladder" is stopped at that μ for which $\tilde{t}_\mu < 1$ or $\tilde{r}_\mu < 1$.

REMARK 3.1. Note that for $a_1 \neq a_2$, i.e. $h_1(\rho,t) \not\equiv h_2(\rho,t)$,

$$\begin{aligned} \tilde{t}_{n-1} &\neq \tau_{n-1} & \tilde{t}_{n+1} &\neq \tau_{n+1}, \\ \tilde{r}_{n-1} &\neq r_{n-1}\ , & \tilde{r}_{n+1} &\neq r_{n+1}. \end{aligned}$$

□

Analogously a zero tuple of $h_2(r,t)$ with $r_n = r^-(t_n), t_n = t^+(r_n)$ and $t_n > 1$ or $r_n > 1$ generates a ladder on $h_2(r,t) = 0$ and induces a ladder on $h_1(\tilde{\rho},\tilde{\tau}) = 0$, see Figure 3.3.

FIGURE 3.3

These ladders are defined analogously to those in Figure 3.2. Actually interchange in (3.9) and (3.4) ρ and r and also τ and t. It should be noted that the ladder in (3.8) with top (ρ_n, τ_n) is identical with the ladder generated on $h_1(\rho,\tau) = 0$ with top (ρ_{n+1}, τ_{n+1}) if $\tau_{n+1} = \tau^+(\rho_{n+1}), \rho_{n+1} = \rho^+(\tau_n)$.

However, the ladder on $h_2(r,t)$ induced by (ρ_n, τ_n) generally differs from that induced on $h_2(r,t)$ by (ρ_{n+1}, τ_{n+1}).

We shall denote by

$$\begin{array}{lll} i. & l(\tau_n) & \text{with } h_1(\rho_n,\tau_n)=0 \text{ the ladder generated by } (\rho_n,\tau_n) \text{ on } h_1(\rho,\tau)=0; \\ ii. & \tilde{l}(\tau_n) & \text{the ladder induced by } (\rho_n,\tau_n) \text{ on } h_2(\tilde{r},\tilde{t})=0 \text{ with } h_1(\rho_n,\tau_n)=0. \end{array} \tag{3.12}$$

Analogously the ladders $l(t_n)$ and $\tilde{l}(t_n)$ are defined.

REMARK 3.2. Note that every point of $h_1(\rho,\tau)$ with $\rho > 1$ or $\tau > 1$ induces a ladder on $h_2(r,t)$ and conversely. □

It is further readily seen from (3.9) and (3.10) that, cf. Figure 3.2. and 3.3:

$$\begin{aligned} \tau_{n+1} &= \tau^+(\rho^+(\tau_n)) \quad , \quad \rho_{n+1} = \rho^+(\tau^+(\rho_n)), \\ t_{n+1} &= t^+(r^+(t_n)) \quad , \quad r_{n+1} = r^+(t^+(r_n)), \end{aligned} \tag{3.13}$$

$$\tau_{n+m} \to \infty,\ \rho_{n+m} \to \infty,\ t_{n+m} \to \infty,\ r_{n+m} \to \infty,$$

$$\text{for } m \to \infty \text{ and } \tau_n > 1, t_n > 1.$$

Next we introduce a notation to describe all the ladder points on the upladders on $h_1(\rho,t)=0$ as well as on $h_2(r,t)=0$ generated by a point

$$(\rho_0,\tau_0) \text{ of } h_1(\rho,\tau)=0.$$

Define for $m = 0,1,2,\dots,$ the binary numbers

$$\begin{aligned} &\delta_m := \delta_{m1}\delta_{m2}\cdots\delta_{mm} \text{ with } \delta_{mj} \in \{0,1\},\ j=1,\dots,m, \\ &\mathcal{B}_m := \{\delta : \delta = \delta_j,\ j=0,\dots,2^m-1\} \equiv \{\delta : \delta \in \{0,1,2,\dots,2^m-1\}\}. \end{aligned} \tag{3.14}$$

The tree generated by (τ_0,ρ_0) is defined as follows:
its nodes at the n-th level, $n = 0,1,\dots,$ are

$$\begin{array}{llll} i. & \tau^{(n)}_{\delta^{(n)}_j} & \text{with } \delta^{(n)}_j < \delta^{(n)}_{j+1}, & \delta^{(n)}_j, \delta^{(n)}_{j+1} \in \mathcal{B}_n,\ j=0,\dots,2^n-1, \\ ii. & \tau^{(n+1)}_{\delta^{(n+1)}_j} & := \tau^+(\rho^+(\tau^{(n)}_{\delta^{(n)}_j})) & \text{for} \quad \delta^{(n+1)}_j = 2\delta^{(n)}_j, \\ & & := t^+(r^+(\tau^{(n)}_{\delta^{(n)}_j})) & \quad ,, \quad\ \ \delta^{(n+1)}_j = 2\delta^{(n)}_j + 1. \\ iii. & \tau^{(0)}_0 & = \tau_0 = \tau^+(\rho_0). & \end{array} \tag{3.15}$$

In Figure 3.4 the nodes at the levels 0, 1, 2 and 3 for the tree generated by (τ_0,ρ_0) are shown.

FIGURE 3.4

From the definitions above it is readily seen that the tree so constructed contains all the ladder points on $h_1(\rho,\tau)=0$ and $h_2(r,t)=0$ generated by τ_0 with $h_1(\rho_0,\tau_0)=0$; note that $l(\tau_0)$is the set of nodes on the left branch of the tree, $\tilde{l}(\tau_0)$ that on the right branch of the tree.

The tree generated by (t_0,r_0) is defined analogously interchange the symbols τ and t and also ρ and r.

We conclude this section with the derivation of some asymptotic results. From (3.1) it is seen that

$$\begin{aligned} R_1^{\pm} &:= \lim_{\tau\to\infty}\frac{\rho^{\pm}(\tau)}{\tau} = \frac{1}{2a_2}(b\pm d_1). \\ R_2^{\pm} &:= \lim_{t\to\infty}\frac{r^{\pm}(t)}{t} = \frac{1}{2a_1}(b\pm d_2), \\ T_1^{\pm} &:= \lim_{\rho\to\infty}\frac{\tau^{\pm}(\rho)}{\rho} = \frac{1}{2a_1a_2}(b\pm d_1), \\ T_2^{\pm} &:= \lim_{r\to\infty}\frac{t^{\pm}(r)}{r} = \frac{1}{2a_1a_2}(b\pm d_2), \end{aligned} \tag{3.16}$$

with

$$d_1 := \sqrt{b^2-4a_1a_2^2}\ ,\ d_2 := \sqrt{b^2-4a_1^2a_2}.$$

Note that, cf. Lemma 3.1, for $j=1,2$,

$$\begin{aligned} &R_j^+ > 1\ ,\ T_j^+ > 1\ ,\ R_j^{\pm}T_j^{\mp} = 1, \\ &a_1a_2(T_1^{\pm})^2 - bT_1^{\pm} + a_2 = 0, \quad a_2(R_1^{\pm})^2 - bR_1^{\pm} + a_1a_2 = 0, \\ &a_2(R_1^{\pm}-1)^2 - (b-2a_2)(R_1^{\pm}-1) - a_1 = 0. \end{aligned} \tag{3.17}$$

Further from (2.4) by using (3.16) and (3.17),

$$\begin{aligned} \lim_{\rho\to\infty}\frac{\mathrm{d}\tau^{\pm}(\rho)}{\mathrm{d}\rho} &= \frac{bT_1^{\pm}-2a_2}{2a_1a_2T_1^{\pm}-b} = [R_1^{\mp}]^{-1}, \\ \lim_{\tau\to\infty}\frac{\mathrm{d}\rho^{\pm}(\tau)}{\mathrm{d}\tau} &= \frac{2a_1a_2-bR_1^{\pm}}{b-2a_2R_1^{\pm}} = T_1^{\mp}. \end{aligned} \tag{3.18}$$

Because

$$R_1^- < 1 < R_2^+ \quad \text{and} \quad R_1^-R_1^+ = a_1,$$

we have

$$R_1^- < \min(1,a_1) \quad , \quad R_1^+ > \max(1,a_1), \tag{3.19}$$

further

$$a_1 - R_1^{\pm} = \frac{a_1}{a_2}\frac{R_1^{\pm}}{1-R_1^{\pm}}. \tag{3.20}$$

4. Analytic continuation

In this section we shall consider the analytic continuation of the functions $\Omega_1(s),\Omega_2(s),B(s)$ and $\Phi_0(s)$ out from $|s|\le 1$ into $|s|>1$.

LEMMA 4.1. *The functions $\Omega_1(s),\Omega_2(s),B(s)$ and $\Phi_0(s)$ can be continued meromorphically out from $|s|\le 1$ into $|s|>1$.*

For the proof of this lemma see Appendix $\mathcal{B}$.

REMARK 4.1. The lemma does not imply that these continuations are meromorphic functions, i.e. have only a finite number of poles in every finite domain, but it implies that their only singularities are poles or accumlation points of poles. □

ASSUMPTION 4.1. Henceforth it will be assumed that the meromorphic continuations of $B(\cdot), \Phi_0(\cdot), \Omega_1(\cdot)$ and $\Omega_2(\cdot)$ are all meromorphic functions.

REMARK 4.2. Whenever the Assumption 4.1 leads to the construction of functions $B(\cdot), \Phi_0(\cdot), \Omega_1(\cdot)$ and $\Omega_2(\cdot)$ which satisfy the conditions (2.10) then our problem has been solved because there exists only one set of such functions, cf. Remark 2.2. □

The meromorphic continuations of $B(\cdot), \Phi_0(\cdot), \Omega_1(\cdot)$ and $\Omega_2(\cdot)$ will be indicated by the same symbols, respectively.

It is further shown in Appendix $\mathcal{B}$ that the functional equations (2.9) can be extended into the domains $|\tau| > 1, |\rho| > 1, |t| > 1$ and $|r| > 1$. Actually it is shown that the following relations hold for all those τ, ρ, r, t, for which $B(\cdot), \Phi_0(\cdot), \Omega_1(\cdot), \Omega_2(\cdot)$ are finite.

$$\Omega_2(\rho) + \frac{a_1\tau^{\pm}(\rho)}{\rho - \tau^{\pm}(\rho)} B(\tau^{\pm}(\rho)) + k_1(\rho, \tau^{\pm}(\rho))\Phi_0(\tau^{\pm}(\rho)) = 0 \quad \text{for} \quad h_1(\rho, \tau^{\pm}(\rho)) = 0, \tag{4.1}$$

$$\Omega_1(r) - \frac{a_2 t^{\pm}(r)}{r - t^{\pm}(r)} B(t^{\pm}(r)) + k_2(r, t^{\pm}(r))\Phi_0(t^{\pm}(r)) = 0, \quad \text{for} \quad h_2(r, t^{\pm}(r)) = 0, \tag{4.2}$$

$$\Omega_2(\rho^{\pm}(\tau)) + \frac{a_1\tau}{\rho^{\pm}(\tau) - \tau} B(\tau) + k_1(\rho^{\pm}(\tau), \tau)\Phi_0(\tau) = 0, \quad \text{for} \quad h_1(\rho^{\pm}(\tau), \tau) = 0, \tag{4.3}$$

$$\Omega_1(r^{\pm}(t)) - \frac{a_2 t}{r^{\pm}(t) - t} B(t) + k_2(r^{\pm}(t)), t)\Phi_0(t) = 0, \quad \text{for} \quad h_2(r^{\pm}(t), t) = 0. \tag{4.4}$$

REMARK 4.3. From the analysis in Appendix $\mathcal{B}$ it is seen that the relations (4.2),...,(4.5), are independent of assumption 4.1. Note that (4.2) and (4.4), and similarly, (4.3) and (4.5), are not independent.

□

ASSUMPTION 4.2. Henceforth it will be assumed that the poles of $B(\cdot), \Phi_0(\cdot), \Omega_1(\cdot), \Omega_2(\cdot)$ are all simple poles.

REMARK 4.4. Concerning the introduction of the latter assumption it is noted that Remark 4.2 also applies here. □

Put for finite t and ρ:

$$\begin{aligned} &b(t) := \lim_{s\to t}(s-t)B(s), \qquad \phi_0(t) := \lim_{s\to t}(s-t)\Phi_0(s), \\ &\omega_i(\rho) := \lim_{r\to\rho}(r-\rho)\Omega_i(r), \quad i = 1, 2. \end{aligned} \tag{4.5}$$

Assumption 4.2 implies that these limits exist. Obviously, cf. Assumption 4.2,

$$\begin{aligned} &|b(t)| \neq 0 \quad \Leftrightarrow \quad t \quad \text{is a pole of } B(\cdot), \\ &|\phi_0(t)| \neq 0 \quad \Leftrightarrow \quad t \quad \text{is a pole of } \Phi_0(\cdot), \\ &|\omega_i(\rho)| \neq 0 \quad \Leftrightarrow \quad t \quad \text{is a pole of } \Omega_i(\cdot), \quad i = 1, 2. \end{aligned} \tag{4.6}$$

Note that (2.10) implies:

$$b(t) = 0 \text{ and } \phi_0(t) = 0 \text{ for } |t| \leq 1, \tag{4.7}$$

$$\omega_i(\rho) = 0 \text{ for } |\rho| \leq 1,\ i = 1, 2.$$

From (4.2) ,. . .,(4.6), it follows readily that, cf.(3.4),
for $|\rho| \neq \rho^{\pm},\ h_1(\rho, \tau^{\pm}(\rho)) = 0$,

$$\omega_2(\rho) + [\frac{a_1\tau^{\pm}(\rho)}{\rho - \tau^{\pm}(\rho)} b(\tau^{\pm}(\rho)) + k_1(\rho, \tau^{\pm}(\rho))\phi_0(\tau^{\pm}(\rho))][\frac{\mathrm{d}\tau^{\pm}(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\rho} = 0; \tag{4.8}$$

for $|r| \neq r^{\pm}, h_2^{\pm}(r, t^{\pm}(r)) = 0$,

$$\omega_1(r) + [\frac{-a_2 t^{\pm}(r)}{r - t^{\pm}(r)} b(t^{\pm}(r)) + k_2(r, t^{\pm}(r))\phi_0(t^{\pm}(r))][\frac{\mathrm{d}t^{\pm}(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=r} = 0; \tag{4.9}$$

for $|\tau| \neq \tau^{\pm}, h_1(\rho^{\pm}(\tau), \tau) = 0$,

$$\omega_2(\rho^{\pm}(\tau))[\frac{\mathrm{d}\rho^{\pm}(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tau} + \frac{a_1\tau}{\rho^{\pm}(\tau) - \tau} b(\tau) + k_1(\rho^{\pm}(\tau), \tau)\phi_0(\tau) = 0; \tag{4.10}$$

for $|t| \neq t^{\pm}, h_2(r^{\pm}(t), t) = 0$,,

$$\omega_1(r^{\pm}(t))[\frac{\mathrm{d}r^{\pm}(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tau} + \frac{-a_2 t}{r^{\pm}(t) - t} b(t) + k_2(r^{\pm}(t), t)\phi_0(t) = 0. \tag{4.11}$$

In Sections 6, 7, 8 and 9 the relations (4.9) ,. . .,(4.12) will be used to calculate the residues of $B(\cdot), \Phi_0(\cdot), \Omega_1(\cdot), \Omega_2(\cdot)$ at their various poles. Note that the relations (4.11) and (4.12) on the one hand and (4.9) and (4.10) on the other hand are dependent of each other.

5. The equation for the top of the tree

In this section we derive a relation for the in absolute value smallest pole of $B(\cdot)$ and of $\Phi_0(\cdot)$.

From (4.4) and (4.5) we have:

for $|\tau| > 1,\ h_1(\rho^-(\tau), \tau) = 0,\ h_2(r^-(\tau), \tau) = 0$:

$$\frac{a_1\tau}{\rho^-(\tau) - \tau} B(\tau) \ +\ k_1(\rho^-(\tau), \tau)\Phi_0(\tau) = -\Omega_2(\rho^-(\tau)), \tag{5.1}$$

$$\frac{-a_2\tau}{r^-(\tau) - \tau} B(\tau) \ +\ k_2(r^-(\tau), \tau)\Phi_0(\tau) = -\Omega_1(r^-(\tau)). \tag{5.2}$$

Put

$$D(\rho(\tau), r(\tau), \tau) := \begin{vmatrix} \dfrac{a_1\tau}{\rho(\tau) - \tau} & k_1(\rho(\tau), \tau) \\ \dfrac{-a_2\tau}{r(\tau) - \tau} & k_2(r(\tau), \tau) \end{vmatrix}. \tag{5.3}$$

A simple calculation using (2.3), (2.8) and (5.1) shows that

$$\begin{aligned} &\frac{1}{\tau}(\rho^-(\tau) - \tau)(r^-(\tau) - \tau)D(\rho^-(\tau), r^-(\tau), \tau) = \\ &[b - a_1a_2(\pi_1 r^-(\tau) + \pi_2\rho^-(\tau))]\tau - a_1 r^-(\tau) - a_2\rho^-(\tau). \end{aligned} \tag{5.4}$$

Because $\rho^-(\tau) - \tau \neq 0,\ r^-(\tau) - \tau \neq 0$ for $|\tau| > 1$ it follows

$$D(\rho^-(\tau), r^-(\tau), \tau) = 0 \Leftrightarrow \tau = \frac{a_1 r^-(\tau) + a_2\rho^-(\tau)}{b - a_1a_2[\pi_1 r^-(\tau) + \pi_2\rho^-(\tau)]}. \tag{5.5}$$

From (2.3) and (3.7) it is seen that

$$\frac{b}{a_1 a_2} - \pi_1 r^-(1) - \pi_2 \rho^-(1) > 0. \tag{5.6}$$

LEMMA 5.1.

i. The equation

$$\tau = \frac{\dfrac{r^-(\tau)}{a_2} + \dfrac{\rho^-(\tau)}{a_1}}{\dfrac{b}{a_1 a_2} - \{\pi_1 r^-(\tau) + \pi_2 \rho^-(\tau)\}}$$

has in $\tau > 1$ a unique root $\tau = T$, say, it has multiplicity one,

ii. for $\dfrac{1}{a_1} + \dfrac{1}{a_2} > 1$ it is given by

$$T = (\frac{1}{a_1} + \frac{1}{a_2})^2,$$

further

$$\rho^-(T) = r^-(T) = \frac{1}{a_1} + \frac{1}{a_2}.$$

PROOF. It is readily seen that the equation is equivalent with

$$\tau = \frac{1 + \dfrac{1}{a_2}(1 - \dfrac{r^-(\tau)}{\tau}) + \dfrac{1}{a_1}(1 - \dfrac{\rho^-(\tau)}{\tau})}{1 - \pi_1(1 - \dfrac{r^-(\tau)}{\tau}) - \pi_2(1 - \dfrac{\rho^-(\tau)}{\tau})}. \tag{5.7}$$

Because, cf.Lemma 3.1 and (3.16),

$$0 < \frac{r^-(\tau)}{\tau} < 1, \quad 0 < \frac{\rho^-(\tau)}{\tau} < 1 \quad \text{for } \tau > 1,$$

$$\frac{r^-(\tau)}{\tau} \downarrow R_2^-, \qquad \frac{\rho^-(\tau)}{\tau} \downarrow R_1^- \quad \text{for } \tau : 1 \to \infty,$$

and because the righthand side cf.(5.7) is readily seen to be larger than one for $\tau \geq 1$, the first statement of the lemma follows. It is simply verified that $\tau = T, \rho^-(\tau) = \rho^-(T),\ r^-(\tau) = r^-(T)$ satisfy (5.7). It remains to show that, cf.(3.1)ii and Lemma 3.1,

$$\begin{aligned} &i. \quad \rho^-(T) < T, \quad r^-(T) < T, \\ &ii. \quad h_1(\rho^-(T), T) = 0, \quad h_2(r^-(T), T) = 0. \end{aligned} \tag{5.8}$$

Because $\dfrac{1}{a_1} + \dfrac{1}{a_2} > 1$, (5.8)i follows. Further by noting that $T = [\rho^-(T)]^2$ it is readily seen that (5.8)ii holds. □

ASSUMPTION 5.1. Henceforth it will be assumed that, cf. also Assumption 6.1.

$$|\Omega_2(\rho^-(T))| < \infty, |\Omega_1(r^-(T))| < \infty.$$

REMARK 5.1. Concerning the introduction of the latter Assumption it is noted that Remark 4.2 also applies here. □

Since $\tau = T$ is a simple zero of the determinant in (5.3) it follows from (5.2), and Assumption 5.1 that

$\tau = T$ *is a simple pole of* $B(\tau)$ *and also of* $\Phi_0(\tau)$. (5.9)

In Section 8, Remark 8.2, it will be shown that T is the in absolute value smallest pole of $B(\cdot)$ and also of $\Phi_0(\cdot)$.

From (4.6), (5.2) and Assumption 5.1 we obtain:

$$\begin{aligned} &\frac{a_1 T}{\rho^-(T) - T} b(T) + k_1(\rho^-(T), T)\phi_0(T) = 0, \\ &\frac{-a_2 T}{r^-(T) - T} b(T) + k_2(r^-(T), T)\phi_0(T) = 0, \end{aligned} \tag{5.10}$$

note that these relations are linearly dependent.

REMARK 5.2. A simple calculation shows that for $\tau > 1$ the determinant

$$\begin{vmatrix} \dfrac{a_1\tau}{\rho^+(\tau)-\tau} & k_1(\rho^+(\tau),\tau) \\ \dfrac{a_1\tau}{\rho^-(\tau)-\tau} & k_1(\rho^-(\tau),\tau) \end{vmatrix} \neq 0 \text{ with } h_1(\rho^\pm(\tau),\tau) = 0. \tag{5.11}$$

Similarly,

$$\begin{vmatrix} \dfrac{-a_2 t}{r^+(t)-t} & k_2(r^+(t),t) \\ \dfrac{-a_2 t}{r^-(t)-t} & k_2(r^-(t),t) \end{vmatrix} \neq 0 \text{ with } h_2(r^\pm(\tau),\tau) = 0, \tag{5.12}$$

and by using (5.7)

$$\begin{vmatrix} \dfrac{-a_1 T}{\rho^+(T)-T} & k_1(\rho^+(T),T) \\ \dfrac{-a_2 T}{r^+(T)-T} & k_2(r^+(T),T) \end{vmatrix} \neq 0, \text{ with } h_1(\rho^+(T),T) = 0,\ h_2(r^+(T),T) = 0\ . \tag{5.13}$$

REMARK 5.3. From Lemma 5.1 it is seen that T is independent of the value of π_1, and that it depends only on the sum $\dfrac{1}{a_1} + \dfrac{1}{a_2}$. □

6. Poles and residues for the case $a_1 \neq a_2$

In this section we determine the poles of $B(\cdot), \Phi_0(\cdot), \Omega_i(\cdot), i = 1, 2$, and derive equations for the residues at these poles. From these equations it will be seen that these residues can be calculated recursively and that they all contain the factor $\phi_0(T)$.

With

$$\tau_0^{(0)} := T, \quad \rho_0^{(0)} := \rho^-(\tau_0^{(0)}), \quad r_0^{(0)} = r^-(\tau_0^{(0)}), \tag{6.1}$$

and T as defined in Lemma 5.1, we have

$$\begin{aligned} &i. \quad \frac{a_1 \tau_0^{(0)}}{\rho_0^{(0)} - \tau_0} b(\tau_0^{(0)}) \quad + k_1(\rho_0^{(0)}, \tau_0^{(0)})\phi_0(\tau_0^{(0)}) = 0, \\ &ii. \quad \frac{-a_2 \tau_0^{(0)}}{r_0^{(0)} - \tau_0} b(\tau_0^{(0)}) \quad + k_2(r_0^{(0)}, \tau_0^{(0)})\phi_0(\tau_0^{(0)}) = 0. \end{aligned} \tag{6.2}$$

With

$$\rho_0^{(1)} = \rho^+(\tau_0^{(0)}), \quad r_1^{(1)} := r^+(\tau_0^{(0)}), \tag{6.3}$$

we obtain from (4.11) and (4.12),

$$\begin{aligned} &i. \quad \frac{a_1\tau_0^{(0)}}{\rho_0^{(1)}-\tau_0^{(0)}}b(\tau_0^{(0)}) \;+\; k_1(\rho_0^{(1)},\,\tau_0^{(0)})\phi_0(\tau_0^{(0)}) \quad +\omega_2(\rho_0^{(1)})[\frac{\mathrm{d}\rho^+(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tau_0^{(0)}}=0,\\ &ii. \quad \frac{-a_2\tau_0^{(0)}}{r_1^{(1)}-\tau_0^{(0)}}b(\tau_0^{(0)}) \;+\; k_2(r_1^{(1)},\,\tau_0^{(0)})\phi_0(\tau_0^{(0)}) \quad +\omega_1(r_1^{(1)})[\frac{\mathrm{d}r^+(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tau_0^{(0)}}=0. \end{aligned} \tag{6.4}$$

From (2.8) and (3.4) it is readily verified that the derivative in (6.4)i is finite and nonzero. Further the determinant formed by the coefficients of $b(\tau_0^{(0)})$ and $\phi_0(\tau_0^{(0)})$ in (6.2)i and (6.4)i is nonzero, cf.(5.11). Hence from (6.2)i and (6.4)i it is seen that $\omega_2(\rho_0^{(1)})$ is nonzero and finite and proportional to $\phi_0(\tau_0^{(0)})$, analogously for $\omega_1(r_1^{(1)})$. Consequently

$$\begin{aligned} &i. \quad \rho_0^{(1)}=\rho^+(\tau_0^{(0)}) \text{ is a simple pole of } \Omega_2(\cdot),\\ &ii. \quad r_1^{(1)}=r^+(\tau_0^{(0)}) \text{ is a simple pole of } \Omega_1(\cdot),\\ &iii. \quad \omega_2(\rho_0^{(1)}) \text{ and } \omega_1(r_1^{(1)}) \text{ contain } \phi_0(\tau_0^{(0)}) \text{ as a factor.} \end{aligned} \tag{6.5}$$

FIGURE 6.1

In Figure 6.1 several nodes of the tree generated by $\tau_0^{(0)}=T$ are shown and

$$\rho_{00}^{(2)}=\rho^+(\tau_0^{(1)}),\; r_{11}^{(2)}=r^+(\tau_1^{(1)})\,,\quad r_{01}^{(2)}=r^+(\tau_0^{(1)})\,,\quad \rho_{10}^{(1)}=\rho^+(\tau_1^{(1)}), \tag{6.6}$$

note that

$(r^-(\tau_0^{(1)}),\ \tau_0^{(1)})$ and $(r_{01}^{(2)},\ \tau_0^{(1)})$ are zero-tuples of $h_2(r,t)=0$ induced by $\tau_0^{(1)}$,
$(r^-(\tau_0^{(1)})$ is not shown in Figure 6.1),
$(\rho^-(\tau_1^{(1)}),\tau_1^{(1)})$ and $(\rho_{10}^{(1)},\tau_1^{(1)})$ are zero-tuples of $h_1(\rho,\tau)=0$ induced by $\tau_1^{(1)}$,
$(\rho^-(\tau_1^{(1)})$ is not shown in Figure 6.1). (6.7)

Next we consider (4.12) for the zero-tuple $(r^-(\tau_0^{(1)}),\tau_0^{(1)})$ of $h_2(r,t)=0$ and (4.11) for the zero-tuple $(\rho^-(\tau_1^{(1)}),\tau_1^{(1)})$ of $h_1(\rho,\tau)=0$, i.e.

$$\begin{aligned} &i. \quad \omega_1(r^-(\tau_0^{(1)}))[\frac{\mathrm{d}r^-(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tau_0^{(1)}} \quad + \quad \frac{-a_2\tau_0^{(1)}}{r^-(\tau_0^{(1)})-\tau_0^{(1)}}b(\tau_0^{(1)})+k_2(r^-(\tau_0^{(1)}),\tau_0^{(1)})\phi_0(\tau_0^{(1)})=0\,,\\ &ii. \quad \omega_2(\rho^-(\tau_1^{(1)}))[\frac{\mathrm{d}\rho^-(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tau_1^{(1)}} \quad + \quad \frac{a_1\tau_1^{(1)}}{\rho^-(\tau_1^{(1)})-\tau_1^{(1)}}b(\tau_1^{(1)})+k_1(\rho^-(\tau_1^{(1)}),\tau_1^{(1)})\phi_0(\tau_1^{(1)})=0\,. \end{aligned} \tag{6.8}$$

Below we introduce Assumption 6.1 which implies that for the case $a_1 \neq a_2$,

$$\omega_1(r^-(\tau_0^{(1)})) = 0 \quad , \quad \omega_2(\rho^-(\tau_1^{(1)})) = 0. \tag{6.9}$$

Hence from (4.11) and (6.8)i we have

$$\begin{aligned} \omega_2(\rho_0^{(1)})[\frac{\mathrm{d}\rho^-(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tau_0^{(1)}} \quad &+ \frac{a_1\tau_0^{(1)}}{\rho_0^{(1)} - \tau_0^{(1)}} b(\tau_0^{(1)}) + k_1(\rho_0^{(1)}, \tau_0^{(1)})\phi_0(\tau_0^{(1)}) = 0, \\ &- \frac{a_2\tau_0^{(1)}}{r^-(\tau_0^{(1)}) - \tau_0^{(1)}} b(\tau_0^{(1)}) + k_2(r^-(\tau_0^{(1)}), \tau_0^{(1)})\phi_0(\tau_0^{(1)}) = 0. \end{aligned} \tag{6.10}$$

Here $\omega_2(\rho_0^{(1)})$ is given by (6.4)i. The main determinant of the two equations (6.10) with unknowns $b(\tau_0^{(1)}), \phi_0(\tau_0^{(1)})$ is $D(\rho^-(\tau_0^{(1)}), r^-(\tau_0^{(1)}), \tau_0^{(1)})$, cf.(5.3). The unique zero of the determinant (5.3) in $\tau > 1$ is $\tau = T = \tau_0^{(0)}$, cf.Lemma 5.1, so that it is nonzero, because $\tau_0^{(1)} > \tau_0^{(0)} > 1$. The derivative in (6.10) is also finite and nonzero, hence (6.10) has a nonzero and finite solution $b(\tau_0^{(1)}), \phi_0(\tau_0^{(1)})$.

Analogously the equations

$$\begin{aligned} \omega_1(r_1^{(1)})[\frac{\mathrm{d}r^-(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tau_1^{(1)}} \quad &- \frac{a_2\tau_1^{(1)}}{r_1^{(1)} - \tau_1^{(1)}} b(\tau_1^{(1)}) + k_2(r_1^{(1)}, \tau_1^{(1)})\phi_0(\tau_1^{(1)}) = 0\,, \\ &\frac{a_1\tau_1^{(1)}}{\rho^-(\tau_1^{(1)}) - \tau_1^{(1)}} b(\tau_1^{(1)}) + k_1(\rho^-(\tau_1^{(1)}), \tau_1^{(1)})\phi_0(\tau_1^{(1)}) = 0, \end{aligned} \tag{6.11}$$

has a nonzero and finite solution $b(\tau_1^{(1)}), \phi_0^{(1)}(\tau_1^{(1)})$. Consequently,

$$\begin{aligned} &i. \quad \tau_0^{(1)} \text{ and } \tau_1^{(1)} \text{ are simple poles of } B(\cdot) \text{ and also of } \Phi_0(\cdot), \\ &ii. \quad b(\tau_0^{(1)}), b(\tau_1^{(1)}), \phi_0(\tau_0^{(1)}), \phi_0(\tau_1^{(1)}) \text{ all contain } \phi_0(\tau_0^{(1)}) \text{ as a factor.} \end{aligned} \tag{6.12}$$

With $b(\tau_0^{(1)}), \phi_0(\tau_0^{(1)})$ defined above as the solution of the equations (6.10) and $b(\tau_1^{(1)}), \phi_0(\tau_1^{(1)})$ as the solution of the equations (6.11) we obtain from (4.4) and (4.12),

$$\begin{aligned} &i. \quad \omega_2(\rho_{00}^{(2)})[\frac{\mathrm{d}\rho^+(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tau_0^{(1)}} \quad + \frac{a_1\tau_0^{(1)}}{\rho_{00}^{(2)} - \tau_0^{(1)}} b(\tau_0^{(1)}) + k_1(\rho_{00}^{(2)}, \tau_0^{(1)})\phi_0(\tau_0^{(1)}) = 0, \\ &ii. \quad \omega_1(r_{01}^{(2)})[\frac{\mathrm{d}r^+(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tau_0^{(1)}} \quad + \frac{-a_2\tau_0^{(1)}}{r_{01}^{(2)} - \tau_0^{(1)}} b(\tau_0^{(1)}) + k_2(r_{01}^{(2)}, \tau_0^{(1)})\phi_0(\tau_0^{(1)}) = 0, \end{aligned} \tag{6.13}$$

$$\begin{aligned} &i. \quad \omega_2(\rho_{10}^{(2)})[\frac{\mathrm{d}\rho^+(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tau_1^{(1)}} \quad + \frac{a_1\tau_1^{(1)}}{\rho_{10}^{(2)} - \tau_1^{(1)}} b(\tau_1^{(1)}) + k_1(\rho_{10}^{(2)}, \tau_1^{(1)})\phi_0(\tau_1^{(1)}) = 0, \\ &ii. \quad \omega_1(r_{11}^{(2)})[\frac{\mathrm{d}r^+(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tau_1^{(1)}} \quad + \frac{-a_2\tau_1^{(1)}}{r_{11}^{(2)} - \tau_1^{(1)}} b(\tau_1^{(1)}) + k_2(r_{11}^{(2)}, \tau_1^{(1)})\phi_0(\tau_1^{(1)}) = 0. \end{aligned} \tag{6.14}$$

From (6.13) and (6.14) it is seen that:

$$\begin{aligned} &\rho_{00}^{(2)}, \rho_{10}^{(2)} \text{ and } r_{01}^{(2)}, r_{11}^{(2)} \text{ are simple poles of } \Omega_2(\cdot) \text{ and } \Omega_1(\cdot), \text{ respectively,} \\ &\text{for which the residues follow from (6.13) and (6.14).} \end{aligned} \tag{6.15}$$

The relations (6.4) and (6.8) represent the relations for the poles and residues at the zero-level of the tree, cf. also (6.3), the relations (6.10), (6.11), (6.13) and (6.14) describe the relations for the poles and residues at the first level of the tree generated by $\tau_0^{(0)}$. To obtain those relations at the nth level of the tree we introduce, cf.(6.9), the following

ASSUMPTION 6.1. For the case $a_1 \neq a_2$ let $\tau_\delta^{(n)}, \delta \in \{0, 1, \dots, 2^n - 1\}$ be an element of the tree generated by $\tau_0^{(0)} = T$, see Figure 3.4. Henceforth it will be assumed, cf.Remark 2.1,

$$\begin{aligned} &i. \quad \text{for } \delta \text{ even :} \\ &\qquad \omega_1(r^-(\tau_\delta^{(n)})) = 0 \quad \text{with} \quad h_2(r^-(\tau_\delta^{(n)}), \tau_\delta^{(n)}) = 0, \\ &ii. \quad \text{for } \delta \text{ odd :} \\ &\qquad \omega_2(\rho^-(\tau_\delta^{(n)})) = 0 \quad \text{with} \quad h_1(\rho^-(\tau_\delta^{(n)}),\ \tau_\delta^{(n)}) = 0. \end{aligned} \tag{6.16}$$

□

REMARK 6.1. Concerning the introduction of the latter assumption it is noted that remark 4.2 also applies here. □

Consider Figure 6.2.

FIGURE 6.2

with the symbols defined by

$$\begin{aligned} \rho_\delta^{(n)} &:= \rho^-(\tau_\delta^{(n)}), \qquad r_\delta^{(n)} := r^-(\tau_\delta^{(n)}), \\ \rho_{2\delta}^{(n+1)} &:= \rho^+(\tau_\delta^{(n)}), \quad r_{2\delta+1}^{(n+1)} = r^+(\tau_\delta^{(n)}), \end{aligned} \tag{6.17}$$

and

$$\delta \in \{0, \dots, 2^n - 1\}$$

written as a binary number, cf.(3.14).

Suppose for the present that

$$\begin{aligned} &\text{for } \delta \text{ even :} \quad \rho_\delta^{(n)} \quad \text{is a simple pole of} \quad \Omega_2(\cdot), \\ &\text{for } \delta \text{ odd :} \quad r_\delta^{(n)} \quad \text{is a simple pole of} \quad \Omega_1(\cdot). \end{aligned} \tag{6.18}$$

We consider first the case that

$$\delta \textit{ is even.} \tag{6.19}$$

For this case we have from (4.11) and (4.12) because assumption 6.1 implies that $\omega_1(r_\delta^{(n)}) = 0$:

$$\begin{aligned}
&i.\quad \omega_2(\rho_\delta^{(n)})[\frac{\mathrm{d}\rho^-(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tau_\delta^{(n)}} && +\frac{a_1\tau_\delta^{(n)}}{\rho_\delta^{(n)}-\tau_\delta^{(n)}}b(\tau_\delta^{(n)})+k_1(\rho_\delta^{(n)},\ \tau_\delta^{(n)})\phi_0(\tau_\delta^{(n)})=0,\\
& && -\frac{a_2\tau_\delta^{(n)}}{r_\delta^{(n)}-\tau_\delta^{(n)}}b(\tau_\delta^{(n)})+k_2(r_\delta^{(n)},\ \tau_\delta^{(n)})\phi_0(\tau_\delta^{(n)})=0;\\
&ii.\quad \omega_2(\rho_{2\delta}^{(n+1)})[\frac{\mathrm{d}\rho^+(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tau_\delta^{(n)}} && +\frac{a_1\tau_\delta^{(n)}}{\rho_{2\delta}^{(n+1)}-\tau_\delta^{(n)}}b(\tau_\delta^{(n)})+\ k_1(\rho_{2\delta}^{(n+1)},\ \tau_\delta^{(n)})\phi_0(\tau_\delta^{(n)})=0,\\
&iii.\quad \omega_1(r_{2\delta+1}^{(n+1)})[\frac{\mathrm{d}r^+(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tau_\delta^{(n)}} && -\frac{a_2\tau_\delta^{(n)}}{r_{2\delta+1}^{(n+1)}-\tau_\delta^{(n)}}b(\tau_\delta^{(n)})+k_2(r_{2\delta+1}^{(n+1)},\ \tau_\delta^{(n)})\phi_0(\tau_\delta^{(n)})=0.
\end{aligned}\tag{6.20}$$

Next the case

$$\delta\quad \textit{is odd}\tag{6.21}$$

then analogously we have because $\omega_2(\rho_\delta^{(n)})=0$:

$$\begin{aligned}
&i.\quad \omega_1(r_\delta^{(n)})[\frac{\mathrm{d}r^-(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tau_\delta^{(n)}} && +\frac{-a_2\tau_\delta^{(n)}}{r_\delta^{(n)}-\tau_\delta^{(n)}}b(\tau_\delta^{(n)})+k_2(r_\delta^{(n)},\ \tau_\delta^{(n)})\phi_0(\tau_\delta^{(n)})=0,\\
& && \frac{a_1\tau_\delta^{(n)}}{\rho_\delta^{(n)}-\tau_\delta^{(n)}}b(\tau_\delta^{(n)})+\ k_1(\rho_\delta^{(n)},\ \tau_\delta^{(n)})\phi_0(\tau_\delta^{(n)})=0;\\
&ii.\quad \omega_1(r_{2\delta+1}^{(n+1)})[\frac{\mathrm{d}r^+(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tau_\delta^{(n)}} && +\frac{-a_2\tau_\delta^{(n)}}{r_{2\delta+1}^{(n+1)}-\tau_\delta^{(n)}}b(\tau_\delta^{(n)})+k_2(r_{2\delta+1}^{(n+1)},\tau_\delta^{(n)})\phi_0(\tau_\delta^{(n)})=0.\\
&iii.\quad \omega_2(\rho_{2\delta}^{(n+1)})[\frac{\mathrm{d}\rho^+(\sigma)}{\mathrm{d}\sigma}]_{\sigma=\tau_\delta^{(n)}} && +\frac{a_1\tau_\delta^{(n)}}{\rho_{2\delta}^{(n+1)}-\tau_\delta^{(n)}}b(\tau_\delta^{(n)})+k_1(\rho_{2\delta}^{(n+1)},\ \tau_\delta^{(n)})\phi_0(\tau_\delta^{(n)})=0.
\end{aligned}\tag{6.22}$$

The relations (6.20)i are two equations for $b(\tau_\delta^{(n)}),\phi_0(\tau_\delta^{(n)})$ for δ even. Again it is readily verified that the main determinant is nonzero and that the derivative in (6.20)i is finite and nonzero see below (6.10). From (4.6) and (6.18) it is seen that $\omega_2(\rho_\delta^{(n)})$ is finite and nonzero. Then (6.17)i has a nonnull solution $b(\tau_\delta^{(n)}),\phi_0(\tau_\delta^{(n)})$, i.e., (cf.(4.6),

for δ even:
$\tau_\delta^{(n)}$ is a simple pole of $B(\cdot)$ and also of $\Phi_0(\cdot)$. (6.23)

From the solution of (6.20)i it follows from (6.21)ii and iii that, cf.(4.6),

for δ even:
$\rho_{2\delta}^{(n+1)}$ and $r_{2\delta+1}^{(n+1)}$ are simple poles of $\Omega_2(\cdot)$ and $\Omega_1(\cdot)$, respectively, (6.24)
and their residues are calculated from (6.20)ii and (6.20)iii.

Further from (4.6) and (6.18) it follows that

for δ odd:
$\tau_\delta^{(n)}$ is a simple pole of $B(\cdot)$ and $\Phi_0(\cdot)$,
$\rho_{2\delta}^{(n+1)}$ and $r_{2\delta+1}^{(n+1)}$ are simple poles of $\Omega_2(\cdot)$ and $\Omega_1(\cdot)$, respectively, (6.25)
and their residues are calculated from (6.22)ii and (6.22)iii.

It remains to consider the hypothesis, cf.(6.18), that $\omega_2(\rho_\delta^{(n)})$ for δ even and $\omega_1(r_\delta^{(n)})$ for δ odd are both finite and nonzero. By induction it is seen from (6.20) and (6.22) for $n=1,2,\dots,$ and (6.10), (6.11), (6.13) and (6.14) that these hypotheses are indeed valid. Note that these relations show that all residues are zero or no one is zero; the latter case is impossible see last but one paragraph of appendix $\mathcal{B}$.

LEMMA 6.1.*For the case* $a_1 \neq a_2$ *and with* T *as defined in Lemma 5.1:*

i. every element $\tau_\delta^{(n)}, n = 0, 1, 2, \ldots; \delta \in \{0, 1, \ldots, 2^n - 1\}$, *of the tree generated by* $\tau_0^{(0)} = T$ *is a simple pole of* $B(\cdot)$ *and also of* $\Phi_0(\cdot)$,

ii. $\rho_\delta^{(n)} \equiv \rho^-(\tau_\delta^{(n)})$ *is for* δ *even a simple pole of* $\Omega_2(\cdot)$,

iii. $r_\delta^{(n)} \equiv r^-(\tau_\delta^{(n)})$ *is for* δ *odd a simple pole of* $\Omega_1(\cdot)$,

iv. the residues $b(\tau_\delta^{(n)}), \phi_0(\tau_\delta^{(n)})$ *of* $B(\cdot)$ *and* $\Phi_0(\cdot)$ *are obtained by solving for each* $\tau_\delta^{(n)}$ *two linear equation, viz.(6.2) for* $n = 0$, *(6.10) and (6.11) for* $n = 1$, *and (6.20) and (6.22) for* $n = 2, 3, \ldots$;

v. for δ *even the residues* $\omega_2(\rho_{2\delta}^{(n+1)})$ *and* $\omega_1(r_{2\delta+1}^{(n+1)})$ *are determined by (6.8) for* $n = 0$, *by (6.13) for* $n = 1$, *by (6.20)ii, iii for* $n = 2, 3, \ldots$,

vi. for δ *odd the residues* $\omega_2(\rho_{2\delta}^{(n+1)})$ *and* $\omega_1(r_{2\delta+1}^{(n+1)})$ *are determined by (6.14) for* $n = 1$, *and by (6.22)ii, iii for* $n = 2, 3, \ldots$,

vii. these residues can be calculated recursively, except for $\phi_0(\tau_0^{(0)})$ *which is a factor of every residue.*

PROOF. The proof follows immediately from the above analysis in this section. □

Lemma 6.1 describes the equations for the residues at all nodes of the tree generated by $\tau_0^{(0)} = T$. But as we have seen in Section 3 every node $\tau_\delta^{(n)}$ with δ even induces on $h_2(r, t) = 0$ a ladder, and analogously for δ odd a ladder on $h_1(\rho, \tau) = 0$, see Figures 3.2 and 3.3. So we have to consider also the equations (4.9),...,(4.12) for the residues at the points of the down-ladders of such induced ladders, see below (3.9).

LEMMA 6.2. *For the case* $a_1 \neq a_2$ *the only poles of* $B(\cdot), \Phi_0(\cdot), \Omega_1(r)$ *and* $\Omega_2(\rho)$ *are those described in Lemma 6.1.*

PROOF. Let (τ_n, ρ_n) be a zero-tuple of $h_1(\rho, \tau)$ and consider the down-ladder induced by τ_n on $h_2(r, t) = 0$, see Figure 3.2.

With

$$\tau_n = \tilde{t}_n \ , \ \tilde{r}_n = r^-(\tilde{t}_n), \ \tilde{t}_{n-1} = t^-(\tilde{r}_n), \ \tilde{r}_{n-1} = r^-(\tilde{t}_{n-1}), \ldots,$$

it follows from (4.10) that

$$\begin{aligned} -\omega_1(\tilde{r}_n) = \quad & [\frac{-a_2\tilde{t}_n}{\tilde{r}_n - \tilde{t}_n} b(\tilde{t}_n) + k_2(\tilde{r}_n, \tilde{t}_n)\phi_0(\tilde{t}_n)][\frac{\mathrm{d}t^+(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tilde{r}_n} = \\ & [\frac{-a_2\tilde{t}_{n-1}}{\tilde{r}_n - \tilde{t}_{n-1}}]b(\tilde{t}_{n-1}) + k_2(\tilde{r}_n, \tilde{t}_{n-1})\phi_0(\tilde{t}_{n-1})[\frac{\mathrm{d}t^-(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tilde{r}_n}, \end{aligned} \tag{6.26}$$

and

$$\begin{aligned} -\omega_1(\tilde{r}_i) = \quad & [\frac{-a_2\tilde{t}_i}{\tilde{r}_i - \tilde{t}_i} b(\tilde{t}_i) + k_2(\tilde{r}_i, \tilde{t}_i)\phi_0(\tilde{t}_i)][\frac{\mathrm{d}t^+(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tilde{r}_i} = \\ & \frac{-a_2\tilde{t}_{i-1}}{\tilde{r}_i - \tilde{t}_{i-1} b(\tilde{t}_{i-1}) + k_2(\tilde{r}_i, \tilde{t}_{i-1})\phi_0(\tilde{t}_{i-1})}[\frac{\mathrm{d}t^-(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\tilde{r}_{i-1}}, \end{aligned} \tag{6.27}$$

for $i = n-1, n-2, \ldots, \nu$, with ν the index at which the down-ladder is stopped (cf. below (3.9)) i.e. the index for which

$$0 < \tilde{t}_\nu < 1 \text{ or } 0 < \tilde{r}_\nu < 1;$$

so that, cf.(4.8),

$$b(\tau_\nu) = 0,\ \phi_0(\tau_\nu) = 0 \ \text{ or } \ -\omega_1(\tilde{r}_\nu) = 0, \tag{6.28}$$

and, cf. Assumption 6.1,

$$\omega_1(\tilde{r}_n) = 0. \tag{6.29}$$

The set of relations (6.27), (6.28) and (6.29) is insufficient to determine the unknown residues $\omega_1(\tilde{r}_i), b(\tilde{t}_i)$ and $\phi_0(\tilde{t}_i)$. However, assumption 4.1 leads to the conclusion that the only solution of this set of relations is the zero-solution, i.e.

at all induced down-ladders the residues at the elements of these down-ladders are zero, so that these elements cannot be poles of $B(\cdot), \Phi_0(\cdot), \Omega_1(\cdot)$ and $\Omega_2(\cdot)$.

To see this first note that every element of the tree generated by $\tau_0^{(0)} = T$ induces down-ladders on $h_1(\rho, \tau) = 0$ or on $h_2(r, t) = 0$, and elements of down-ladders again induce down-ladders. Since the tree generated by $\tau_0^{(0)}$ consists of an infinite, but countable number of nodes, it follows that the finite part of $h_1(\rho, \tau) = 0$ with $\rho \in [\rho^-(1), \rho^+(1)]$ contains an infinitely countable set of elements (ρ, τ) stemming from the induced down-ladders, similarly for $h_2(r, t) = 0$. Hence Assumption 4.1 requires that the residues at these elements are all zero, because a meromorphic function can have at most a finite number of poles in a finite domain. It is readily seen (cf.(6.27), (6.28)) from Assumption 6.1 together with the relations (4.9),...,(4.12), that all residues at points of the down-ladders *induced* by any point of the tree are indeed zero. Consequently for the various assumptions so far introduced, cf. Assumptions 4.1, 4.2, 5.1 and 6.1, the only poles of $B(\cdot), \Phi_0(\cdot), \Omega_i(\cdot), i = 1, 2$, are those mentioned in Lemma 6.1. □

7. Asymptotic behaviour of the residues, $a_1 \neq a_2$

For the further analysis of the functions $B(\cdot), \Phi_0(\cdot), \Omega_1(\cdot)$ and $\Omega_2(\cdot)$, we require the asymptotic behaviour for $n \to \infty$ of the residues of these functions at their poles, see the preceding section.

First we consider the ladder generated by $\tau_0^{(0)} = T$ on $h_1(\rho, \tau) = 0$, i.e. the set of nodes $\tau^{(n)}_{\delta_j^{(n)}}, \delta_j^{(n)} = 0, n = 0, 1, 2, \ldots$, which is the left-most branch of the tree generated by $(T, \rho^-(T))$.

Put cf. Figure 3.2, for $n = 0, 1, 2, \ldots$,

$$\begin{aligned}
&\tau_n := \tau_0^{(n)},\ \tau_0^{(0)} = T, \\
&\rho_n := \ \rho^-(\tau_n),\ \rho_{n+1} = \rho^+(\tau_n) = \rho^-(\tau_{n+1}), \quad h_1(\rho_n, \tau_n) = 0, \\
&\tau_n = \tau^+(\rho_n),\ \tau_{n+1} = \tau^+(\rho_{n+1}), \qquad\qquad\quad h_1(\rho_{n+1}, \tau_{n+1}) = 0.
\end{aligned} \tag{7.1}$$

From (4.9) we obtain

$$\omega_2(\rho_{n+1}) + [\frac{a_1 \tau_{n+1}}{\rho_{n+1} - \tau_{n+1}} b(\tau_{n+1}) + k_1(\rho_{n+1}, \tau_{n+1}) \Phi_0(\tau_{n+1})][\frac{\mathrm{d}\tau^+(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma = \rho_{n+1}} = 0, \tag{7.2}$$

$$\omega_2(\rho_{n+1}) + [\frac{a_1 \tau_n}{\rho_{n+1} - \tau_n} b(\tau_n) + k_1(\rho_{n+1}, \tau_n) \phi_0(\tau_n)][\frac{\mathrm{d}\tau^-(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma = \rho_{n+1}} = 0. \tag{7.3}$$

Elimination of $\omega_2(\rho_{n+1})$ yields

$$
\begin{aligned}
&[\frac{a_1\tau_{n+1}}{\rho_{n+1}-\tau_{n+1}}b(\tau_{n+1})+ \quad \frac{k_1(\rho_{n+1},\tau_{n+1})}{\tau_{n+1}}\tau_{n+1}\phi_0(\tau_{n+1})][\frac{\mathrm{d}\tau^+(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\rho_{n+1}} = \\
&[\frac{a_1\tau_n}{\rho_{n+1}-\tau_n}b(\tau_n)+ \qquad \frac{k_1(\rho_{n+1},\tau_n)}{\tau_n}\tau_n\phi_0(\tau_n)][\frac{\mathrm{d}\tau^-(\sigma)}{\mathrm{d}\sigma}]^{-1}_{\sigma=\rho_{n+1}}.
\end{aligned} \tag{7.4}
$$

Further with

$$\tilde{r}_{n+1} = r^-(\tau_{n+1}), \quad h_2(\tilde{r}_{n+1},\tau_{n+1}) = 0,$$

from (6.15),

$$\frac{-a_2\tau_{n+1}}{\tilde{r}_{n+1}-\tau_{n+1}}b(\tau_{n+1}) + \frac{k_2(\tilde{r}_{n+1},\tau_{n+1})}{\tau_{n+1}}\tau_{n+1}\phi_0(\tau_{n+1}) = 0. \tag{7.5}$$

For $n\to\infty$ we have, cf.(2.8), (3.16),..., (3.19), since $\tau_n\to\infty$ implies $\rho_n\to\infty$ and $\tilde{r}_n\to\infty$:

$$
\begin{aligned}
&\frac{a_1\tau_{n+1}}{\rho_{n+1}-\tau_{n+1}} \to \frac{a_1}{R_1^- - 1}, \quad \frac{k_1(\rho_{n+1},\tau_{n+1})}{\tau_{n+1}} \to a_2\pi_2(a_1-R_1^-),\\
&\frac{a_1\tau_n}{\rho_{n+1}-\tau_n} \to \frac{a_1}{R_1^+ - 1}, \quad \frac{k_1(\rho_{n+1},\tau_n)}{\tau_n} \to a_2\pi_2(a_1-R_1^+),\\
&\frac{a_2\tau_{n+1}}{\tilde{r}_{n+1}-\tau_{n+1}} \to \frac{a_2}{R_2^- - 1}, \quad \frac{k_2(\tilde{r}_{n+1},\tau_{n+1})}{\tau_{n+1}} \to a_1\pi_1(a_2-R_2^-),
\end{aligned} \tag{7.6}
$$

$$\frac{\mathrm{d}\tau^\pm(\sigma)}{\mathrm{d}\sigma}\,|_{\sigma=\rho_{n+1}} \to \frac{bT_1^\pm - 2a_2}{2a_1a_2T_1^\pm - b},$$

$$\frac{\tau_{n+1}}{\tau_n} = \frac{\tau^+(\rho^+(\tau_n))}{\rho^+(\tau_n)}\,\frac{\rho^+(\tau_n)}{\tau_n} \to \frac{R_1^+}{R_1^-} = \frac{T_1^+}{T_1^-} = R_1^+T_1^+ > 1.$$

LEMMA 7.1. *For the elements (ρ_n,τ_n) of the left-most ladder of the tree generated by $(T,\rho_0(T))$, cf.(7.1), holds:*

for $n = 0,1,2,\dots,$

$$
\begin{aligned}
&i. \quad \lim_{m\to\infty}\frac{b(\tau_{m+n})}{b(\tau_m)} = \mu_1^n,\\
&ii. \quad \lim_{m\to\infty}\frac{\tau_{m+n}\phi_0(\tau_{m+n})}{\tau_m\phi_0(\tau_m)} = \mu_1^n, \quad \lim_{m\to\infty}\frac{\phi_0(\tau_{m+1})}{\phi_0(\tau_m)} = \mu_1\lambda_1,\\
&iii. \quad \lim_{m\to\infty}\frac{\omega_2(\rho_{m+n})}{\omega_2(\rho_m)} = \mu_1^n, \quad \lim_{m\to\infty}\frac{\phi_0(\tau_{m+n})}{\phi_0(\tau_m)} = (\mu_1\lambda_1)^n,
\end{aligned} \tag{7.7}
$$

with

$$iv. \quad 0 < \lambda_1 := \frac{R_1^-}{R_1^+} < 1, \quad \mu_1 = \lambda_1^{-1}\frac{R_1^- - 1}{R_1^+ - 1}\frac{\pi_1R_2^- + \pi_2R_1^+}{\pi_1R_2^- + \pi_2R_1^-} < 0,$$

$$v. \quad \lim_{n\to\infty}\frac{b(\tau_n)}{\tau_n\phi_0(\tau_n)} = -\pi_1R_2^-,$$

$$vi. \quad \lim_{n\to\infty}\frac{\omega_2(\rho_n)}{\tau_n\phi_0(\tau_n)} = \frac{a_1R_1^-}{(R_1^- - 1)}(\pi_1R_2^- + \pi_2R_1^-) < 0;$$

for the elements (r_n,t_n) of the right-most ladder of the tree generated by $(T,r^-(T))$ holds: for $n = 0,1,2,\dots,$

$$
\begin{aligned}
&i. && \lim_{m\to\infty}\frac{b(t_{m+n})}{b(t_m)}=\mu_2^n, \\
&ii. && \lim_{m\to\infty}\frac{t_{m+n}\phi_0(t_{m+n})}{t_m\phi_0(t_m)}=\mu_2^n, \quad \lim_{m\to\infty}\frac{\phi_0(t_{m+1})}{\phi_0(t_m)}=\mu_2\lambda_2, \\
&iii. && \lim_{m\to\infty}\frac{\omega_1(r_{m+n})}{\omega_1(r_m)}=\mu_2^n, \qquad \lim_{m\to\infty}\frac{\phi_0(t_{m+n})}{\phi_0(t_m)}=(\mu_2\lambda_2)^n,
\end{aligned}
\tag{7.8}
$$

with

$$
\begin{aligned}
&iv. && 0<\lambda_2:=\frac{R_2^-}{R_2^+}<1, \quad \mu_2=\lambda_2^{-1}\frac{R_2^- -1}{R_2^+ -1}\frac{\pi_2R_1^- +\pi_1R_2^+}{\pi_2R_1^- +\pi_1R_2^-}<0, \\
&v. && \lim_{n\to\infty}\frac{b(t_n)}{t_n\phi_0(t_n)} = -\pi_2R_1^-, \\
&vi. && \lim_{n\to\infty}\frac{\omega_1(r_n)}{\tau_n\phi_0(\tau_n)} = \frac{a_2R_2^-}{R_2^- -1}(\pi_2R_1^- +\pi_1R_2^-)<0.
\end{aligned}
$$

PROOF. From (7.5) and (7.6) it follows that the following limit exists and is given by, cf.(3.20),

$$\lim_{n\to\infty}\frac{b(\tau_{n+1})}{\tau_{n+1}\phi_0(\tau_{n+1})}=\frac{a_1}{a_2}\pi_1(R_2^- -1)(a_2-R_2^-)=-\pi_1R_2^-,$$

so that (7.7)v has been proved.

From (7.4), (7.6) and (7.7)v it is seen that the first limit in (7.7)ii exists for $n=1$ and from this result the relation for the second limit in (7.7)ii follows by using the definition of λ_1, cf.(7.7)iv, and the last relation of (7.6). The second relation in (7.7)iii follows immediately from the first one in (7.7)ii.

Next we let $n\to\infty$ in (7.4). We then obtain by using (3.16), (3.18), (7.1) and (7.7)v,

$$
\begin{aligned}
&[\frac{a_1}{R_1^- -1}(-\pi_1R_2^-)+a_2\pi_2(a_1-R_1^-)]R_1^-\mu_1 = \\
&\frac{a_1}{R_1^+ -1}(-\pi_1R_2^-)+a_2\pi_2(a_1-R_1^+)]R_1^+.
\end{aligned}
$$

By using (3.20) the second relation in (7.7)iv is readily obtained, since $0<R_1^-<1, R_1^+>1$. The lefthand inequality of the first relation of (7.7)iv follows from $R_1^+>R_1^->0$.
From (7.7)ii, v, and from

$$
\begin{aligned}
-\pi_1R_2^- &= \lim_{n\to\infty}\frac{b(\tau_{n+1})}{b(\tau_n)}\frac{b(\tau_n)}{\tau_n\phi_0(\tau_n)}\frac{\tau_n\phi_0(\tau_n)}{\tau_{n+1}\phi_0(\tau_{n+1})} = \\
&= -\pi_1R_2^-\frac{1}{\mu_1}\lim_{n\to\infty}\frac{b(\tau_{n+1})}{b(\tau_n)},
\end{aligned}
$$

the relation (7.7)i readily follows.

From (7.2) we obtain for $n\to\infty$,

$$\lim_{n\to\infty}\frac{\omega_2(\rho_{n+1})}{\tau_{n+1}\phi_0(\tau_{n+1})}+\frac{a_1}{R_1^- -1}(-\pi_1R_2^-)+a_2\pi_2(a_1-R_1^-)R_1^-=0,$$

so by using (3.20) we obtain (7.7)vi. From

$$\frac{\omega_2(\rho_{n+1})}{\tau_{n+1}\phi_0(\tau_{n+1})}=\frac{\omega_2(\rho_{n+1})}{\omega_2(\rho_n)}\frac{\omega_2(\rho_n)}{\tau_n\phi_0(\tau_n)}\frac{\tau_n\phi_0(\tau_n)}{\tau_{n+1}\phi_0(\tau_{n+1})}$$

it follows, because the first and third quotient have the same limit, cf.(7.7), whereas that of the last quotient is equal to μ_1^-, that

$$\lim_{n\to\infty}\frac{\omega_2(\rho_{n+1})}{\omega_2(\rho_n)}=\mu_1,$$

and so the first relation in (7.7)iii follows. Hence (7.7) has been proved, and (7.8) follows from the symmetry. □

The lemma above describes only the asymptotic behaviour of the various residues at those nodes of the tree generated by $(T,\rho^-(T))$ which belong to the left- or right-most branch of that tree, i.e. at the nodes

$$\tau_0^{(m+n)}\quad\text{and}\quad\tau_\delta^{(n)}\quad\text{with}\quad\delta=2^n-1.$$

To consider the asymptotics for $n\to\infty$ of the residues at a generic point

$$\tau_\delta^{(m+n)},\delta\in\{0,1,\dots,2^{m+n}-1\},$$

we write

$$\begin{aligned}\delta=\ & e_m2^n+d_n,\\ & e_m\in\{0,\dots,2^m-1\},\quad d_n\in\{0,1,\dots,2^n-1\}.\end{aligned}\tag{7.9}$$

Hence $\tau_\delta^{(m+n)},n=0,1,2,\dots$, is the tree generated by $\tau_{e_m}^{(m)}$, it is a subtree of that generated by T. Now write d_n as a binary number and in this binary representation denote by

$$\begin{aligned}&d_1^{(n)}\text{ the number of zeros },\\ &d_2^{(n)}\text{ the number of ones },\end{aligned}\tag{7.10}$$

$$d_1^{(n)}+d_2^{(n)}=n.\tag{7.11}$$

It readily follows from (7.6), (7.9) and (7.10) that for every finite n: for $m\to\infty$,

$$\frac{\tau_\delta^{(m+n)}}{\tau_{e_m}^{(m)}}\to\{T_1^+R_1^+\}^{d_1^{(n)}}\{T_2^+R_2^+\}^{d_2^{(n)}}=\lambda_1^{-d_1^{(n)}}\lambda_2^{-d_2^{(n)}}>1.\tag{7.12}$$

LEMMA 7.2. *Let $\tau_\delta^{(m+n)},m=0,1,2,\dots;n=0,1,2,\dots$, be a generic element of the tree generated by $(T,\rho^-(T))$ then: for $n=0,1,2,\dots$,*

$$\begin{aligned}
&i.\quad \lim_{m\to\infty}\frac{b(\tau_\delta^{(m+n)})}{b(\tau_{e_m}^{(m)})}=\mu_1^{d_1^{(n)}}\mu_2^{d_2^{(n)}},\\
&ii.\quad \lim_{m\to\infty}\frac{\tau_\delta^{(m+n)}\phi_0(\tau_\delta^{(m+m)})}{\tau_{e_m}^{(m)}\phi_0(\tau_{e_m}^{(m)})}=\mu_1^{d_1^{(n)}}\mu_2^{d_2^{(n)}},\\
&iii.\quad \lim_{m\to\infty}\frac{\omega_i(\rho_\delta^{(m+n)})}{\omega_i(\rho_{e_m}^{(m+n)})}=\mu_1^{d_1^{(n)}}\mu_2^{d_2^{(n)}},\quad i=1,2,
\end{aligned}\tag{7.13}$$

with $e_m,d_n,d_1^{(n)},d_2^{(n)}$ for given $\delta\in\{0,1,\dots,2^{m+n}-1\}$ as defined in (7.9) and (7.10), and, cf.(6.17),

$$\begin{aligned}\rho_\delta^{(m+n)}\ &=\rho^-(\tau_\delta^{(m+n)})\quad \textit{for}\quad \delta\ \textit{even},\\ &=r^-(\tau_\delta^{(m+n)})\quad ,,\quad \delta\ \textit{odd}.\end{aligned}$$

PROOF. For $\delta=e_m2^n+d$ consider the tree generated by $\tau_{e_m}^{(m)}$ which is a subtree of the tree generated by $\tau_0^{(0)}=T$. Apply for this subtree Lemma 7.1 with $n=1$. Next apply again this lemma with $n=1$ for the elements

$$\tau_\delta^{(m+1)} \text{ with } \delta = 2e_m + d_1\ ,\ d_1 \in \{0,1\}; \tag{7.14}$$

then (7.13)i for $n = 1$ follows from (7.7)i and (7.8)i. Apply Lemma 7.1 again with $n = 1$ for the subtree gnerated by $\tau_\delta^{(m+1)}$ as given in (7.14), then (7.13)i follows for

$$\tau_\delta^{(m+2)} \text{ with } \delta = 2^2 e_m + d_2\ ,\ d_2 \in \{0,1,2,3\}.$$

Repeating this procedure leads to (7.13)i. The statements (7.13)ii and iii are similarly proved. □

8. The solution for the case $a_1 \neq a_2$

We introduce the following meromorphic functions.

For $\tau_\delta^{(n)}, \delta \in (0,1,2,\dots,2^n-1), n = 0,1,2\dots$, a node of the nth level of the tree generated by $\tau_0^{(0)} = T$, cf. Section 3, and with

$$\begin{aligned} &\rho_\delta^{(n)} = \rho^-(\tau_\delta^{(n)})\ , \quad h_1(\rho^-(\tau_\delta^{(n)}), \tau_\delta^{(n)}) = 0, \quad \delta \text{ even}\ ,\\ &r_\delta^{(n)} = r^-(\tau_\delta^{(n)})\ , \quad h_2(r^-(\tau_\delta^{(n)}),\ \tau_\delta^{(n)}) = 0, \quad \delta \text{ odd}\ , \end{aligned} \tag{8.1}$$

put for nonnegative integers m_b, m_ϕ, m_2, m_1:

$$\begin{aligned} &i. \quad \tilde{B}(\tau) := \sum_{n=0}^{\infty} \sum_{\delta\in\mathcal{B}_n} \frac{b_\delta^{(n)}}{\tau - \tau_\delta^{(n)}} \Big(\frac{\tau}{\tau_\delta^{(n)}}\Big)^{m_b},\\ &ii. \quad \tilde{\Phi}_0(\tau) := \sum_{n=0}^{\infty} \sum_{\delta\in\mathcal{B}_n} \frac{\phi_\delta^{(n)}}{\tau - \tau_\delta^{(n)}} \Big(\frac{\tau}{\tau_\delta^{(n)}}\Big)^{m_\phi},\\ &iii. \quad \tilde{\Omega}_2(\rho) := \sum_{n=1}^{\infty} \sum_{\delta\in\mathcal{B}_n} \frac{\omega_{2\delta}^{(n)}}{\rho - \rho_\delta^{(n)}} \Big(\frac{\rho}{\rho_\delta^{(n)}}\Big)^{m_2},\\ &iv. \quad \tilde{\Omega}_1(r) := \sum_{n=1}^{\infty} \sum_{\delta\in\mathcal{B}_n} \frac{\omega_{1\delta}^{(n)}}{r - r_\delta^{(n)}} \Big(\frac{r}{r_\delta^{(n)}}\Big)^{m_1}, \end{aligned} \tag{8.2}$$

with, cf.(3.14),

$$\mathcal{B}_n = \{0,1,2,\dots,2^n-1\}, \tag{8.3}$$

and where, cf.(4.6),

$$b_\delta^{(n)} \equiv b(\tau_\delta^{(n)}),\ \phi_\delta^{(n)} \equiv \phi_0(\tau_\delta^{(n)}),\ \omega_{2\delta}^{(n)} \equiv \omega_2(\rho_\delta^{(n)}),\ \omega_{1\delta}^{(n)}) \equiv \omega_1(r_\delta^{(n)}). \tag{8.4}$$

First we have to determine the values of $m_{\dots}$ for which the righthand sides of (8.2) are well-defined meromorphic functions, cf.[14].

Consider first the function in (8.2)i. From (7.12) it is readily seen that $\tilde{B}(\tau)$ as defined in (8.2)i has only a finite number of poles in any finite interval. From Lemma 7.2 and (7.12) we have with

$$\delta = e_k 2^n + d_n,\ d_1^{(n)}, d_2^{(n)},\ \text{as defined in (7.10)},$$

for $k \to \infty$,

$$\frac{b_\delta^{(k+n)}}{[\tau_\delta^{(k+n)}]^{m+1}} \Big/ \frac{b_{e_k}^{(k)}}{[\tau_{e_k}^{(k)}]^{m+1}} \to \frac{\mu_1^{d_1^{(n)}} \mu_2^{d_2^{(n)}}}{[\lambda_1^{-d_1^{(n)}} \lambda_2^{-d_2^{(n)}}]^{m+1}} = [\mu_1\lambda_1^{m+1}]^{d_1^{(n)}} [\mu_2\lambda_2^{m+1}]^{d_2^{(n)}}. \tag{8.5}$$

Because $0 < \lambda_i < 1, i = 1,2$, and $d_1^{(n)} + d_2^{(n)} = n$, we have

$$\sum_{d_1^{(n)}=0}^{n} \binom{n}{d_1^{(n)}} [\mu_1\lambda_1^{m+1}]^{d_1^{(n)}} [\mu_2\lambda_2^{m+1}]^{d_2^{(n)}} = [\mu_1\lambda_1^{m+1} + \mu_2\lambda_2^{m+1}]^n.$$

Obviously, a unique, finite nonnegative number $\tilde{M}$ may be and is defined by

$$\begin{aligned}
&i. \quad |\mu_1\lambda_1^{\tilde{M}+1}+\mu_2\lambda_2^{\tilde{M}+1}|<1<|\mu_1\lambda_1^{\tilde{M}}+\mu_2\lambda_2^{\tilde{M}}| \text{ if } |\mu_1\lambda_1+\mu_2\lambda_2|\geq 1,\\
&ii. \quad \tilde{M}=0 \text{ if } |\mu_1\lambda_1+\mu_2\lambda_2|<1,
\end{aligned} \tag{8.6}$$

note that μ_1 and μ_2 are both negative, cf.(7.7)iv, (7.8)iv.
Hence for $m_b \geq \tilde{M}$ it is seen that the righthand side of (8.2)i converges absolutely for every τ with $|\tau| < R$, for every finite R, whenever terms with poles $\tau_\delta^{(n)}, |\tau_\delta^{(n)}| < R$, are deleted from this sum. Consequently, the sum in the righthand side of (8.2)i is a well-defined meromorphic function for $m_b \geq \tilde{M}$. The same conclusion is reached for the sums in (8.2)iii and (8.2)iv, i.e. $m_2 \geq \tilde{M}, m_1 \geq \tilde{M}$. Next consider (8.2)ii. A calculation analogous to that in (8.5) yields by using (7.7)ii that the meromorphic function in (8.2)ii is well-defined for $m_\phi \geq \max(0, \tilde{M}-1)$.

LEMMA 8.1. *For $a_1 \neq a_2, \dfrac{1}{a_1}+\dfrac{1}{a_2} > 1$, the functions $\tilde{B}(\cdot), \tilde{\Phi}_0(\cdot), \tilde{\Omega}_i, i=1,2$, are well-defined meromorphic functions for*

$$m_b \geq \tilde{M}, m_2 \geq \tilde{M}, m_1 \geq \tilde{M},\ m_\phi \geq \max(0, \tilde{M}-1),$$

with $\tilde{M}$ as defined in (8.6);
$\tilde{B}(\tau)$ and $\tilde{\Phi}_0(\tau)$ are both regular for $|\tau| < T = (\dfrac{1}{a_1}+\dfrac{1}{a_2})^2, \tilde{\Omega}_2(\rho)$ is regular for $|\rho| < \rho^+(\tau_0^{(0)}) =$

$a_1(\dfrac{1}{a_1}+\dfrac{1}{a_2})^3+\dfrac{a_1}{a_2}(\dfrac{1}{a_1}+\dfrac{1}{a_2})$, and $\tilde{\Omega}_1(r)$ for $|r| < a_2(\dfrac{1}{a_1}+\dfrac{1}{a_2})^3+\dfrac{a_2}{a_1}(\dfrac{1}{a_2}+\dfrac{1}{a_2})$.

PROOF. The first statement has been proved above, the other statements follow from Lemma 6.2 and $\tau_0^{(0)} = T > 1, \tau_\delta^{(n)} > T$,

$\rho_\delta^{(n)} \geq \rho^+(\tau_0^{(0)}) = a_1(\dfrac{1}{a_1}+\dfrac{1}{a_2})^3+\dfrac{a_1}{a_2}+(\dfrac{1}{a_1}+\dfrac{1}{a_2}),\ \delta \in \mathcal{B}_n,\ n=1,2,\ldots$, and analogously for $\tilde{\Omega}_1(r)$.□

Next we introduce four polynomials, viz. $\hat{B}(\cdot), \hat{\Phi}_0(\cdot), \hat{\Omega}_i(\cdot)$, with degrees $\hat{N}_b, \hat{N}_\phi, \hat{N}_i, i=1,2$, and put

$$\begin{aligned}
B(\tau) &= \hat{B}(\tau)+\tilde{B}(\tau) && \text{with} && \hat{B}(\tau)=\sum_{k=0}^{\hat{N}_b}\hat{B}_k\tau^k,\\
\Phi_0(\tau) &= \hat{\Phi}_0(\tau)+\tilde{\Phi}_0(\tau) && ,, && \hat{\Phi}_0(\tau)=\sum_{k=0}^{\hat{N}_\phi}\hat{\Phi}_{00}\tau^k,\\
\Omega_2(\rho) &= \hat{\Omega}_2(\rho)+\tilde{\Omega}_2(\rho) && ,, && \hat{\Omega}_2(\rho)=\sum_{k=0}^{\hat{N}_2}\hat{\Omega}_{20}\rho^k,\\
\Omega_1(r) &= \hat{\Omega}_1(r)+\tilde{\Omega}_1(r) && ,, && \hat{\Omega}_1(r)=\sum_{k=0}^{\hat{N}_1}\hat{\Omega}_{10}r^k,
\end{aligned} \tag{8.7}$$

and $\tilde{B}(\cdot), \tilde{\Phi}_0(\cdot), \tilde{\Omega}_i(\cdot), i=1,2$, given by (8.2), cf. also Lemma 8.1. It will be shown that these polynomials may be determined in such a way that the functions in the lefthand sides of (8.7) satisfy the conditions (2.10).

LEMMA 8.2. *The functions $B(\cdot), \Phi_0(\cdot), \Omega_i(\cdot), i=1,2$, as given by (8.7), satisfy the condition (2.10)i.*

PROOF. The statement of the lemma follows immediately from Lemma 8.2, note that $T > 1, \rho^+(T) > 1$ and $r^+(T) > 1$, cf.Lemmas 3.1 and 5.1. □

For the functions $B(\cdot), \Phi_0(\cdot), \Omega_i(\cdot), i = 1,2$ as described in (8.7) denote by:

$$\begin{aligned} &F_2^{\pm}(\tau) \quad \text{the lefthand side of (4.4)},\\ &F_1^{\pm}(\tau) \quad \text{the lefthand side of (4.5)}. \end{aligned} \tag{8.8}$$

From (8.2), (8.7) and (8.8) we then have (note $\omega_{20}^{(0)} = 0$)

$$\begin{aligned} F_2^{\pm}(\tau) &= \hat{\Omega}_2(\rho^{\pm}(\tau)) + \frac{a_1\tau}{\rho^{\pm}-\tau}\hat{B}(\tau) + \frac{1}{\tau}k_1(\rho^{\pm}(\tau),\tau)\tau\hat{\Phi}_0(\tau)\\ &+ \textstyle\sum_{n=0}^{\infty} \sum_{\delta\in\mathcal{B}_n} \displaystyle\frac{1}{\tau-\tau_\delta^{(n)}}[\omega_{2\delta}^{(n)}\frac{\tau-\tau_\delta^{(n)}}{\rho^{\pm}(\tau)-\rho_\delta^{(n)}}\{\frac{\rho^{\pm}(\tau)}{\rho_\delta^{(n)}}\}^{m_2} + b_\delta^{(n)}\frac{a_1\tau}{\rho^{\pm}(\tau)-\tau}\{\frac{\tau}{\tau_\delta^{(n)}}\}^{m_b} +\\ &+\tau\phi_\delta^{(n)}\frac{k_1(\rho^{\pm}(\tau),\tau)}{\tau}\{\frac{\tau}{\tau_\delta^{(n)}}\}^{m_\phi}], \end{aligned} \tag{8.9}$$

note that the sum of the three terms inside the square brackets is zero for $\tau = \tau_\delta^{(n)}$, cf. (4.11). For $\tau \to \infty$ it follows from (2.8), (3.16), (8.7) and (8.9) since $F_2^{\pm}(\tau)$ is regular for $|\tau| \geq 1$, that

$$\begin{aligned} F_2^{\pm}(\tau) &= \frac{\hat{\Omega}_2(\rho^{\pm}(\tau))}{[\rho^{\pm}(\tau)]^{\hat{N}_2}}[\frac{\rho^{\pm}(\tau)}{\tau}]^{\hat{N}_2}\tau^{\hat{N}_2} + \frac{a_1}{R_1^{\pm}-1}\frac{\hat{B}(\tau)}{[\tau]^{\hat{N}_b}}\tau^{\hat{N}_b} +\\ &a_2\pi_2(a_1 - R_1^{\pm})\frac{\hat{\Phi}_0(\tau)}{[\tau]^{\hat{N}_\phi}}\tau^{\hat{N}_\phi+1} + \tilde{\gamma}_2\tau^{\tilde{m}_2} + \mathrm{O}(\tau^{\tilde{m}_2-1}),\ \tau\to\infty, \end{aligned} \tag{8.10}$$

with $\tilde{\gamma}_2$ a nonzero constant and

$$\tilde{m}_i := \max(m_i - 1, m_b - 1, m_\phi),\ i = 1,2. \tag{8.11}$$

Put

$$\hat{\mu}_i := \max(\hat{N}_i, \hat{N}_b, \hat{N}_\phi + 1,\ m_2 - 1, m_b - 1, m_\phi),\ i = 1,2, \tag{8.12}$$

with $\hat{N}_\phi + 1$ deleted if $\hat{\Phi}_0(\cdot) \equiv 0$.
From (8.9), (8.10) and (8.12) it is readily verified that

$$|F_2^{\pm}(\tau)| \sim \tilde{\gamma}_2|\tau|^{\hat{\mu}_2} \text{ for } |\tau| \to \infty. \tag{8.13}$$

Because $\tau^{\pm}$ are the only branch points of $\rho^{+}(\tau)$ and $\rho^{-}(\tau)$ it follows from (4.4) and Lemma 6.1 that $F_2^{+}(\tau) + F_2^{-}(\tau)$ is regular for all τ. Consequently, Liouville's theorem implies that $F_2^{+}(\tau) + F_2^{-}(\tau)$ is a polynomial of degree $\hat{\mu}_2$. Such a polynomial contains $\hat{\mu}_2 + 1$ coefficients. Because $F_2^{+}(\tau) + F_2^{-}(\tau)$ should be zero for all τ, cf.(4.4), and (8.8), we thus obtain conditions for the coefficients of the polynomials $\hat{B}(\cdot), \hat{\Phi}_0(\cdot), \hat{\Omega}_i(\cdot), i = 1,2$, since analogous conclusions hold for $F_1^{\pm}(t)$.

Note that next to these conditions we have the two conditions which stem from the definitions (2.1), see also (2.10). Further is should be mentioned that the set of Kolmogorov equations which are equivalent with the conditions (2.10) contains one dependent equation. So in total the coefficients of the polynomials have to satisfy $\hat{\mu}_1 + 1 + \hat{\mu}_2 + 1 + 2 - 1 = \hat{\mu}_1 + \hat{\mu}_2 + 3$ conditions. Consequently, we have, cf.(8.7) and Lemma 8.1,

$$\begin{aligned} &i. \quad m_b \geq \tilde{M}, m_2 \geq \tilde{M}, m_1 \geq \tilde{M},\quad m_\phi \geq \max(0, \tilde{M}-1),\\ &ii. \quad \hat{N}_b \geq 0,\ \hat{N}_2 \geq 0,\ \hat{N}_1 \geq 0\ ,\ \hat{N}_\phi \geq 0,\\ &iii. \quad |\hat{B}_0| \geq 0, |\hat{\Omega}_{20}| \geq 0, |\hat{\Omega}_{10}| \geq 0, |\hat{\Phi}_{00}| \geq 0,\\ &iv. \quad \hat{\mu}_1 + \hat{\mu}_2 + 3 \quad \textit{conditions have to be satisfied.} \end{aligned} \tag{8.14}$$

The determination of the polynomials in (8.7) for given $m_b, m_i, m_\phi, \hat{N}_b, \hat{N}_i, \hat{N}_\phi, i = 1, 2$, proceeds in principle as follows. From (8.12) $\hat{\mu}_i, i = 1, 2$, is determined, so that we need $\hat{\mu}_2 + 1$ relations to guarantee that $F_2^+(\tau) \equiv 0$, and analogously $\hat{\mu}_1 + 1$ relations in order that $F_1^+(t) \equiv 0$. These relations are obtained by chosing $\hat{\mu}_2 + 1$ zero-tuples $(\rho^+(\tau_j), \tau_j), j = 1, \dots, \hat{\mu}_2 + 1$, of $h_1(\rho, \tau) = 0$. Insertion of these zero-tuples in (4.4) leads to $\hat{\mu}_2 + 1$ inhomogeneous linear equations for the coefficients of the polynomials in (8.7). Analogously $\hat{\mu}_1 + 1$ zero-tuples $(r(t_j), t_j)$ of $h_2(r, t)$ are chosen and substituted in (4.5). From the structure of the relations (4.4) and (4.5) it is seen that the τ_j and t_j may always be chosen in such a way that the resulting set of linear equations together with $\Omega_i(0) = \Phi_0(0)$, cf. (2.10), is sufficient and hence leads to a solution. Once the polynomials in (8.7) have been determined the lefthand sides in (8.7) are known. From the analysis given so far it then follows that the functions given by (8.7) satisfy (2.9) for all $|\tau| > 0$.

Obviously, we have quite some freedom in chosing the exponents in (8.2) and the degrees of the polynomials in (8.7). This freedom is not so surprising because in general a meromorphic function has not a unique decomposition, cf.[14], p.304, see also Remark 8.1.

The available freedom will be used to choose the numbers in (8.14)i and ii as small as possible, with $\tilde{M}$ being defined in (8.6). Before discussing this point we first consider several zero-tuples which are most appropriate for the determination of the polynomials in (8.7).

Denote by $(\hat{\rho}, \hat{\tau})$ and $(\hat{r}, \hat{t})$ a zero-tuple of $h_1(\rho, \tau) = 0$ and $h_2(r, t) = 0$, respectively.

For

$$(\hat{\rho}, \hat{\tau}) = (0, -\frac{1}{a_2}) \text{ and } (\hat{r}, \hat{t}) = (0, -\frac{1}{a_1}), \tag{8.15}$$

it follows from (2.9) that

$$\begin{aligned} &i. \quad \Omega_2(0) - a_1 B(-\frac{1}{a_2}) - (1 + a_2\pi_2)\Phi_0(-\frac{1}{a_2}) = 0, \\ &ii. \quad \Omega_1(0) + a_2 B(-\frac{1}{a_1}) - (1 + a_2\pi_1)\Phi_0(-\frac{1}{a_1}) = 0; \end{aligned} \tag{8.16}$$

note that $\tau = -\frac{1}{a_2} > \tau_0^{(0)} = T$ and $\tau = -\frac{1}{a_1} > \tau_0^{(0)} = T$, so that $\tau = -\frac{1}{a_2}$ and $\tau = -\frac{1}{a_1}$ are not poles of $B(\cdot)$ and of $\Phi_0(\cdot)$, cf. Lemmas 6.1 and 6.2. From the definitions (2.1) we have

$$\Omega_i(0) = \Phi_0(0) \ , \ 1, 2. \tag{8.17}$$

Hence from (8.16),

$$\begin{aligned} &i. \quad a_1 B(-\frac{1}{a_2}) + (1 + a_1\pi_2)\Phi_0(-\frac{1}{a_2}) - \Phi_0(0) = 0, \\ &ii. \quad -a_2 B(-\frac{1}{a_1}) + (1 + a_2\pi_1)\Phi_0(-\frac{1}{a_1}) - \Phi_0(0) = 0. \end{aligned} \tag{8.18}$$

Comparison of the relations (8.18) with (2.1)iv shows that

$$\begin{aligned} &\mathrm{E}\{r^{\mathbf{x}_1}(\mathbf{x}_1 = \mathbf{x}_2 + 1)\} \quad \text{is finite for } r = -\frac{1}{a_1}, \\ &\mathrm{E}\{r^{\mathbf{x}_2}(\mathbf{x}_2 = \mathbf{x}_1 + 1)\} \quad \text{is finite for } r = -\frac{1}{a_2}. \end{aligned} \tag{8.19}$$

For

$$(\hat{\rho}, \hat{\tau}) = (0, 0) \text{ and } (\hat{r}, \hat{t}) = (0, 0), \tag{8.20}$$

we have

$$\frac{\mathrm{d}\tau}{\mathrm{d}\rho}/_{\substack{\rho=0\\ \tau=0}} = 0 \ , \ \frac{\mathrm{d}t}{\mathrm{d}r}/_{\substack{r=0\\ t=0}} = 0,$$

and so from (2.9),

$$\begin{aligned} &i. \quad \Omega_2(0) + a_1 B(0) - \Phi_0(0) = 0, \\ &ii. \quad \Omega_1(0) - a_2 B(0) - \Phi_0(0) = 0. \end{aligned} \tag{8.21}$$

Hence, from (8.17) and (8.21)i or (8.21)ii:

$$B(0) = 0, \tag{8.22}$$

obviously, here the dependency of the set of Kolmogorov equations is manifested.

For

$$(\hat{\rho}, \hat{\tau}) = (1, \frac{1}{a_1}) \ \text{and} \ (\hat{r}, \hat{t}) = (1, \frac{1}{a_2}), \tag{8.23}$$

we have from (2.9),

$$\begin{aligned} &i. \quad \Omega_2(1) + \frac{a_1}{a_1 - 1} B(\frac{1}{a_1}) - \Phi_0(\frac{1}{a_1}) = 0, \quad a_1 \neq 1, \\ &ii. \quad \Omega_1(1) - \frac{a_2}{a_2 - 1} B(\frac{1}{a_2}) - \Phi_0(\frac{1}{a_2}) = 0, \quad a_2 \neq 1. \end{aligned} \tag{8.24}$$

For

$$(\hat{\rho}, \hat{\tau}) = (1, 1) \ \text{and} \ (\hat{r}, \hat{t}) = (1, 1), \tag{8.25}$$

we have from (2.9),

$$\begin{aligned} &\Omega_2(1) + a_1 \lim_{\rho \to \tau = 1} \frac{B(\tau)}{\rho - \tau} + [-1 + a_2 \pi_2 (a_1 - 1)] \Phi_0(1) = 0, \\ &\Omega_1(1) - a_2 \lim_{r \to t = 1} \frac{B(t)}{r - t} + [-1 + a_1 \pi_1 (a_2 - 1)] \Phi_0(1) = 0. \end{aligned} \tag{8.26}$$

Hence, since $\Omega_i(1)$ is finite,

$$B(1) = 0. \tag{8.27}$$

From (3.7) we have

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} - 1 = \frac{-a_1}{a_1 a_2 + a_1 - a_2}, \quad \frac{\mathrm{d}r}{\mathrm{d}t} - 1 = \frac{-a_2}{a_1 a_2 + a_1 - a_2}.$$

Hence from (8.26),

$$\begin{aligned} &\Omega_2(1) - (a_1 a_2 + a_1 - a_2) \frac{\mathrm{d}}{\mathrm{d}\tau} B(\tau) \mid_{\tau=1} + [-1 + a_2 \pi_2 (a_1 - 1)] \Phi_0(1) = 0, \\ &\Omega_1(1) + (a_1 a_2 + a_1 - a_2) \frac{\mathrm{d}}{\mathrm{d}\tau} B(\tau) \mid_{\tau=1} + [-1 + a_1 \pi_1 (a_2 - 1)] \Phi_0(1) = 0. \end{aligned} \tag{8.28}$$

Next we consider the zero-tuples, cf.(3.4),

$$(\hat{\rho}, \hat{\tau}) = (\rho(\tau^+), \tau^+) \ \text{and} \ (\hat{r}, \hat{t}) = (r(t^+), t^+). \tag{8.29}$$

From (3.4) it is seen that $\rho(\tau^+)$ is a zero with multiplicity two of $h_1(\rho, \tau^+)$. Consequently, it follows from (4.4) that $\rho(\tau^+)$ should be a zero of multiplicity two of (4.4) with $\tau = \tau^+$, since $\tau = \tau^+$ is not a pole of $B(\tau)$ and $\Phi_0(\tau)$, and $\rho = \rho(\tau^+)$ is not a pole of $\Omega_2(\rho)$. Hence from (2.9)i:

$$\begin{aligned}
&i. \quad [\Omega_2(\rho) + \frac{a_1\tau^+}{\rho-\tau^+}B(\tau^+) + k_1(\rho,\tau^+)\Phi_0(\tau^+)]_{\rho=\rho(\tau^+)} &= 0,\\
&ii. \quad [\frac{\mathrm{d}}{\mathrm{d}\rho}\Omega_2(\rho) + B(\tau^+)\frac{\mathrm{d}}{\mathrm{d}\rho}\frac{a_1\tau^+}{\rho-\tau^+} + \Phi_0(\tau^+)\frac{\mathrm{d}}{\mathrm{d}\rho}k_1(\rho,\tau^+)]_{\rho=\rho(\tau^+)} &= 0,\\
&iii. \quad [\Omega_1(r) - \frac{a_2t^+}{r-t^+}B(t^+) + k_2(r,t^+)\Phi_0(t^+)]_{r=r(t^+)} &= 0,\\
&iv. \quad [\frac{\mathrm{d}}{\mathrm{d}r}\Omega_1(r) - B(\tau^+)\frac{\mathrm{d}}{\mathrm{d}r}\frac{a_2t^+}{r-t^+} + \Phi_0(t^+)\frac{\mathrm{d}}{\mathrm{d}r}k_2(r,t^+)]_{r=r(t^+)} &= 0;
\end{aligned} \tag{8.30}$$

note that $\rho(\tau^+) - \tau^+ \neq 0$.

Next note that (2.4)i for $r_2 = 0, r_1 \neq 0$, leads to (8.17) and so the zero tuple $(\hat\rho,\hat\tau) = (\rho(\tau^-),\tau^-) = (0,0)$ needs no further attention.

Finally we consider the zero-tuples, cf.(3.4),

$$(\hat\rho,\hat\tau) = (\rho^\pm,\tau(\rho^\pm)) \text{ and } (\hat r,\hat t) = (r^\pm, t(r^\pm)). \tag{8.31}$$

For $\rho = \rho^+$ it is seen from (3.4) that $\hat\tau = \tau(\rho^+)$ is a zero of multiplicity two of $h_1(\rho^+,\tau)$, also $\hat\tau = \tau(\rho^-)$ is a zero of multiplicity two of $h_1(\rho^-,\tau)$. As before we obtain from (2.9):
for $\rho^+ \neq 1$,

$$\begin{aligned}
&i. \quad [\Omega_2(\rho^\pm) + \frac{a_1\tau}{\rho^\pm-\tau}B(\tau) \quad + \quad k_1(\rho^\pm,\tau)\Phi_0(\tau)]_{\tau=\tau(\rho^\pm)} = 0,\\
&ii. \quad [\frac{\mathrm{d}}{\mathrm{d}\tau}\{\frac{a_1\tau}{\rho^\pm-\tau}B(\tau)\} + \frac{\mathrm{d}}{\mathrm{d}\tau}\{k_1(\rho^\pm,\tau)\Phi_0\}]_{\tau=\tau(\rho^\pm)} = 0,
\end{aligned} \tag{8.32}$$

and for $r^+ \neq 1$,

$$\begin{aligned}
&i. \quad [\Omega_1(r^\pm) - \frac{a_2t}{r^\pm-t}B(t) + k_2(r^\pm,t)\Phi_0(t)]_{t=t(r^\pm)} = 0,\\
&ii. \quad \frac{\mathrm{d}}{\mathrm{d}t}\{\frac{-a_2t}{r^\pm-t}B(t)\} + \frac{\mathrm{d}}{\mathrm{d}t}\{k_2(r^\pm,t)\Phi_0(t)\}]_{t=t(r^\pm)} = 0;
\end{aligned} \tag{8.33}$$

note that, cf.(3.4),

$$\rho^+ = 1 \Leftrightarrow a_1 = 1 \text{ and } r^+ = 1 \Leftrightarrow a_2 = 1. \tag{8.34}$$

The case $a_1 = 1$ has to excluded in (8.24)i and (8.30)i, similarly $a_2 = 1$ in (8.24)ii and (8.30)ii. If $a_1 = 1$ then the second terms in (8.24)i and (8.30)i have to replaced by their limits for $a_1 \to 1$.

We proceed with the determination of the polynomials in (8.8). With regard to the available freedom mentioned above we shall try to choose the degrees of the polynomials in (8.7) as small as possible.

First we consider the case

$$\tilde M = 1. \tag{8.35}$$

Take for the present case

$$m_b - 1 = m_2 - 1 = m_1 - 1 = m_\phi = \hat N_b = \hat N_2 = \hat N_1 = \hat N_\phi = 0, \hat B_0 = 0, \hat\Phi_{00} = 0. \tag{8.36}$$

This choice is consistent with (8.14), and it follows from (8.2) and (8.12) that

$$\begin{aligned}
&i. \quad \tilde B(0) = \tilde\Omega_2(0) = \tilde\Omega_1(0) = 0, \quad \tilde\Phi_0 = -\sum_{n=0}^{\infty}\sum_{\delta\in\mathcal{B}_n}\frac{\phi_\delta^{(n)}}{\tau_\delta^{(n)}},\\
&ii. \quad \hat\mu_1 = \hat\mu_2 = 0.
\end{aligned} \tag{8.37}$$

From (8.37)ii it is seen that we need three coefficients. From (8.8), (8.17), (8.22) and (8.37) we obtain

$$\hat{\Omega}_i(\cdot) = \hat{\Omega}_{i0} = \tilde{\Phi}_0(0), i = 1, 2; \tag{8.38}$$

and so the three nonzero coefficients of the polynomials in (8.8) have been determined. The results so far obtained lead to the following

THEOREM 8.1. *For* $\frac{1}{a_1} + \frac{1}{a_1} > 1, a_1 \neq a_2,\ \tilde{M} = 1$, *cf.(8.6), the functions* $B(\cdot), \Phi_0(\cdot), \Omega_i(\cdot), i = 1, 2$, *which satisfy the conditions (2.10) are given by, cf.(8.2),*

$$\begin{aligned}
&i. \quad B(\tau) = \sum_{n=0}^{\infty} \sum_{\delta \in \mathcal{B}_n} \frac{b_\delta^{(n)}}{\tau - \tau_\delta^{(n)}} \frac{\tau}{\tau_\delta^{(n)}}, && |\tau| < T = (\frac{1}{a_1} + \frac{1}{a_2})^2,\\
&ii. \quad \Phi_0(\tau) = \sum_{n=0}^{\infty} \sum_{\delta \in \mathcal{B}_n} \frac{\phi_\delta^{(n)}}{\tau - \tau_\delta^{(n)}}, && |\tau| < T,\\
&iii. \quad \Omega_2(\rho) = -\sum_{n=0}^{\infty} \sum_{\delta \in \mathcal{B}_n} \frac{\phi_\delta^{(n)}}{\tau_\delta^{(n)}} + \sum_{n=1}^{\infty} \sum_{\delta \in \mathcal{B}_n} \frac{\omega_{2\delta}^{(n)}}{\rho - \rho_\delta^{(n)}} \frac{\rho}{\rho_\delta^{(n)}}, && |\rho| < \rho^+(T),\\
&iv. \quad \Omega_1(r) = -\textstyle\sum_{n=0}^{\infty} \sum_{\delta \in \mathcal{B}_n} \displaystyle\frac{\phi_\delta^{(n)}}{\tau_\delta^{(n)}} + \sum_{n=1}^{\infty} \sum_{\delta \in \mathcal{B}_n} \frac{\omega_{1\delta}^{(n)}}{r - r_\delta^{(n)}} \frac{r}{r_\delta^{(n)}}, && |r| < r^+(T);
\end{aligned} \tag{8.39}$$

these functions have meromorphic continuations in the whole complex plane, which are given by (8.39). The residues $b_\delta^{(n)}, \phi_\delta^{(n)}, \omega_{i\delta}^{(n)}, i, 1, 2$, *can be calculated recursively, see Lemma 6.2, they all contain the factor* $\phi_0(\tau_0^{(0)})$, *which is uniquely determined by*

$$\frac{1}{a_1}\Omega_2(1) + \frac{1}{a_2}\Omega_1(1) = \frac{1}{a_1} + \frac{1}{a_2} - 1. \tag{8.40}$$

The generating functions $E\{r^{\mathbf{x}_1}(\mathbf{x}_1 = \mathbf{x}_2 + 1)\}, E\{r^{\mathbf{x}_2}(\mathbf{x}_2 = \mathbf{x}_1 + 1)\}, |r| < (\frac{1}{a_1} + \frac{1}{a_2})^2$ *are determined by (2.1)iv, (8.39)i,ii, and* $E\{r_1^{\mathbf{x}_1} r_2^{\mathbf{x}_2}(\mathbf{x}_2 > \mathbf{x}_1)\}, E\{r_1^{\mathbf{x}_1} r_2^{\mathbf{x}_2}(\mathbf{x}_1 > \mathbf{x}_2)\}$ *are obtained from (2.4) and (8.39).*
The queue length process $\{\mathbf{x}_1(t), \mathbf{x}_2(t), t > 0\}$ *is positive recurrent if and only if* $\frac{1}{a_1} + \frac{1}{a_2} > 1$.

PROOF. From (8.35), ..., (8.38) and the analysis given above it follows that the functions in (8.39) satisfy (4.4) for all $|\tau| \geq 0$ and (4.5) for all $|t| \geq 0$. So by using Lemma 8.2 they satisfy the conditions (2.10)i and (2.9) of (2.10)ii. The determination of $\mathrm{E}\{r^{\mathbf{x}_1}(\mathbf{x}_1 = \mathbf{x}_2 + 1)\}$ follows from (2.1)iv and (8.39). It is further seen, cf.(8.19), that the radius of convergence of the latter generating function is larger than one, analogously for $\mathrm{E}\{r^{\mathbf{x}_2}(\mathbf{x}_2 = \mathbf{x}_1 + 1)\}$. From (2.4) and (8.39) the bivariate generating functions $\mathrm{E}\{r_1^{\mathbf{x}_1} r_2^{\mathbf{x}_2}(\mathbf{x}_2 > \mathbf{x}_1)\}$ and $\mathrm{E}\{r_1^{\mathbf{x}_1} r_2^{\mathbf{x}_2}(\mathbf{x}_1 > \mathbf{x}_2)\}$ are obtained. It is readily seen that the domain of convergence of these bivariate generating functions contains the product unit disks $|r_1| \leq 1$, $|r_2| \leq 1$, as a true subset. Hence the coefficients in the series expansions of the bivariate generating function $\mathrm{E}\{r_1^{\mathbf{x}_1} r_2^{\mathbf{x}_2}\}$ is an absolutely convergent solution of the Kolmogorov equations. Hence Foster's criterion, cf.Remark 2.2, implies that the queue length process $\{\mathbf{x}_1(t),\ \mathbf{x}_2(t),\ t > 0\}$ is positive recurrent for $\frac{1}{a_1} + \frac{1}{a_2} > 1$; (2.6) shows that this condition is also necessary. Because all generating function contain $\phi_0(T)$ as a linear factor, cf.Lemma 6.1, it follows from (8.40), cf.(2.1) and (2.5), and so it is uniquely defined. For the uniqueness of the solution constructed for the conditions of the theorem see Remark 2.2. □

Next we consider the case

$$\tilde{M} = 2,\ \frac{1}{a_1} + \frac{1}{a_2} > 1,\ a_1 \neq a_2. \tag{8.41}$$

Take

$$m_b - 1 = m_2 - 1 = m_1 - 1 = m_\phi = 1,$$
$$\hat{N}_b = 1,\ \hat{N}_2 = \hat{N}_1 = \hat{N}_\phi = 0. \tag{8.42}$$

This choice is again consistent with (8.14) and it follows that

$$\hat{\mu}_1 = \hat{\mu}_2 = 1. \tag{8.43}$$

Hence we need four coefficients. We have $\hat{N}_b = 1$ and further $\hat{\Phi}_{00}$ should be nonzero since $\tilde{\Phi}_0(0) = 0$, and $\Phi_0(0)$ should be positive for a positive recurrent queue length process, cf.(2.1)i. Note $\tilde{B}(0) = 0$ for $m_b = 1$, so that $\hat{B}_0(0) = 0$, cf.(8.17). From the two equations (8.18), $\hat{B}_1$ and $\hat{\Phi}_{00}$ can be determined, their main determinant is nonzero. Then from (8.17) we obtain $\hat{\Omega}_{i0},\ i = 1, 2$. The explicit equations for $\hat{B}_1$ and $\hat{\Phi}_{00}$ read, cf.(8.18),

$$-\frac{a_1}{a_2}\hat{B}_1 + a_1\pi_2\hat{\Phi}_{00} = -a_1\tilde{B}(-\frac{1}{a_2}) - (1 + a_1\pi_2)\tilde{\Phi}_0(-\frac{1}{a_2}),$$
$$+\frac{a_2}{a_1}\hat{B}_1 + a_2\pi_1\hat{\Phi}_{00} = a_2\tilde{B}(-\frac{1}{a_1}) - (1 + a_2\pi_1)\tilde{\Phi}_0(-\frac{1}{a_1}). \tag{8.44}$$

Hence with $\hat{B}_1$ and $\hat{\Phi}_{00}$ determined by (8.44) we have for the present case (8.41):

$$B(\tau) = \tau\hat{B}_1 + \sum_{n=0}^{\infty}\sum_{\delta\in\mathcal{B}_n}\frac{b_\delta{}^{(n)}}{\tau - \tau_\delta^{(n)}}[\frac{\tau}{\tau_\delta^{(n)}}]^2,$$
$$\Phi(\tau) = \hat{\Phi}_{00} + \sum_{n=0}^{\infty}\sum_{\delta\in\mathcal{B}_n}\frac{\phi_\delta^{(n)}}{\tau - \tau_\delta^{(n)}}\frac{\tau}{\tau_\delta^{(n)}},$$
$$\Omega_2(\rho) = \hat{\Phi}_{00} + \sum_{n=1}^{\infty}\sum_{\delta\in\mathcal{B}_n}\frac{\omega_{2\delta}^{(n)}}{\rho - \rho_\delta^{(n)}}[\frac{\rho}{\rho_\delta^{(n)}}]^2,$$
$$\Omega_1(r) = \hat{\Phi}_{00} + \sum_{n=1}^{\infty}\sum_{\delta\in\mathcal{B}_n}\frac{\omega_{1\delta}^{(n)}}{r - r_\delta^{(n)}}[\frac{r}{r_\delta^{(n)}}]^2. \tag{8.45}$$

Note that (8.27) implies:

$$\hat{B}_1 = -\sum_{n=0}^{\infty}\sum_{\delta\in B_n}\frac{b_\delta^{(n)}}{1 - \tau_\delta^{(n)}}\frac{1}{(\tau_\delta^{(n)})^2}. \tag{8.46}$$

For the conditions (8.41) and with (8.39) replaced by (8.45) in Theorem 8.1 we obtain the relevant theorem for the case $\tilde{M} = 2, a_1 \neq a_2,\ \frac{1}{a_1} + \frac{1}{a_2} > 1$. It is fully analogous and its explicit formulation is, therefore, omitted.

The determination of the polynomials in (8.7) for $\tilde{M} \geq 3,\ \frac{1}{a_1} + \frac{1}{a_2} > 1,\ a_1 \neq a_2$, proceeds along the same lines as for the cases $\tilde{M} = 1, 2$. The relations (8.17), (8.18), (8.22), (8.24), (8.27), (8.28), (8.30), (8.32) and (8.34) yield in general twenty-two equations, except for $a_1 = 1$ or $a_2 = 1$, cf.(8.24), (8.28) and (8.34), so their number suffices for rather large $\tilde{M}$, the more so since cases with $\tilde{M} \geq 3$ seem hardly to occur in the applications.

REMARK 8.1. The degrees of the polynomials and the exponents of the meromorphic functions have been introduced in (8.7) and (8.2). They have to be determined in such a way that (8.14) is satisfied and $F_i^+(\tau) + F_i^-(\tau),\ i = 1, 2$, are zero at $\hat{\mu}_i + 1$ points. In this determination it is no objection to replace m_b by $m_b + h_b,\ m_\phi$ by $m_\phi + h_\phi,\ m_2$ by $m_2 + h_2$ and m_1 by $m_1 + h_1$, with $h_b,\ h_\phi,\ h_2,\ h_1$, positive integers (and $\tilde{M}$ defined by (8.6)). Such a change when compared with the case that $h_b,\ h_\phi,\ h_2,\ h_1$ are all zero actually amounts to substraction of a polynomial from the meromorphic function and addition of that polynomial to the "$\wedge$" polynomial. In fact this also occurs by noting that the solution given by (8.44) also holds for the case $\tilde{M} = 1$. □

REMARK 8.2. From (8.2) and (8.7) it is readily seen that $T = \tau_0^{(0)} = (\frac{1}{a_1} + \frac{1}{a_2})^2 > 1$ is the smallest pole of $\Phi_0(\cdot)$ and also of $B(\cdot)$. Hence T determines the asymptotic behaviour of $\Pr\{\mathbf{x}_1 = \mathbf{x}_2 = n\}$ for $n \to \infty$, i.e.

$$\Pr\{\mathbf{x}_1 = \mathbf{x}_2 = n\} \sim -\frac{\phi_0(T)}{T^{n+1}} \quad \text{for} \quad n \to \infty.$$

Similarly it is seen that

$$\rho_0^{(1)} = \rho^+(\tau_0^{(0)}) = a_1(\frac{1}{a_1} + \frac{1}{a_2})^3 + \frac{a_1}{a_2}(\frac{1}{a_1} + \frac{1}{a_2}),$$
$$r_0^{(1)} = r^+(\tau_0^{(0)}) = a_2(\frac{1}{a_1} + \frac{1}{a_2})^3 + \frac{a_2}{a_1}(\frac{1}{a_1} + \frac{1}{a_2}),$$

are the smallest poles of $\Omega_2(\cdot)$ and $\Omega_1(\cdot)$, respectively, and so they determine the leading term in the asymptotic behaviour of $\Pr\{\mathbf{x}_2 = n,\ \mathbf{x}_1 = 0\}$ and $\Pr\{\mathbf{x}_1 = n,\ \mathbf{x}_2 = 0\}$ for $n \to \infty$. □

REMARK 8.4. Numerical calculations indicate that always $|\lambda_1\mu_1 + \lambda_2\mu_2| > 1$. For quite a few cases this has been proved in Appendix $\mathcal{C}$. However, a complete proof of $\tilde{M} \geq 1$ has not been obtained; actually, this is not very important, because if it can happen indeed that $\tilde{M} = 0$, then theorem 8.1 also implies. Note that then may be written

$$\sum_{n=0}^{\infty} \sum_{\delta \in \mathcal{B}_n} \frac{b_\delta^{(n)}}{\tau - \tau_\delta^{(n)}} \frac{\tau}{\tau_\delta^{(n)}} = \sum_{n=0}^{\infty} \sum_{\delta \in \mathcal{B}_n} \frac{b_\delta^{(n)}}{\tau_\delta^{(n)}} + \sum_{n=0}^{\infty} \sum_{\delta \in \mathcal{B}_n} \frac{b_\delta^{(n)}}{\tau - \tau_\delta^{(n)}},$$

since for $\tilde{M} = 0$ the first sum in the righthand side converges absolutely and the second sum is a well-defined meromorphic function, analogously for the similar sums in (8.39). □

9. THE SOLUTION FOR THE CASE $a_1 = a_2 < 2,\ \pi_1 \neq \pi_2$

In the preceeding section the solution has been described for the case $a_1 \neq a_2$. In this section we derive the solution for the case

$$a := a_1 = a_2 < 2, \pi_1 \neq \pi_2,\ 0 < \pi_1 = 1 - \pi_2 < 1. \tag{9.1}$$

Again T is defined as in section 5, so cf.Lemma 5.1:

$$T = \frac{4}{a^2},\ \ \rho^-(T) = r^-(T) = \frac{2}{a}, \tag{9.2}$$

note that (9.1) implies $h_1(\rho, \tau) = h_2(\rho, \tau)$, so

$$r^{\pm}(\tau) = \rho^{\pm}(\tau), \quad \tau^{\pm}(\rho) = t^{\pm}(\rho). \tag{9.3}$$

Again Assumption 5.1 is here introduced, and as in (5.9) it follows that

$$\tau = T \textit{ is a simple pole of } B(\tau) \textit{ and also of } \Phi_0(\tau). \tag{9.4}$$

It follows, cf. the derivations of (5.10), that

$$\begin{aligned} &\frac{aT}{\rho^-(T) - T} b(T) + k_1(\rho^-(T), T)\phi_0(T) = 0, \\ &\frac{-aT}{\rho^-(T) - T} b(T) + k_2(\rho^-(T), T)\phi_0(T) = 0, \end{aligned} \tag{9.5}$$

and that the two relations in (9.5) are linearly dependent because (9.2) implies that the main determinant of the system (9.5) is equal to zero.

From (9.1) and (9.3) it is seen that the zero-tuple of the ladder (3.8) generated by the zero-tuple T on $h_1(\rho,\tau)$ induces on $h_2(r,t)=0$ ladders which are all congruent with the ladder, cf.fig. 3.2,

$$\begin{aligned}&(\rho_\nu,\tau_\nu),\dots,(\rho_{n-1},\tau_{n-1}),\ (\rho_n,\tau_n),\ (\rho_{n+1},\tau_{n+1}),\dots,\\&\tau_0:=T,\quad \rho_0:=\rho^-(T),\end{aligned}\tag{9.6}$$

witn ν as defined below (3.9), and $\rho_n,\ \tau_n$ recursively defined as in (3.9).

Consider for the present case the relations (6.2) and (6.4), i.e.

$$\begin{aligned}&\frac{aT}{\rho_1-T}b(T)+k_1(\rho_1,T)\phi_0(T)+\omega_2(\rho_1)\left[\frac{\mathrm{d}\rho^+(\sigma)}{\mathrm{d}\sigma}\right]^{-1}_{\sigma=T}=0,\\&\frac{aT}{\rho_0-T}b(T)+k_2(\rho_0,T)\phi_0(T)=0.\end{aligned}\tag{9.7}$$

As in section 6, cf.(6.5), it follows that

$$\rho_1 \textit{ is a simple pole of } \Omega_i(\rho),\ 0<|\omega_i(\rho_1)|<\infty,\ i=1,2. \tag{9.8}$$

Next consider for the present case the relations (4.9)i and (4.10)i,

$$\begin{aligned}&\omega_2(\rho_1)+[\frac{a\tau_1}{\rho_1-\tau_1}b(\tau_1)+k_1(\rho_1,\tau_1)\phi_0(\tau_1)]\left[\frac{\mathrm{d}\tau^+(\sigma)}{\mathrm{d}\sigma}\right]^{-1}_{\sigma=\rho_1}=0,\\&\omega_1(\rho_1)+[\frac{-a\tau_1}{\rho_1-\tau_1}b(\tau_1)+k_2(\rho_1,\tau_1)\phi_0(\tau_1)]\left[\frac{\mathrm{d}\tau^+(\sigma)}{\mathrm{d}\sigma}\right]^{-1}_{\sigma=\rho_1}=0.\end{aligned}\tag{9.9}$$

Via (9.7) $\omega_2(\rho_1)$ can be expressed uniquely as a linear function of $\phi_0(T)$, note that the determinant formed by the coefficients of $b(T)$ and $\phi_0(T)$ is nonzero, cf.(5.13), (9.5), and that $\rho_0=\rho^-(T)$. Analogously $\omega_1(\rho_1)$ is determined. It is now readily seen that the system (9.9) for the unknowns $b(\tau_1)$ and $\phi_0(\tau_1)$ has a solution $b(\tau_1)\neq 0,\ \phi_0(\tau_1)\neq 0$, since its main determinant is nonzero, cf.(5.13) for $a_1=a_2,\ \pi_1\neq\pi_2$. Consequently:

$$\tau_1 \textit{ is a simple pole of } B(\tau) \textit{and also of } \Phi_0(\tau). \tag{9.10}$$

Next we consider the relations (4.11) and (4.12) for the present case, i.e.

$$\begin{aligned}&\omega_2(\rho_2)\left[\frac{\mathrm{d}\rho^+(\sigma)}{\mathrm{d}\sigma}\right]^{-1}_{\sigma=\tau_1}+\frac{a\tau_1}{\rho_2-\tau_1}b(\tau_1)+k_1(\rho_2,\tau_1)\phi_0(\tau_1)=0,\\&\omega_1(\rho_2)\left[\frac{\mathrm{d}\rho^+(\sigma)}{\mathrm{d}\sigma}\right]^{-1}_{\sigma=\tau_1}+\frac{-a\tau_1}{\rho_2-\tau_1}b(\tau_1)+k_2(\rho_2,\tau_1)\phi_0(\tau_1)=0.\end{aligned}\tag{9.11}$$

Hence, since $b(\tau_1)$ and $\phi_0(\tau_1)$ are determined by (9.9), it follows readily that $\omega_2(\rho_2)$ and $\omega_1(\rho_2)$ are both finite and nonzero. Consequenly,

$$\rho_2 \textit{ is a simple pole of } \Omega_i(\rho),\ i=1,2. \tag{9.12}$$

By repeating the argumentation above it is readily verified that the following lemma holds; its detailed proof is therefore omitted.

LEMMA 9.1. *For the case* $a_1=a_2,\ \pi_1\neq\pi_2$ *with* $\tau_0=T$:

i. $\tau_n,\ n=0,1,2,\dots,$ *are simple poles of* $B(\tau)$ *and also of* $\Phi_0(\tau)$;

ii. $\rho_n,\ n=1,2,\dots,$ *are simple poles of* $\Omega_i(\rho),\ i=1,2$;

iii. *the residues* $b(\tau_n),\ \phi_0(\tau_n),\ \omega_i(\rho_n),\ i=1,2,$ *are recursively determined by*

$$\frac{aT}{\rho_0 - T} b(T) + k_1(\rho_0, T)\phi_0(T) = 0, \tag{9.13}$$

and for $n = 1, 2, \ldots,$ *by*

$$\begin{aligned}
&i. \quad \omega_2(\rho_n) + [\frac{a\tau_{n-1}}{\rho_n - \tau_{n-1}} b(\tau_{n-1}) + k_1(\rho, \tau_{n-1})\phi_0(\tau_{n-1})] \left[\frac{\mathrm{d}\tau^-(\sigma)}{\mathrm{d}\sigma}\right]^{-1}_{\sigma=\rho_0} = 0,\\
&ii. \quad \omega_1(\rho_n) + [\frac{-a\tau_{n-1}}{\rho_n - \tau_{n-1}} b(\tau_{n-1}) + k_2(\rho, \tau_{n-1})\phi_0(\tau_{n-1})] \left[\frac{\mathrm{d}\tau^-(\sigma)}{\mathrm{d}\sigma}\right]^{-1}_{\sigma=\rho_n} = 0,\\
&iii. \quad \omega_2(\rho_n) + [\frac{a\tau_n}{\rho_n - \tau_n} b(\tau_n) + \frac{1}{\tau_n} k_1(\rho_n, \tau_n)\tau_n\phi_0(\tau_n)] \left[\frac{\mathrm{d}\tau^+(\sigma)}{\mathrm{d}\sigma}\right]^{-1}_{\sigma=\rho_n} = 0,\\
&iv. \quad \omega_1(\rho_n) + [\frac{-a\tau_n}{\rho_n - \tau_n} b(\tau_n) + \frac{1}{\tau_n} k_2(\rho_n, \tau_n)\tau_n\phi_0(\tau_n)] \left[\frac{\mathrm{d}\tau^+(\sigma)}{\mathrm{d}\sigma}\right]^{-1}_{\sigma=\rho_n} = 0.
\end{aligned} \tag{9.14}$$

Next we consider the asymptotic behaviour of $b(\tau_n)$, $\phi_0(\tau_n)$ and $\omega_i(\rho_n)$, $i = 1, 2$, for $n \to \infty$.

LEMMA 9.2. *For the case* $a_1 = a_2 = a$, $\pi_1 \neq \pi_2$ *with* $\tau_0 = T$:

$$\begin{aligned}
&\psi := \lim_{n\to\infty} \frac{b(\tau_{n+1})}{b(\tau_n)} = \lim_{n\to\infty} \frac{\phi_0(\tau_{n+1})}{\phi_0(\tau_n)} = \lim_{n\to\infty} \frac{\omega_i(\rho_{n+1})}{\omega_i(\rho_n)}, i = 1, 2,\\
&\psi = \frac{R^- - 1}{R^+ - 1}\frac{R^+}{R^-} = \lambda^{-1}\frac{R^- - 1}{R^+ - 1} < 0,
\end{aligned} \tag{9.15}$$

with, since $a_1 = a_2$,

$$R_2^- = R_1^- \equiv R^-,\ R_2^+ = R_1^+ \equiv R^+, \quad \lambda = \lambda_1 = \lambda_2 = \frac{R^-}{R^+}. \tag{9.16}$$

PROOF. Add (9.14)i and ii and also (9.14)iii and iv. Next eliminate from the resulting expressions $\omega_2(\rho_n) + \omega_1(\rho_n)$, this leads to

$$\begin{aligned}
&[k_1(\rho_n, \tau_n) + k_2(\rho_n, \tau_n)]\phi_0(\tau_n) \left[\frac{\mathrm{d}\tau^+(\sigma)}{\mathrm{d}\sigma}\right]^{-1}_{\sigma=\rho_n} =\\
&[k_1(\rho_n, \tau_{n-1}) + k_2(\rho_n, \tau_{n-1}]\phi_0(\tau_{n-1}) \left[\frac{\mathrm{d}\tau^-(\sigma)}{\mathrm{d}\sigma}\right]^{-1}_{\sigma=\rho_n}.
\end{aligned} \tag{9.17}$$

By using (3.16), (3.17), (3.18) and (3.20) it follows for $n \to \infty$ that

$$\frac{[R^-]^2}{1 - R^-}\tau_n\phi_0(\tau_n) - \frac{[R^+]^2}{1 - R^+}\tau_{n-1}\phi_0(\tau_{n-1}) \to 0; \tag{9.18}$$

so that by using, cf. (7.6), $\tau_{n+1}/\tau_n \to \lambda^{-1} = R^+/R^-$, it is seen that

$$\frac{\phi_0(\tau_n)}{\phi_0(\tau_{n-1})} \to \frac{R^+}{R^-}\frac{1 - R^-}{1 - R^+} < 0.$$

Substraction of (9.14)i and ii, and also (9.14)iii and iv yields after elimination of $\omega_2(\rho_n) - \omega_1(\rho_n)$,

$$\begin{aligned}
&\frac{2a\tau_n}{\rho_n - \tau_n}\left[\frac{\mathrm{d}\tau^+(\sigma)}{\mathrm{d}\sigma}\right]^{-1}_{\sigma=\rho_n} b(\tau_n) + \frac{2a\tau_{n-1}}{\rho_n - \tau_{n-1}}\left[\frac{\mathrm{d}\tau^-(\sigma)}{\mathrm{d}\sigma}\right]^{-1}_{\sigma=\rho_n} b(\tau_{n-1}) =\\
&\frac{1}{\tau_n}[-k_1(\rho_n, \tau_n) + k_2(\rho_n, \tau_n)]\tau_n\phi_0(\tau_n) \left[\frac{\mathrm{d}\tau^+(\sigma)}{\mathrm{d}\sigma}\right]^{-1}_{\sigma=\rho_n} +\\
&\frac{1}{\tau_{n-1}}[-k_1(\rho_n, \tau_{n-1}) + k_2(\rho_n, \tau_{n-1})]\tau_{n-1}\phi_0(\tau_{n-1}) \left[\frac{\mathrm{d}\tau^-(\sigma)}{\mathrm{d}\sigma}\right]^{-1}_{\sigma=\rho_n}.
\end{aligned}$$

Again by using (3.16), (3.17), (3.18) and (3.20) it follows for $n \to \infty$:

$$\frac{2aR^-}{R^- - 1} b(\tau_n) + \frac{2aR^+}{R^+ - 1} b(\tau_{n-1}) = -a(\pi_2 - \pi_1)\{\frac{[R^-]^2}{1 - R^-}\tau_n \phi_0(\tau_n) - \frac{[R^+]^2}{1 - R^+}\tau_{n-1}\phi_0(\tau_{n-1})\} \to 0.$$

Hence

$$\frac{b(\tau_n)}{b(\tau_{n-1})} \;\to\; \frac{R^+}{R^-}\frac{1 - R^-}{1 - R^+}.$$

From (9.14)i with n replaced by $m + n$, it follows for m sufficiently large again by using (3.17), (3.18), (3.20) and the asymptotic relations for $b(\tau_m)$ and $\phi_0(\tau_m)$ obtained above that for $n = 1, 2, \ldots$,

$$\omega_2(\rho_{m+n}) = -\frac{aR^-}{R^- - 1}\{b(\tau_m)\left[\frac{R^+}{R_-}\frac{R^- - 1}{R^+ - 1}\right]^n + a\pi_2 R^- \tau_m \phi_0(\tau_m)\left[\frac{R^+}{R^-}\frac{1 - R^-}{1 - R^+}\right]^n\};$$

from which it follows that

$$\frac{\omega_2(\rho_{n+1})}{\omega_2(\rho_n)} \;\to\; \frac{R^+}{R^-}\frac{1 - R^-}{1 - R^+}.$$

Hence (9.15) has been proved. □

As in (8.2) we introduce for the present case the meromorphic functions

$$\begin{aligned}
\tilde{B}(\tau) &:= \sum_{n=0}^{\infty} \frac{b(\tau_n)}{\tau - \tau_n}(\frac{\tau}{\tau_n})^{m_b}, \\
\tilde{\Phi}_0(\tau) &:= \sum_{n=0}^{\infty} \frac{\phi_0(\tau_n)}{\tau - \tau_n}(\frac{\tau}{\tau_n})^{m_\phi}, \\
\tilde{\Omega}_i(\tau) &:= \sum_{n=1}^{\infty} \frac{\omega_i(\rho_n)}{\rho - \rho_n}(\frac{\rho}{\rho_n})^{m_i},
\end{aligned} \tag{9.19}$$

with for $n = 1, 2, \ldots$,

$$\tau_0 = T,\; \rho_{n+1} = \rho^+(\tau_n),\; \tau_n = \tau^+(\rho_n), \tag{9.20}$$

and

$$m_b \geq \tilde{M},\; m_\phi \geq \tilde{M},\; m_i \geq \tilde{M},\; i = 1, 2;$$

here $\tilde{M}$ is uniquely defined as the nonnegative integer such that (note $0 < \lambda < 1$).

$$\begin{aligned}
&i. \quad \tilde{M} = 0 && \text{for } |\psi\lambda| < 1, \\
&ii. \quad |\psi\lambda^{\tilde{M}+1}| < 1 \leq |\psi\lambda^{\tilde{M}}| && \text{otherwise.}
\end{aligned} \tag{9.21}$$

By using (7.6) and Lemma 9.1 it is readily seen, since for k large,

$$\frac{b(\tau_{n+k})}{[\tau_{n+k}]^{m+1}} \;\sim\; \frac{b(\tau_k)}{[\tau_k]^{m+1}}(\psi\lambda^{m+1})^n,$$

that for $m \geq \tilde{M}$ the function $\tilde{B}(\tau)$ is a well-defined meromorphic function which is regular in $|\tau| \leq 1$. Similarly for the other functions defined in (9.19).

Because, cf.(9.15),

$$\lambda\psi = \frac{R^- - 1}{R^+ - 1},$$

and since it is readily verified by using (2.3) and (3.16) that $|\lambda\psi| < 1$ we take from now on in (9.19),

$$m = \tilde{M} = 0. \tag{9.22}$$

THEOREM 9.1. *For $a_1 = a_2 = a < 2$, $\pi_1 \neq \pi_2 \neq 1$, the functions $B(\cdot)$, $\Phi_0(\cdot)$, $\Omega_i(\cdot)$, $1, 2$, which satisfy the conditions (2.10) are given by*

$$\begin{aligned} &i. \quad B(s) = -\tilde{B}(0) + \tilde{B}(s), \\ &ii. \quad \Phi_0(s) = \tilde{\Phi}(s), \\ &iii. \quad \Omega_i(s) = \tilde{\Phi}(0) - \tilde{\Omega}_i(0) + \tilde{\Omega}_i(s),\ i = 1, 2, \end{aligned} \tag{9.23}$$

here $\tilde{B}(\cdot)$, $\tilde{\Phi}_0(\cdot)$, $\tilde{\Omega}_i(\cdot)$, $i = 1, 2$, are given by (9.19) with $m_b = m_\phi = m_2 = m_1 = 0$, they all contain $\phi_0(T)$ as a factor, which is determined by

$$\Omega_2(1) + \Omega_1(1) = 2 - a;$$

if $a \geq 2$ no stationary joint distribution exists.

PROOF. As in section 8 introduce the polynomials $\hat{B}(\cdot)$, $\hat{\Phi}_0(\cdot)$, $\hat{\Omega}_i(\cdot)$, $i = 1, 2$. The degrees of these functions are determined by the same arguments as used in section 8 for the case $\tilde{M} = 1$; then the proof is accomplished as in the proof of Theorem 8.1 and is therefore omitted here. Note that for the present case $\hat{\mu}_i = 0$, $i = 1, 2$, cf. (8.12). □

Theorem 9.1 provides all the results needed for the evaluation of the characteristics of the stationary joint distribution of the queue lengths $(\mathbf{x}_1, \mathbf{x}_2)$. The following analysis provides some detailed information on the influence of the probabilities π_i, $i = 1, 2$, $\pi_1 + \pi_2 = 1$, cf. (2.3).

Put

$$\Omega(r) := \frac{1}{2}\{\Omega_1(r) + \Omega_2(r)\}, \tag{9.24}$$

Elimination of $B(\cdot)$ from (4.4) and (4.5) yields for the present case, i.e., $a_1 = a_2 = a$, $\pi_1 \neq \pi_2$,

$$\Omega(r^{\pm}(t)) + [-1 + \frac{1}{2}a^2 t - \frac{1}{2}ar^{\pm}(t)]\Phi_0(t) = 0, \tag{9.25}$$

with $(r^{\pm}(t),\ t)$ a zero-tuple of

$$h(r, t) \equiv at^2 + [1 - (2 + a)r]t + r^2 = 0. \tag{9.26}$$

The relations (9.25), (9.26) formulate a functional equation which is identical with that of the *symmetrical shortest* queue, cf. (3.6) of [2]. Hence the solution constructed in [2] can be used here.

Put, cf. (3.2), (3.3) and also [2],

$$\begin{aligned} &i. \quad r_{n+1}^+ := r^+(t_n^+),\ t_n^+ := t^+(r_n^+),\ n = 0, 1, \ldots, \\ &\text{with} \\ &\qquad t_0^+ = \frac{4}{a^2},\ r_0^+ = \frac{2}{a}, \\ &ii. \quad t_{n+1}^- = t^+(r_n^-),\ r_n^- = r^+(t_n^-),\ n = 0, 1, \ldots, \\ &\text{with} \\ &\qquad t_0^- = -\frac{1}{a},\ r_0^- = -1 - \frac{2}{a}. \end{aligned} \tag{9.27}$$

The solution of (9.25) and (9.26) is then expressed by, cf. formula (4.7) of [2]:

$$\begin{aligned}\Phi_0(t) &= \Phi(1)\,\frac{\Pi_{n=1}^{\infty}(1-\frac{t}{t_n^-})}{\Pi_{n=1}^{\infty}(1-\frac{1}{t_n^-})}\,\frac{\Pi_{n=0}^{\infty}(1-\frac{1}{t_n^+})}{\Pi_{n=0}^{\infty}(1-\frac{t}{t_n^+})},\\ \Omega(r) &= \Omega(1)\,\frac{\Pi_{n=0}^{\infty}(1-\frac{r}{r_n^-})}{\Pi_{n=0}^{\infty}(1-\frac{1}{r_n^-})}\,\frac{\Pi_{n=1}^{\infty}(1-\frac{1}{r_n^+})}{\Pi_{n=1}^{\infty}(1-\frac{r}{r_n^+})},\end{aligned} \tag{9.28}$$

with

$$\Omega(1)=\frac{1}{2}(2-a),\ \Phi_0(1)=\frac{1}{1+a}. \tag{9.29}$$

Here the first relation in (9.29) follows from (2.5) and (9.1) and the second one is obtained from (9.25) for the zero-tuple $(r,t)=(1,1)$. Because the zero-tuples in (9.27) are independent of $\pi_i,\ i=1,2$, it is seen that

$$\Phi_0(t)\ \textit{and}\ \Omega(r)\ \textit{are independent of}\ \pi_i,\ i=1,2. \tag{9.30}$$

Consequently, the stationary distribution of $\mathbf{x}_1+\mathbf{x}_2$, that of $\max(\mathbf{x}_1,\ \mathbf{x}_2)$ and of $\min(\mathbf{x}_1,\ \mathbf{x}_2)$ are all independent of $\pi_i,\ i=1,2$, cf. [2].

10. Some expressions for probabilities and moments, $a_1 \neq a_2$

In this section we derive some expressions for several characteristics of the queue lengths.

We consider first the case

$$a_1\neq 1,\ a_2\neq 1,\ a_1\neq a_2,\ \frac{1}{a_1}+\frac{1}{a_2}>1, \tag{10.1}$$

since we have to discuss separately the case that one of the a_i is equal to one.

From (2.4) and appendix $\mathcal{D}$ we have

$$\begin{aligned}\mathrm{E}\{r^{\mathbf{x}_1}(\mathbf{x}_2>\mathbf{x}_1)\} &= \frac{1}{1-a_1r}\Omega_2(1)-\frac{1}{1-a_1r}\Phi_0(r)-\frac{a_1r}{1-a_1r}\frac{B(r)}{r-1},\\ \mathrm{E}\{r^{\mathbf{x}_2}(\mathbf{x}_2>\mathbf{x}_1)\} &= a_2r\pi_2\Phi_0(r)-a_2r\frac{B(r)}{r-1}.\end{aligned} \tag{10.2}$$

From which we obtain, cf. appendix (d.4),

$$\begin{aligned}&i.\quad \mathrm{E}\{(\mathbf{x}_2>\mathbf{x}_1)\}=\frac{1}{1-a_1}\Omega_2(1)-\frac{1}{1-a_1}\Phi_0(1)-\frac{a_1}{1-a_1}\frac{\mathrm{d}}{\mathrm{d}r}B(r)|_{r-1},\\ &ii.\quad \mathrm{E}\{(\mathbf{x}_2>\mathbf{x}_1)\}=a_2\pi_2\Phi_0(1)-a_2\frac{\mathrm{d}}{\mathrm{d}r}B(r)|_{r=1},\\ &iii.\quad \mathrm{E}\{(\mathbf{x}_1=\mathbf{x}_2)\}=\Phi_0(1),\\ &iv.\quad \mathrm{E}\{(\mathbf{x}_1>\mathbf{x}_2)\}=a_1\pi_1\Phi_0(1)+a_1\frac{\mathrm{d}}{\mathrm{d}r}B(r)|_{r=1};\end{aligned} \tag{10.3}$$

here (10.3)iv is obtained from (10.3)ii by interchanging a_1 and a_2 and by changing the sign of the term with $B(\cdot)$, cf.(2.4)i and (2.4)ii.

It follows from (10.3) ii,..., iv,

$$1 = \{1 + a_2\pi_2 + a_1\pi_1\}\Phi_0(1) + (a_1 - a_2)\frac{\mathrm{d}}{\mathrm{d}r}B(r)|_{r=1},$$

so that

$$\begin{aligned}
\Phi_0(1) &= \frac{1}{1 + a_1\pi_1 + a_2\pi_2}\{1 - (a_1 - a_2)\frac{\mathrm{d}}{\mathrm{d}r}B(r)|_{r=1}\},\\
\mathrm{E}\{(\mathbf{x}_2 > \mathbf{x}_1)\} &= \frac{a_2}{a_1 - a_2}[-1 + (1 + a_1)\Phi_0(1)],\\
\mathrm{E}\{(\mathbf{x}_1 > \mathbf{x}_2)\} &= \frac{a_1}{a_2 - a_1}[-1 + (1 + a_2)\Phi_0(1)].
\end{aligned} \tag{10.4}$$

From (d.6) and (d.4) we obtain for the present case, cf.(10.1),

$$\begin{aligned}
&i. \quad \mathrm{E}\{\mathbf{x}_2(\mathbf{x}_2 > \mathbf{x}_1)\} = a_2\pi_2\Phi_0(1) + a_2\pi_2\frac{\mathrm{d}}{\mathrm{d}r}\Phi_0(r)|_{r=1}\\
&\qquad - a_2\frac{\mathrm{d}}{\mathrm{d}r}B(r)|_{r=1} - \frac{1}{2}a_2\frac{\mathrm{d}^2}{\mathrm{d}r^2}B(r)|_{r=1},\\
&ii. \quad \mathrm{E}\{\mathbf{x}_1(\mathbf{x}_1 > \mathbf{x}_2)\} = a_1\pi_1\Phi_0(1) + a_1\pi_1\frac{\mathrm{d}}{\mathrm{d}r}\Phi_0(r)|_{r=1}\\
&\qquad + a_1\frac{\mathrm{d}}{\mathrm{d}r}B(r)|_{r=1} + \frac{1}{2}a_1\frac{\mathrm{d}^2}{\mathrm{d}r^2}B(r)|_{r=1},\\
&iii. \quad \mathrm{E}\{\mathbf{x}_1(\mathbf{x}_1 = \mathbf{x}_2)\} = \frac{\mathrm{d}}{\mathrm{d}r}\Phi_0(r)|_{r=1},\\
&iv. \quad \mathrm{E}\{\mathbf{x}_1(\mathbf{x}_2 > \mathbf{x}_1)\} = \frac{a_1}{(1 - a_1)^2}\{\Omega_2(1) - \Phi_0(1) - \frac{\mathrm{d}}{\mathrm{d}r}B(r)|_{r=1}\}\\
&\qquad - \frac{1}{1 - a_1}\frac{\mathrm{d}}{\mathrm{d}r}\Phi_0(r)|_{r=1} - \frac{1}{2}\frac{a_1}{1 - a_1}\frac{\mathrm{d}^2}{\mathrm{d}r^2}B(r)|_{r=1}.
\end{aligned} \tag{10.5}$$

By summing (10.5),ii, iii and iv the expression for $\mathrm{E}\{\mathbf{x}_1\}$ is obtained, that for $\mathrm{E}\{\mathbf{x}_2\}$ then follows by interchanging a_1 and a_2 and changing the signs of the terms containing $B(\cdot)$ in the expression for $\mathrm{E}\{\mathbf{x}_1\}$.
Next we consider the case

$$a_1 = 1,\ a_2 \neq 1,\ \frac{1}{a_1} + \frac{1}{a_2} > 1. \tag{10.6}$$

By noting that the relations (10.3)ii, iii, iv and the relations (10.4) have been all derived from (d.6) in which $1 - a_1$ does not occur it is seen that these relations also apply for the present case with $a_1 = 1$.
From (d.10) we obtain for the present case (10.6):

$$\begin{aligned}
&i. \quad \mathrm{E}\{\mathbf{x}_1(\mathbf{x}_2 > \mathbf{x}_1)\} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\Phi_0(r)|_{r=1} + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}B(r)|_{r=1},\\
&ii. \quad \mathrm{E}\{\mathbf{x}_1(\mathbf{x}_1 = \mathbf{x}_2)\} = \frac{\mathrm{d}}{\mathrm{d}r}\Phi_0(r)|_{r=1},\\
&iii. \quad \mathrm{E}\{\mathbf{x}_1(\mathbf{x}_1 > \mathbf{x}_2)\} = \pi_1\Phi_0(1) + \pi_1\frac{\mathrm{d}}{\mathrm{d}r}\Phi_0(r)|_{r=1} + \frac{\mathrm{d}}{\mathrm{d}r}B(r)|_{r=1} + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}B(r)|_{r=1},
\end{aligned} \tag{10.7}$$

here (10.7)iii follows from (10.5)ii with $a_1 = 1$, note that (10.5)ii has been derived from (d.6).

Appendix $\mathcal{A}$.

Let $(\boldsymbol{\xi}, \boldsymbol{\eta})$ be two stochastic variables with joint distribution given by

$$\Pr\{\boldsymbol{\xi}=2,\ \boldsymbol{\eta}=0\} = \frac{a_1 a_2}{b}, \tag{a.1}$$

$$\Pr\{\boldsymbol{\xi}=1,\ \boldsymbol{\eta}=0\} = \frac{a_1}{b},$$

$$\Pr\{\boldsymbol{\xi}=0,\ \boldsymbol{\eta}=2\} = \frac{a_2}{b}.$$

Hence we have from (2.8),

$$h_1(\rho,\tau)=0 \iff \mathrm{E}\{\tau^{\boldsymbol{\xi}}\rho^{\boldsymbol{\eta}}\} = \rho\tau. \tag{a.2}$$

Put $\tau = p\rho$ in (a.2), so that

$$h_1(\rho, p\rho) \iff p = \mathrm{E}\{p^{\boldsymbol{\xi}}\rho^{\boldsymbol{\xi}+\boldsymbol{\eta}-2}\}. \tag{a.3}$$

Note that (a.1) implies $\Pr\{\boldsymbol{\xi}+\boldsymbol{\eta}-2 \le 0\} = 1$, so that for $|\rho| \ge 1,\ \rho \ne 1$ and $|p| = 1$,

$$|\mathrm{E}\{p^{\boldsymbol{\xi}}\rho^{\boldsymbol{\xi}+\boldsymbol{\eta}-2}\}| \le \mathrm{E}\{|\rho|^{\boldsymbol{\xi}}+\boldsymbol{\eta}-2\} < 1 = |p|. \tag{a.4}$$

For fixed $\rho, |\rho| \ge 1$, the last term in (a.3) is regular for $|p| < 1$, continuous for $|p| \le 1$; it follows by applying Rouché's theorem that $h_1(\rho, p\rho)$ has for $|\rho| \ge 1,\ \rho \ne 1$ exactly one zero in $|p| < 1$, and so the other zero of $h_1(\rho, p\rho)$ lies in $|p| > 1$, cf. (a.4), note that $h_1(\rho, p\rho)$ is a quadratic function in p. This proves the final statement of Lemma 3.1, the proof of the other ones is similar. □

Appendix $\mathcal{B}$.

In this appendix we prove Lemma 4.1 and the relations (4.2). From (2.9) for the zero-tuple $|\rho| = 1,\ \tau = \tau^-(\rho)$, note $|\tau^-(\rho)| \le |\rho| = 1$, cf. (3.2), we have

$$-\Omega_2(\rho) = \frac{a_1\tau^-(\rho)}{\rho-\tau^-(\rho)} B(\tau^-(\rho)) + k_1(\rho,\tau^-(\rho))\Phi_0(\tau^-(\rho)), \quad |\rho| = 1. \tag{b.1}$$

The function $\Omega_2(\rho)$ is regular in $|\rho| < 1$, continuous in $|\rho| \le 1$. Consequently the righthand side of (b.1) can be continued analytically into $|\rho| < 1$. Consider this analytic continuation of the righthand side of (b.1) along a simple contour in $|\rho| \le 1$, starting at a point σ_0 with $|\sigma_0| = 1,\ \sigma_0 \ne 1$, and such that it intersects the interval $[\rho^-, \rho^+]$, cf. (3.4), only once at an interior point σ_1, say, with $\rho^- < \sigma_1 < \rho^+$. Because $\tau^-(\sigma_1) = \tau^+(\sigma_1)$, it is seen that the analytic continuation of the righthand side along this simple contour leads at its return to σ_0 to

$$-\Omega_2(\sigma_0) = \frac{a_1\tau^+(\sigma_0)}{\rho-\tau^+(\sigma_0)} B(\tau^+(\sigma_0)) + k_1(\sigma_0,\tau^+(\sigma_0))\Phi_0(\tau^+(\sigma_0)). \tag{b.2}$$

This relation holds for all $|\sigma_0| = 1,\ \sigma_0 \ne 1$; and so by continuity it also holds for $\sigma_0 = 1$.

Hence we obtain the following set of relations of which the last three are motivated by analogous arguments as those used in deriving the first one:

$$\Omega_2(\rho) + \frac{a_1\tau^{\pm}(\rho)}{\rho-\tau^{\pm}(\rho)} B(\tau^{\pm}(\rho)) + k_1(\rho,\ \tau^{\pm}(\rho))\Phi_0(\tau^{\pm}(\rho)) = 0, \quad |\rho| = 1, \tag{b.3}$$

$$\Omega_1(r) - \frac{a_2 t^{\pm}(r)}{r - t^{\pm}(r)} B(t^{\pm}(r)) + k_2(r,\ t^{\pm}(r))\Phi_0(t^{\pm}(r)) = 0, \quad |r| = 1, \tag{b.4}$$

$$\Omega_2(\rho^{\pm}(\tau)) + \frac{a_1 \tau}{\rho^{\pm}(\tau) - \tau} B(\tau) + k_1(\rho^{\pm}(\tau), \tau)\Phi_0(\tau) = 0, \quad |\tau| = 1, \tag{b.5}$$

$$\Omega_1(r^{\pm}(t)) - \frac{a_2 t}{r^{\pm}(t) - t} B(t) + k_2(r^{\pm}(t), t)\Phi_0(t) = 0, \quad |t| = 1, \tag{b.6}$$

with

$$\begin{array}{llll} (\rho, \tau^{\pm}(\rho)) & and \quad (\rho^{\pm}(\tau), \tau) & zero-tuples\ of & h_1(\rho, \tau), \\ (r, t^{\pm}(r)) & ,, \quad (r^{\pm}(t), t) & ,, \quad ,, & h_2(r, t). \end{array}$$

Because, cf. (3.2), $|\tau^+(\rho)| > 1$ for $|\rho| = 1,\ \rho \neq 1$, it is seen from (b.3) that a domain in $|\tau| > 1$ exists where $\Phi_0(\tau)$, and similarly, $B(\tau)$ is regular, note that $k_1(\rho, \tau)$ is regular in $|\rho| \geq 1,\ |\tau| \geq 1$, also $\tau^+(\rho)$ is regular in $|\rho| \geq 1$, with $|\rho|$ and $|\tau|$ finite. Further it is seen that in $|\rho| \geq 1$ domains exist where $\Omega_2(\rho)$ and $\Omega_1(\rho)$ are regular. Next take τ in the domain where $\Phi_0(\tau)$ and $B(t)$ are regular, i.e. in the domain defined by $\{\tau : \tau = \tau^+(\rho),\ |\rho| = 1\}$. For such τ's it is seen from (b.5) that here $\Omega_2(\rho^+(\tau))$ is regular. Since $|\rho^+(\tau)| > |\tau|,\ \tau \neq 1$ it follows that the domain outside $|\rho| = 1$ where $\Omega_2(\rho)$ is regular can be again extended. So by repeatedly using the relations (b.3) and (b.5) the domains of regularity of $\Phi_0(\tau)$, $B_0(\tau)$ and $\Omega_2(\rho)$ can be recursively extended, analogously for $\Omega_1(r)$. Because $|\rho^+(\tau)| > |\tau|$ in $|\tau| > 1$, $|\tau^+(\rho)| > |\rho|$ in $|\rho| > 1$, it follows that the domain in $|\tau| > 1$ where $\Phi_0(\tau)$ is regular is unbounded, similarly for $B(\tau)$, and analogously for $\Omega_2(\rho)$ and $\Omega_1(r)$.

The singularities of $\Phi_0(\tau)$ in $|\tau| > 1$ can be only poles, because $k_1(\rho, \tau^{\pm}(\rho))$ and $k_2(\rho^{\pm}(\tau), \tau)$ are regular in $|\rho| > 1$ and $|\tau| > 1$, respectively, note (3.4), and similarly for the other coefficients in (b.3),..., (b.6). Further $\Phi_0(\tau)$ has at least one pole in $\{\tau : 1 < |\tau| \leq \infty\}$, because if $\Phi_0(\tau)$ would be regular here, then, since it is also regular for $|\tau| < 1$, cf (2.10), it is necessarily a constant, as Liouville's theorem implies. Analogously, for $B(\tau)$, $\Omega_2(\rho)$ and $\Omega_1(r)$. Consequently, Lemma 4.1 has been proved. □

From the analytic continuations discussed above it is seen that the relations (b.3), ... , (b.6) hold for all those τ, t, ρ and r where the functions in (b.3), ..., (b.6) are finite. Consequently, it is seen that the validity of the relations (4.2), ... , (4.5) has been established.

Appendix $\mathcal{C}$

The integer $\tilde{M} \geq 0$ has been defined in (8.6). Numerical results indicate that $\tilde{M}$ is always larger than zero. A proof of $\tilde{M} > 0$ seems to be rather lengthy and intricate. Below we discuss some cases for which the proof is fairly simple.

The case

$$a_1 = a_2 = a < 2, \quad \pi_1 = \pi_2 = \frac{1}{2}. \tag{c.1}$$

From (c.1), (3.17) and (7.7)iv it follows that

$$\lambda_1 \mu_1 = \frac{R_1^- - 1}{R_1^+ - 1} \frac{R_1^- + R_1^+}{2R_1^-} = \frac{R_1^- - 1}{R_1^+ - 1} \frac{b/a}{2R_1^-}. \tag{c.2}$$

From (3.16), (3.20), (c.1) and (c.2) it results

$$-2\lambda_1\mu_2 = (2+a)(R_1^+ - 1)(a - R_1^-) = \frac{2+a}{2-a-(1-a)R_1^+}, \tag{c.3}$$

$$R_1^+ = \frac{1}{2}(2 + a + \sqrt{4+a^2}).$$

From (c.3) it is not difficult to verify, since $\lambda_1\mu_1 = \lambda_2\mu_2$ for the present case that

$$|\lambda_1\mu_1 + \lambda_2\mu_2| > 1 \text{ for } 0 < a < 2,$$

so that for the present case $\tilde{M} > 0$.

Because $\lambda_1\mu_1$ is a continuous function in each of the parameters $a_1,\ a_2, \pi_1 = 1 - \pi_2$ it follows from the results so far obtained that $\tilde{M}$ is larger than zero for $|a_1 - a_2| < \epsilon_1, |\pi_1 - \frac{1}{2}| < \epsilon_2,\ 1/a_1 + 1/a_2 > 1$, with ϵ_1 and ϵ_2 sufficiently small.

Next we consider the case

$$a_1 > 1,\ \pi_1 = \pi_2 = \frac{1}{2}. \tag{c.4}$$

From (3.17), (7.7)iv and (c.4) we have

$$\begin{aligned} \lambda_1\mu_1 &= \frac{(R_1^- - 1)(R_2^- + R_1^+)}{(R_1^+ - 1)(R_2^- + R_1^-)} = \frac{R_1^-R_2^- + \frac{1}{a_1} - R_2^- - R_1^+}{R_1^+R_2^- + \frac{1}{a_1} - R_2^- - R_1^-} = \\ &= -1 + \frac{R_2^-(R_1^+ + R_1^-) + \frac{2}{a_1} - 2R_2^- - (R_1^+ + R_1^-)}{R_1^+R_2^- + \frac{1}{a_1} - R_2^- - R_1^-} \\ &= -1 + \frac{(R_2^- - 1)(\frac{b}{a_2} - 2) + 2(\frac{1}{a_1} - 1)}{R_1^+R_2^- + \frac{1}{a_1} - 2R_2^-}. \end{aligned} \tag{c.5}$$

Because $R_1^+ - 1 > 0$, the denominators in (c.5) are positive and the numerator in the last term of (c.5) is not positive for $a_1 \geq 1$, note

$$\frac{b}{a_2} - 2 = -1 + a_1 + \frac{a_1}{a_2} > 0 \quad \text{for} \quad a_1 \geq 1,$$

it follows that $\lambda_1\mu_1 < -1$, hence from $\lambda_2\mu_2 < 0$ we obtain $|\lambda_1\mu_1 + \lambda_2\mu_2| > 1$, i.e. $\tilde{M} > 0$.
Analogously $\tilde{M} > 0$ for $a_2 \geq 1,\ \pi_1 = \pi_2 = 1/2$. So far for the present case.
Finally consider the case $\pi_1 = \pi_2 = \frac{1}{2}$ and $a_2 \downarrow 0$. It is readily verified that $R_2^- \to 0$ for $a_2 \downarrow 0$ and so it is seen from (c.5) that for a_2 sufficiently small the numerator in the last term of (c.5) will be negative and it follows again that $\lambda_1\mu_1 < -1$ and so $\tilde{M} > 0$ since $\lambda_2\mu_2 < 0$.

Appendix $\mathcal{D}$

From (2.4) it is seen that for $|r| \leq 1$,

$$\mathrm{E}\{r^{\mathbf{x}_1}(\mathbf{x}_2 > \mathbf{x}_1)\}\frac{1-a_1r}{a_1r}(1-r) - \frac{1-r}{a_1r}\Omega_2(1) + \frac{1-r}{a_1r}\Phi_0(r) - B(r) = 0, \tag{d.1}$$

$$\mathrm{E}\{r^{\mathbf{x}_2}(\mathbf{x}_2 > \mathbf{x}_1)\}\frac{1-r}{a_2r} - (1-r)\pi_2\Phi_0(r) - B(r) = 0. \tag{d.2}$$

So that by letting $r \to 1$ it follows, cf. (2.10), that

$$B(1) = 0. \tag{d.3}$$

Because the in absolute value smallest pole of $B(\tau)$ as well as of $\Phi_0(\tau)$ is $\tau = \tau_0^{(0)} = T > 1$, cf. Lemma 5.1, (8.2) and (8.7), it follows that there exists a neighbourhood of $\tau = 1$ where $B(\cdot)$ and $\Phi_0(\cdot)$ are both regular, so that all derivatives of $B(\cdot)$ and $\Phi_0(\cdot)$ at $r = 1$ are finite, so we may write, cf. d.3: for $r \sim 1$,

$$B(r) = (r-1)\frac{\mathrm{d}}{\mathrm{d}r}B(r)|_{r=1} + \frac{1}{2}(r-1)^2\frac{\mathrm{d}}{\mathrm{d}r^2}B(r)|_{r=1} + \frac{1}{6}(r-1)^3\frac{\mathrm{d}^3}{\mathrm{d}r^3}B(r)|_{r=1} + \mathrm{O}((r-1)^4), \tag{d.4}$$

$$\Phi_0(r) = \Phi_0(1) + (r-1)\frac{\mathrm{d}}{\mathrm{d}r}\Phi_0(r)|_{r=1} + \frac{1}{2}(r-1)^2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\Phi_0(r)|_{r=1} + \mathrm{O}((r-1)^3).$$

From (d.1) we have: for $r \sim 1$,

$$\mathrm{E}\{r^{\mathbf{x}_1}(\mathbf{x}_2 > \mathbf{x}_1)\} = \frac{1}{1-a_1 r}\Omega_2(1) - \frac{1}{1-a_1 r}[\Phi_0(1) + (r-1)\frac{\mathrm{d}}{\mathrm{d}r}\Phi_0(r)|_{r=1}$$

$$+\frac{1}{2}(r-1)^2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\Phi_0(r)|_{r=1} + \frac{1}{6}(r-1)^3\frac{\mathrm{d}^3}{\mathrm{d}r^3}\Phi_0(r)|_{r=1} + \mathrm{O}\left((r-1)^4\right)]$$

$$-\frac{a_1 r}{1-a_1 r}[\frac{\mathrm{d}}{\mathrm{d}r}B(r)|_{r=1} + \frac{1}{2}(r-1)\frac{\mathrm{d}^2}{\mathrm{d}r^2}B(r)|_{r=1} + \frac{1}{6}(r-1)^2\frac{\mathrm{d}^3}{\mathrm{d}r^3}B(r)|_{r=1} + \mathrm{O}\left((r-1)^3\right)], \tag{d.5}$$

and from (d.2),

$$\mathrm{E}\{r^{\mathbf{x}_2}(\mathbf{x}_2 > \mathbf{x}_1)\} = a_2\pi_2 r\Phi_0(r) - a_2 r\frac{B(r)}{r-1}. \tag{d.6}$$

For, cf. Theorem 8.1,

$$a_1 = 1,\ a_2 \neq 1,\ \frac{1}{a_1} + \frac{1}{a_2} > 1, \tag{d.7}$$

we have from (d.5): for $r \sim 1$,

$$\mathrm{E}\{r^{\mathbf{x}_1}(\mathbf{x}_2 > \mathbf{x}_1)\} = \frac{1}{1-r}[\Omega_2(1) - \Phi_0(1) - (r-1)\frac{\mathrm{d}}{\mathrm{d}r}\Phi_0(r)|_{r=1} - \frac{1}{2}(r-1)^2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\Phi_0(r)|_{r=1} \tag{d.8}$$

$$-\frac{1}{6}(r-1)^3\frac{\mathrm{d}^3}{\mathrm{d}r^3}\Phi_0(r)|_{r=1} + \mathrm{O}((1-r)^4)]$$

$$-\frac{r}{1-r}[\frac{\mathrm{d}}{\mathrm{d}r}B(r)|_{r=1} + \frac{1}{2}(r-1)\frac{\mathrm{d}^2}{\mathrm{d}r^2}B(r)|_{r=1} + \frac{1}{6}(r-1)^2\frac{\mathrm{d}^3}{\mathrm{d}r^3}B(r)|_{r=1} + \mathrm{O}((1-r)^3)].$$

From (d.7) and (d.8) we have:

$$a_1 = 1,\ a_2 \neq 1,\ \frac{1}{a_1} + \frac{1}{a_2} > 1 \implies \Omega_2(1) - \Phi_0(1) - \frac{\mathrm{d}}{\mathrm{d}r}B(r)|_{r=1} = 0, \tag{d.9}$$

because the lefthand side is bounded by 1 for $r = 1$. Hence if (d.7) applies then: for $r \sim 1$,

$$\mathrm{E}\{r^{\mathbf{x}_1}(\mathbf{x}_2 > \mathbf{x}_1)\} = \frac{\mathrm{d}}{\mathrm{d}r}\Phi_0(r)|_{r=1} + \frac{1}{2}(r-1)\frac{\mathrm{d}^2}{\mathrm{d}r^2}\Phi_0(r)|_{r=1} + \frac{\mathrm{d}}{\mathrm{d}r}B(r)|_{r=1}$$
$$+\frac{1}{2}r\frac{\mathrm{d}^2}{\mathrm{d}r^2}B(r)|_{r=1} + \frac{1}{6}r(r-1)\frac{\mathrm{d}^3}{\mathrm{d}r^3}B(r)|_{r=1} + \mathrm{O}\left((1-r)^2\right). \tag{d.10}$$

Appendix $\mathcal{E}$

In Section 7 we have defined λ_1 and λ_2. For the determination of $\tilde{M}$, cf. (8.6), we need some more detailed information. It is obtained in this appendix, further some relations between λ_1 and a_1, a_2 are deduced. These relations are helpful in the numerical evaluation of the queueing characteristics.

With

$$A := \frac{b}{2a_1a_2} = \frac{1}{2}(\frac{1}{a_1} + \frac{1}{a_2} + 1), \tag{e.1}$$

so that since for $i = 1, 2$,

$$a_iA^2 > 1, \tag{e.2}$$

put

$$\hat{\delta}_i = a_iA^2 > 1. \tag{e.3}$$

Hence from (3.16), (7.7) and (7.8): for $i = 1, 2$,

$$0 < \lambda_i = \sqrt{\hat{\delta}_i} - \sqrt{\hat{\delta}_i - 1} < 1. \tag{e.4}$$

It follows that

$$\lambda_i - 2\hat{\delta}_i\lambda_i^{1/2} + 1 = 0.$$

Consider the hyperbola $y^2 - 2yx + 1 = 0$. It has its center at the point (0, 0) and asymptotes $y = 0$, $y = 2x$ the point (1, 1) lies on it. It is readily seen that for $i = 1, 2$,

$$\hat{\delta}_i > 0 \implies 0 < \lambda_i < 1, \tag{e.5}$$

and λ_i decreases monotonically from 1 to 0 for $\hat{\delta} : 1 \to \infty$.

From (e.5) it is seen that a positive integer n may exist such that

$$\lambda_1^n + \lambda_2^n > 1.$$

However, for n sufficient large this inequality cannot hold for any $\hat{\delta}_i > 1$, $i = 1, 2$.

To derive the relation between a_i and $\hat{\delta}_i$ note that from (e.1) and (e.3) we have

$$a_1^2\left(\frac{\hat{\delta}_2}{\hat{\delta}_1 + \hat{\delta}_2}\right)^2 + 2(1-2\epsilon)a_1\frac{\hat{\delta}_2}{\hat{\delta}_1 + \hat{\delta}_2} + 1 = 0, \tag{e.6}$$

with

$$\epsilon := \frac{\hat{\delta}_1\hat{\delta}_2}{\hat{\delta}_1 + \hat{\delta}_2} - 1 > 0. \tag{e.7}$$

From (e.6) and (e.7) it follows readily that: for $i = 1, 2$,

$$a_i^{\pm} = \frac{\hat{\delta}_i}{1+\epsilon}\{1 + 2\epsilon \pm 2\sqrt{\epsilon(1+\epsilon)}\}, \tag{e.8}$$

so that

$$\frac{1}{a_2^{\pm}} + \frac{1}{a_2^{\pm}} = [1 + 2\epsilon \pm 2\sqrt{\epsilon(\epsilon+1)}]^{-1}.$$

From which it follows that

$$\frac{1}{a_1^{+}} + \frac{1}{a_2^{+}} < 1, \quad \frac{1}{a_1^{-}} + \frac{1}{a_2^{-}} > 1. \tag{e.9}$$

Hence if, cf. (2.6) and remark (2.1),

$$\frac{1}{a_1} + \frac{1}{a_2} > 1,$$

then the relation between a_i and $\hat{\delta}_i$ is given by

$$a_i = \frac{\hat{\delta}_i}{1+\epsilon}\{1 + 2\epsilon - 2\sqrt{\epsilon(\epsilon+1)}\}, \quad i = 1, 2, \tag{e.10}$$

$$\frac{1}{a_1} + \frac{1}{a_2} = 1 + 2\epsilon + 2\sqrt{\epsilon(\epsilon+1)}.$$

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